

Weyl and transverse diffeomorphism invariant spin-2 models in $D = 2 + 1$

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Abstract There are two covariant descriptions of massless spin-2 particles in $D = 3 + 1$ via a symmetric rank-2 tensor: the linearized Einstein–Hilbert (LEH) theory and the Weyl plus transverse diffeomorphism (WTDIFF) invariant model. From the LEH theory one can obtain the linearized new massive gravity (NMG) in $D = 2 + 1$ via Kaluza–Klein dimensional reduction followed by a dual master action. Here we show that a similar route takes us from the WTDIFF model to a linearized scalar–tensor NMG which belongs to a larger class of consistent spin-0 modifications of NMG. We also show that a traceless master action applied to a parity singlet furnishes two new spin-2 self-dual models. Moreover, we examine the singular replacement $h_{\mu\nu} \rightarrow h_{\mu\nu} - \eta_{\mu\nu}h/D$ and prove that it leads to consistent massive spin-2 models in $D = 2 + 1$. They include linearized versions of unimodular topologically massive gravity (TMG) and unimodular NMG. Although the free part of those unimodular theories are Weyl invariant, we do not expect any improvement in the renormalizability. Both the linearized K-term (in NMG) and the linearized gravitational Chern–Simons term (in TMG) are invariant under longitudinal reparametrizations $\delta h_{\mu\nu} = \partial_\mu \partial_\nu \zeta$, which is not a symmetry of the WTDIFF Einstein–Hilbert term. Therefore, we still have one degree of freedom whose propagator behaves like $1/p^2$ for large momentum.

1 Introduction

The covariant description of massless spin-2 particles is very constrained; see for instance [1, 2]. By far the most popular

model is the massless limit of the massive Fierz–Pauli (FP) theory [3, 4]. It is equivalent to the LEH theory.¹ It is invariant under linearized diffeomorphisms $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$. The second way is the WTDIFF model, (see [1, 5, 6] for earlier references), which is invariant under linearized diffeomorphisms and Weyl transformations, i.e., $\delta h_{\mu\nu} = \partial_\mu \xi_\nu^T + \partial_\nu \xi_\mu^T + \eta_{\mu\nu} \lambda$ where $\partial^\mu \xi_\mu^T = 0$. The WTDIFF model is the linearized truncation of unimodular gravity [7–10] which, in its turn, corresponds to the Einstein–Hilbert theory with the replacement $g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu}/(-\hat{g})^{1/D}$.

The WTDIFF model can be obtained from the usual LEH theory by the singular replacement $h_{\mu\nu} \rightarrow h_{\mu\nu} - \eta_{\mu\nu}h/D$. The reason why this replacement is successful is not obvious. An argument is given in [11]. Namely, we first introduce a harmless Stueckelberg scalar field altogether with a trivial Weyl symmetry in the LEH model via $h_{\mu\nu} \rightarrow h_{\mu\nu} + \eta_{\mu\nu}\phi$, thus defining a conformal model which is now invariant under full linearized reparametrizations and Weyl symmetry, namely, $\delta\phi = -\lambda$ and $\delta h_{\mu\nu} = \partial_\mu \xi_\nu^T + \partial_\nu \xi_\mu^T + \partial_\mu \partial_\nu \Lambda + \eta_{\mu\nu} \lambda$. Now on the one hand, we can choose to break only the Weyl symmetry by setting $\phi = 0$ which brings us back to the LEH model while on the other hand, we can break only the longitudinal reparametrizations picking up $\phi = -h/D$. In the latter case we are left with WTDIFF symmetry. Therefore, the LEH and WTDIFF models correspond to two differently partially broken versions of the same conformal theory. This is not a proof of equivalence, since the partial symmetry breaking conditions are implemented at action level. This is not the usual gauge fixing procedure. According to [12], the equivalence between a general gauge theory and its broken version (at action level) requires that the symmetry breaking proce-

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¹ Throughout this work we use $\eta_{\mu\nu} = \text{diag}(-, +, \dots, +)$ and $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \eta_{\mu\nu}h/D$. Moreover, we use the acronyms LEH, DIFF, TDIFF and WTDIFF, standing for linearized Einstein–Hilbert, diffeomorphisms, transverse diffeomorphisms and Weyl and transverse diffeomorphisms, respectively.

ture be complete which is not the case here. The key point is that the symmetry breaking at action level leads to less equations of motion. This is not equivalent in general to first derive the full set of equations of motion and impose those conditions afterwards.

Mainly motivated by the accelerated expansion of the universe, but also as a matter of principle, we are interested here in massive gravitational theories. They have been a subject of intense work in the last decade, (see [13, 14] and the review works [15, 16]). The modern massive gravities are built on top of the massive FP model, so we might wonder whether massive WTDIFF models do exist, or even before that, we must search for WTDIFF massive spin-2 theories. Naive addition of mass terms to the massless WTDIFF model breaks unitarity [2]. In [11] the reader can find a recent discussion in that direction via dimensional reduction.

Notice that the argument of [11] cannot be used in order to derive a WTDIFF version of the *massive* FP model. The condition $\phi = -h/D$ is not allowed, since there is no reparametrization symmetry to be broken or partially broken in the massive case. The best we can do is to replace $h_{\mu\nu} \rightarrow h_{\mu\nu} + \eta_{\mu\nu}\phi$ and carry out a field redefinition $\phi \rightarrow \phi - h/D$. We end up with the Weyl symmetry $\delta h_{\mu\nu} = \eta_{\mu\nu}\Lambda$ but the new ϕ is not pure gauge anymore. It remains in the theory as an extra degree of freedom [11]. This is the typical situation for massive WTDIFF models; extra fields are required in general.

In $D = 2 + 1$ the situation is different. We may have massive spin-2 models still invariant under reparametrization. This is the case of the second-, third- and fourth-order self-dual models of helicity $+2$ or -2 (parity singlets) and the linearized version of the new massive gravity (NMG) [17] with both helicities ± 2 (parity doublet). This raises the question of defining WTDIFF versions of those models according to the argument of [11] and eventually building unimodular versions of the corresponding massive gravitational theories. This issue is specially interesting from the point of view of renormalizability because the highest derivative term of topologically massive gravity (TMG) and of NMG is Weyl invariant at linearized level, contrary to the lower derivative term (Einstein–Hilbert). It would be interesting to have both terms Weyl invariant in order to make sure that all degrees of freedom have their large momentum behavior ruled by the highest derivative term. We examine that question here.

In Sect. 2 we show the consistency of the WTDIFF version of the linearized NMG model and of the second, third- and fourth-order spin-2 self-dual models SD_n ($n = 1, 2, 3$). We also comment on possible unimodular massive gravities and the issue of renormalizability. In Sect. 3, by means of a traceless master action approach we derive a new scalar–tensor self-dual model of second order NSD2 and also a new and unusual fourth-order model NSD4 from NSD2. In Sect. 3.2 a traceless master action gives rise to a new scalar–tensor NMG

model which is shown to be a specific subcase of a more general class of consistent spin-0 (scalar–tensor) deformations of NMG. In Sect. 4 we present our final comments.

