Eur. Phys. J. C (2017) 77:539
DOI 10.1140/epjc/s10052-017-5100-6

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Neutrino induced vorticity, Alfvén waves and the normal modes

Jitesh R. Bhatt[1,a], **Manu George**[1,2,b]

[1] Theory Division, Physical Research Laboratory, Ahmedabad 380 009, India
[2] Department of Physics, Indian Institute of Technology, Gandhinagar, Ahmedabad 382 424, India

Received: 16 January 2017 / Accepted: 21 July 2017 / Published online: 12 August 2017

Abstract We consider a plasma consisting of electrons and ions in the presence of a background neutrino gas and develop the magnetohydrodynamic equations for the system. We show that the electron neutrino interaction can induce vorticity in the plasma even in the absence of any electromagnetic perturbations if the background neutrino density is left–right asymmetric. This induced vorticity supports a new kind of Alfvén wave whose velocity depends on both the external magnetic field and on the neutrino asymmetry. The normal mode analysis show that in the presence of neutrino background the Alfvén waves can have different velocities. We also discuss our results in the context of dense astrophysical plasma such as magnetars and show that the difference in the Alfvén velocities can be used to explain the observed pulsar kick. We discuss also the relativistic generalisation of the electron fluid in presence of an asymmetric neutrino background.

1 Introduction

It is important to study the characteristics of the plasma in the presence of neutrinos, since such systems are important in understanding various physical phenomena during the evolution of the early Universe as well as the systems like core-collapsing supernovae and magnetars (see e.g. [1] for a brief overview). The presence of the cosmic neutrino background can influence cosmic microwave anisotropy and matter clustering [2,3] and it can also influence dynamics of the primordial magnetic field [4–6]. There exist several studies in the literature where the neutrino plasma interaction has been analysed in a variety of physical situations. Non-linear coupling of intense neutrino flux with collective plasma oscillations is studied in Ref. [7]. The authors have shown that a neutrino flux as intense as that in supernovae core can cause parametric instabilities in the surrounding plasma. The effect of a neutrino medium in the evolution of the lepton plasma had been studied invoking a ponderomotive description [8,9]. In these cases it was shown that the ponderomotive force is proportional to the gradient of neutrino density and the electrons are repelled from the regions where neutrino density is large. Interaction of very large number of neutrinos with collective plasma and oscillation and the excitation of the plasma turbulence is considered in Ref. [10]. Different kinds of the plasma–neutrino interactions using the ponderomotive force description and the effect on collective plasma properties can be found in Refs. [11–17]. In the above-mentioned ponderomotive force description, it was assumed that the neutrino field satisfies the naive Klein–Gordon equation with appropriate interaction terms. Thus in this formalism the information about the chiral structure of the weak interaction is absent. Here we note that, by Silva et al. in Refs. [18,19], the problem of neutrino driven streaming instability, which in turn can cause significant energy transfer from neutrino to the plasma, was considered in the kinetic theory formalism. A formulation to study the plasma interaction with intense neutrino beam using the field theory techniques is developed in [20]. The photon polarisation tensor in a medium consistent with gauge and Lorentz invariance can be found in [21]. In this work it is shown that, in the presence of a medium, the photon polarisation tensor can have an anti-symmetric part, indicating P and CP violations. Further studies of such effect in the presence of neutrinos for different physical scenarios are found in [22,23].

In the context of the early Universe, it has been shown by Shukla et al. [8] that the ponderomotive force of a non-uniform intense neutrino beam can be responsible for a large-scale quasi-stationary magnetic field. In fact, this was first to suggest the magnetic field generation in the plasma due to plasma–neutrino interactions. Further, large-scale magnetic field generation at the time of neutrino decoupling due to the evolution of the plasma in the presence of an asymmetric neutrino background is studied in [24,25]. This field

