

Unified first law and some general prescription: a redefinition of surface gravity

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Abstract The paper contains an extensive study of the unified first law (UFL) in the Friedmann–Robertson–Walker spacetime model. By projecting the UFL along the Kodama vector the second Friedmann equation can be obtained. Also studying the UFL on the event horizon it is found that the Clausius relation cannot be obtained from the UFL by projecting it along the tangent to the event horizon as it can be for the trapping horizon. However, it is shown in the present work that Clausius relation can be obtained by projecting the UFL along the Kodama vector on the horizon and the result is found to be true for any horizon. Finally motivated by the Unruh temperature for the Rindler observer, surface gravity is redefined and a Clausius relation is obtained from the UFL by projecting it along a vector analogous to the Kodama vector.

1 Introduction

In the 1970s Hawking [1] showed that a black hole (BH) is not totally black; rather it emits thermal radiation by a combined application of quantum mechanics and general relativity at semi-classical level. Interestingly, the temperature of the radiation (known as the Hawking temperature) and the entropy of the horizon (known as the Bekenstein entropy) have a certain universality in the sense that surface gravity (proportional to the Hawking temperature) and the horizon area (proportional to the Bekenstein entropy) [1, 2] are purely geometric entity characterized by the spacetime geometry. Also this entropy and temperature are related to the BH mass through the first law of BH thermodynamics: $dM = T dS$ [3]. Moreover, this fantastic discovery gave rise to (i) a speculation for a deep interrelationship between gravity theories

and thermodynamics and (ii) a clue to the nature of quantum gravity.

However, the first possibility came true in 1995 when Jacobson [4] derived Einstein equations from the Clausius relations: $\delta Q = T dS$ for the whole local Rindler causal horizon through a spacetime point ($\delta Q \rightarrow$ the energy flux, $T \rightarrow$ Unruh temperature seen by the accelerated observer just inside the horizon). Subsequently, Padmanabhan [5, 6] was able to show the first law of thermodynamics on the horizon, starting from the Einstein equations, for a general static spherically symmetric spacetime.

Assuming the universe to be a thermodynamical system, this nice interrelation between Einstein equations and thermodynamic laws has been extended in the context of cosmology. For a homogeneous and isotropic FRW model the Hawking temperature at the apparent horizon: $T_A = \frac{1}{2\pi R_A}$ ($R_A =$ geometric radius of the apparent horizon) was first derived by Cai et al. [7], it was found [8–10] that the Friedmann equations are equivalent to the first law of thermodynamics on the apparent horizon having Hawking temperature T_A and Bekenstein entropy $S_A = \frac{\pi R_A^2}{G}$. Then in higher dimensional spacetime, this equivalence was established for gravity with the Gauss–Bonnet term [11, 12] and for Lovelock gravity [13, 14].

On the contrary, the situation is totally different for a universe bounded by the event horizon (which exists only in an accelerating phase of the expansion). Wang et al. [15] showed that a universe bounded by an apparent horizon is a perfect thermodynamical system as both first and second law of thermodynamics hold for a perfect fluid with constant equation of state and holographic dark energy models. However, according to them both thermodynamical laws failed to be satisfied on the event horizon. Then assuming the first law, Mazumdar and Chakraborty [16–18] were able to satisfy the second law of thermodynamics on the event horizon with some realistic restrictions. In analogy with the apparent horizon, the entropy and temperature at the event horizon were

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chosen as $S_E = \frac{\pi R_E^2}{G}$ and $T_E = \frac{1}{2\pi R_E}$. Later, it was found [19,20] that the temperature taken on the event horizon (i.e. $T_E = \frac{1}{2\pi R_E}$) is not correct and taking the corrected form (i.e. $T_E^{(m)} = \frac{R_E}{2\pi R_A^2}$) the thermodynamics on the event horizon has been studied. It has been shown [21] that for the two choices

$$(a) S_E^{(B)} = \frac{\pi R_E^2}{G}, \quad T_E = T_E^{(g)} = \alpha T_E^{(m)} = \frac{\alpha R_E}{2\pi R_A^2},$$