2 WTDIF invariant models in $D = 2 + 1$

2.1 $m = 0$

In order to point out the subtleties of gauge fixing procedures at action level, it is instructive to first look at the massless case. It is well known that the Einstein–Hilbert theory has no propagating degrees of freedom in $D = 2 + 1$. At linearized level we have

$$\begin{aligned} S_{\text{LEH}}[h_{\mu\nu}] &= \int d^3x (\sqrt{-g}R)_{hh} \\ &= (1/4) \int d^3x h_{\rho\gamma} E^{\rho\delta} E^{\gamma\sigma} h_{\delta\sigma}, \end{aligned} \quad (1)$$

where the transverse operators

$$E^{\rho\delta} \equiv \epsilon^{\rho\delta\sigma} \partial_\sigma; \quad \square \theta_{\rho\sigma} \equiv \square \eta_{\rho\sigma} - \partial_\rho \partial_\sigma \quad (2)$$

are such that

$$E^{\mu\nu} E^{\alpha\beta} = \square (\theta^{\mu\beta} \theta^{\nu\alpha} - \theta^{\mu\alpha} \theta^{\nu\beta}). \quad (3)$$

The equations of motion $E^{\rho\delta} E^{\gamma\sigma} h_{\delta\sigma} = 0$ are equivalent (multiply by $\epsilon_{\rho\mu\nu} \epsilon_{\gamma\alpha\beta}$) to a vanishing linearized Riemann curvature $R^L_{\mu\nu\alpha\beta}(h) = 0$ (flat space). The general solution is pure gauge $h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$.

On the other hand, following the argument of [11], if we make the Stueckelberg replacement $h_{\mu\nu} \rightarrow h_{\mu\nu} + \eta_{\mu\nu}\phi$ in (1) followed by the symmetry breaking condition $\phi = -h/3$ at the action level, we have a WTDIFF invariant model $S_{\text{LEH}}[\bar{h}_{\mu\nu}]$ whose equations of motion are traceless:

$$E^{\rho\delta} E^{\gamma\sigma} \bar{h}_{\delta\sigma} - \eta^{\rho\gamma} \partial^\mu \partial^\nu \bar{h}_{\mu\nu} / 3 = 0. \quad (4)$$

Applying ∂_ρ we show that the linearized scalar curvature is an arbitrary constant, not necessarily vanishing anymore, i.e., $R^L = \partial^\mu \partial^\nu \bar{h}_{\mu\nu} = c$. The integration constant c cannot be fixed by the equations of motion. Contracting (4) with $\epsilon_{\rho\mu\nu} \epsilon_{\gamma\alpha\beta}$ we have a maximally symmetric space in general, not necessarily flat:

$$R^L_{\mu\nu\alpha\beta}(\bar{h}) = \frac{c}{6} (\eta_{\mu\beta} \eta_{\nu\alpha} - \eta_{\mu\alpha} \eta_{\nu\beta}). \quad (5)$$

The solution to (5) is given by

$$\bar{h}_{\mu\nu} = \partial_\mu \xi_\nu^T + \partial_\nu \xi_\mu^T + \frac{c}{10} \left(x_\mu x_\nu - \eta_{\mu\nu} \frac{x^2}{3} \right). \quad (6)$$

Except for the c -dependent term, the solution is pure gauge. So the number of propagating degrees of freedom still vanishes. We have one extra parameter (c , a global one) to be

fixed and the geometry has been changed as if we had a cosmological constant.² On the other hand, if after the Stueckelberg substitution in (1) we first derive the field equations with respect to $h_{\mu\nu}$ and ϕ and then require $\phi = -h/3$, then we would have obtained $R^L_{\mu\nu\alpha\beta}(\bar{h}) = 0$ and consequently $R^L = \partial^\mu \partial^\nu \bar{h}_{\mu\nu} = 0$, which corresponds to $c = 0$. We learn that imposing symmetry breaking conditions at action level is nontrivial, specially regarding gravitational theories. Thus, whenever we do it, as in the next section, we must explicitly check the consistency of the resulting model. We cannot take physical equivalence for granted. In the following subsections we turn to massive models which are still gauge invariant in $D = 2 + 1$ dimensions.

2.2 WTDIFF linearized new massive gravity (parity doublet)

The linearized version of the new massive gravity [17] (LNMG) can be written in the compact Fierz–Pauli form

$$S_{\text{LNMG}}[h] = \int d^3x \sqrt{-g} \left[\frac{1}{m^2} \left(R^2_{\mu\nu} - \frac{3}{8} R^2 \right) - R \right]_{hh}, \tag{7}$$

$$= (1/4) \int d^3x \left[h_{\rho\gamma} E^{\rho\delta} E^{\gamma\sigma} h^*_{\delta\sigma} - m^2 (h^{\mu\nu} h^*_{\mu\nu} - h h^*) \right], \tag{8}$$

where the dual field is given by [19]

$$h^*_{\mu\nu}[h] = \frac{1}{m^2} \left(E_{\mu\rho} E_{\nu\sigma} h^{\rho\sigma} + \frac{1}{2} \eta_{\mu\nu} \square \theta_{\rho\sigma} h^{\rho\sigma} \right), \tag{9}$$

and identically

$$\partial^\mu h^*_{\mu\nu} = \partial_\nu h^*. \tag{10}$$

If we replace $h^*_{\mu\nu}$ by $h_{\mu\nu}$ in (8) we recover the usual massive FP model. The theory $S_{\text{LNMG}}[h]$ is DIFF invariant. The highest derivative term of (8) is invariant under Weyl symmetry $\delta h_{\mu\nu} = \lambda \eta_{\mu\nu}$ which is broken by the EH term. Repeating in (8) the procedure of the last subsection, which amounts to the replacement $h_{\mu\nu} \rightarrow \bar{h}_{\mu\nu}$ at action level, we derive a WTDIFF version of LNMG: $S_{\text{WLNMG}}(h) = S_{\text{LNMG}}(\bar{h})$. Let us check the particle content of S_{WLNMG} . The equations of motion $\delta S_{\text{WLNMG}}/\delta h^{\mu\nu} = 0$ are traceless, namely,

$$E_\mu^\rho E_\nu^\sigma h^*_{\rho\sigma}[\bar{h}] + \frac{1}{3} \eta_{\mu\nu} \square \theta^{\rho\sigma} h^*_{\rho\sigma} = m^2 \left(h^*_{\mu\nu}[\bar{h}] - \frac{1}{3} \eta_{\mu\nu} h^*[\bar{h}] \right) = m^2 \bar{h}^*_{\mu\nu}[\bar{h}]. \tag{11}$$

² In $D = 2 + 1$ the Riemann tensor is proportional to the Ricci tensor; see e.g. [18].