[a] e-mail: jeet@prl.res.in
[b] e-mail: manu@prl.res.in

can act as a seed for generation of the galactic magnetic field via the galactic dynamo mechanism (see e.g [26] for the galactic dynamo mechanism). It is to be noted that at finite lepton/baryon density the loop corrections to the photon polarisation tensor are non-vanishing. With these corrections the photon polarisation tensor acquires a non-zero parity-odd contribution $\Pi_2(\boldsymbol{k})$ where $\boldsymbol{k}$ is the wave vector. A finite and non-zero value of $\Pi_2(\boldsymbol{k})$ in the photon polarisation tensor means that there can be single field derivative terms in the effective Lagrangian and free energy, which dominates the kinetic energy part of the free energy which is having a double derivative term. For e.g. the free energy for a static gauge field can be written as $\mathcal{F}[A] = \int \mathrm{d}^3 p A_i(k)\Pi_{ij}(k)A_j(-k)$ and with parity violating interactions $\Pi_{ij}(k)$ can have a contribution $i\Pi_2(p^2)\epsilon_{ijl}k_l$. Thus a non-zero value of $\Pi_2(0)$ means a term $\Pi_2(0)\boldsymbol{A}\cdot\nabla\times\boldsymbol{A}$ in the expression for the free energy. This in turn means that there can be a generation of a large-scale ($k \to 0$) magnetic field by an instability arising due to non-zero values of parity-odd contributions $\Pi_2(0)$ to the polarisation tensor [24]. In Ref. [27], thermal field theory calculations were carried out to study the corrections to the photon polarisation in the presence of a background neutrino which is asymmetric in left–right number densities. The authors have shown that the axial part Π_2 is proportional to the neutrino asymmetry parameter and argued that the contribution to Π_2 due to the plasma which is interacting with the neutrino gas is $\sim 10^1$ times larger than the contribution to Π_2 through the correction due to the virtual process. In Ref. [28] using a kinetic theory approach it was shown that the photon polarisation tensor can have a parity-odd contribution $\Pi_2(\boldsymbol{k})$ due to the asymmetric neutrino background in both the collision-less and the collision dominated regime. In the collision dominated regime the result for $\Pi_2(\boldsymbol{k})$ using the kinetic approach agrees with that in Ref. [27]. In a recent work [29] the authors have calculated the effective potential or refractive index for the cosmic neutrino background (CNB) and future experimental implications have been discussed.

Further, recent theoretical calculations showed that the asymmetry in the neutrino density can be transmuted to the fluid helicity for sufficiently large electron neutrino interaction [30]. This neutrino induced vorticity can act as axial chemical potential for the chiral electrons. This phenomenon can induce the *helical plasma instability*, generating a strong magnetic field [30]. In this work the plasma particles are considered to be massless and chirally polarised. Moreover, it was assumed, in this work, that the neutrino mean free path l_ν is much smaller than the system dynamics at the length scale L i.e. $L \gg l_\nu$. This allows one to write the equations for the neutrino hydrodynamics [30]. Though this assumption is justifiable for a core collapsing supernova, it is hard to satisfy in other scenarios like the early Universe. Electroweak plasma in a rotating matter is studied in [31]. In this work it is shown that the electric current can be induced in the direction of the rotation axis due to the parity violating nature of the interaction. This phenomenon is called the *galvano-rotational effect* (GRE). In a recent work [32], a spin paramagnetic deformation of neutron star has been studied and the authors have calculated the ellipticity of a strongly magnetised neutron star using the spin magnetohydrodynamic equations developed in [33].

In the present work we are interested in developing a magnetohydrodynamic description of the plasma in the presence of the left–right asymmetric neutrino background. The expression of the interaction Lagrangian of a charged lepton field and the asymmetric neutrinos suggests that the neutrino can couple with the spin of the electron [27,34]. It is interesting to note here that there exists a lot of literature in on the usual electron–ion plasmas where the dynamics of spin degree can play a significant role. For example it was suggested that a spin polarised plasma in a fusion reactor can yield a higher nuclear reaction cross section [35] and the spin depolarisation process in the plasma can remain small [36]. The effect of spin dynamics using a single particle description, valid for a dilute gas, is studied in the context of laser plasma interaction in Ref. [37]. The collective effects within the framework of spin-magnetohydrodynamics has been studied in Refs. [33,38] (for a general discussion see [39]). This work can have applications in studying environments with a strong external magnetic field like pulsars and magnetars. In the present work we generalise the spin-magnetohydrodynamics considered in [33] to incorporate the effect of the asymmetric neutrino background.

