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Supersymmetric $AdS_2 \times \Sigma_2$ solutions from tri-sasakian truncation

Parinya Karndumri[a]

String Theory and Supergravity Group, Department of Physics, Faculty of Science, Chulalongkorn University, 254 Phayathai Road, Pathumwan, Bangkok 10330, Thailand

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Abstract A class of $AdS_2 \times \Sigma_2$, with Σ_2 being a two-sphere or a hyperbolic space, solutions within four-dimensional $N = 4$ gauged supergravity coupled to three-vector multiplets with dyonic gauging is identified. The gauged supergravity has a non-semisimple $SO(3) \ltimes (\mathbf{T}^3, \hat{\mathbf{T}}^3)$ gauge group and can be obtained from a consistent truncation of 11-dimensional supergravity on a tri-sasakian manifold. The maximally symmetric vacua contain AdS_4 geometries with $N = 1, 3$ supersymmetry corresponding to $N = 1$ and $N = 3$ superconformal field theories (SCFTs) in three dimensions. We find supersymmetric solutions of the form $AdS_2 \times \Sigma_2$ preserving two supercharges. These solutions describe twisted compactifications of the dual $N = 1$ and $N = 3$ SCFTs and should arise as near horizon geometries of dyonic black holes in asymptotically AdS_4 space-time. Most solutions have hyperbolic horizons although some of them exhibit spherical horizons. These provide a new class of $AdS_2 \times \Sigma_2$ geometries with known M-theory origin.

1 Introduction

Apart from giving deep insight to strongly coupled gauge theories and string/M-theory compactifications in various dimensions, the AdS/CFT correspondence has been recently used to explain the entropy of asymptotically AdS_4 black holes [1–3]. In this context, the black hole entropy is computed using topologically twisted index of three-dimensional superconformal field theories (SCFTs) compactified on a Riemann surface Σ_2 [4–8]. In the dual gravity solutions, the black holes interpolate between the asymptotically AdS_4 and the near horizon $AdS_2 \times \Sigma_2$ geometries. These can be interpreted as RG flows from three-dimensional SCFTs in the form of Chern–Simons–Matter (CSM) theories possibly with flavor matters to superconformal quantum mechanics corresponding to the AdS_2 geometry.

Along this line of research, BPS black hole solutions in four-dimensional gauged supergravity, in particular near horizon geometries, with known higher-dimensional origins are very useful. Most of the solutions have been studied within $N = 2$ gauged supergravities [9–15], for recent results; see [16,17]. Many of these solutions can be uplifted to string/M-theory since these $N = 2$ gauged supergravities can be obtained either from truncations of the maximal $N = 8$ gauged supergravity, whose higher-dimensional origin is known, or direct truncations of M-theory on Sasaki–Einstein manifolds.

In this work, we give evidence for a new class of BPS black hole solutions in the half-maximal $N = 4$ gauged supergravity with known higher-dimensional origin by finding a number of new $AdS_2 \times \Sigma_2$ solutions. This gauged supergravity has $SO(3) \ltimes (\mathbf{T}^3, \hat{\mathbf{T}}^3)$ gauge group and can be obtained from a compactification of M-theory on a tri-sasakian manifold [18]. Holographic RG flows and supersymmetric Janus solutions, describing $(1 + 1)$-dimensional interfaces in the dual SCFTs have recently appeared in [19]. In the present paper, we will look for supersymmetric solutions of the form $AdS_2 \times \Sigma_2$ within this tri-sasakian compactification.

Apart from giving this type of solutions in gauged supergravity with more supersymmetry, to the best of the authors' knowledge, the results are the first example of $AdS_2 \times \Sigma_2$ solutions from the truncation of M-theory on a tri-sasakian manifold. The truncation given in [18] gives a reduction ansatz for 11-dimensional supergravity on a generic tri-sasakian manifold including massive Kaluza–Klein modes. Among this type of manifolds, N^{010} with isometry $SU(2) \times SU(3)$ is of particular interest. In this case, there is a non-trivial two-form giving rise to an extra vector multiplet; see [20,21] for the Kaluza–Klein spectrum of $AdS_4 \times N^{010}$. This background, discovered long ago in [22], preserves $N = 3$ out of the original $N = 4$ supersymmetry. There is another supersymmetric AdS_4 vacuum with $SO(3)$ symmetry and $N = 1$ supersymmetry, the one broken by $AdS_4 \times N^{010}$.

[a] e-mail: parinya.ka@hotmail.com

This vacuum corresponds to $AdS_4 \times \tilde{N}^{010}$ geometry, with $\tilde{N}^{010}$ being a squashed version of N^{010}.

Not much is known about the dual $N = 1$ SCFT, but the dual $N = 3$ SCFT has been proposed in a number of previous works [23,24]; see also [25,26]. This SCFT takes the form of a CSM theory with $SU(N) \times SU(N)$ gauge group. It is a theory of interacting three hypermultiplets transforming in a triplet of the $SU(3)$ flavor symmetry, and each hypermultiplet transforms as a bifundamental under the $SU(N) \times SU(N)$ gauge group and as a doublet of the $SU(2)_R \sim SO(3)_R$ R-symmetry. There are also a number of previous works giving holographic studies of this theory both in 11-dimensional context and in the effective $N = 3$ and $N = 4$ gauged supergravities [19,27–31]. Solutions given in these works are holographic RG flows, Janus solutions and supersymmetric $AdS_2 \times \Sigma_2$ solutions with magnetic charges.

In this work, we consider $N = 4$ gauged supergravity constructed in the embedding tensor formalism in [32]. This construction is the most general supersymmetric gaugins of $N = 4$ supergravity in which both the "electric" vector fields, appearing in the ungauged Lagrangian, and their magnetic duals can participate. Therefore, magnetic and dyonic gaugings are allowed in this formulation. Furthermore, this formulation contains the "purely electric" gauged $N = 4$ supergravity constructed long time ago in [33] and the non-trivial $SL(2, \mathbb{R})$ phases of [34,35] as special cases. We will look for supersymmetric $AdS_2 \times \Sigma_2$ solutions in the $N = 4$ gauged supergravity with a dyonic gauging of the non-semisimple group $SO(3) \ltimes (\mathbf{T}^3, \hat{\mathbf{T}}^3)$. The solutions are required to preserve $SO(2) \subset SO(3)_R$, so only a particular combination of vector fields corresponding to this $SO(2)$ residual gauge symmetry appears in the gauge covariant derivative. The strategy is essentially similar to the wrapped brane solutions of [36], implementing the twist by canceling the spin connections on Σ_2 by the $SO(2)$ gauge connection.

These $AdS_2 \times \Sigma_2$ solutions should appear as near horizon geometries of supersymmetric black holes in asymptotically AdS_4 space-time. Since the $N = 4$ gauged supergravity admits two supersymmetric AdS_4 vacua with unbroken $SO(3)_R$ symmetry and $N = 1, 3$ supersymmetries, the $AdS_2 \times \Sigma_2$ solutions should be RG fixed points in one dimension of the dual $N = 1, 3$ SCFTs. Although the structure of the dual $N = 1$ SCFT is presently not clear, we expect that there should be RG flows between these twisted $N = 1, 3$ SCFTs on Σ_2 to one-dimensional superconformal quantum mechanics dual to the $AdS_2 \times \Sigma_2$ solutions. In this sense, the entropy of these black holes would possibly be computed from the topologically twisted indices of the dual $N = 1, 3$ SCFTs. Furthermore, these solutions should provide a new class of AdS_2 geometries within M-theory.

The paper is organized as follows. In Sect. 2, we review $N = 4$ gauged supergravity coupled to vector multiplets and relevant formulas for uplifting the resulting solutions to 11 dimensions. The analysis of BPS equations for $SO(2) \subset SO(3)_R$ singlet scalars and Yang–Mills equations, for static black hole ansatze consistent with the symmetry of Σ_2, will be carried out in Sect. 3. In Sect. 4, we will explicitly give $AdS_2 \times \Sigma_2$ solutions to the BPS flow equations. We separately consider the $N = 1$ and $N = 3$ cases and end the section with the uplift formulas for embedding the solutions in 11 dimensions. We finally give some conclusions and comments on the results in Sect. 5. In the appendix, we collect the convention regarding 't Hooft matrices and give the explicit form of the Yang–Mills and BPS equations.

2 $N = 4$ gauged supergravity with dyonic gauging

In this section, we review $N = 4$ gauged supergravity in the embedding tensor formalism following [32]. We mainly focus on the bosonic Lagrangian and supersymmetry transformations of fermions which provide basic ingredients for finding supersymmetric solutions. Since the gauged supergravity under consideration is known to arise from a trisasakian truncation of 11-dimensional supergravity, we will also give relevant formulas which are useful to uplift four-dimensional solutions to 11 dimensions. The full detail of this truncation can be found in [18].

2.1 $N = 4$ gauged supergravity coupled to vector multiplets

In the half-maximal $N = 4$ supergravity in four dimensions, the supergravity multiplet consists of the graviton $e^{\hat{\mu}}_\mu$, four gravitini ψ^i_μ, six vectors A^m_μ, four spin-$\frac{1}{2}$ fields χ^i and one complex scalar τ. The complex scalar can be parametrized by the $SL(2, \mathbb{R})/SO(2)$ coset. The supergravity multiplet can couple to an arbitrary number n of vector multiplets containing a vector field A_μ, four gaugini λ^i and six scalars ϕ^m. The scalar fields can be parametrized by the $SO(6, n)/SO(6) \times SO(n)$ coset.

Space-time and tangent space indices are denoted, respectively, by $\mu, \nu, \ldots = 0, 1, 2, 3$ and $\hat{\mu}, \hat{\nu}, \ldots = 0, 1, 2, 3$. Indices $m, n = 1, \ldots, 6$ and $i, j = 1, 2, 3, 4$, respectively, describe the vector and spinor representations of the $SO(6)_R \sim SU(4)_R$ R-symmetry or equivalently a second-rank anti-symmetric tensor and fundamental representations of $SU(4)_R$. The n vector multiplets are labeled by indices $a, b = 1, \ldots, n$. All the fields in the vector multiplets will accordingly carry an additional index in the form of $(A^a_\mu, \lambda^{ia}, \phi^{ma})$.

All fermionic fields and the supersymmetry parameters transform in the fundamental representation of $SU(4)_R$ R-symmetry with the chirality projections

$$\gamma_5 \psi^i_\mu = \psi^i_\mu, \quad \gamma_5 \chi^i = -\chi^i, \quad \gamma_5 \lambda^i = \lambda^i. \tag{1}$$

Similarly, for the fields transforming in the anti-fundamental representation of $SU(4)_R$, we have

$$\gamma_5\psi_{\mu i} = -\psi_{\mu i}, \quad \gamma_5\chi_i = \chi_i, \quad \gamma_5\lambda_i = -\lambda_i. \tag{2}$$

General gaugings of the matter-coupled $N = 4$ supergravity can be efficiently described by the embedding tensor Θ, which encodes the information as regards the embedding of any gauge group G_0 in the global or duality symmetry $SL(2,\mathbb{R}) \times SO(6,n)$. There are two components of the embedding tensor $\xi^{\alpha M}$ and $f_{\alpha MNP}$ with $\alpha = (+,-)$ and $M, N = (m, a) = 1, \ldots, n+6$ denoting fundamental representations of $SL(2,\mathbb{R})$ and $SO(6,n)$, respectively. The electric vector fields $A^{M+} = (A^m_\mu, A^a_\mu)$, appearing in the ungauged Lagrangian, together with their magnetic dual A^{M-} form a doublet under $SL(2,\mathbb{R})$. These are denoted collectively by $A^{M\alpha}$. In general, a subgroup of both $SL(2,\mathbb{R})$ and $SO(6,n)$ can be gauged, and the magnetic vector fields can also participate in the gauging. However, in this paper, we only consider gaugings with only $f_{\alpha MNP}$ non-vanishing. We then set $\xi^{\alpha M}$ to zero from now on. This also considerably simplifies many formulas given below.

The full covariant derivative can be written as

$$D_\mu = \nabla_\mu - gA^{M\alpha}_\mu f_{\alpha M}{}^{NP} t_{NP} \tag{3}$$

where ∇_μ is the space-time covariant derivative including the spin connections. t_{MN} are $SO(6,n)$ generators which can be chosen as

$$(t_{MN})_P{}^Q = 2\delta^Q_{[M}\eta_{N]P}, \tag{4}$$

with η_{MN} being the $SO(6,n)$ invariant tensor. The gauge coupling constant g can be absorbed in the embedding tensor Θ. The original gauging considered in [33] only involves electric vector fields and is called electric gauging. In this case, only f_{+MNP} are non-vanishing. In the following discussions, we will consider dyonic gauging involving both electric and magnetic vector fields. In this case, both A^{M+} and A^{M-} enter the Lagrangian, and $f_{\alpha MNP}$ with $\alpha = \pm$ are non-vanishing. Consistency requires the presence of two-form fields when magnetic vector fields are included. In the present case with $\xi^{\alpha M} = 0$, these two-forms transform as an anti-symmetric tensor under $SO(6,n)$ and will be denoted by $B^{MN}_{\mu\nu} = B^{[MN]}_{\mu\nu}$. The two-forms modify the gauge field strengths to

$$\mathcal{H}^{M\pm} = dA^{M\pm} - \frac{1}{2}\eta^{MQ} f_{\alpha QNP} A^{N\alpha} \wedge A^{P\pm} \pm \frac{1}{2}\eta^{MQ} f_{\mp QNP} B^{NP}. \tag{5}$$

Note that for non-vanishing f_{-MNP} the field strengths of electric vectors $\mathcal{H}^{M+}$ have a contribution from the two-form fields.