$$\alpha = \frac{\dot{R}_A/R_A}{\dot{R}_E/R_E}, \quad (1)$$

$$(b) S_E^{(m)} = \beta S_E^{(B)}, \quad T_E = T_E^{(m)}, \quad \beta = \frac{2}{R_E^2} \int R_E^2 \frac{dR_A}{R_A},$$

both thermodynamical laws are satisfied on the event horizon. Also for infinitesimal thermal fluctuation, there is a logarithmic correction to the Bekenstein entropy in the second choice [21]. However, there are no other motivations and proof for the above two choices in Eq. (1).

On the other hand, in the context of a dynamical BH, Hayward [22–25] introduced the notion of a trapping horizon and proposed a method to deal with the thermodynamics associated with a trapping horizon. According to him, for spherically symmetric spacetimes, the Einstein equations can be rewritten in a form termed the “unified first law”. Then projecting this UFL along a trapping horizon, the first law of thermodynamics was derived. Further, from the point of view of universal thermodynamics we consider our universe as a non-stationary gravitational system and the FRW model may be considered as a dynamical spherically symmetric spacetime. Moreover, in the FRW model we have only an inner trapping horizon, which coincides with the apparent horizon [22–27] and the Friedmann equations are equivalent to the UFL on the apparent horizon [26,28]. Also the projection of the UFL along the tangent to the apparent horizon gives the Clausius relation [26].

Further, there is no preferred time coordinate in an evolving time dependent spacetime as there is no longer any (asymptotically timelike) Killing vector field. To resolve this problem, Kodama [29] came forward with a geometrically natural divergence-free vector field which exists in any time-dependent spherically symmetric spacetime. This vector in the literature is commonly known as the Kodama vector, and it identifies a natural timelike direction outside a dynamic BH. Also there is a conserved current associated with this vector field [29,30].

We have organized this paper as follows. In Sect. 2 we derive the Friedmann equations from the UFL projecting along the Kodama vector. In Sect. 3 we obtain the Clausius relation using the Kodama vector. Section 4 leads to a redefinition of surface gravity. Section 5 is devoted to conclusions.

2 Friedmann equations from the UFL

The line element for FRW spacetime can be written as [31]

$$ds^2 = h_{ab} dx^a dx^b + R^2 d\Omega_2^2,$$

$$= -dt^2 + \frac{a^2}{1-kr^2} dr^2 + R^2 d\Omega_2^2, \quad (2)$$

where $R = ar$ is the area radius, $h_{ab} = \text{diag}(-1, \frac{a^2}{1-kr^2})$ is the metric on the two-space orthogonal to the spherical symmetry. Using null coordinates (ξ_+, ξ_-) the above metric can be written as

$$ds^2 = -2 d\xi_+ d\xi_- + R^2 d\Omega_2^2, \quad (3)$$

with

$$\frac{\partial}{\partial \xi_{\pm}} = -\sqrt{2} \left(\frac{\partial}{\partial t} \mp \frac{\sqrt{1-kr^2}}{a} \frac{\partial}{\partial r} \right),$$

as future pointing null vectors.

Now according to Hayward [22–25], the trapping horizon (R_T) is defined as $(\partial_t R)_{R=R_T} = 0$, i.e.

$$R_T = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}} = R_A, \quad k = 0, \pm 1 \quad (4)$$

where R_A is the geometric radius of the apparent horizon. For the trapping horizon (i.e. the apparent horizon for FRW model) the surface gravity is defined as [32]

$$\kappa_T = \frac{1}{2\sqrt{-h}} \partial_a (\sqrt{-h} h^{ab} \partial_b R) \Big|_{R=R_T}. \quad (5)$$