From (10) we see that $\square \theta^{\rho\sigma} h^*_{\rho\sigma} = 0$. Applying ∂^μ on (11) we have $\partial^\mu \bar{h}^*_{\mu\nu}[\bar{h}] = 0$. Using the identities (3) and (10) we see that (11) is equivalent to the Klein–Gordon equations $(\square - m^2) \bar{h}^*_{\mu\nu}[\bar{h}] = 0$. Therefore, $\bar{h}^*_{\mu\nu}[\bar{h}]$ is transverse and traceless, and it satisfies the Klein–Gordon equations. Moreover, it is invariant under the WTDIFF gauge symmetry of S_{WLNMG} , i.e., $\delta h_{\mu\nu} = \partial_\mu \xi_\nu^T + \partial_\nu \xi_\mu^T + \eta_{\mu\nu} \phi$. So S_{WLNMG} correctly describes massive spin-2 particles. From (10) and $\partial^\mu \bar{h}^*_{\mu\nu} = 0$ we have $\partial_\mu h^* = 0$, so $h^* = \eta^{\alpha\beta} h^*_{\alpha\beta}$ becomes an integration constant which plays no role from the point of view of the particle content of S_{WLNMG} but from the point of view of a linearized gravitational theory works like a cosmological constant.

Although LNMG can be obtained from the usual massive FP model via master action, see for instance [19], we have not been able to derive the WLNMG model from any second-order theory. The WLNMG model contains both helicities +2 and -2, in the next subsection we look at parity singlets of helicity +2 or -2 described in terms of a symmetric traceless tensor.

2.3 WTDIFF self-dual models (parity singlets)

Free helicity +2 or -2 states can be described by the so called spin-2 self-dual models (SDn), of n th order in derivatives with $n = 1, 2, 3, 4$. The SDn model can be obtained from $\text{SD}(n - 1)$ via a consecutive Noether gauge embedding procedure as shown in [19]. The equivalence among all those models can be proved by means of a master action approach [20]; see [19, 21], which also furnishes a dual map $e_{\mu\nu} \rightarrow e^*_{\mu\nu}$ responsible for the equivalence of correlation functions of $e_{\mu\nu}$ in the SD1 model with correlation functions of $e^*_{\mu\nu}$ in the higher-order SDn models. All SDn models can be written in a compact way³ like the first-order model of Aragone and Khoudair [24], which was the first one to be suggested, namely

$$\mathcal{L}_{\text{SDn}}^{(2)} = \frac{m}{2} e_\mu{}^\nu E^{\mu\alpha} e^*_{\alpha\nu} - \frac{m^2}{2} (e^{\mu\nu} e^*_{\nu\mu} - e e^*) \tag{13}$$

where

$$\begin{aligned} e^*_{\mu\nu}(n = 1) &= e_{\mu\nu}, \\ e^*_{\mu\nu}(n = 2) &= (2 E_\nu{}^\alpha e_{\alpha\mu} + \eta_{\mu\nu} E^{\alpha\beta} e_{\alpha\beta}) / (2m), \\ e^*_{\mu\nu}(n = 3) &= (2 E_\mu{}^\alpha E_\nu{}^\beta e_{(\alpha\beta)} + \eta_{\mu\nu} \square \theta^{\alpha\beta} e_{\alpha\beta}) / (2m^2), \\ e^*_{\mu\nu}(n = 4) &= (E_\mu{}^\alpha \square \theta_\nu{}^\beta + E_\nu{}^\alpha \square \theta_\mu{}^\beta) e_{(\alpha\beta)} / (2m^3) \end{aligned} \tag{14}$$

³ Similarly for the spin-1 case where $n = 1, 2$. The SD2 model is the Maxwell–Chern–Simons theory of [22] and SD1 was suggested in [23]. With the maps $A^*_\mu = A_\mu$ ($n = 1$) and $A^*_\mu = E_{\mu\nu} A^\nu / m$ ($n = 2$) we have

$$\mathcal{L}_{\text{SDn}}^{(1)} = \frac{m}{2} A_\mu E^{\mu\nu} A^*_\nu - \frac{m^2}{2} A^\mu A^*_\mu. \tag{12}$$

The equations of motion from (13) are then given by

$$E_\mu^\alpha e_{\alpha\nu}^* - m(e_{\nu\mu}^* - \eta_{\mu\nu}e^*) = 0 \tag{15}$$

Applying ∂^μ in (15) we have

$$\partial^\mu e_{\nu\mu}^* = \partial_\nu e^*. \tag{16}$$

Notice that (16) holds identically for the higher-order cases, $n = 2, 3, 4$, as a consequence of a local vector symmetry in those cases. Next, by acting with $\epsilon_{\mu\nu\lambda}$ on (15) we conclude that $e_{[\mu\nu]}^* = 0$. If we take the trace of (15) we obtain $e = 0$. Therefore $\partial^\mu e_{\mu\nu}^* = 0$. Then we can rewrite (15) in the form of the Pauli–Lubanski equation which specifies the helicity

$$E_\mu^\alpha e_{\alpha\nu}^* + E_\nu^\alpha e_{\alpha\mu}^* + 2m e_{(\mu\nu)}^* = 0 \tag{17}$$

Now if we apply E_σ^μ in (17) and use it recursively we obtain the Klein–Gordon equation:

$$(\square - m^2)e_{(\sigma\nu)}^* = 0 \tag{18}$$

Therefore $\mathcal{L}_{SDn}^{(2)}$ represent a massive particle of helicity $+2$ for all cases $n = 1, 2, 3$ and 4 .

The SDn models, with $n = 2, 3, 4$, are invariant under the following respective gauge transformations:

$$\begin{aligned} \delta_2 e_{\mu\nu} &= \partial_\mu V_\nu; & \delta_3 e_{\mu\nu} &= \partial_\mu V_\nu + \Lambda_{[\mu\nu]}; \\ \delta_4 e_{\mu\nu} &= \partial_\mu V_\nu + \Lambda_{[\mu\nu]} + \eta_{\mu\nu}\phi \end{aligned} \tag{19}$$

where $\Lambda_{[\mu\nu]} = -\Lambda_{[\nu\mu]}$ stand for arbitrary antisymmetric shifts. If we replace $e_{\mu\nu}$ by its traceless part $\bar{e}_{\mu\nu} = e_{\mu\nu} - \eta_{\mu\nu}e/3$ in $\mathcal{L}_{SDn}^{(2)}$, the models will be invariant under transverse diffeomorphisms and Weyl transformations, i.e.,

$$\delta_W e_{\mu\nu} = \partial_\mu V_\nu^T + \eta_{\mu\nu}\phi \tag{20}$$

with $\partial^\mu V_\mu^T = 0$. So we can define the models:

$$\mathcal{L}_{WSDn}^{(2)}(e_{\mu\nu}) = \mathcal{L}_{SDn}^{(2)}(\bar{e}_{\mu\nu}), \quad n = 2, 3, 4, \tag{21}$$

which lead to the following traceless equations of motion:

$$E_\mu^\alpha e_{\alpha\nu}^*(\bar{e}) + \frac{\eta_{\mu\nu}}{3} E^{\alpha\beta} e_{\alpha\beta}^*(\bar{e}) + m \left[e_{\nu\mu}^*(\bar{e}) - \frac{\eta_{\mu\nu}}{3} e^*(\bar{e}) \right] = 0. \tag{22}$$

Due to (16), which holds identically for $n = 2, 3, 4$, after applying $\epsilon_{\mu\nu\beta}$ on (22) it follows that $e_{\alpha\beta}^*(\bar{h}) = e_{\beta\alpha}^*(\bar{h})$. This implies $E^{\alpha\beta} e_{\alpha\beta}^*(\bar{h}) = 0$. By applying ∂^μ on (22) we have $\partial^\mu e^* = 0$, so e^* must be constant. Thus, the trace of the original equations of motion of the usual models \mathcal{L}_{SDn} , i.e. $e^* = 0$ is recovered up to an integration constant. This is typical for WTDIFF modifications of diffeomorphisms invariant theories. Notice however, that (22) is equivalent to (15) when $e_{\mu\nu}^*$ is replaced by $\bar{e}_{\mu\nu}^* = e_{\mu\nu}^* - \eta_{\mu\nu}e^*/3$. Consequently, we deduce the Klein–Gordon equations, the helicity equation (17) and the Fierz–Pauli conditions, ensuring that \mathcal{L}_{WSDn} have the same particle content of the \mathcal{L}_{SDn} models.