Before moving to the Lagrangian, we explicitly give the parametrization of the scalar coset manifold $SL(2,\mathbb{R})/SO(2) \times SO(6,n)/SO(6) \times SO(n)$. The first factor can be described by a coset representative

$$\mathcal{V}_\alpha = \frac{1}{\sqrt{\mathrm{Im}\tau}} \begin{pmatrix} \tau \\ 1 \end{pmatrix} \tag{6}$$

or equivalently by a symmetric matrix

$$M_{\alpha\beta} = \mathrm{Re}(\mathcal{V}_\alpha \mathcal{V}^*_\beta) = \frac{1}{\mathrm{Im}\tau} \begin{pmatrix} |\tau|^2 & \mathrm{Re}\tau \\ \mathrm{Re}\tau & 1 \end{pmatrix}. \tag{7}$$

It should also be noted that $\mathrm{Im}(\mathcal{V}_\alpha \mathcal{V}^*_\beta) = \epsilon_{\alpha\beta}$. The complex scalar τ can also be written in terms of the dilaton ϕ and the axion χ as

$$\tau = \chi + ie^\phi. \tag{8}$$

For the $SO(6,n)/SO(6)\times SO(n)$ factor, we can introduce the coset representative $\mathcal{V}_M{}^A$ transforming by left and right multiplications under $SO(6,n)$ and $SO(6)\times SO(n)$, respectively. The $SO(6)\times SO(n)$ index will be split as $A = (m, a)$ according to which the coset representative can be written as $\mathcal{V}_M{}^A = (\mathcal{V}_M{}^m, \mathcal{V}_M{}^a)$. As an element of $SO(6,n)$, the matrix $\mathcal{V}_M{}^A$ also satisfies the relation

$$\eta_{MN} = -\mathcal{V}_M{}^m \mathcal{V}_N{}^m + \mathcal{V}_M{}^a \mathcal{V}_N{}^a. \tag{9}$$

As in the $SL(2,\mathbb{R})/SO(2)$ factor, the $SO(6,n)/SO(6) \times SO(n)$ coset can also be parametrized in term of a symmetric matrix defined by

$$M_{MN} = \mathcal{V}_M{}^m \mathcal{V}_N{}^m + \mathcal{V}_M{}^a \mathcal{V}_N{}^a. \tag{10}$$

The bosonic Lagrangian of the $N = 4$ gauged supergravity is given by

$$\begin{aligned} e^{-1}\mathcal{L} &= \frac{1}{2}R + \frac{1}{16}\mathcal{D}_\mu M_{MN}\mathcal{D}^\mu M^{MN} \\ &\quad - \frac{1}{4(\mathrm{Im}\tau)^2}\partial_\mu\tau\partial^\mu\tau^* - V \\ &\quad - \frac{1}{4}\mathrm{Im}\,\tau M_{MN}\mathcal{H}^{M+}_{\mu\nu}\mathcal{H}^{N+\mu\nu} \\ &\quad - \frac{1}{8}\mathrm{Re}\,\tau e^{-1}\epsilon^{\mu\nu\rho\sigma}\eta_{MN}\mathcal{H}^{M+}_{\mu\nu}\mathcal{H}^{N+}_{\rho\sigma} \\ &\quad - \frac{1}{2}e^{-1}\epsilon^{\mu\nu\rho\sigma}\Big[f_{-MNP}A^{M-}_\mu A^{N+}_\nu \partial_\rho A^{P-}_\sigma \\ &\quad + \frac{1}{4}f_{\alpha MNR}f_{\beta PQS}\eta^{RS}A^{M\alpha}_\mu A^{N+}_\nu A^{P\beta}_\rho A^{Q-}_\sigma \\ &\quad - \frac{1}{4}f_{-MNP}B^{NP}_{\mu\nu} \\ &\quad \times \left(2\partial_\rho A^{M-}_\sigma - \frac{1}{2}\eta^{MS}f_{\alpha SQR}A^{Q\alpha}_\rho A^{R-}_\sigma\right) \\ &\quad - \frac{1}{16}f_{+MNR}f_{-PQS}\eta^{RS}B^{MN}_{\mu\nu}B^{PQ}_{\rho\sigma}\Big] \end{aligned} \tag{11}$$

where e is the vielbein determinant. The scalar potential is given by

$$V = \frac{g^2}{16}\Big[f_{\alpha MNP} f_{\beta QRS} M^{\alpha\beta} \times \Big[\frac{1}{3}M^{MQ}M^{NR}M^{PS} + \Big(\frac{2}{3}\eta^{MQ} - M^{MQ}\Big)\eta^{NR}\eta^{PS}\Big] - \frac{4}{9} f_{\alpha MNP} f_{\beta QRS}\epsilon^{\alpha\beta} M^{MNPQRS}\Big] \quad (12)$$

where M^{MN} is the inverse of M_{MN}, and M^{MNPQRS} is defined by

$$M_{MNPQRS} = \epsilon_{mnpqrs}\mathcal{V}_M{}^m\mathcal{V}_N{}^n\mathcal{V}_P{}^p\mathcal{V}_Q{}^q\mathcal{V}_R{}^r\mathcal{V}_S{}^s \quad (13)$$

with indices raised by η^{MN}. The covariant derivative of M_{MN} is defined by

$$\mathcal{D}M_{MN} = \mathrm{d}M_{MN} + 2A^{P\alpha}\eta^{QR} f_{\alpha QP(M}M_{N)R}. \quad (14)$$

It should be pointed out here that the magnetic vectors and the two-forms do not have kinetic terms. They are auxiliary fields whose field equations give rise to the duality relation between two-forms and scalars and the electric-magnetic duality of A^{M+} and A^{M-}, respectively. Together with the Yang–Mills equations obtained from the variation with respect to A^{M+}, these equations are given by

$$\eta_{MN} * \mathcal{D}\mathcal{H}^{N-} = -\frac{1}{4} f_{+MP}{}^N M_{NQ}\mathcal{D}M^{QP}, \quad (15)$$

$$\eta_{MN} * \mathcal{D}\mathcal{H}^{N+} = \frac{1}{4} f_{-MP}{}^N M_{NQ}\mathcal{D}M^{QP}, \quad (16)$$

$$\mathcal{H}^{M-} = \mathrm{Im}\,\tau M^{MN}\eta_{NP} * \mathcal{H}^{P+} - \mathrm{Re}\,\tau\mathcal{H}^{M+}, \quad (17)$$

where we have used differential form language for later computational convenience. By substituting $\mathcal{H}^{M-}$ from (17) in (15), we obtain the usual Yang–Mills equations for $\mathcal{H}^{M+}$ while Eq. (16) simply gives the relation between the Hodge dual of the three-form field strength and the scalars due to the usual Bianchi identity of the gauge field strengths

$$\mathcal{F}^{M\pm} = dA^{M\pm} - \frac{1}{2}\eta^{MQ} f_{\alpha QNP} A^{N\alpha}\wedge A^{P\pm}. \quad (18)$$

In this paper, we are interested in $N = 4$ gauged supergravity coupled to three vector multiplets. The gauge group of interest here is a non-semisimple group $SO(3)\ltimes(\mathbf{T}^3, \hat{\mathbf{T}}^3) \subset SO(6,3)$ described by the following components of the embedding tensor:

$$\begin{aligned} f_{+IJ,K+6} &= -f_{+I+3,J+6,K+6} \\ &= -2\sqrt{2}\epsilon_{IJK},\ I, J, K = 1, 2, 3, \\ f_{+I+6,J+6,K+6} &= 6\sqrt{2}k\epsilon_{IJK}, \\ f_{-I,J+6,K+6} &= -4\epsilon_{IJK}. \end{aligned} \quad (19)$$

The constant k is related to the four-form flux along the four-dimensional space-time; see Eq. (122).

We should also remark that we follow the convention of [18] in all of the computations carried out here. In particular, the $SO(6,3)$ tensor η_{MN} is off-diagonal

$$\eta_{MN} = \begin{pmatrix} -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \end{pmatrix} \quad (20)$$

where $\mathbf{0}_3$ and $\mathbf{I}_3$ denote 3×3 zero and identity matrices, respectively. As a result, the computation of M_{MNPQRS} in (13) and parts of the supersymmetry transformations given below which involve $\mathcal{V}_M{}^m$ and $\mathcal{V}_M{}^a$ must be done with the projection to the negative and positive eigenvalues of η_{MN}, respectively. This can be achieved by using the projection matrix

$$P = \begin{pmatrix} \mathbf{0}_3 & \sqrt{2}\tilde{P}_3 & \mathbf{0}_3 \\ -\tilde{P}_3 & \mathbf{0}_3 & \tilde{P}_3 \\ \tilde{P}_3 & \mathbf{0}_3 & \tilde{P}_3 \end{pmatrix} \quad (21)$$

where the 3×3 matrix $\tilde{P}_3$ is given by

$$\tilde{P}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (22)$$

We now turn to the supersymmetry transformations of fermionic fields. These are given by

$$\delta\psi^i_\mu = 2D_\mu\epsilon^i - \frac{2}{3}gA_1^{ij}\gamma_\mu\epsilon_j + \frac{i}{4}(\mathcal{V}_\alpha)^*\mathcal{V}_M{}^{ij}\mathcal{H}^{M\alpha}_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu\epsilon_j, \quad (23)$$

$$\delta\chi^i = i\epsilon^{\alpha\beta}\mathcal{V}_\alpha D_\mu\mathcal{V}_\beta\gamma^\mu\epsilon^i - \frac{4}{3}igA_2^{ij}\epsilon_j + \frac{i}{2}\mathcal{V}_\alpha\mathcal{V}_M{}^{ij}\mathcal{H}^{M\alpha}_{\mu\nu}\epsilon_j, \quad (24)$$

$$\delta\lambda^i_a = 2i\mathcal{V}_a{}^M D_\mu\mathcal{V}_M{}^{ij}\gamma^\mu\epsilon_j + 2igA_{2aj}{}^i\epsilon^j - \frac{1}{4}\mathcal{V}_\alpha\mathcal{V}_{Ma}\mathcal{H}^{M\alpha}_{\mu\nu}\gamma^{\mu\nu}\epsilon^i. \quad (25)$$

The fermion shift matrices are defined by

$$\begin{aligned} A_1^{ij} &= \epsilon^{\alpha\beta}(\mathcal{V}_\alpha)^*\mathcal{V}_{kl}{}^M\mathcal{V}_N{}^{ik}\mathcal{V}_P{}^{jl} f_{\beta M}{}^{NP}, \\ A_2^{ij} &= \epsilon^{\alpha\beta}\mathcal{V}_\alpha\mathcal{V}_{kl}{}^M\mathcal{V}_N{}^{ik}\mathcal{V}_P{}^{jl} f_{\beta M}{}^{NP}, \\ A_{2ai}{}^j &= \epsilon^{\alpha\beta}\mathcal{V}_\alpha\mathcal{V}^M{}_a\mathcal{V}^N{}_{ik}\mathcal{V}_P{}^{jk} f_{\beta MN}{}^P \end{aligned} \quad (26)$$

where $\mathcal{V}_M{}^{ij}$ is defined in terms of the 't Hooft matrices G^{ij}_m and $\mathcal{V}_M{}^m$ as

$$\mathcal{V}_M{}^{ij} = \frac{1}{2}\mathcal{V}_M{}^m G^{ij}_m \quad (27)$$

and similarly for its inverse

$$\mathcal{V}^M{}_{ij} = -\frac{1}{2}\mathcal{V}_M{}^m(G^{ij}_m)^*. \quad (28)$$

G^{ij}_m satisfy the relations

$$G_{mij} = (G^{ij}_m)^* = \frac{1}{2}\epsilon_{ijkl}G^{kl}_m. \quad (29)$$

The explicit form of these matrices is given in the appendix. It should also be noted that the scalar potential can be written in terms of A_1 and A_2 tensors as

$$V = -\frac{1}{3}A_1^{ij}A_{1ij} + \frac{1}{9}A_2^{ij}A_{2ij} + \frac{1}{2}A_{2ai}{}^{j}A_{2a}{}^{i}{}_{j}. \tag{30}$$