However, in the present work we are examining the equivalence of the UFL and the Friedmann equations over any horizon and, consequently, we are assuming the above definition of surface gravity (i.e. Eq. (5)) is true for any horizon, i.e. for any horizon (having area radius R) the surface gravity is defined as

$$\kappa = \frac{1}{2\sqrt{-h}} \partial_a (\sqrt{-h} h^{ab} \partial_b R),$$

or in explicit form

$$\kappa = - \left(\frac{R}{R_A} \right)^2 \left(\frac{1 - \dot{R}_A/2HR_A}{R} \right). \quad (6)$$

Now the total energy inside the horizon is a purely geometric quantity, related to the structure of the spacetime and to the Einstein equations [32]. According to Misner and Sharp [22–27], the total energy is given by

$$E = \frac{R}{2G} (1 - h^{ab} \partial_a R \partial_b R), \quad (7)$$

which on simplification gives

$$E = \frac{R^3}{2G} \left(H^2 + \frac{k}{a^2} \right) = \frac{R^3}{2GR_A^2}. \tag{8}$$

According to Hayward [22–25], the UFL,

$$dE = A\psi + WdV, \tag{9}$$

is nothing but a rearrangement of the Einstein equations. In the above, A and V stand for the area and volume bounded by the horizon, and the work density,

$$W = -\frac{1}{2}T^{ab}h_{ab}, \tag{10}$$

is regarded as the work done by a change of the horizon, and the energy-supply term

$$\Psi_a = T_a^b \partial_b R + W \partial_a R, \tag{11}$$

determines the total energy flow (i.e. $\delta Q = A\psi$) through the horizon and $\psi = \Psi_a dx^a$.

We now introduce the Kodama vector for the present FRW model. It is defined as [29,33]

$$K^a = \epsilon^{ab} \nabla_b R, \tag{12}$$

where ϵ^{ab} is the usual Levi-Civita tensor in the 2D radial-temporal plane (i.e. normal to the spherical symmetry). For the present homogeneous and isotropic FRW model

$$\epsilon_{lm} = a(t)(dt)_l \wedge (dr)_m, \tag{13}$$

and

$$K^b = \left[-a \left(\frac{\partial}{\partial t} \right)^b + HR \left(\frac{\partial}{\partial r} \right)^b \right]. \tag{14}$$

Note that the Kodama vector is very similar to the Killing vector $\left(\frac{\partial}{\partial t} \right)^a$ in the de Sitter space. Also Kodama vector takes the role of the timelike Killing vector (in stationary BH spacetime) for dynamical BH and FRW spacetime. Further, it can be used as a preferred time evolution vector field in spherically symmetric dynamical systems.

From Eqs. (10) and (11) the explicit forms of the work density and energy-supply one form for the present model are

$$W = \frac{1}{2}(\rho - p), \tag{15}$$

and

$$\psi = \left(\frac{\rho + p}{2} \right) \{-HRdt + adr\}. \tag{16}$$

Hence we have

$$WdV = 2\pi R^2(\rho - p) \{HRdt + adr\}, \tag{17}$$

and

$$A\psi = 2\pi R^2(\rho + p) \{-HRdt + adr\}. \tag{18}$$

Also, from Eq. (8) we obtain

$$dE = \frac{1}{2GR_A^3} \left[R^3 (3HR_A - 2\dot{R}_A) dt + 3R^2 R_A a dr \right]. \tag{19}$$

We shall now show that by projecting the UFL along the Kodama vector gives the second Friedmann equation, in general.

For the above one forms using the scalar product with the Kodama vector we have

$$\langle dE, K^b \rangle = -\frac{aR^3 H}{G} \left(\dot{H} - \frac{k}{a^2} \right). \tag{20}$$

Now

$$\langle A\psi, K^b \rangle = 4\pi R^3 Ha(\rho + p), \tag{21}$$

and

$$\langle WdV, K^b \rangle = 0.$$

Thus, projecting the UFL along the Kodama vector gives

$$\dot{H} - \frac{k}{a^2} = -4\pi G(\rho + p), \tag{22}$$

which is nothing but the second Friedmann equation on any arbitrary horizon.