2.4 A note on renormalizability

The models S_{LNMG} , SD3 and SD4 have gravitational nonlinear completions; they correspond, respectively, to NMG, topologically massive gravity (TMG) and higher derivative topologically massive gravity (HDTMG). In the cases of S_{LNMG} and SD3 the highest derivative term, of fourth and third order, respectively, is invariant under WDIFF $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \eta_{\mu\nu}\phi$ while the lowest derivative term (linearized Einstein–Hilbert) is only invariant under DIFF. As argued in [25] this is an obstruction to the renormalizability of their nonlinear completions, since there will always be one metric degree of freedom (absent in the highest derivative term due to the Weyl symmetry) whose propagator is governed by the Einstein–Hilbert term and behaves unfortunately like $1/p^2$ for large momentum.

On the other hand, in the last subsections we have shown that WLNMG and WSD3 correctly describe free massive spin-2 particles. They are obtained from LNMG and SD3 by the replacement $h_{\mu\nu} \rightarrow \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \eta_{\mu\nu}h/3$ which ensures that the Weyl symmetry is present in all sectors of the Lagrangian. In fact, they are invariant under WTDIFF transformations. The nonlinear version of such a replacement, i.e., $g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} \equiv g_{\mu\nu}/(-g)^{1/3}$ leads to unimodular theories $\hat{g} = -1$ which are invariant under Weyl transformations and volume preserving diffeomorphisms ($\nabla^\mu \xi_\mu = 0$). Now we can be sure that both highest and lowest derivative terms in the quadratic part of the action are invariant under WTDIFF by construction. So we may hope that all degrees of freedom behave like $1/P^4$ in the case of unimodular NMG or $1/P^3$ for unimodular topologically massive gravity, respectively. However, there is a subtlety. Due to their Weyl symmetry, the highest derivative terms are unchanged by the replacement $h_{\mu\nu} \rightarrow \bar{h}_{\mu\nu}$. So, they remain invariant under full WDIFF while the Einstein–Hilbert term is only invariant under WTDIFF. Consequently, the linearized K-term (NMG case) and the linearized gravitational Chern–Simons term (TMG case) still have one more local symmetry than the EH term, namely, they are invariant under longitudinal diffeomorphisms: $\delta h_{\mu\nu} = \partial_\mu \partial_\nu \zeta$. Indeed, such symmetry can be used in order to obtain the WSD4 model, which is equivalent to SD4, from the WSD3 model via Noether gauge embedding just like the Weyl symmetry is used to get from SD3 to SD4 as shown in [19]. Therefore, the pure longitudinal sector of the metric will behave like $1/P^2$. So there is no improvement in the renormalizability as we go to the unimodular theories.

The case of HDTMG [19,26], i.e., the nonlinear completion of SD4, is even worse from the point of view of perturbative quantum field theory. Both terms of the quadratic (free) piece of HDTMG, i.e., the linearized K-term and the linearized gravitational Chern–Simons term are invariant under linearized WDIFF while the cubic and higher vertices are only invariant under DIFF. Thus, there is one metric degree of

freedom which only appears in the vertices without any free propagator. At quantum level it gives rise to a nonlinear constraint whose role is unclear. A similar problem also appears in the massless limit of NMG as discussed in [27]. The replacement $g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu}$, which amounts to $h_{\mu\nu} \rightarrow \bar{h}_{\mu\nu}$ in the quadratic ($\mathcal{O}(h^2)$) piece of the theory, turns the DIFF symmetry into Weyl plus volume preserving diffeomorphisms or WTDIFF at linearized level. However, the quadratic theory is invariant under the larger WDIFF transformations, so the pure longitudinal degree of freedom ($\partial_\mu \partial_\nu \zeta$) of the metric only appears in the vertices, leading to an awkward constraint again. The only hope is to start with the SD4 model and examine the addition of cubic and higher vertices invariant under the full set of WDIFF.

3 New massive spin-2 models via a traceless master action

3.1 Self-dual models

Let us consider the first-order self-dual model originally proposed by [24]:

$$S_{SD1}[f] = \int d^3x \left[-\frac{m}{2} f_{\mu\nu} E^{\mu\alpha} f_\alpha{}^\nu - \frac{m^2}{2} (f_{\mu\nu} f^{\nu\mu} - f^2) \right] \tag{23}$$

We can split the non-symmetrical field $f_{\mu\nu}$ into its traceless and trace full parts by making $f_{\mu\nu} = \bar{f}_{\mu\nu} + \eta_{\mu\nu}\phi$ where ϕ is a fundamental scalar field. After that we have

$$S_{SD1}[\bar{f}, \phi] = \int d^3x \left[-\frac{m}{2} \bar{f}_{\mu\nu} E^{\mu\alpha} \bar{f}_\alpha{}^\nu - m \bar{f}_{\mu\nu} E^{\mu\nu} \phi - \frac{m^2}{2} \bar{f}_{\mu\nu} \bar{f}^{\nu\mu} + 3m^2 \phi^2 \right] \tag{24}$$

The traceless Chern–Simons like term is invariant under $\delta \bar{f}_{\mu\nu} = \partial_\mu \xi_\nu^T$ with $\partial^\nu \xi_\nu^T = 0$. Moreover, it is possible to show that it is trivial, it has no particle content by itself. This fact tells us that it might be used as a mixing term in order to construct a master action:

$$S_M[\bar{f}, \bar{e}, \phi] = S_{SD1}[\bar{f}, \phi] + \frac{m}{2} \int d^3x (\bar{f}_{\mu\nu} - \bar{e}_{\mu\nu}) E^{\mu\alpha} (\bar{f}_\alpha{}^\nu - \bar{e}_\alpha{}^\nu). \tag{25}$$