With the explicit form of $\mathcal{V}_\alpha$ given in (6) and Eq. (17), it is straightforward to derive the following identities:

$$i\mathcal{V}_\alpha \mathcal{V}_M{}^{ij}\mathcal{H}^{M\alpha}_{\mu\nu}\gamma^{\mu\nu} = -(\mathcal{V}_-)^{-1}\mathcal{V}_M{}^{ij}\mathcal{H}^{M+}_{\mu\nu}\gamma^{\mu\nu}(1-\gamma_5), \tag{31}$$

$$i\mathcal{V}_\alpha \mathcal{V}_M{}^{a}\mathcal{H}^{M\alpha}_{\mu\nu}\gamma^{\mu\nu} = -(\mathcal{V}_-)^{-1}\mathcal{V}_M{}^{a}\mathcal{H}^{M+}_{\mu\nu}\gamma^{\mu\nu}(1+\gamma_5), \tag{32}$$

$$i(\mathcal{V}_\alpha)^*\mathcal{V}_M{}^{ij}\mathcal{H}^{M\alpha}_{\mu\nu}\gamma^{\mu\nu}\gamma_\rho = (\mathcal{V}_-)^{-1}\mathcal{V}_M{}^{ij}\mathcal{H}^{M+}_{\mu\nu}\gamma^{\mu\nu}\gamma_\rho(1-\gamma_5). \tag{33}$$

In obtaining these results, we have used the following relations for the $SO(6,n)$ coset representative [33]:

$$\begin{aligned} \eta_{MN} &= -\frac{1}{2}\epsilon_{ijkl}\mathcal{V}_M{}^{ij}\mathcal{V}_N{}^{kl} + \mathcal{V}_M{}^{a}\mathcal{V}_N{}^{a}, \\ \mathcal{V}_M{}^{a}\mathcal{V}^M{}_{ij} &= 0, \\ \mathcal{V}_M{}^{ij}\mathcal{V}^M{}_{kl} &= -\frac{1}{2}(\delta^i_k\delta^j_l - \delta^i_l\delta^j_k), \quad \mathcal{V}_M{}^{a}\mathcal{V}^M{}_{b} = \delta^a_b. \end{aligned} \tag{34}$$

These relations are useful in simplifying the BPS equations resulting from the supersymmetry transformations. Note also that these relations are slightly different from those given in [32] due to a different convention on $\mathcal{V}_\alpha$ in terms of the scalar τ. In more detail, $\mathcal{V}_\alpha$ used in this paper and in [18] satisfies $\mathcal{V}_+/\mathcal{V}_- = \tau$ while $\mathcal{V}_\alpha$ used in [32] gives $\mathcal{V}_+/\mathcal{V}_- = \tau^*$. This results in some sign changes in the above equations compared to those of [32].

2.2 Uplift formulas to 11 dimensions

As mentioned above, four-dimensional $N=4$ gauged supergravity coupled to three vector multiplets with $SO(3)\ltimes(\mathbf{T}^3,\hat{\mathbf{T}}^3)$ gauge group has been obtained from a truncation of 11-dimensional supergravity on a tri-sasakian manifold in [18]. We will briefly review the structure and relevant formulas focusing on the reduction ansatz which will be useful for embedding four-dimensional solutions. Essentially, we simply collect some formulas without giving detailed explanations for which we refer the interested reader to [18].

The 11-dimensional metric can be written as

$$\begin{aligned} ds^2_{11} &= e^{2\varphi}ds^2_4 + e^{2U}ds^2(B_{\text{QK}}) \\ &\quad + g_{IJ}(\eta^I + A_1^I)(\eta^J + A_1^J). \end{aligned} \tag{35}$$

The three-dimensional internal metric g_{IJ} can be written in terms of the vielbein as

$$g = Q^T Q. \tag{36}$$

Following [18], we will parametrize the matrix Q in terms of a product of a diagonal matrix V and an $SO(3)$ matrix O

$$Q = VO, \quad V = \text{diag}(e^{V_1}, e^{V_2}, e^{V_3}). \tag{37}$$

The scalar φ is chosen to be

$$\varphi = -\frac{1}{2}(4U + V_1 + V_2 + V_3) \tag{38}$$

in order to obtain the Einstein frame action in four dimensions. B_{QK} denotes a four-dimensional quaternionic Kähler manifold whose explicit metric is not needed in the following discussions.

The ansatz for the four-form field is given by

$$\begin{aligned} G_4 &= H_4 + H_{3I}\wedge(\eta+A_1)^I + \frac{1}{2}\epsilon_{IJK}\tilde{H}^I_2\wedge(\eta+A_1)^J \\ &\quad \wedge(\eta+A_1)^K + 4\text{Tr}c\,\text{vol(QK)}H_{1IJ} \\ &\quad \wedge(\eta+A_1)^I\wedge J^I + \frac{1}{6}\epsilon_{IJK} \\ &\quad \times d\chi\wedge(\eta+A_1)^I\wedge(\eta+A_1)^J\wedge(\eta+A_1)^K \\ &\quad + H_{2I}\wedge J^I + \epsilon_{IJL}[(\chi+\text{Tr}c)\delta_{LK} \\ &\quad - 2c_{(LK)}](\eta+A_1)^I\wedge(\eta+A_1)^J\wedge J^K. \end{aligned} \tag{39}$$

c_{IJ} is a 3×3 matrix and $\text{Tr}c = \delta^{IJ}c_{IJ}$. The volume form of B_{QK}, vol(QK), can be written in terms of the two-forms J^I as

$$\text{vol(QK)} = \frac{1}{6}J^I\wedge J^I. \tag{40}$$

Various forms in the above equation are defined by

$$\begin{aligned} H_4 &= dc_3 + c_{2I}\wedge F^I_2, \\ H_{3I} &= Dc_{2I} + \epsilon_{IJK}F^J_2\wedge\tilde{c}_{1K}, \\ \tilde{H}_{2I} &= D\tilde{c}_{1I} - 2c_{2I} + \chi F_{2I}, \\ H_{2I} &= Dc_{1I} + 2c_{2I} + c_{JI}F^J_2, \\ H_{1IJ} &= Dc_{IJ} + 2\epsilon_{IJK}(c_{1K} + \tilde{c}_{1K}) \end{aligned} \tag{41}$$

with the $SO(3)$ covariant derivative

$$Dc_{I_1\ldots I_n} = dc_{I_1\ldots I_n} + 2\sum_{l=1}^{n}\epsilon_{JI_lK}A^J_1\wedge c_{I_1\ldots K\ldots I_n}. \tag{42}$$

The $SO(3)_R$ field strengths are defined by

$$F^I_2 = dA^I_1 - \epsilon_{IJK}A^J_1\wedge A^K_1. \tag{43}$$

It is useful to note here that the $SL(2,\mathbb{R})/SO(2)$ scalars are given by

$$\tau = \chi + ie^{V_1+V_2+V_3}. \tag{44}$$

Although we will not directly need the explicit form of $ds^2(B_{\text{QK}})$ and η^I's in the remaining parts of this paper, it is useful to give some information on the N^{010} tri-sasakian manifold. N^{010} is a 7-manifold with $SU(2)\times SU(3)$ isometry. The $SU(2)$ is identified with the R-symmetry of the dual $N=3$ SCFT while $SU(3)$ is the flavor symmetry. A simple description of N^{010} can be obtained in terms of a coset manifold $SU(3)/U(1)$. With the standard Gell-Mann matrices, the $SU(3)$ generators can be chosen to be $-\frac{i}{2}\lambda_\alpha$, $\alpha = 1, \ldots, 8$. The coset and $U(1)$ generators are accordingly identified as

$$K_i = -\frac{i}{2}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7),$$
$$H = -\frac{i\sqrt{3}}{2}\lambda_8. \quad (45)$$

The vielbein on N^{010} can eventually be obtained from the decomposition of the Maurer–Cartan one-form

$$L^{-1}\mathrm{d}L = e^i K_i + \omega H \quad (46)$$

where L is the coset representative for $SU(3)/U(1)$, and ω is the corresponding $U(1)$ connection.

Following [18], we can use the tri-sasakian structures of the form

$$\eta^I = \frac{1}{2}(e^1, e^2, e^7),$$
$$J^I = \frac{1}{8}(e^4 \wedge e^5 - e^3 \wedge e^6, -e^3 \wedge e^5 - e^4 \wedge e^6, e^5 \wedge e^6 - e^3 \wedge e^4). \quad (47)$$

From these, we find the metric on the quaternionic–Kähler base B_{QK} to be

$$\mathrm{d}s^2(B_{\mathrm{QK}}) = \frac{1}{256}[(e^3)^2 + (e^4)^2 + (e^5)^2 + (e^6)^2] \quad (48)$$

with the volume form given by

$$\mathrm{vol(QK)} = \frac{1}{6}J^I \wedge J^I = -\frac{1}{64}e^3 \wedge e^4 \wedge e^5 \wedge e^6. \quad (49)$$

As mentioned before, all of the fields appearing in the reduction of [18] are $SU(3)$ singlets.

3 BPS flow equations

In this section, we perform the analysis of Yang–Mills equations and supersymmetry transformations in order to obtain BPS equations for the flows between AdS_4 vacua and possible $AdS_2 \times \Sigma_2$ geometries. We set all fermions to zero and truncate the bosonic fields to $SO(2) \subset SO(3)_R$ singlets. This $SO(2)$ is generated by

$$\hat{X} = X_{9+} + X_{6+} + X_{3-} \quad (50)$$

where the gauge generators are defined by

$$X_{M\alpha} = -f_{\alpha MNP}t^{NP}. \quad (51)$$

We see that a combination of the electric vectors A^{9+}, A^{6+} and the magnetic vector A^{3-} becomes the corresponding $SO(2)$ gauge field.

We are interested in supersymmetric solutions of the form $AdS_2 \times \Sigma_2$ with $\Sigma_2 = S^2, H^2$. We will then take the ansatz for the four-dimensional metric to be

$$\mathrm{d}s_4^2 = -e^{2f(r)}\mathrm{d}t^2 + \mathrm{d}r^2 + e^{2g(r)}(\mathrm{d}\theta^2 + F(\theta)^2\mathrm{d}\phi^2) \quad (52)$$

with

$$F(\theta) = \sin\theta \quad \text{and} \quad F(\theta) = \sinh\theta \quad (53)$$

for the S^2 and H^2, respectively. We will also use the parameter $\kappa = \pm 1$ to denote the S^2 and H^2 cases. The functions $f(r)$, $g(r)$ and all other fields only depend on the radial coordinate r for static solutions. With the obvious vielbein

$$e^{\hat{t}} = e^f \mathrm{d}t, \quad e^{\hat{r}} = \mathrm{d}r,$$
$$e^{\hat{\theta}} = e^g \mathrm{d}\theta, \quad e^{\hat{\phi}} = e^g F \mathrm{d}\phi, \quad (54)$$

it is now straightforward to compute the spin connections of the above metric

$$\omega^{\hat{t}\hat{r}} = f' e^{\hat{t}}, \quad \omega^{\hat{\theta}\hat{r}} = g' e^{\hat{\theta}},$$
$$\omega^{\hat{\phi}\hat{r}} = g' e^{\hat{\phi}}, \quad \omega^{\hat{\theta}\hat{\phi}} = \frac{F'(\theta)}{F(\theta)}e^{-g}e^{\hat{\phi}}. \quad (55)$$

In the above expressions, we have used the hat to denote "flat" indices while $'$ stands for the r-derivative with the only exception that $F'(\theta) = \frac{\mathrm{d}F(\theta)}{\mathrm{d}\theta}$. The ansatz for electric and magnetic vector fields are given by

$$A^{M+} = \mathcal{A}_t^M \mathrm{d}t - p^M F'(\theta)\mathrm{d}\phi, \quad (56)$$
$$A^{M-} = \tilde{\mathcal{A}}_t^M \mathrm{d}t - e_M F'(\theta)\mathrm{d}\phi \quad (57)$$

where we have chosen the gauge such that $A_r^{M\alpha} = 0$. p^M and e_M correspond to magnetic and electric charges, respectively. In the present case, only $A^{M\alpha}$ with $M = 3, 6, 9$ are relevant.