We shall now show that the first Friedmann equation can also be obtained from the UFL by projecting it along a vector orthogonal to the Kodama vector, namely

$$U^\mu = \left(-\frac{1}{a^2 R}, \frac{1}{Ha^3 R^2}, 0, 0 \right).$$

Clearly the vector U^μ lies on the radial-temporal plane and it has the following properties:

- (i) The vector may be spacelike, timelike or null depending on R .
- (ii) It is divergence-free in nature (i.e. $\nabla_\mu U^\mu = 0$) and there is a current associated with the vector U^μ given by the relation $\xi^\mu = G^{\mu\nu} U_\nu$. So from the Bianchi identity (i.e. $\nabla_\mu G^{\mu\nu} = 0$) the vector ξ^μ is conserved, i.e.

$$\nabla_\mu \xi^\mu = 0.$$

The scalar product of the individual one-form terms on both sides of the UFL with U^μ gives

$$\langle dE, U^\mu \rangle = \frac{R^2}{2GR_A^3 a^2} \left[- (3HR_A - 2\dot{R}_A) + \frac{3R_A}{HR^2} \right], \tag{23}$$

$$\langle A\psi, U^\mu \rangle = \frac{2\pi R^2}{a^2} (\rho + p) \left\{ H + \frac{1}{HR^2} \right\}, \tag{24}$$

and

$$\langle WdV, U^\mu \rangle = \frac{2\pi R^2}{a^2} (\rho - p) \left\{ -H + \frac{1}{HR^2} \right\}. \tag{25}$$

Hence projecting the UFL along U^μ and after some algebra we obtain the first Friedmann equation, i.e.

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho. \tag{26}$$

3 Clausius relation from the UFL

Further, it has been shown in the literature that the first law of BH thermodynamics can be obtained by projecting the UFL along the trapping horizon [22–26], i.e.

$$\langle A\psi, z \rangle = \frac{\kappa}{8\pi G} \langle dA, z \rangle, \tag{27}$$

where z is a vector tangential to the trapping horizon.

We shall now show that the situation is not so easy in the case of an event horizon (EH). The area radius of the EH is given by

$$R_E = a \int_t^\infty \frac{dt}{a}. \tag{28}$$

(Note that the improper integral converges for an accelerating phase of the universe.)

The normal vector to the null hypersurface $R - a \int_t^\infty \frac{dt}{a} = 0$ is given by $n_a = (-1, a, 0, 0)$, a null vector.

From the property of the null vector, n_a is also tangential to the (null) event horizon hypersurface. Then one can easily see that the Clausius relation, i.e. Eq. (27) is not satisfied for the event horizon. Thus the result [22–26] namely the first law of thermodynamics on the trapping horizon, obtained by projecting the UFL along the tangent to the trapping horizon, can be generalized to any other horizon by projecting along the Kodama vector.

Note that Eq. (21) gives the rate of energy across the horizon. Thus the energy flux across the event horizon during infinitesimal time dt is

$$dQ = 4\pi H R_E^3 (\rho + p) dt, \tag{29}$$

or using the second Friedmann equation ($\dot{H} - \frac{k}{a^2} = -4\pi G(\rho + p)$)

$$dQ = -\frac{H R_E^3}{G} \left(\dot{H} - \frac{k}{a^2} \right) dt = \frac{R_E^3}{G R_A^3} \dot{R}_A dt, \tag{30}$$

where $R_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}}$ has been used in the last equality.

Recently, a notion of generalized Hawking temperature [21] (see Eq. (1))

$$T_E^{(G)} = \alpha \frac{R_E}{2\pi R_A^2}, \tag{31}$$

has been introduced on the event horizon for the validity of the thermodynamical laws. So in the present context using the first choice in Eq. (1) we have

$$T_E^{(G)} dS_E = \frac{R_E^3}{G R_A^3} \dot{R}_A dt. \tag{32}$$

Thus we obtain the Clausius relation $\delta Q = T_E^{(G)} dS_E$ on the event horizon, by projecting the UFL along the Kodama vector on the horizon. It is interesting to note that the present approach to obtain the Clausius relation (i.e. the first law of thermodynamics) from the UFL is a general prescription and it holds in any horizon even in the trapping horizon.