Then it might be possible to interpolate between the first-order self-dual model and alternative traceless descriptions. In order to implement it we define a generating functional by adding a source term to the field $f_{\mu\nu}$,

$$W[f, \phi] = \int \mathcal{D}\bar{f}_{\mu\nu} \mathcal{D}\bar{e}_{\mu\nu} \mathcal{D}\phi \exp i \left\{ S_M[\bar{f}, \bar{e}, \phi] + \int d^3x [\bar{f}_{\mu\nu} \bar{T}^{\nu\mu} + \phi T] \right\} \tag{26}$$

where one can easily see that after the shift $\bar{e}_{\mu\nu} \rightarrow \bar{e}_{\mu\nu} + \bar{f}_{\mu\nu}$ we have basically the first-order self-dual model, since we end up with a completely decoupled trivial Chern–Simons term. On the other hand without any shifts, we would have

$$S_M[\bar{f}, \bar{e}, \phi] = \int d^3x \left[\frac{m}{2} \bar{e}_{\mu\nu} E^{\mu\alpha} \bar{e}_\alpha{}^\nu - m \bar{f}_{\mu\nu} E^\mu{}_\alpha (\bar{e}^{\alpha\nu} + \eta^{\alpha\nu} \phi) - \frac{m^2}{2} \bar{f}_{\mu\nu} \bar{f}^{\nu\mu} + 3m^2 \phi^2 + \bar{f}_{\mu\nu} \bar{T}^{\nu\mu} + \phi T \right] \tag{27}$$

After functionally integrating over $\bar{f}_{\mu\nu}$ and shifting,

$$\bar{f}_{\mu\nu} \rightarrow \bar{f}_{\mu\nu} - \frac{1}{m} E_\nu{}^\lambda (\bar{e}_{\lambda\mu} + \eta_{\lambda\mu} \phi) - \frac{1}{3m} \eta_{\mu\nu} E^{\lambda\sigma} \bar{e}_{\lambda\sigma} + \frac{\bar{T}_{\mu\nu}}{m^2}, \tag{28}$$

we can obtain the new second-order self-dual model given by

$$S_{NSD2}[\bar{e}, \phi] = \int d^3x \left[\frac{m}{2} \bar{e}_{\mu\nu} E^{\mu\alpha} \bar{e}_\alpha{}^\nu + \frac{1}{2} \bar{e}_{\mu\nu} (E^{\mu\beta} E^{\nu\alpha} + \frac{1}{3} E^{\mu\nu} E^{\alpha\beta}) \bar{e}_{\alpha\beta} - \bar{e}_{\mu\nu} \square \theta^{\mu\nu} \phi - \phi (\square - 3m^2) \phi + \bar{e}_{\mu\nu}^* (\bar{e}, \phi) \bar{T}^{\nu\mu} + \phi T + \mathcal{O}(\bar{T}^2) \right] \tag{29}$$

where we have neglected quadratic contributions in the source term, which lead to contact terms when we are comparing correlation functions between SD1 and NSD2. As a byproduct we have obtained the following dual maps:

$$\bar{f}_{\mu\nu} \leftrightarrow \bar{e}_{\mu\nu}^* (\bar{e}, \phi) = -\frac{1}{m} E_\nu{}^\lambda (\bar{e}_{\lambda\mu} + \eta_{\lambda\mu} \phi) - \frac{1}{3m} \eta_{\mu\nu} E^{\lambda\sigma} \bar{e}_{\lambda\sigma}; \quad \phi \leftrightarrow \phi. \tag{30}$$

The model that we have found in (29) is invariant under the gauge transformations $\delta \bar{e}_{\mu\nu} = \partial_\mu \xi_\nu^T$ and $\delta \phi = 0$. Surprisingly one can also note that the set of second-order terms in (29) is invariant under the gauge transformations:

$$\delta \bar{e}_{\mu\nu} = \partial_\mu A_\nu + \partial_\nu B_\mu - \frac{1}{3} \eta_{\mu\nu} \partial^\alpha (A_\alpha + B_\alpha); \quad \delta \phi = \frac{1}{3} \partial^\alpha (A_\alpha + B_\alpha). \tag{31}$$

The second-order sector of (29) is free of particle content, which can be seen by means of a Hamiltonian analysis and also by studying its propagator. Thus, we can use it as mixing terms (S_{mix}) in order to construct another master action with the following structure:

$$S_M = S_{NSD2}(\bar{e}, \phi) - S_{\text{mix}}(\bar{e}_{\mu\nu} - \bar{f}_{\mu\nu}, \phi - \chi), \tag{32}$$

which after the shifts $\bar{f}_{\mu\nu} \rightarrow \bar{f}_{\mu\nu} - \bar{e}_{\mu\nu}$ and $\chi \rightarrow \chi - \phi$ takes us back to the NSD2 model thanks to the triviality of the second-order sector. On the other hand we have

$$\begin{aligned}
 S_M = & -\frac{1}{2} \bar{f}_{\mu\nu} \left(E^{\mu\beta} E^{\nu\alpha} + \frac{1}{3} E^{\mu\nu} E^{\alpha\beta} \right) \bar{f}_{\alpha\beta} \\
 & + (\bar{f}_{\mu\nu} - \bar{e}_{\mu\nu}) \square \theta^{\mu\nu} \chi + \chi \square \chi + \frac{m}{2} \bar{e}_{\mu\nu} E^{\mu\alpha} \bar{e}_{\alpha}{}^{\beta} \\
 & + \bar{e}_{\mu\nu} \left(E^{\mu\beta} E^{\nu\alpha} + \frac{1}{3} E^{\mu\nu} E^{\alpha\beta} \right) \bar{f}_{\alpha\beta} \\
 & + 3m^2 \phi^2 - \phi \square \theta^{\mu\nu} \bar{f}_{\mu\nu} - 2\phi \square \chi + \phi T \\
 & - \frac{1}{m} \bar{e}_{\mu\nu} E^{\mu}{}_{\alpha} \bar{T}^{\alpha\nu} + \frac{1}{m} \phi E_{\mu\nu} \bar{T}^{\mu\nu}. \tag{33}
 \end{aligned}$$

After functionally integrating over $\bar{e}_{\mu\nu}$ and the scalar ϕ and then defining $f_{\mu\nu} = \bar{f}_{\mu\nu} + \eta_{\mu\nu} \chi$ we arrive at a new, and unusual, new self-dual model which contains second-, third- and fourth-order terms in derivatives:

$$\begin{aligned}
 S_{\text{NSD4}} = & -\frac{1}{2} f_{\mu\nu} \left(E^{\mu\beta} E^{\nu\alpha} + \frac{1}{3} E^{\mu\nu} E^{\alpha\beta} \right) f_{\alpha\beta} \\
 & - \frac{1}{2m} f_{\mu\nu} \square \left(\theta^{\mu\alpha} E^{\nu\beta} - \frac{2}{3} \theta^{\mu\nu} E^{\alpha\beta} \right) f_{\alpha\beta} \\
 & - \frac{1}{12m^2} f_{\mu\nu} \square^2 \theta^{\mu\nu} \theta^{\alpha\beta} f_{\alpha\beta} + f_{\mu\nu}^* T^{\nu\mu}, \tag{34}
 \end{aligned}$$

where we have defined the dual field:

$$\begin{aligned}
 f_{\mu\nu}^* = & \frac{1}{m^2} \left(E_{\mu\alpha} E_{\nu\beta} + \frac{1}{3} E_{\nu\mu} E_{\alpha\beta} \right) f^{\alpha\beta} \\
 & - \frac{1}{6m^3} \square E_{\nu\mu} \theta_{\alpha\beta} f^{\alpha\beta} + \frac{1}{2m^2} \square \eta_{\mu\nu} \theta_{\alpha\beta} f^{\alpha\beta}. \tag{35}
 \end{aligned}$$

One can check that correlation functions of $e_{\mu\nu}$ in the first-order self-dual model of [24] coincide with correlation functions of the dual field $f_{\mu\nu}^*$ in the model S_{NSD4} up to contact terms. The model (34) is invariant under the gauge transformation $\delta f_{\mu\nu} = \partial_{\mu} A_{\nu} + \partial_{\nu} B_{\mu}$, which leaves $f_{\mu\nu}^*$ also invariant.