We finally give the explicit form of the scalar coset representative for $SO(6,3)/SO(6) \times SO(3)$. The parametrization of [18] which is directly related to the higher-dimensional origin is given by

$$\mathcal{V} = \mathcal{C}\mathcal{Q} \quad (58)$$

where the matrices $\mathcal{Q}$ and $\mathcal{C}$ are defined by

$$\mathcal{Q} = \begin{pmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & e^{-2U}Q^{-1} & \mathbf{I}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & e^{2U}Q^T \end{pmatrix},$$
$$\mathcal{C} = \exp\begin{pmatrix} \mathbf{0}_3 & \sqrt{2}c^T & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \sqrt{2}c & a & \mathbf{0}_3 \end{pmatrix}. \quad (59)$$

For $SO(2)$ invariant scalars, the 3×3 matrices c and a are given by

$$c = \begin{pmatrix} Z_1 & Z_3 & 0 \\ -Z_3 & Z_1 & 0 \\ 0 & 0 & Z_2 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & \Phi & 0 \\ -\Phi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (60)$$

while Q can be obtained from (37) with $V_2 = V_1$ and O being

$$O = \exp\begin{pmatrix} 0 & \beta & 0 \\ -\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{61}$$

This is a generalization of the coset representative of the $SO(3)_R$ singlet scalars used in [19] in which $\Phi = \beta = Z_3 = 0$, $Z_1 = Z_2$ and $V_1 = V_2 = V_3$. In the following, we will rename the scalars $V_3 \to V_2$ such that the complex scalar τ becomes

$$\tau = \chi + i e^{2V_1+V_2}. \tag{62}$$

We now give the scalar potential for $SO(2)$ singlet scalars,

$$\begin{aligned} V = {} & e^{-3(4U+2V_1+V_2)}[e^{4(U+V_2)}(e^{4U} + 2e^{4V_1}) \\ & +9k^2 + 4\chi^2 e^{4U+2V_1} \\ & -4e^{6U+4V_1+2V_2}(6 + e^{2(U-V_1)} \\ & -e^{-2(U-V_1)}) + 24k\chi Z_1 + 16\chi^2 Z_1^2 \\ & +8\chi Z_2 e^{4U+2V_1} - 12k\chi Z_2 \\ & +(16\chi^2 - 24k)Z_1 Z_2 + 32\chi Z_1^2 Z_2 \\ & +4Z_2^2 e^{4U+2V_1} + 4\chi^2 Z_2^2 + 8\chi Z_1 Z_2^2 \\ & +16 Z_1^2 Z_2^2 - 4\chi Z_2^3 - 8 Z_1 Z_2^3 \\ & +6k Z_2^2 + Z_2^4 + 2e^{2V_2} \\ & [e^{4U}(\chi + 2Z_1 - Z_2)^2 + 2e^{4V_1}(2Z_1 + Z_2)^2]]. \end{aligned} \tag{63}$$

The scalars β, Φ and Z_3 do not appear in the potential. It can also be checked that setting $\beta = \Phi = Z_3 = 0$ is a consistent truncation. In fact, β never appears in any equations, so we can set it to zero. On the other hand, the Yang–Mills equations, to be given later, demand that Φ and Z_3 must be constant. Since we are interested in the flow solutions interpolating between $AdS_2 \times \Sigma_2$ and AdS_4 vacua, and at supersymmetric AdS_4 critical points, both Φ and Z_3 vanish. We then choose $Z_3 = \Phi = 0$.

The kinetic terms for the remaining scalars read

$$\begin{aligned} \mathcal{L}_{\text{kin}} = {} & -6U'^2 - 2U'(2V_1' + V_2') - 2V_1'^2 - V_1'V_2' \\ & -\frac{1}{4}[3V_2'^2 + e^{-2(2V_1+V_2)}\chi'^2 + 4e^{-2(2U+V_1)} \\ & \times Z_1'^2 + 2e^{-2(2U+V_2)}Z_2'^2]. \end{aligned} \tag{64}$$

We now redefine the scalars such that the kinetic terms are diagonal

$$\tilde{V} = 2V_1 + V_2, \quad \tilde{U}_1 = 2U + V_1, \quad \tilde{U}_2 = 2U + V_2, \tag{65}$$

in terms of which we find

$$\begin{aligned} \mathcal{L}_{\text{kin}} = {} & -\frac{1}{4}(4\tilde{U}_1'^2 + 2\tilde{U}_2'^2 + \tilde{V}'^2 \\ & +e^{-2\tilde{V}}\chi'^2 + 4e^{-2\tilde{U}_1}Z_1'^2 + 2e^{-2\tilde{U}_2}Z_2'^2). \end{aligned} \tag{66}$$

These new scalars will also be useful in the analysis of the BPS equations below.

The above scalar potential admits two supersymmetric AdS_4 vacua with $N = 1$ and $N = 3$ supersymmetries [18]. At these vacua the symmetry is enhanced from $SO(2)$ to $SO(3)$. For convenience, before carry out the analysis of the Yang–Mills and BPS equations, we review the $N = 3$ and $N = 1$ AdS_4 critical points in terms of the new scalars defined above:

$$\begin{aligned} & N = 3: \ \tilde{V} = \tilde{U}_1 = \tilde{U}_2 = \frac{1}{2}\ln k, \\ & V_0 = -12|k|^{-\frac{3}{2}}, \quad k > 0, \end{aligned} \tag{67}$$

$$\begin{aligned} & N = 1: \ \tilde{U}_1 = \tilde{U}_2 = \ln 5 + \frac{1}{2}\ln\left[-\frac{k}{15}\right], \\ & \tilde{V} = \frac{1}{2}\ln\left[-\frac{k}{15}\right], \\ & V_0 = -12|k|^{-\frac{3}{2}}\sqrt{\frac{3^7}{5^5}}, \quad k < 0. \end{aligned} \tag{68}$$

V_0 is the cosmological constant related to the AdS_4 radius by

$$L^2 = -\frac{3}{V_0}. \tag{69}$$

3.1 The analysis of Yang–Mills equations

We now solve the equations of motion for the gauge fields given in (15)–(17). We should emphasize that, in the reduction of [18], the magnetic vectors A^{M-} with $M = 4, 5, 6$ do not appear in the reduction ansatz. These might arise from the reduction of the dual internal seven-dimensional metric. Furthermore, in this reduction, the two-form fields corresponding to these magnetic vectors do not appear.

Although the present analysis involves A^{6+}, we will truncate out the A^{6-} in order to use the reduction ansatz of [18] to uplift the resulting solutions to 11 dimensions. This amounts to setting e_6 and $\tilde{A}_t^6$ in (57) to zero. It turns out that this truncation is consistent provided that the two-form fields are properly truncated. Therefore, we will set $e_6 = \tilde{\mathcal{A}}_t^6 = 0$ in the following analysis. Note also that the vanishing of A^{6-} does not mean the covariant field strength $\mathcal{H}^{6-}$ vanishes although the usual gauge field strength $\mathcal{F}^{6-}$ vanishes. This is due to the fact that $\mathcal{H}^{6-}$ gets a contribution from the two-form fields.

In order to consistently remove A^{6-}, we truncate the two-form fields to only B^{18} and B^{78}. With the symmetry of $AdS_2 \times \Sigma_2$ background and a particular choice of tensor gauge transformations

$$B^{MN} \to B^{MN} + d\Xi^{MN}, \tag{70}$$

we will take the ansatz for the two-forms to be

$$B^{78} = B(r)F(\theta)d\theta \wedge d\phi,$$

$$B^{18} = \tilde{B}(r)F(\theta)\mathrm{d}\theta \wedge \mathrm{d}\phi. \quad (71)$$

With the explicit form of the embedding tensor, we can compute the covariant field strengths

$$\begin{aligned}
\mathcal{H}^{3+} &= \mathcal{A}_t^{3\prime}\mathrm{d}r \wedge \mathrm{d}t + (p^3 + 4B)F(\theta)\mathrm{d}\theta \wedge \mathrm{d}\phi,\\
\mathcal{H}^{6+} &= \mathcal{A}_t^{6\prime}\mathrm{d}r \wedge \mathrm{d}t + (p^6 - 4\tilde{B})F(\theta)\mathrm{d}\theta \wedge \mathrm{d}\phi,\\
\mathcal{H}^{9+} &= \mathcal{A}_t^{9\prime}\mathrm{d}r \wedge \mathrm{d}t + p^9 F(\theta)\mathrm{d}\theta \wedge \mathrm{d}\phi,\\
\mathcal{H}^{3-} &= \tilde{\mathcal{A}}_t^{3\prime}\mathrm{d}r \wedge \mathrm{d}t + (e_3 - 2\sqrt{2}\tilde{B})F(\theta)\mathrm{d}\theta \wedge \mathrm{d}\phi,\\
\mathcal{H}^{6-} &= -6\sqrt{2}kBF(\theta)\mathrm{d}\theta \wedge \mathrm{d}\phi,\\
\mathcal{H}^{9-} &= \tilde{\mathcal{A}}_t^{9\prime}\mathrm{d}r \wedge \mathrm{d}t + (e_9 - 2\sqrt{2}B)F(\theta)\mathrm{d}\theta \wedge \mathrm{d}\phi. \quad (72)
\end{aligned}$$

Note the non-vanishing covariant field strength $\mathcal{H}^{6-}$, as mentioned above, due to the contribution from the two-form fields despite $A^{6-} = 0$.

Equations arising from (15) and (16) are explicitly given in the appendix. They can be solved by imposing the following conditions:

$$\begin{aligned}
&Z_3' = 0, \quad \Phi' = 2Z_1 Z_3' - 2Z_3 Z_1',\\
&B' F(\theta)\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi = \sqrt{2}e^{-4(2U+V_1)}\\
&\quad \times(3k * A^{9+} + *A^{6+} - \sqrt{2} * A^{3-}),\\
&\tilde{B}' F(\theta)\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi = 4Z_1 e^{-4(2U+V_1)}\\
&\quad \times(3k * A^{9+} + *A^{6+} - \sqrt{2} * A^{3-}). \quad (73)
\end{aligned}$$

The first condition implies that Z_3 is constant. As mentioned above, this allows one to set $Z_3 = 0$. The second condition then requires that Φ is constant. We can also set $\Phi = 0$. Together with $\beta = 0$, we are left with only six scalars $(U, V_1, V_2, \chi, Z_1, Z_2)$ or equivalently $(\tilde{U}_1, \tilde{U}_2, \tilde{V}, \chi, Z_1, Z_2)$.

We move to the last two conditions in (73). First of all, the $\mathrm{d}t \wedge \mathrm{d}r \wedge \mathrm{d}\theta$ component gives

$$3kp^9 + p^6 - \sqrt{2}e_3 = 0 \quad (74)$$

while the $\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi$ component leads to first-order differential equations for B and $\tilde{B}$

$$B' = \sqrt{2}e^{-4(2U+V_1)+2g-f}(3k\mathcal{A}_t^9 + \mathcal{A}_t^6 - \sqrt{2}\tilde{\mathcal{A}}_t^3), \quad (75)$$

$$\tilde{B}' = -4Z_1 e^{-4(2U+V_1)+2g-f}(3k\mathcal{A}_t^9 + \mathcal{A}_t^6 - \sqrt{2}\tilde{\mathcal{A}}_t^3). \quad (76)$$

After solving all of the Yang–Mills equations and Bianchi identities, we now consider the duality equation for electric and magnetic vector fields. These equations whose explicit form is given in the appendix lead to the relations between $(\mathcal{A}_t^{M\prime}, \tilde{\mathcal{A}}_t^{M\prime})$ and scalars. We can accordingly express the former in terms of the latter. These relations are given by

$$\begin{aligned}
\mathcal{A}_t^{3\prime} &= e^{f-2g-2(U+V_1)-3V_2}[e^{4U+2V_2}\\
&\quad [e_3 + \sqrt{2}e_9 Z_2 - 4BZ_2\\
&\quad +\chi(p^3 + 4B + \sqrt{2}Z_2)]\\
&\quad +Z_2^2[2(e_3 + p^3\chi) + \sqrt{2}Z_2(e_9 + p^9\chi)]\\
&\quad -4Z_2 B(3k - 2\chi Z_2 + Z_2^2)\\
&\quad -2\sqrt{2}\tilde{B}(e^{4U+2V_2} + 2Z_2\chi + 2Z_2^2) + \sqrt{2}p^6 Z_2\chi], \quad (77)
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_t^{6\prime} &= e^{f-2g-2(2U+V_1)-3V_2}\\
&\quad [(2\sqrt{2}B - e_9 - p^9\chi)e^{8U+4V_2} - p^6 Z_2^2\chi\\
&\quad -e^{4U+2V_2}Z_2[\sqrt{2}e_3 - 4\tilde{B} + 2e_9 Z_2 + \chi(\sqrt{2}p^3 + 2p^9 Z_2)]\\
&\quad +4\tilde{B}Z_2^2(\chi + Z_2) - Z_2^3[\sqrt{2}(e_3 + p^3\chi) + Z_2(e_9 + p^9\chi)]\\
&\quad +4\sqrt{2}BZ_2 e^{4U+2V_2}(Z_2 - \chi) + 2\sqrt{2}BZ_2^2(3k - 2\chi Z_2 + Z_2^2)], \quad (78)
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_t^{9\prime} &= -e^{f-2g-2(2U+V_1)-3V_2}\\
&\quad [Z_2(\sqrt{2}e_3 - 4\tilde{B} + e_9 Z_2) - 2\sqrt{2}B(3k - 2\chi Z_2 + Z_2^2)\\
&\quad +\chi(p^6 - 4\tilde{B} + \sqrt{2}Z_2 + p^9 Z_2^2)], \quad (79)
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{A}}_t^{3\prime} &= \frac{e^{f-2g-2V_1-V_2}}{Z_2}[-e^{4V_1+2V_2}[\sqrt{2}e^{4U+2V_2}p^9\\
&\quad +Z_2(p^3 + 4B + \sqrt{2}p^9 Z_2)]\\
&\quad +\chi Z_2[e_3 - 2\sqrt{2}\tilde{B} + \sqrt{2}e_9 Z_2 - 4BZ_2\\
&\quad +\chi(p^3 + 4B + \sqrt{2}p^9 Z_2)]\\
&\quad +\chi e^{4U+2V_2}[\sqrt{2}(e_9 + p^9\chi) - 4B]], \quad (80)
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{A}}_t^{9\prime} &= \frac{e^{f-2g-2V_1-V_2}}{Z_2^2}[e^{4(U+V_1+V_2)}p^9 - e^{4U+2V_1}\chi(e_9 - 2\sqrt{2}B + p^9\chi)\\
&\quad -\chi Z_2[\sqrt{2}e_3 - 4\tilde{B} + 4\sqrt{2}B(\chi - Z_2) + 2e_9 Z_2\\
&\quad +\chi(\sqrt{2}p^3 + 2p^9 Z_2)]\\
&\quad +e^{4V_1+2V_2}Z_2(\sqrt{2}p^3 + 4\sqrt{2}B + 2p^9 Z_2)]. \quad (81)
\end{aligned}$$

It turns out that only $\mathcal{A}_t^9$, $\mathcal{A}_t^6$ and $\tilde{\mathcal{A}}_t^3$ appear in other equations, while the remaining ones only appear through their derivatives. Therefore, these fields can be integrated out.