4 A redefinition of surface gravity

Finally, we redefine the surface gravity motivated by the concept of a Rindler observer. We have seen that at the local Rindler causal horizon, the Unruh temperature is the effective temperature experienced by a uniformly accelerating detector in a vacuum field. This temperature ($T = \frac{\hbar f}{2\pi c k_B}$) is proportional to the local acceleration (f) and has the same form as the Hawking temperature of a black hole [34]. So in analogy with the Unruh temperature we assume that the temperature T_{RH} (the redefined Hawking temperature) should be proportional to the acceleration of the model (in units $\hbar = 1 = c = k_B$), i.e.

$$T_{RH} = \frac{A}{2\pi}, \tag{33}$$

where the acceleration (A) of the FRW model (equivalently, it is the acceleration of a hypothetical observer moving with the expansion of the universe) can be defined as

$$A = a_0 R \ddot{a} = a_0 R (\dot{H} + H^2). \tag{34}$$

Here, a_0 is a dimensionless constant of proportionality (to be determined from the present context) and R is introduced on dimensional grounds. Then using the Einstein equations, the redefined Hawking temperature becomes

$$T_{RH} = -\frac{2a_0RG}{3}(\rho + 3p). \quad (35)$$

By introducing the vector (ad hoc, in analogy with the Kodama vector, having the properties given below)

$$V^\alpha = \left(-\frac{3}{R^3}, \frac{H}{aR^2}, 0, 0 \right), \quad (36)$$

we obtain

$$\langle dE, V^\alpha \rangle = 4\pi H(\rho + 3p),$$

and thus we have the Clausius relation

$$\langle dE, V^\alpha \rangle = \langle T_{RH}dS, V^\alpha \rangle,$$

$$\text{i.e. } \delta Q = T_{RH}dS,$$

provided $a_0 = \frac{3}{2}$ and S is the usual Bekenstein entropy. Therefore, we have the Clausius relation obeying the temperature–acceleration relation due to Rindler.

It is worth mentioning that the vector V^α (termed a modified Kodama vector) has properties similar to the Kodama vector namely

- (i) V^α lies in the radial–temporal plane, neither parallel nor orthogonal to the Kodama vector as well as to U^μ .
- (ii) Depending on the choice of R , the vector may be space-like, timelike or null in nature.
- (iii) Due to the divergence-free nature of V^α (i.e. $\nabla_\alpha V^\alpha = 0$), there exists a conserved current $C^\alpha = G^{\alpha\beta}V_\beta$ associated with it, i.e. $\nabla_\alpha C^\alpha = 0$.

5 Conclusion

So we have the following conclusions from Sects. 2–4:

- (a) Projecting the UFL along the Kodama vector, the second Friedmann equation is always obtained.
- (b) Projecting the UFL along a vector orthogonal to Kodama vector (having other properties same as Kodama vector), the first Friedmann equation can be obtained.
- (c) Projecting the UFL along the tangent to the horizon to obtain the first law of thermodynamics is valid only for the trapping horizon.
- (d) First law of thermodynamics on any horizon can be obtained from the UFL by projecting it along the Kodama vector on the horizon.

- (e) Clausius relation has been obtained on any horizon considering Rindler’s temperature–acceleration relation.

Thus we have shown that the Hayward–Kodama definition of surface gravity [defined in Eq. (5)] is valid for dynamical models. However, projecting with the Kodama vector we cannot only obtain both Friedmann equations from the UFL but also are able to obtain the Clausius relation on the event horizon (or any horizon) with temperature as generalized Hawking temperature. Also the first Friedmann equation can be obtained from the UFL by projecting along the orthogonal direction of the Kodama vector. Finally, in analogy with the Unruh temperature the surface gravity is redefined as proportional to acceleration in FRW model and it is possible to obtain the Clausius relation by projecting the UFL along V^α (i.e. the modified Kodama vector).

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