Remarkably, the model S_{NSD4} can be written in the compact form (13) with the help of (35). Although the fourth-order term of (34) has no particle content, we have not been able to produce any higher (than four) self-dual model out of S_{NSD4} . It seems that the highest number of derivatives in spin-2 models in $D = 2 + 1$ is indeed four.

3.2 Scalar–tensor new massive gravities

One way of obtaining the new massive gravity of [17] is to start with the massless linearized Einstein–Hilbert (LEH) theory in $D = 3 + 1$ and perform its Kaluza–Klein (KK) dimensional reduction leading to the massive Fierz–Pauli theory in $D = 2 + 1$ from which we obtain NMG as a dual model via a master action technique [20] where the mixing term between old and new (dual) fields is the full Einstein–Hilbert theory; see [19]. If we replace the LEH by the WTD-

IFF model in $D = 3 + 1$, the KK dimensional reduction leads to a massive model where one of the Stueckelberg fields cannot be gauged away; see [11]. We may choose to end up with a lower dimensional theory which corresponds to the FP model after the replacement $h_{\mu\nu} \rightarrow \bar{h}_{\mu\nu} + \eta_{\mu\nu} \phi$. This is physically equivalent to the usual FP model, since it could have been obtained by first introducing a scalar Stueckelberg field $h_{\mu\nu} \rightarrow h_{\mu\nu} + \eta_{\mu\nu} \phi$, altogether with a Weyl symmetry, followed by the harmless shift $\phi \rightarrow \phi - h/3$. This new form of the FP theory inspires us to define a new master action with a traceless mixing term:

$$\begin{aligned}
 \mathcal{L}_M = & \frac{1}{2} (\bar{h}_{\mu\nu} + \eta_{\mu\nu} \phi) E^{\mu\alpha} E^{\nu\beta} (\bar{h}_{\alpha\beta} + \eta_{\alpha\beta} \phi) \\
 & - \frac{m^2}{2} \left[(\bar{h}_{\mu\nu} + \eta_{\mu\nu} \phi)^2 - (3\phi)^2 \right] \\
 & - \frac{1}{2} (\bar{h}_{\mu\nu} - \bar{f}_{\mu\nu}) E^{\mu\alpha} E^{\nu\beta} (\bar{h}_{\alpha\beta} - \bar{f}_{\alpha\beta}). \tag{36}
 \end{aligned}$$

Since the traceless LEH theory has no propagating degree of freedom, after the shift $\bar{f}_{\mu\nu} \rightarrow \bar{f}_{\mu\nu} + \bar{h}_{\mu\nu}$ the fields decouple and it is clear that the particle content of (36) is the same as the one of the massive FP model, i.e., one helicity doublet +2 and -2. On the other hand, after integrating over $\bar{h}_{\mu\nu}$ in (36) we have a quadratic scalar–tensor model depending upon $(\phi, \bar{f}_{\mu\nu})$. If we suppose that such a theory comes from the singular replacement $f_{\mu\nu} \rightarrow \bar{f}_{\mu\nu}$ of a full reparametrization invariant model, its simplest nonlinear completion would be

$$\begin{aligned}
 \mathcal{L}_{\text{SNMG}} = & 2\sqrt{-g} \left[-R + \frac{1}{m^2} \left(R_{\mu\nu}^2 - \frac{1}{3} R^2 \right) \right. \\
 & \left. + \frac{1}{2} \phi r(\square) R + \frac{1}{2} \phi s(\square) \phi \right] \tag{37}
 \end{aligned}$$

where $g_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu}$ and

$$r(\square) = -\frac{\square}{3m^2}; \quad s(\square) = 3m^2 - \square + \frac{\square^2}{3m^2}. \tag{38}$$

The model (37) is a scalar modification of NMG. This becomes clearer after introducing an auxiliary scalar field which allows us, using (38), to rewrite (37) in the local form:

$$\begin{aligned}
 \mathcal{L}_{\text{SNMG}} = & 2\sqrt{-g} \left[-R + \frac{1}{m^2} \left(R_{\mu\nu}^2 - \frac{3}{8} R^2 \right) \right. \\
 & \left. + \chi (R - \square\phi) - \frac{3}{2} m^2 \chi^2 - \frac{1}{2} \phi (\square - 3m^2) \phi \right]. \tag{39}
 \end{aligned}$$

Following [17] we can eventually introduce an auxiliary symmetric field and bring (39) to a fully second-order form.

The NMG itself corresponds to (37) with $(r, s) = (1, 3m^2)$. In what follows we perform an analysis of the analytic structure of the linearized version of (37) in search for other viable (unitary and non-tachyonic) scalar deformations of NMG. The linearized version of (37), using the more common notation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, can be conveniently written

as

$$\mathcal{L} = -\frac{h_{\mu\nu}\square h^{\mu\nu}}{2} + \frac{h\square h}{2} - (\partial^\mu h_{\mu\nu})^2 + \partial^\mu h \partial^\alpha h_{\alpha\mu} + h_{\mu\nu} \frac{\square^2}{2m^2} (P_{SS}^{(2)})^{\mu\nu\alpha\beta} h_{\alpha\beta} + A (\partial^\mu \partial^\nu h_{\mu\nu} - \square h)^2 + \phi s(\square)\phi + \phi r(\square) (\partial^\mu \partial^\nu h_{\mu\nu} - \square h) \tag{40}$$

where $s(\square)$ and $r(\square)$ are now arbitrary functions of the d'Alembertian, while A is an arbitrary constant. In the case of (38) we had $A = 1/12m^2$. We have used

$$\left[\frac{2\sqrt{-g}}{m^2} \left(R_{\mu\nu}^2 - \frac{3}{8} R^2 \right) \right]_{hh} = h_{\mu\nu} \frac{\square^2}{2m^2} (P_{SS}^{(2)})^{\mu\nu\alpha\beta} h_{\alpha\beta} \tag{41}$$

with the spin-2 and spin-0 (for later use) projection operators given by

$$\begin{aligned} (P_{SS}^{(2)})^{\lambda\mu}_{\alpha\beta} &= \frac{1}{2} (\theta^\lambda_\alpha \theta^\mu_\beta + \theta^\mu_\alpha \theta^\lambda_\beta) \\ &\quad - \frac{\theta^{\lambda\mu} \theta_{\alpha\beta}}{D-1}, \quad (P_{SS}^{(0)})^{\lambda\mu}_{\alpha\beta} \\ &= \frac{\theta^{\lambda\mu} \theta_{\alpha\beta}}{D-1}. \end{aligned} \tag{42}$$