3.2 BPS equations for $SO(2)$ invariant scalars

We now use the ansatz for all the fields given in the previous section to set up the BPS equations for finding supersymmetric solutions. We will use Majorana representation for the gamma matrices in which all γ_μ are real, and

$$\gamma_5 = i\gamma_{\hat{0}}\gamma_{\hat{r}}\gamma_{\hat{\theta}}\gamma_{\hat{\phi}} \quad (82)$$

is purely imaginary. We then have, for example,

$$\begin{aligned}
\epsilon^i &= \frac{1}{2}(1 + \gamma_5)\epsilon_M^i,\\
\epsilon_i &= \frac{1}{2}(1 - \gamma_5)\epsilon_M^i \quad (83)
\end{aligned}$$

with ϵ_M^i being four-component Majorana spinors. It follows that $\epsilon_i = (\epsilon^i)^*$.

We first consider the gravitino transformations. As in other holographic solutions involving twisted compactifications of the dual SCFTs, the strategy is to use the gauge connection to cancel the spin connection on Σ_2. Equations from $\delta\psi^i_{\hat{\theta}} = 0$ and $\delta\psi^i_{\hat{\phi}} = 0$ then reduce to the same equation. The gauge connection enters the covariant derivative of ϵ^i through the composite connection ${Q_j}^i$. With the $SO(2)$ singlet scalars, we find that $Q_j{}^i$ takes the form of

$$Q_j{}^i = \frac{1}{2}\hat{A}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (84)$$

where $\hat{A}$ is given by

$$\hat{A} = \sqrt{2}e^{-2(2U+V_1)}(3kA^{9+} + A^{6+} - \sqrt{2}A^{3-} - 4e^{4U+2V_1}A^{9+}). \quad (85)$$

From the form of $Q_i{}^j$, we can see that supersymmetry corresponding to $\epsilon^{3,4}$ is broken for spherical and hyperbolic Σ_2 since we cannot cancel the spin connections along $\epsilon^{3,4}$. The $N = 4$ supersymmetry is then broken to $N = 2$.

After using the condition (74) in the $Q_{\hat{\phi}i}{}^j$ components, the twist is achieved by imposing the projection

$$\gamma^{\hat{\theta}\hat{\phi}}\epsilon^{\hat{i}} = \epsilon^{\hat{i}}{}_{\hat{j}}\epsilon^{\hat{j}} \quad (86)$$

provided that we impose the following twist condition:

$$2\sqrt{2}\kappa p^9 = 1. \quad (87)$$

Indices $\hat{i}, \hat{j} = 1, 2$ denote the Killing spinors corresponding to the unbroken supersymmetry. From Eq. (86), the chirality condition on $\epsilon^{\hat{i}}$ implies that

$$\gamma^{\hat{0}\hat{r}}\epsilon^{\hat{i}} = -i\epsilon^{\hat{i}}{}_{\hat{j}}\epsilon^{\hat{j}}. \quad (88)$$

With these projections, we can write the $\delta\psi^i_{\hat{\theta}} = 0$ equation, which is the same as $\delta\psi^i_{\hat{\phi}}$ equation, as

$$g'\gamma_{\hat{r}}\epsilon^{\hat{i}} - \frac{2}{3}A_1^{\hat{i}\hat{j}}\epsilon_{\hat{j}} + \frac{i}{2}(\mathcal{V}_\alpha)^*\mathcal{V}_M{}^{\hat{i}\hat{j}} \times (i\mathcal{H}^{M\alpha}_{\hat{0}\hat{r}} - \mathcal{H}^{M\alpha}_{\hat{\theta}\hat{\phi}})\epsilon_{\hat{j}}{}^{\hat{k}}\epsilon_{\hat{k}} = 0 \quad (89)$$

where we have multiplied the resulting equation by $\gamma^{\hat{\theta}}$. We further impose the projector

$$\gamma_{\hat{r}}\epsilon^{\hat{i}} = e^{i\Lambda}\delta^{\hat{i}\hat{j}}\epsilon_{\hat{j}} \quad (90)$$

in which $e^{i\Lambda}$ is an r-dependent phase. By Eq. (88), this projector implies

$$\gamma_{\hat{0}}\epsilon^{\hat{i}} = ie^{i\Lambda}\epsilon^{\hat{i}\hat{j}}\epsilon_{\hat{j}}. \quad (91)$$

It should be noted that there are only two independent projectors given in (86) and (90). Therefore, the entire flows preserve $\frac{1}{4}$ supersymmetry. On the other hand, the $AdS_2 \times \Sigma_2$ vacua is $\frac{1}{2}$ supersymmetric since the $\gamma_{\hat{r}}$ projection is not needed for constant scalars.

As a next step, we introduce the "superpotential" $\mathcal{W}$ and "central charge" $\mathcal{Z}$ defined, respectively, by the eigenvalues of

$$\frac{2}{3}A_1^{\hat{i}\hat{j}} = \mathcal{W}_{\hat{i}}\delta^{\hat{i}\hat{j}} \quad (92)$$

and

$$-\frac{i}{2}(\mathcal{V}_\alpha)^*\mathcal{V}_M{}^{\hat{i}\hat{j}}(i\mathcal{H}^{M\alpha}_{\hat{0}\hat{r}} - \mathcal{H}^{M\alpha}_{\hat{\theta}\hat{\phi}})\epsilon_{\hat{j}}{}^{\hat{k}} = \mathcal{Z}_{\hat{i}}\delta^{\hat{i}\hat{k}}. \quad (93)$$

It should be emphasized that no summation is implied in the above two equations.

With all these, we obtain the BPS equation from $\delta\psi^{\hat{i}}_{\hat{\theta}} = 0$ by the equation

$$e^{i\Lambda}g' - \mathcal{W}_i - \mathcal{Z}_i = 0 \quad (94)$$

which gives

$$g' = |\mathcal{W}_i + \mathcal{Z}_i| \quad \text{and} \quad e^{i\Lambda} = \frac{\mathcal{W}_i + \mathcal{Z}_i}{|\mathcal{W}_i + \mathcal{Z}_i|}. \quad (95)$$

Using all of these results, we find that the equation $\delta\psi^{\hat{i}}_{\hat{0}} = 0$ gives

$$e^{i\Lambda}(f' + i\hat{A}_t e^{-f}) - \mathcal{W}_i + \mathcal{Z}_i = 0. \quad (96)$$

Taking the real and imaginary parts leads to the following BPS equations:

$$f' = \text{Re}[e^{-i\Lambda}(\mathcal{W}_i - \mathcal{Z}_i)] \quad (97)$$

and

$$\hat{A}_t = e^f\text{Im}[e^{-i\Lambda}(\mathcal{W}_i - \mathcal{Z}_i)]. \quad (98)$$

We now come to the equation $\delta\psi^{\hat{i}}_{\hat{r}} = 0$, which gives the r-dependence of the Killing spinors. When combined with the equation $\delta\psi^{\hat{i}}_{\hat{0}} = 0$, this equation reads

$$2\epsilon^{\hat{i}\prime} - f' - i\hat{A}_t e^{-f}\epsilon^{\hat{i}} = 0, \quad (99)$$

which can be solved by

$$\epsilon^{\hat{i}} = e^{\frac{f}{2} + \frac{i}{2}\int \hat{A}_t e^{-f}dr}\tilde{\epsilon}^{\hat{i}}. \quad (100)$$

$\tilde{\epsilon}^{\hat{i}}$ are constant spinors satisfying the projections

$$\gamma_{\hat{r}}\tilde{\epsilon}^{\hat{i}} = \delta^{\hat{i}\hat{j}}\tilde{\epsilon}_{\hat{j}}, \quad \gamma_{\hat{\theta}\hat{\phi}}\tilde{\epsilon}^{\hat{i}} = \epsilon^{\hat{i}}{}_{\hat{j}}\tilde{\epsilon}^{\hat{j}}. \quad (101)$$

Using the $\gamma_{\hat{r}}$ projector, we obtain the following BPS equations from $\delta\chi^i$ and $\delta\lambda^i_a$:

$$-e^{i\Lambda}\epsilon^{\alpha\beta}\mathcal{V}_\alpha\mathcal{V}'_\beta\delta_{\hat{i}\hat{j}} - \frac{4i}{3}A_2^{\hat{j}\hat{i}} + i\mathcal{V}_\alpha\mathcal{V}_M{}^{\hat{i}\hat{k}}\epsilon^{\hat{k}\hat{j}}(i\mathcal{H}^{M\alpha}_{\hat{0}\hat{r}} + \mathcal{H}^{M\alpha}_{\hat{\theta}\hat{\phi}}) = 0, \quad (102)$$

$$\mathcal{V}_a{}^M\mathcal{V}_M{}^{ij\prime}e^{-i\Lambda} + \frac{1}{4}\mathcal{V}_\alpha\mathcal{V}_{Ma}(\mathcal{H}^{M\alpha}_{\hat{0}\hat{r}} + i\mathcal{H}^{M\alpha}_{\hat{\theta}\hat{\phi}})\delta^i_{\hat{i}}\delta^j_{\hat{j}}\epsilon^{\hat{i}\hat{j}} + A_{2aj}{}^i = 0. \quad (103)$$

Note that there are four equations from $\delta\lambda^i_a$ for each value of $a = 1, 2, 3$, but $\delta\lambda^{i=3,4}_a$ we do not get any contribution from the gauge fields. However, the scalars appearing in these equations cannot be consistently set to zero since ${A_{2aj}}^i$ is not diagonal in ij indices.

It should be pointed out that the $N = 3$ supersymmetric AdS_4 vacuum corresponds to the Killing spinors $\epsilon^{2,3,4}$ while ϵ^1 is the Killing spinor of the $N = 1$ AdS_4 critical point. In the next section, we will look for possible $AdS_2 \times \Sigma_2$ solutions to the above BPS equations. As mentioned before, in the twist given above, the supersymmetry corresponding to $\epsilon^{3,4}$ is broken. Therefore, the resulting $AdS_2 \times \Sigma_2$ solutions will preserve only two supercharges or half of the $N = 1$ supersymmetry corresponding to either ϵ^1 or ϵ^2. We will analyze these two cases separately.

4 Supersymmetric $AdS_2 \times \Sigma_2$ solutions

In this section, we look for the $AdS_2 \times \Sigma_2$ fixed points of the above BPS flow equations with constant scalars. These solutions should correspond to IR fixed points of the RG flows from twisted compactifications of the dual $N = 3$ and $N = 1$ SCFTs in three dimensions. They also describe near horizon geometries of BPS black holes arising from M2-branes wrapped on Σ_2. Before giving the solutions, we first discuss the conditions for obtaining the AdS_2 fixed points.

At the $AdS_2 \times \Sigma_2$ geometries, the scalars are constant, and we can choose the gauge in which $A_t^{M\alpha} \sim 0$. Furthermore, the warped factor $g(r)$ is required to be constant, $g'(r) = 0$. Let r_h be the position of the horizon, we can summarize the conditions for $AdS_2 \times \Sigma_2$ solutions and their properties as follows:

$$
\begin{aligned}
& f(r_h) = \frac{r_h}{L_{\mathrm{AdS}_2}}, \quad e^{g(r_h)} = L_{\Sigma_2}, \\
& \mathrm{Im}[e^{-i\Lambda}(\mathcal{W}_i - \mathcal{Z}_i)] = 0, \\
& |\mathcal{W}_i + \mathcal{Z}_i| = 0, \quad \frac{4}{3} A_2^{\hat{i}\hat{j}} = \mathcal{V}_\alpha {\mathcal{V}_M}^{\hat{i}\hat{k}} \epsilon^{\hat{k}\hat{j}} (i\mathcal{H}^{M\alpha}_{\hat{0}\hat{r}} + \mathcal{H}^{M\alpha}_{\hat{\theta}\hat{\phi}}), \\
& \frac{i}{4} \mathcal{V}_\alpha \mathcal{V}_{Ma} (-i\hat{H}^{M\alpha}_{\hat{0}\hat{r}} + \mathcal{H}^{M\alpha}_{\hat{\theta}\hat{\phi}}) \epsilon^{\hat{i}\hat{j}} = -{A_{2a\hat{j}}}^{\hat{i}} \\
& \quad {A_{2a\tilde{j}}}^{\hat{i}} = 0, \quad \tilde{j} = 3, 4,
\end{aligned} \tag{104}
$$

where L_{AdS_2} and L_{Σ_2} are, respectively, the radii of AdS_2 and Σ_2. These conditions can be viewed as attractor equations for the scalars at the black hole horizon.

4.1 Solutions in the $N = 3$ case

We begin with the $N = 3$ case. The $AdS_2 \times \Sigma_2$ solutions will describe the fixed points of the RG flows from $N = 3$ SCFTs dual to the N^{010} compactification of 11-dimensional supergravity to supersymmetric CFT_1's dual to the $AdS_2 \times \Sigma_2$ geometries. These flows are examples of the twisted compactifications of the $N = 3$ SCFT on Σ_2.

In this case, the superpotential and central charge are given in terms of the redefined scalars $(\tilde{U}_1, \tilde{U}_2, \tilde{V})$ by

$$
\begin{aligned}
\mathcal{W}_2 = {} & \frac{1}{2} e^{-\frac{1}{2}(4\tilde{U}_1 + 2\tilde{U}_2 + \tilde{V})} [e^{2\tilde{U}_2} + 4e^{\tilde{U}_1 + \tilde{U}_2} - 2e^{\tilde{U}_2 + \tilde{V}} + 4e^{\tilde{U}_1 + \tilde{V}} \\
& -3k + 2iZ_2 e^{\tilde{U}_2} + 4iZ_2 e^{\tilde{U}_1} - 4iZ_1(e^{\tilde{U}_2} + e^{\tilde{V}} + iZ_2) \\
& -2iZ_2 e^{\tilde{V}} - Z_2^2 + 2\chi(2ie^{\tilde{U}_1} - ie^{\tilde{U}_2} + 2Z_1 + Z_2)],
\end{aligned} \tag{105}
$$

$$
\begin{aligned}
\mathcal{Z}_2 = {} & \frac{1}{4} e^{-\frac{1}{2}(4g + 2\tilde{U}_2 + \tilde{V})} [2e_3 e^{\tilde{U}_2} - \sqrt{2} i e_9 e^{2\tilde{U}_2} + 2ie_3\chi + 2p^3\chi e^{\tilde{U}_2} \\
& -\sqrt{2} i p^9 \chi (e^{2\tilde{U}_2} + 3k) - 4\sqrt{2}\tilde{B}[e^{\tilde{U}_2} + e^{\tilde{V}} + i(\chi + Z_2)] \\
& +2ie_3 Z_2 + 2\sqrt{2} e_9 Z_2 e^{\tilde{U}_2} + 2ip^3\chi Z_2 + 2\sqrt{2} p^9 \chi Z_2 e^{\tilde{U}_2} \\
& +\sqrt{2} i (e_9 + p^9\chi) Z_2^2 + 4iB(e^{2\tilde{U}_2} - 2e^{\tilde{U}_2 + \tilde{V}} - 3k) \\
& +4B[2\chi(e^{\tilde{U}_2} + iZ_2) + Z_2(e^{\tilde{V}} - 2e^{\tilde{U}_2} - iZ_2)] \\
& +e^{\tilde{V}}(2e_3 - 3\sqrt{2}p^9 - \sqrt{2}p^9 e^{2\tilde{U}_2} + 2p^3 Z_2 + \sqrt{2}p^9 Z_2^2) \\
& -2ie^{\tilde{U}_2 + \tilde{V}}(p^3 + \sqrt{2}p^9 Z_2)]
\end{aligned} \tag{106}
$$

in which the subscript 2 on $\mathcal{W}_2$ and $\mathcal{Z}_2$ refers to the superpotential and central charge associated to the Killing spinor ϵ^2.

The BPS equations are given by

$$
\begin{aligned}
& f' = \mathrm{Re}[e^{-i\Lambda}(\mathcal{W}_2 - \mathcal{Z}_2)], \\
& e^{i\Lambda} = \frac{\mathcal{W}_2 + \mathcal{Z}_2}{|\mathcal{W}_2 + \mathcal{Z}_2|},
\end{aligned} \tag{107}
$$

$$
g' = |\mathcal{W}_2 + \mathcal{Z}_2|, \tag{108}
$$

$$
\begin{aligned}
& e^{i\Lambda}\tilde{V}' - ie^{-\tilde{V} + i\Lambda}\chi' \\
& = \frac{1}{2}\Big[e^{-\frac{\tilde{V}}{2} - \tilde{U}_2 - 2\tilde{U}_1}[2e^{\tilde{U}_2} + 8e^{2\tilde{U}_1} - 6k + Z_2(8Z_1 - 2Z_2)] \\
& \quad -e^{-2g - 2\tilde{U}_1 + \frac{\tilde{V}}{2}}[4e^{2g} + 2e^{2\tilde{U}_1}(p^3 + 4B + \sqrt{2}p^9 Z_2)] \\
& \quad +4\chi(2Z_1 + Z_2)e^{-\frac{\tilde{V}}{2} - \tilde{U}_2 - 2\tilde{U}_1} + \sqrt{2} e_9 e^{\tilde{U}_2 - 2g - \frac{\tilde{V}}{2}} \Big] \\
& \quad +\frac{1}{2} e^{-2g - \tilde{U}_2 - \frac{\tilde{V}}{2}}[\sqrt{2} Z_2(4\tilde{B} - e_9 Z_2) - 2e_3(\chi + Z_2) + 4\sqrt{2}\chi\tilde{B} \\
& \quad -4B(e^{2\tilde{U}_2} - 3k + 2\chi Z_2 - Z_2^2) + \sqrt{2}p^9\chi(e^{\tilde{U}_2} + 3k) \\
& \quad -Z_2\chi(2p^3 + \sqrt{2}p^9 Z_2)] \\
& \quad -\frac{i}{2} e^{-\tilde{U}_2 - \frac{\tilde{V}}{2}}[4e^{\tilde{U}_2 - 2\tilde{U}_1}(Z_2 - 2Z_1 - \chi) - 4e^{\tilde{V} - 2\tilde{U}_1}(2Z_1 + Z_2) \\
& \quad -2e^{\tilde{U}_2 - 2g}[Z_2(\sqrt{2}e_9 - 4B - 2\sqrt{2}\tilde{B}) + \chi(p^3 + 4B + \sqrt{2}p^9 Z_2)] \\
& \quad +e^{\tilde{V} - 2g}[2e_3 - 4\sqrt{2}\tilde{B} - \sqrt{2}p^9(3k + e^{2\tilde{U}_2}) - 4\sqrt{2}\tilde{B} \\
& \quad +Z_2(2p^3 + 8B + \sqrt{2}p^9 Z_2)] - 2e^{\tilde{U}_2 - 2g}e_3],
\end{aligned} \tag{109}
$$

$$
\begin{aligned}
& e^{-i\Lambda}\tilde{U}_2' + ie^{-\tilde{U}_2 - i\Lambda}Z_2' \\
& = \frac{1}{2} e^{-2g - \tilde{U}_2 - 2\tilde{U}_1 - \frac{\tilde{V}}{2}}[2e^{2(g + \tilde{U}_2)} + \sqrt{2} i e_9 e^{2(\tilde{U}_1 + \tilde{U}_2)} + 6ke^{2g} \\
& \quad -2ie_3\chi e^{2\tilde{U}_1} + \sqrt{2} i p^9 \chi e^{2(\tilde{U}_1 + \tilde{U}_2)} + 3\sqrt{2} ikp^9\chi e^{2\tilde{U}_1} \\
& \quad +8iZ_2 e^{2g + \tilde{U}_1} - 2ie_3 Z_2 e^{2\tilde{U}_1} - 4\chi Z_2 e^{2g} - 2ip^3\chi Z_2 e^{2\tilde{U}_1} \\
& \quad -8Z_1 Z_2 e^{2g} + 2Z_2^2 e^{2g} - \sqrt{2} i e_9 Z_2^2 e^{2\tilde{U}_1} - 8\chi Z_1 e^{2g} \\
& \quad -4iBe^{2\tilde{U}_1}[e^{2\tilde{U}_2} - 3k + Z_2(2\chi - Z_2 - 2ie^{\tilde{V}})] + 8i\chi e^{2g + \tilde{U}_1} \\
& \quad +4\sqrt{2}\tilde{B}e^{2\tilde{U}_1}(e^{\tilde{V}} + i\chi + iZ_2) - \sqrt{2} i p^9 \chi Z_2^2 e^{2\tilde{U}_1} \\
& \quad -4ie^{2g + \tilde{V}}(2Z_1 + Z_2) - Z_2 e^{2\tilde{U}_1 + \tilde{V}}(2p^3 + \sqrt{2}p^9 Z_2) \\
& \quad -e^{\tilde{U}_1 + \tilde{V}}[8e^{2g} + e^{\tilde{U}_1}[2e_3 - \sqrt{2}p^9(e^{2\tilde{U}_2} + 3k)]]],
\end{aligned} \tag{110}
$$

$$e^{-i\Lambda}\tilde{U}_1' - ie^{-\tilde{U}_1-i\Lambda}Z_1'$$
$$= e^{-\tilde{U}_2-2\tilde{U}_1-\frac{\tilde{V}}{2}}[2e^{\tilde{U}_2+\tilde{V}} - e^{2\tilde{U}_2} - 2e^{\tilde{U}_1}(e^{\tilde{U}_2}+e^{\tilde{V}}) + 3k$$
$$-4iZ_1(e^{\tilde{U}_2}+e^{\tilde{V}} - iZ_2) + 2i\chi(e^{\tilde{U}_1} - e^{\tilde{U}_2} + 2iZ_1 + iZ_2)$$
$$+2iZ_2(e^{\tilde{U}_2}+e^{\tilde{U}_1} - e^{\tilde{V}}) + Z_2^2] \quad (111)$$

where we have used the relation (74) to express p^6 in terms of p^9 and e_3.

To obtain the complete flow solutions, we have to solve these equations together with the two-form equations (75), (76) and the equations for the gauge fields (77)–(81) as well as the algebraic constraint given by Eq. (98). These equations are very complicated even with the numerical technique not to mention the analytic solutions. In what follows, we will present only the $AdS_2 \times \Sigma_2$ solutions and will not give the numerical flow solutions which may be obtained by suitable boundary conditions. In principle, the horizon is characterized by the values of the scalars as functions of the electric and magnetic charges. However, due to the complexity of the BPS equations, it is more convenient to solve the horizon conditions for the charges in terms of the scalar fields although inverting the solutions to express the scalars in terms of the charges is desirable.

In the present case, although it is straightforward to solve the above equations for $(B, \tilde{B}, \chi, Z_1, p^9, p^3, e_3, e_9)$ in terms of $(\tilde{U}_1, \tilde{U}_2, \tilde{V}, Z_2)$, the resulting expressions turn out to be cumbersome and not very illuminating. Accordingly, we refrain from giving the general result here but instead present some solutions with specific values of the parameters. These are obtained from truncating the full result and represent some examples of $AdS_2 \times \Sigma_2$ geometries within the solution space.

Examples of $AdS_2 \times \Sigma_2$ solutions are as follows:

- We begin with a simple solution with vanishing pseudoscalars. In the M-theory point of view, only scalars coming from the 11-dimensional metric are turned on. The solution is given by

$$k = \frac{1}{5}, \quad \chi = Z_1 = Z_2 = 0, \quad e_9 = 0,$$
$$\tilde{V} = \frac{1}{2}\ln\left[\frac{27}{5}\right], \quad \tilde{U}_1 = \frac{1}{2}\ln\left[\frac{27}{80}\right],$$
$$\tilde{U}_2 = -\frac{1}{2}\ln\left[\frac{5}{3}\right], \quad \tilde{B} = \frac{1}{20}(5\sqrt{2}e_3 - 27p^9),$$
$$g = \frac{1}{2}\ln\left[-\frac{81}{80}\sqrt{\frac{3}{10}}\kappa p^9\right],$$
$$B = -\frac{p^3}{4}, \quad L_{AdS_2} = \frac{3^{\frac{9}{4}}}{32(5)^{\frac{3}{4}}}. \quad (112)$$

It is clearly seen that only the hyperbolic horizon ($\kappa = -1$) is possible otherwise $g(r_h)$ will become complex. Therefore, we find that this is an $AdS_2 \times H^2$ solution.