After a Gaussian integration of the scalar field, we rewrite the Lagrangian as follows:

$$\begin{aligned} \mathcal{L}_{\text{SNMG}} &= -(\partial^\mu h_{\mu\nu})^2 \\ &\quad + \partial^\mu h [1 + 2\square F(\square)] \partial^\alpha h_{\alpha\mu} \\ &\quad + (\partial_\mu \partial_\nu h^{\mu\nu}) F(\square) (\partial_\alpha \partial_\beta h^{\alpha\beta}) \\ &\quad + h \left[\frac{\square}{2} + \square^2 F(\square) \right] h - \frac{h_{\mu\nu} \square h^{\mu\nu}}{2} \\ &\quad + h_{\mu\nu} \left(\frac{\square^2 P_{SS}^{(2)}}{2m^2} \right)^{\mu\nu\alpha\beta} h_{\alpha\beta}, \end{aligned} \tag{43}$$

where

$$F(\square) = A - \frac{r(\square)^2}{4s(\square)}. \tag{44}$$

The Lagrangian (43) can be further written in terms of a four index differential operator $\mathcal{L}_{\text{SNMG}} \equiv h^{\mu\nu} G_{\mu\nu\alpha\beta} h^{\alpha\beta}$. The inverse G^{-1} does not exist due to DIFF symmetry. After adding the de Donder gauge fixing term $\mathcal{L}_{\text{GF}} = \lambda (\partial^\mu h_{\mu\nu} - \partial_\nu h/2)^2$ and suppressing the indices we have

$$G^{-1} = \frac{2m^2 P_{SS}^{(2)}}{\square(\square - m^2)} + \frac{2P_{SS}^{(0)}}{\square[1 + 4\square F(\square)]} + \dots \tag{45}$$

where dots stand for terms which vanish when we saturate G^{-1} with conserved sources and build a gauge invariant amplitude. The NMG case is recovered at $F(\square) = 0$. The two point amplitude in the momentum space is given by

$$\mathcal{A}(k) = -\frac{i}{2} T_{\mu\nu}^*(k) (G^{-1})^{\mu\nu\alpha\beta}(k) T_{\alpha\beta}(k). \tag{46}$$

Here $G^{-1}(k) = G^{-1}(\partial_\mu \rightarrow i k_\mu)$ and $k^\mu T_{\mu\nu} = 0$. More explicitly we have

$$\mathcal{A}(k) = i \left[\frac{S^{(0)}}{k^2 [1 - 4k^2 F(-k^2)]} - \frac{m^2}{k^2(k^2 + m^2)} S^{(2)} \right]. \tag{47}$$

With $k^2 = k_\mu k^\mu$ and

$$S^{(0)} = T_{\mu\nu}^* (P_{SS}^{(0)})^{\mu\nu\alpha\beta} T_{\alpha\beta} = \frac{|T|^2}{2}, \tag{48}$$

$$S^{(2)} = T_{\mu\nu}^* (P_{SS}^{(2)})^{\mu\nu\alpha\beta} T_{\alpha\beta} = T_{\mu\nu}^* T^{\mu\nu} - \frac{|T|^2}{2}, \tag{49}$$

where $T = \eta_{\mu\nu} T^{\mu\nu} = -T_{00} + T_{ii}$ is the trace of the source in momentum space.

The analytic structure of $\mathcal{A}(k)$ determines the particle content of the theory. Physical particles correspond to simple poles with residues with positive imaginary part. First we look at the massless pole $k^2 = 0$. Since both $S^{(2)}$ and $S^{(0)}$ are Lorentz invariant we can choose the convenient frame $k^\mu = (k, \epsilon, k)$, at the end we take $\epsilon \rightarrow 0$. In [28] we have shown that, in such a frame, up to terms of order ϵ and higher, we may write

$$S^{(0)} = S^{(2)} = |T_{11}|^2/2. \tag{50}$$

Therefore, requiring

$$\lim_{k^2 \rightarrow 0} k^2 F(-k^2) = 0 \Leftrightarrow \lim_{k^2 \rightarrow 0} \frac{k^2 [r(-k^2)]^2}{s(-k^2)} = 0, \tag{51}$$

the imaginary part of the residue at $k^2 = 0$ vanishes and we get rid of the massless pole,

$$I_0 = \Im \lim_{k^2 \rightarrow 0} k^2 \mathcal{A}(k) = S^{(0)} - S^{(2)} = 0. \tag{52}$$

The same mechanism works in the NMG case; see [29].

Now we look at possible massive poles $k^2 = -\tilde{m}^2$. We choose the rest frame $k^\mu = (\tilde{m}, 0, 0)$. From $k^\mu T_{\mu\nu} = 0$ one can show [28] that, up to terms of order ϵ and higher,

$$S^{(2)} = 2|T_{12}|^2 + \frac{1}{2}|T_{11} - T_{22}|^2, \tag{53}$$

$$S^{(0)} = |T_{11}|^2 + |T_{22}|^2 - \frac{1}{2}|T_{11} - T_{22}|^2. \tag{54}$$

We see that $S^{(2)} > 0$ while $S^{(0)}$ has no definite sign. Thus, if we have any massive pole coming from $[1 - 4k^2 F(-k^2)] = 0$, with $\tilde{m} \neq m$, its residue will be proportional to $S^{(0)}$ and we are doomed to have a ghost. It is impossible to have a physical massive scalar particle with $\tilde{m} \neq m$. The case $k^2 = \tilde{m}^2 = m^2$ is subtler since the residue acquires contributions from both the spin-2 and the spin-0 sectors. Let us suppose that

$$1 - 4k^2 F(-k^2) \equiv G(k^2)(k^2 + m^2) \tag{55}$$

with some continuous function $G(k^2)$. Consequently, we have the imaginary part of the residue:

$$I_m \equiv \Im \lim_{k^2 \rightarrow -m^2} (k^2 + m^2) \mathcal{A}(k) = S^{(2)} - \frac{S^{(0)}}{m^2 G(-m^2)}. \tag{56}$$

If we take an arbitrary real constant a , we see from (53) and (54) that $S^{(2)} + a S^{(0)} > 0$ whenever $0 \leq a \leq 1$, consequently we must have $G(-m^2) \leq -1/m^2$. On the other hand, from (51) and (55) we get $G(0) = 1/m^2$. From those two points and the continuity of $G(k^2)$ it is clear that $G(-bm^2) = 0$ with some $0 < b < 1$. However, as we have argued before, we are not allowed to have a massive scalar particle with $\tilde{m} \neq m$. So (55) cannot be true and there cannot be any contribution to the residue at $k^2 = -m^2$ coming from the denominator of $S^{(0)}$ in $\mathcal{A}(k)$. Thus, we are left with $I_m = S^{(2)} > 0$ and we are left with only one massive spin-2 particle in the spectrum just like the NMG case.