- We next consider a solution with scalars and pseudoscalars turned on. In the 11-dimensional context, the solution involves scalar fields from both the metric and the four-form field. This solution is characterized by

$$k = 1, \quad Z_1 = Z_2 = \tilde{U} = 0,$$
$$\tilde{U} = \tilde{V} = \ln\left[\frac{12}{7}\right],$$
$$p^3 = \frac{41e_9 + 220p^9}{41\sqrt{2}}, \quad B = -\frac{41e_9 + 136p^9}{164\sqrt{2}},$$
$$\tilde{B} = \frac{e_3}{2\sqrt{2}} - \frac{111}{41}p^9,$$
$$\chi = -\frac{1}{7}, \quad g = \frac{1}{2}\ln\left[-2^{\frac{5}{2}}\kappa p^9\sqrt{\frac{21}{41}}\right],$$
$$L_{AdS_2} = \frac{\sqrt{21}}{19}. \quad (113)$$

This solution is also $AdS_2 \times H^2$.

- As a final example, we consider a solution with more scalars turned on and hence more general than the previous two solutions. This solution is given by

$$Z_1 = 0, \quad Z_2 = -\frac{2\sqrt{k}}{7}, \quad \chi = -\frac{\sqrt{k}}{7},$$
$$\tilde{U}_1 = \tilde{U}_2 = \frac{1}{2}\ln k,$$
$$p^3 = \frac{128,447k - 104,895}{4,116\sqrt{2k}}p^9,$$
$$e_9 = \frac{32,723k - 13,923}{4,116\sqrt{2k}}p^9,$$
$$\tilde{B} = \frac{e_3}{2\sqrt{2}} + \frac{567 - 667k}{98}p^9,$$
$$g = \frac{1}{2}\ln\left[\frac{21(1-k)\sqrt{k}\kappa p^9}{2\sqrt{2}}\right],$$
$$\tilde{V} = \ln(2\sqrt{k}), \quad B = -25p^9\left[\frac{3,809k - 2,961}{16,464\sqrt{2k}}\right],$$
$$L_{AdS_2} = \frac{k^{\frac{3}{4}}}{3\sqrt{2}}. \quad (114)$$

In this case, the flux parameter k is not fixed, and there are two types of solutions, $AdS_2 \times S^2$ and $AdS_2 \times H^2$, depending on the value of k. For $k > 1$, we have an $AdS_2 \times H^2$ solution with $\kappa = -1$ while the solution with $k < 1$ is $AdS_2 \times S^2$ for which $\kappa = 1$.

4.2 Solutions in the $N = 1$ case

We now repeat a similar analysis for the $N = 1$ case in which the $N = 1$ AdS_4 vacuum arises from the squashed N^{010} manifold. This critical point exists only for $k < 0$, and the $AdS_2 \times \Sigma_2$ solutions would be IR fixed points of the twisted compactifications of the dual $N = 1$ SCFT. The superpotential and central charge are given by

$$\begin{aligned}\mathcal{W}_1 = &\frac{1}{2}e^{-\tilde{U}_2-2\tilde{U}_1-\frac{\tilde{V}}{2}}[e^{2\tilde{U}_2} - 4e^{\tilde{U}_1+\tilde{U}_2} - 2e^{\tilde{V}}(e^{\tilde{U}_2} + 2e^{\tilde{U}_1})\\ &+4Z_1(Z_2 - ie^{\tilde{U}_2} - ie^{\tilde{V}})\\ &-3k + iZ_2(2e^{\tilde{U}_2} - 4e^{\tilde{U}_1} - 2e^{\tilde{V}} + iZ_2)\\ &+2\chi(2Z_1 + Z_2 - ie^{\tilde{U}_2} - 2ie^{\tilde{U}_1})],\end{aligned} \tag{115}$$

$$\begin{aligned}\mathcal{Z}_1 = &\frac{1}{4}e^{-2g-\tilde{U}_2-\frac{\tilde{V}}{2}}\\ &\times[2e_3(e^{\tilde{U}_2} + i\chi) - \sqrt{2}ie_9e^{2\tilde{U}_2} + 2p^3\chi e^{\tilde{U}_2} - 3\sqrt{2}ikp^9\chi\\ &-\sqrt{2}ip^9\chi e^{2\tilde{U}_2} - 4\sqrt{2}\tilde{B}(e^{\tilde{U}_2} + e^{\tilde{V}+i\chi+iZ_2}) + 2ie_3Z_2\\ &+2\sqrt{2}e_9Z_2e^{\tilde{U}_2} + 2ip^3\chi Z_2 + 2\sqrt{2}p^9\chi Z_2 + \sqrt{2}ie_9Z_2^2\\ &+\sqrt{2}ip^9\chi Z_2^2 + 4B[2\chi(e^{\tilde{U}_2} + iZ_2) + i(e^{2\tilde{U}_2} - 2e^{\tilde{U}_2+\tilde{V}} - 3k)]\\ &+4BZ_2(2e^{\tilde{V}} - 2e^{\tilde{U}_2} - iZ_2) - 2ie^{\tilde{U}_2+\tilde{V}}(p^3 + \sqrt{2}p^9Z_2)\\ &+e^{\tilde{V}}(2e_3 - 6\sqrt{2}p^9 - \sqrt{2}p^9e^{2\tilde{U}_2} + 2p^3Z_2 + \sqrt{2}p^9Z_2^2)].\end{aligned} \tag{116}$$

The procedure is essentially the same, so we will just present the result of $AdS_2 \times \Sigma_2$ solutions and leave the explicit form of the corresponding BPS equations to the appendix. In this case, it turns out to be more difficult to find the solutions in particular we have not found any solutions without the pseudoscalars turned on. With some effort, we obtain the following solutions:

- We begin with a simple solution in which all scalars have the same value as the $N = 1$ supersymmetric AdS_4 vacuum

$$\begin{aligned}&k = -\frac{18}{11}, \quad Z_1 = Z_2 = \chi = 0,\\ &\tilde{U}_1 = \tilde{U}_2 = \ln 5 - \frac{1}{2}\ln\left[\frac{55}{6}\right], \quad \tilde{V} = -\frac{1}{2}\ln\left[\frac{55}{6}\right],\\ &B = -\frac{p^3}{4}, \quad \tilde{B} = \frac{e_3}{2\sqrt{2}}, \quad e_9 = -\frac{14p^3}{5\sqrt{2}},\\ &g = \frac{1}{2}\ln\left[-\frac{10}{11}\sqrt{\frac{15}{11}\kappa p^9}\right],\\ &L_{AdS_2} = \frac{5^{\frac{5}{4}}}{2^{\frac{5}{4}}\left(3^{\frac{1}{4}}\right)\left(11^{\frac{3}{4}}\right)}.\end{aligned} \tag{117}$$

  The solution is of the $AdS_2 \times H^2$ form.

- We now give a more complicated solution

$$\begin{aligned}&k = -\frac{18}{11}, \quad Z_1 = \chi = 0,\\ &\tilde{U}_1 = \tilde{V} = \ln\left[7\sqrt{-\frac{3k}{319}}\right],\\ &p^3 = \sqrt{\frac{3}{638}}\left(\frac{p^9}{3,190\sqrt{-k}}\right)\\ &\quad(567,365k - 1,002,298),\\ &B = \sqrt{\frac{3}{638}}\left(\frac{p^9}{89,320\sqrt{-k}}\right)\\ &\quad(13,987,355k - 27,368,286),\\ &\tilde{B} = \frac{e_3}{2\sqrt{2}} + \frac{3p^9}{8,932}(63,162 - 32,267k),\\ &Z_2 = -5\sqrt{-\frac{3k}{319}},\\ &g = \ln\left[7\left(\frac{3}{638}\right)^{\frac{1}{4}}\sqrt{(k-2)\sqrt{-k}\kappa p^9}\right],\\ &\tilde{U}_2 = \frac{1}{2}\ln\left[-\frac{588k}{319}\right],\\ &L_{AdS_2} = \frac{21(3^{\frac{1}{4}})}{11}\sqrt{\frac{7}{21}}\left(\frac{2}{29}\right)^{\frac{3}{4}}.\end{aligned} \tag{118}$$

  This solution also gives $AdS_2 \times H^2$ geometry. To show that this leads to real solutions, we explicitly give one example of the possible solutions

$$\begin{aligned}&Z_1 = \chi = 0, \quad e_9 = 54.35, \quad p^3 = -11.56,\\ &\tilde{U}_1 = \tilde{V} = -0.14,\\ &\tilde{U}_2 = 0.55, \quad Z_2 = -0.62, \quad B = 10.66,\\ &\tilde{B} = -13.77 + 0.35e_3,\\ &g = 1.06.\end{aligned} \tag{119}$$

4.3 Uplift formulas

We end this section by giving the uplift formulas for embedding the previously found $AdS_2 \times \Sigma_2$ solutions in 11 dimensions. We first identify the vector and tensor fields in the $N = 4$ gauged supergravity and those obtained from the dimensional reduction of 11-dimensional supergravity on a tri-sasakian manifold

$$\begin{aligned}&A_1^3 = \sqrt{2}A^{9+}, \quad a_1^3 = -\sqrt{2}A^{6+}, \quad c_1^3 = A^{3+},\\ &\tilde{a}_1^3 = -A^{3-},\\ &\tilde{c}_1^3 = \sqrt{2}A^{9-}, \quad a_2^{12} = \sqrt{2}B^{18}, \quad c_2^3 = B^{78}.\end{aligned} \tag{120}$$

With this identification and the ansatz for the scalars and vector fields, the 11-dimensional metric and the four-form field are given by

$$
\begin{aligned}
ds_{11}^2 &= e^{-\frac{1}{3}(4\tilde{U}_1+2\tilde{U}_2+\tilde{V})}[-e^{2f}\mathrm{d}t^2+\mathrm{d}r^2\\
&\quad +e^{2g}(\mathrm{d}\theta^2+F(\theta)^2\mathrm{d}\phi^2)]\\
&\quad +e^{\frac{1}{3}(2\tilde{U}_1+\tilde{U}_2-\tilde{V})}ds^2(B_{\mathrm{QK}})+e^{\frac{2}{3}(\tilde{U}_1-\tilde{U}_2+\tilde{V})}\\
&\quad \times[(\eta^1)^2+(\eta^2)^2]+e^{\frac{2}{3}(\tilde{V}-2\tilde{U}_1-2\tilde{U}_2)}\\
&\quad \times(\eta^3+\sqrt{2}\mathcal{A}^9_t\mathrm{d}t-\sqrt{2}p^9F'(\theta)\mathrm{d}\phi)^2 \qquad (121)
\end{aligned}
$$

and

$$
\begin{aligned}
G_4 &= -[6ke^{-(4\tilde{U}_1+2\tilde{U}_2+\tilde{V})+f+2g}-\sqrt{2}B\mathcal{A}^{9\prime}_t-\sqrt{2}B'\mathcal{A}^9_t]\\
&\quad \times F(\theta)\mathrm{d}t\wedge\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi\\
&\quad +B'F(\theta)\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge\eta^3+dZ_1\wedge(\eta^1\wedge J^1+\eta^2\\
&\quad \wedge J^2)[\sqrt{2}(\tilde{\mathcal{A}}^{9\prime}_t+\chi\mathcal{A}^{9\prime}_t)\mathrm{d}r\wedge\mathrm{d}t+\sqrt{2}(e_9+\chi p^9\\
&\quad -\sqrt{2}B)F(\theta)\mathrm{d}\theta\wedge\mathrm{d}\phi]\wedge\eta^1\wedge\eta^2\\
&\quad \times[(\mathcal{A}^{3\prime}_t+\sqrt{2}Z_2\mathcal{A}^{9\prime}_t)\mathrm{d}r\\
&\quad \wedge\mathrm{d}t+(p^3+\sqrt{2}p^9Z_2+2B)F(\theta)\mathrm{d}\theta\wedge\mathrm{d}\phi]\wedge J^3\\
&\quad +2(\chi+2Z_1)\eta^1\wedge\eta^2\wedge J^3+(dZ_2\wedge J^3+\mathrm{d}\chi\\
&\quad \wedge\eta^2\wedge\eta^2)\wedge(\eta^3-\sqrt{2}p^9F(\theta)\mathrm{d}\phi)\\
&\quad +2[(\mathcal{A}^3_t+\sqrt{2}\tilde{\mathcal{A}}^9_t)\mathrm{d}t\\
&\quad -(\sqrt{2}e_9+p^3)F(\theta)\mathrm{d}\phi+4(2Z_1+Z_2)\mathrm{vol}(B_{\mathrm{QK}})\\
&\quad +(\chi+Z_2)(\eta^3+\sqrt{2}\mathcal{A}^9_t\mathrm{d}t-\sqrt{2}p^9F(\theta)\mathrm{d}\phi)]\\
&\quad \wedge(\eta^1\wedge J^2-\eta^2\wedge J^1). \qquad (122)
\end{aligned}
$$

5 Conclusions

In this paper, we have found a number of $\mathrm{AdS}_2\times\Sigma_2$ solutions in $N=4$ gauged supergravity with $SO(3)\ltimes(\mathbf{T}^3,\hat{\mathbf{T}}^3)$ gauge group. The solutions can be uplifted to M-theory since the $N=4$ gauged supergravity is a consistent truncation of 11-dimensional supergravity on a tri-sasakian manifold. These $\mathrm{AdS}_2\times\Sigma_2$ gemetries are expected to arise from the near horizon limit of certain dyonic BPS black holes which can be identified as holographic RG flows from twisted compactifications of the dual $N=1,3$ SCFTs in the UV to superconformal quantum mechanics corresponding to the AdS_2 geometry in the IR. We have found that most of the solutions have hyperbolic horizons, but some of them have spherical horizons depending on the values of the four-form flux parameter. These solutions provide examples of AdS_2 geometries from M-theory compactified on a tri-sasakian manifold such as N^{010} and are hopefully useful in the holographic study of the $N=1,3$ Chern–Simons–Matter theories in three dimensions. They should also be useful in the study of black hole entropy along the line of recent results in [37–39]. In this aspect, the near horizon solutions given here are enough although we have not constructed the full black hole solutions, numerically. It would be interesting to compute the topologically twisted index in the dual $N=1,3$ SCFTs and compare with the black hole entropy computed from the area of the horizon $A\sim L^2_{\Sigma_2}$.

The solutions found here might constitute only a small number of all possible solutions due to the complexity of the resulting BPS equations. It could be interesting to look for more solutions or even to identify all possible black hole solutions to this $N=4$ gauged supergravity similar to the analysis in $N=2$ gauged supergravity. For the case of N^{010} manifold, there exists an invariant two-form in addition to the universal forms on a generic tri-sasakian manifold. This leads to an additional vector multiplet, called a Betti multiplet, in $N=4$ gauged supergravity. This vector multiplet corresponds to a baryonic symmetry in the dual SCFTs. Finding a reduction that includes the Betti multiplet and $SU(3)$ non-singlet fields would be very useful in order to find more interesting black hole and other holographic solutions. We leave all these issues for future work.

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Appendix A: Useful formulas

In this appendix, we collect some convention on t' Hooft matrices and details on Yang–Mills equations and complicated BPS equations in the $N=1$ case.

A.1: 't Hooft matrices

In converting $SO(6)$ vector indices m,n to chiral spinor indices i,j, we use the following 't Hooft matrices:

$$
G_1^{ij}=\begin{bmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{bmatrix},
$$

$$
G_2^{ij}=\begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix},
$$

$$G_3^{ij} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$
$$G_4^{ij} = \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix},$$
$$G_5^{ij} = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix},$$
$$G_6^{ij} = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}. \qquad (123)$$

A.2: Field equations of gauge fields

In this section, we present the full equations of motion for the gauge fields $A^{M\alpha}$. Equation (15) gives

$$\begin{aligned} -*\mathcal{D}\mathcal{H}^{3-} &= e^{-4(2U+V_1)}\big[4Z_1(\Phi' + 2Z_3Z_1') \\ &\quad -4e^{4U+2V_1}Z_3' - 8Z_1^2Z_3'\big]\mathrm{d}r \\ &\quad -8Z_1e^{-4(2U+V_1)}(2A^{3-} - \sqrt{2}A^{6+} - 3\sqrt{2}kA^{9+}), \end{aligned} \qquad (124)$$
$$\begin{aligned} *\mathcal{D}\mathcal{H}^{6-} &= 3\sqrt{2}ke^{-4(2U+V_1)}(\Phi' + 2Z_3Z_1' - 2Z_1Z_3')\mathrm{d}r \\ &\quad +12ke^{-4(2U+V_1)}(3kA^{9+} + A^{6+} - \sqrt{2}A^{3-}), \end{aligned} \qquad (125)$$
$$\begin{aligned} *\mathcal{D}\mathcal{H}^{9-} &= \sqrt{2}ke^{-4(2U+V_1)}(\Phi' + 2Z_3Z_1' - 2Z_1Z_3')\mathrm{d}r \\ &\quad +4e^{-4(2U+V_1)}(3kA^{9+} + A^{6+} - \sqrt{2}A^{3-}) \end{aligned} \qquad (126)$$

while Eq. (16) leads to

$$\begin{aligned} -*\mathcal{D}\mathcal{H}^{3+} &= 2e^{-4(2U+V_1)}(\Phi' + 2Z_3Z_1' - 2Z_1Z_3')\mathrm{d}r \\ &\quad +4e^{-4(2U+V_1)}(3kA^{9+} + A^{6+} - \sqrt{2}A^{3-}), \end{aligned} \qquad (127)$$
$$\begin{aligned} *\mathcal{D}\mathcal{H}^{6+} &= 4\sqrt{2}ke^{-4(2U+V_1)} \\ &\quad [e^{4U+2V_1}Z_3' + 2Z_1^2Z_3' - Z_1(\Phi' + 2Z_3Z_1')]\mathrm{d}r \\ &\quad -16Z_1e^{-4(2U+V_1)}(3kA^{9+} + A^{6+} - \sqrt{2}A^{3-}), \end{aligned} \qquad (128)$$
$$*\mathcal{D}\mathcal{H}^{9+} = 0 \qquad (129)$$

For equations obtained from (17), it is more convenient to express them in the following combinations:

$$\begin{aligned} &\mathcal{H}^{9-} = e^{-4U+2V_1-V_2}(Z_2^2 * \mathcal{H}^{9+} + *\mathcal{H}^{6+} \\ &\quad +\sqrt{2}Z_2 * \mathcal{H}^{3+}) - \chi\mathcal{H}^{9+}, \end{aligned} \qquad (130)$$
$$\begin{aligned} &Z_2^2\mathcal{H}^{9-} + \mathcal{H}^{6-} + \sqrt{2}Z_2\mathcal{H}^{3-} = e^{4U+2V_1+3V_2} * \mathcal{H}^{9+} \\ &\quad -\chi(Z_2^2\mathcal{H}^{9+} + \mathcal{H}^{6+} + \sqrt{2}Z_2\mathcal{H}^{3+}), \end{aligned} \qquad (131)$$
$$\begin{aligned} &\sqrt{2}Z_2\mathcal{H}^{9-} + \mathcal{H}^{3-} = -\chi(\sqrt{2}Z_2\mathcal{H}^{9+} + \mathcal{H}^{3+}) \\ &\quad -e^{2V_1+V_2}(\sqrt{2}Z_2 * \mathcal{H}^{9+} + *\mathcal{H}^{3+}). \end{aligned} \qquad (132)$$

A.3: BPS equations for the $N = 1$ case

In this section, we collect all the relevant BPS equations in the $N = 1$ case. These are given by

$$\begin{aligned} &e^{-i\Lambda}\tilde{U}_1' + ie^{-\tilde{U}_1 - i\Lambda}Z_1' \\ &\quad = e^{-\tilde{U}_2 - 2\tilde{U}_1 - \frac{\tilde{V}}{2}}[2e^{\tilde{U}_1+\tilde{U}_2} - e^{2\tilde{U}_2} + 2e^{\tilde{V}}(e^{\tilde{U}_2} + e^{\tilde{U}_1}) + 3k \\ &\quad -4iZ_1(e^{\tilde{U}_2} + e^{\tilde{V}} - iZ_2) - 2i\chi(e^{\tilde{U}_2} + e^{\tilde{U}_1} - 2iZ_1 - iZ_2) \\ &\quad +Z_2[Z_2 - 2i(e^{\tilde{V}} + e^{\tilde{U}_1} - e^{\tilde{U}_2})]], \end{aligned} \qquad (133)$$
$$\begin{aligned} &e^{-i\Lambda}\tilde{U}_2' + ie^{-\tilde{U}_2 - i\Lambda}Z_2' \\ &\quad = \frac{1}{2}e^{-2g - \tilde{U}_2 - 2\tilde{U}_1 - \frac{\tilde{V}}{2}}[2e^{2(g+\tilde{U}_2)} + \sqrt{2}ie_9e^{2(\tilde{U}_1+\tilde{U}_2)} + 6ke^{2g} \\ &\quad -2ie_3\chi e^{2\tilde{U}_1} + \sqrt{2}ip^9\chi e^{2(\tilde{U}_1+\tilde{U}_2)} + 3\sqrt{2}ikp^9\chi e^{2\tilde{U}_1} \\ &\quad -8iZ_2e^{2g+\tilde{U}_1} - 2ie_3Z_2e^{2\tilde{U}_1} - 4\chi Z_2e^{2g} - 2ip^3\chi Z_2e^{2\tilde{U}_1} \\ &\quad -8Z_1Z_2e^{2g} + 2Z_2^2e^{2g} - \sqrt{2}ie_9Z_2^2e^{2\tilde{U}_1} - \sqrt{2}ip^9\chi Z_2^2e^{2\tilde{U}_1} \\ &\quad -4iBe^{2\tilde{U}_1}[e^{2\tilde{U}_2} - 3k + Z_2(2\chi - Z_2 - 2ie^{\tilde{V}})] - 8i\chi e^{2g+\tilde{U}_1} \\ &\quad -4ie^{2g+\tilde{V}}(2Z_1 + Z_2) + 4\sqrt{2}\tilde{B}e^{2\tilde{U}_1}[e^{\tilde{V}} + i(\chi + Z_2)] \\ &\quad +e^{\tilde{U}_1+\tilde{V}}[8e^{2g} + e^{\tilde{U}_1}(\sqrt{2}p^9(e^{2\tilde{U}_2} + 3k) - 2e_3)] \\ &\quad -8\chi Z_1e^{2g} - Z_2e^{2\tilde{U}_1+\tilde{V}}(2p^3 + \sqrt{2}p^9Z_2)], \end{aligned} \qquad (134)$$
$$\begin{aligned} &e^{i\Lambda}\tilde{V}' - ie^{-\tilde{V}+i\Lambda}\chi' \\ &\quad = \frac{1}{2}e^{-2g - \tilde{U}_2 - 2\tilde{U}_1 - \frac{\tilde{V}}{2}}[2e^{2(g+\tilde{U}_2)} - 8e^{2g+\tilde{U}_2+\tilde{U}_1} + 2ie_3e^{\tilde{U}_2+2\tilde{U}_1} \\ &\quad +\sqrt{2}e_9e^{2(\tilde{U}_1+\tilde{U}_2)} - 4i\chi e^{2g+\tilde{U}_2} - 8i\chi e^{2g+\tilde{U}_1} - 2e_3\chi e^{2\tilde{U}_1} \\ &\quad +2ip^3\chi e^{\tilde{U}_2+2\tilde{U}_1} + \sqrt{2}p^9\chi e^{2(\tilde{U}_1+\tilde{U}_2)} + 3\sqrt{2}kp^9\chi e^{2\tilde{U}_1} \\ &\quad +4e^{2g}(2\chi Z_1 + iZ_2e^{\tilde{U}_2}) - 2Z_2e^{\tilde{U}_1}(4ie^{2g} + e_3e^{\tilde{U}_1}) - 2Z_2^2e^{2g} \\ &\quad +2\sqrt{2}ie_9Z_2e^{\tilde{U}_2+2\tilde{U}_1} + 4\chi Z_2e^{2g} - 2p^3\chi Z_2e^{2\tilde{U}_1} - 6ke^{2g} \\ &\quad +2\sqrt{2}ip^9\chi Z_2e^{\tilde{U}_2+2\tilde{U}_1} + 8Z_1Z_2e^{2g} - \sqrt{2}e_9Z_2^2e^{2\tilde{U}_1} \\ &\quad +4\sqrt{2}\tilde{B}e^{2\tilde{U}_1}[Z_2 + \chi - i(e^{\tilde{U}_2} - e^{\tilde{V}})] - 4Be^{2\tilde{U}_1}(e^{2\tilde{U}_2+\tilde{V}} - 3k) \\ &\quad -4Be^{2\tilde{U}_1}[2iZ_2(e^{\tilde{U}_2} + e^{\tilde{V}}) - Z_2^2 + 2\chi(Z_2 - ie^{\tilde{U}_2})] \\ &\quad +ie^{2\tilde{U}_1+\tilde{V}}[6\sqrt{2}p^9 - 2e_3 + \sqrt{2}p^9 - 2p^3Z_2 - \sqrt{2}p^9Z_2^2 \\ &\quad +2ie^{\tilde{U}_2}(p^3 + \sqrt{2}p^9Z_2)] + 4e^{2g+\tilde{V}}[e^{\tilde{U}_2} + i(2Z_1 + Z_2)] \\ &\quad +8e^{2g+\tilde{U}_1+\tilde{V}} - 8iZ_1e^{2g+\tilde{U}_2} - \sqrt{2}p^9\chi Z_2^2e^{2\tilde{U}_1}] \end{aligned} \qquad (135)$$

where

$$e^{i\Lambda} = \frac{\mathcal{W}_1 + \mathcal{Z}_1}{|\mathcal{W}_1 + \mathcal{Z}_1|}. \qquad (136)$$

These equations need to be solved together with the following equations:

$$\begin{aligned} &f' = \mathrm{Re}[e^{-i\Lambda}(\mathcal{W}_1 - \mathcal{Z}_1)], \\ &g' = |\mathcal{W}_1 + \mathcal{Z}_1|, \quad \hat{A}_t = e^f\mathrm{Im}[e^{-i\Lambda}(\mathcal{W}_1 - \mathcal{Z}_1)] \end{aligned} \qquad (137)$$

and the two-form equations (75) and (76).

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