The previous arguments amounts to the requirement that the numerator of the function

$$H(\square) \equiv 1 + 4\square F(\square) = \frac{(1 + 4A\square) s(\square) - \square [r(\square)]^2}{s(\square)} \tag{57}$$

be independent of \square . Thus, the polynomials $r(\square)$ and $s(\square)$ must be such that

$$[r(\square)]^2 = 4A s(\square) + \frac{[s(\square) - s_0]}{\square}. \tag{58}$$

Here A is an arbitrary constant and $s_0 = s(\square = 0)$

After integration over the scalar field in (37) using (58), we have the following class of spin-0 nonlocal deformations of NMG:

$$\begin{aligned} \mathcal{L}_{\text{NL-NMG}} = & -\sqrt{-g} R + \frac{1}{m^2} \sqrt{-g} \left(R_{\mu\nu}^2 - \frac{3}{8} R^2 \right) \\ & - \sqrt{-g} R \frac{[s(\square) - s_0]}{8\square s(\square)}. \end{aligned} \tag{59}$$

The case $s(\square) = s_0$ corresponds to the NMG [17]. The reader can check that $r(\square)$, $s(\square)$ and A given in (38) and in the text after (40), respectively, satisfy (58).

Another special case is $s_0 = 0$ where the function $H(\square)$ vanishes. Such momentum independent singularity in G^{-1} indicates the presence of a spin-0 local symmetry, in fact we have a Weyl symmetry. The corresponding model has been found before in our previous work [28]. It corresponds to making the Stueckelberg substitution $h_{\mu\nu} \rightarrow h_{\mu\nu} + \eta_{\mu\nu} \phi$ in the LNMG and then build its simplest nonlinear completion. Since this is not equivalent to first taking the nonlinear NMG and then making $g_{\mu\nu} \rightarrow e^\phi g_{\mu\nu}$, we expect that the linearized unitarity of the $s_0 = 0$ case breaks down at nonlinear level,

since ϕ stops being a pure gauge degree of freedom at non-linear level, so the Weyl symmetry only exists in the linear theory.

Regarding the other solutions to (58), since they are not associated with any local symmetry, it is not clear whether the unitarity of the linearized model is broken in the nonlinear theory (37).

4 Conclusion

Here we have examined different issues regarding the Weyl and transverse diffeomorphism (WTDIFF) symmetry in $D = 2 + 1$ massive spin-2 theories as well as their nonlinear analogues (unimodular theories).

The issue of imposing symmetry breaking conditions at the action level is tricky; see [12]. In particular, the triviality of Einstein–Hilbert gravity in $D = 2 + 1$ is lost in its unimodular ($g = -1$) version as we have briefly commented in the beginning of Sect. 2 using the linearized theory. Instead of flat space we have now a maximally symmetric space in general which may include BTZ black holes [30] in the non-linear case, depending on the sign of an integration constant which plays the role of a cosmological constant.

We have explicitly checked that WTDIFF versions of massive spin-2 theories (one and two helicities) are fully consistent. In the special cases of the third and fourth order (in derivatives) self-dual (one helicity) theories, they correspond to linearized versions of unimodular topologically massive gravity and unimodular higher derivative topologically massive gravity. Likewise, in the case of a parity doublet we have a linearized version of a unimodular new massive gravity.

At the end of Sect. 2 we have examined the issue of renormalizability and Weyl symmetry. We argued that although both highest and lowest derivative terms in the free (quadratic) sector of unimodular TMG and unimodular NMG are Weyl invariant, we still have a mismatch of local symmetries, which is dangerous for the renormalizability as pointed out in [25]. The Einstein–Hilbert theory is only invariant under WTDIFF (linearized theory) while the highest derivative term (gravitational Chern–Simons term or the K-term) is invariant under full WDIFF. Thus, the pure longitudinal degree of freedom $h_{\mu\nu} \sim \partial_\mu \partial_\nu \phi$ only appears in the Einstein–Hilbert term. Consequently, it propagates like $1/p^2$ at large momentum and no improvement is achieved for renormalizability in unimodular theories. The mismatch between the symmetries of the highest derivative term and the lower one seems to be unavoidable. In [31] we have pointed out that there exists a massive spin-2 model in $D = 2 + 1$ described by a nonsymmetric tensor $e_{\mu\nu}$, see [32], where both the second- and the fourth-order terms are Weyl invariant, however, only the fourth-order one is invariant under antisymmetric shifts. The mismatch also occurs in the higher-

dimensional analog of the linearized NMG; see [33] and [34]. This is the higher derivative analog of the usual breakdown of gauge symmetries by mass terms as in the Proca ($s = 1$) and massive Fierz–Pauli ($s = 2$) theories. The only hope is to find a theory where the lowest derivative term has already more than two derivatives.

It is well known that massive theories in D dimensions can be obtained from $D + 1$ massless theories via Kaluza–Klein dimensional reduction. From the massless spin-2 linearized Einstein–Hilbert theory in $D = 3 + 1$ one can obtain the massive spin-2 Fierz–Pauli theory in $D = 2 + 1$. From the latter theory one can derive, via the master action approach of [20], the fourth-order linearized new massive gravity theory [17]. A key point in this approach is the absence of propagating degrees of freedom of the Einstein–Hilbert theory in $D = 2 + 1$, which works like a mixing term between old and new (dual) fields in the master action approach. If, however, we replace the linearized EH theory in $D = 3 + 1$ as starting point by the WTDIFF massless spin-2 theory, its dimensional reduction⁴ [11] leads to the massive FP theory with the replacement $h_{\mu\nu} \rightarrow \bar{h}_{\mu\nu} + \eta_{\mu\nu}\phi$. In Sect. 3, starting from the latter theory we have defined a noncanonical (traceless) master action where the mixing term is the EH action for the traceless field $\bar{h}_{\mu\nu}$. This leads to a scalar–tensor modification of the NMG theory. We have shown it belongs to a more general class of consistent (unitary at quadratic level) scalar–tensor modifications of NMG. The consistency of their nonlinear completion (37) demands further investigations.

From the point of view of dimensional reduction the appearance of an extra scalar field in the massive model is a consequence of the fact that the gauge parameter of the massless higher dimensional theory satisfies the scalar constraint $\partial^M \xi_M^T = 0$ with $M = 0, \dots, D$. This makes the lower dimensional gauge parameters not independent, consequently we are not able to eliminate all the Stueckelberg fields and we may choose to be left with one scalar Stueckelberg field; see [11]. This is similar to the massive spin-3 theory, which requires an extra scalar field besides a totally symmetric rank-3 tensor $\phi_{\alpha\beta\gamma}$ due to the constrained symmetry of the higher dimensional massless theory $\delta\phi_{\alpha\beta\gamma} = \partial_{(\alpha}\bar{\xi}_{\beta\gamma)}$ where $\eta^{\mu\nu}\xi_{\mu\nu} = 0$.

Still in Sect. 3 we have also applied the traceless master action approach on the first-order self-dual model of [24] and derived a new second-order model (NSD2) which, in its turn, has given rise to a new fourth-order model (NSD4). The unusual NSD4 model contains second-, third- and fourth-order terms. Remarkably, the SDn models and also NSD4 can

all be written in the compact form (13), which also works in the spin-1 case (see footnote 3). We believe that such compact formulas may exist also for higher spins, which might help in filling some gaps in the chain of spin-3 and higher spin self-dual models.

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