The report is organised in the following way. In Sect. 2 we consider the low energy Lagrangian for our system and the equations of motion and spin evolution equations are derived invoking the non-relativistic approximations. MHD equations are considered in Sect. 3. Velocity perturbations and electromagnetic perturbations in a magnetised plasma interacting with neutrino background are considered in Sect. 4. In Sect. 5 we apply our theory to a neutron star to calculate the kick and Sect. 6 is for a summary and for our conclusions. We provide a brief summary of a relativistic generalisation of the theory in Appendix A.

2 The Lagrangian and non-relativistic approximation

The Lagrangian density for a lepton field interacting with background neutrinos is given by

$$\mathcal{L} = \bar{\psi}[i\gamma^\mu\partial_\mu\psi - \gamma_\mu(f_L^\mu P_L + f_R^\mu P_R) - m]\psi \tag{1}$$

where m is the mass of the lepton, $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$ are the Dirac matrices and $P_{L,R} = \frac{1\mp\gamma^5}{2}$ are the chiral projection operators with $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. $f^\mu_{L,R} = (f^0_{L,R}, \boldsymbol{f}_{L,R})$ are the neu-

trino currents and they are regarded as external macroscopic quantities.

An explicit form of $f^{\mu}_{L,R}$ can be calculated from the effective Lagrangian [27,34]

$$\mathcal{L}_{\rm eff} = [-\sqrt{2}G_{\rm F}\sum_{\alpha}\bar{\nu}_{\alpha}\gamma^{\mu}\frac{(1-\gamma^5)}{2}\nu_{\alpha}] \times[\bar{\psi}\gamma_{\mu}(a^{\alpha}_{L}P_L + a^{\alpha}_{R}P_R)\psi], \quad (2)$$

where the label α denotes a neutrino species $\alpha = e,\ \mu,\ \tau$ and $G_{\rm F} = 1.17\times10^{-11}\,{\rm MeV}^{-2}$ is the Fermi constant. The coefficients a^{α}_{L} and a^{α}_{R} are given by

$$a^{\alpha}_{L} = \delta_{\alpha,e} + \sin^2\theta_{\rm W} - 1/2,\ a^{\alpha}_{R} = \sin^2\theta_{\rm W}, \quad (3)$$

with $\theta_{\rm W}$ being the Weinberg angle. Next, we assume that $\nu\bar{\nu}$ form an isotropic background gas. This in turn means that in averaging over the neutrino ensemble, the only non-zero quantity will be $\langle\bar{\nu}_{\alpha}\gamma^0(1-\gamma^5)\nu\rangle = 2(n_{\nu_\alpha} - n_{\bar{\nu}_\alpha})$. The number densities of neutrinos and anti-neutrinos can be calculated using the corresponding Fermi–Dirac distribution function,

$$n_{\nu_\alpha,\bar{\nu}_\alpha} = \int\frac{{\rm d}^3p}{(2\pi)^3}\frac{1}{e^{\beta_{\nu_\alpha}(|\boldsymbol{p}|\mp\mu_{\nu_\alpha})}+1}, \quad (4)$$

where β is the inverse temperature. Using Eqs. (1–4) one obtains

$$f^0_L = 2\sqrt{2}G_{\rm F}[\Delta n_{\nu_e} + (\sin^2\theta_{\rm W} - 1/2)\sum_{\alpha}\Delta n_{\nu_\alpha})], \quad (5)$$

$$f^0_R = 2\sqrt{2}G_{\rm F}\sin^2\theta_{\rm W}\sum_{\alpha}\Delta n_{\nu_\alpha}. \quad (6)$$

Thus the equation of motion obtained from Eq. (1) can be written as

$$i\frac{\partial\psi}{\partial t} = [\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}\psi + \beta m - (f^0_L P_L + f^0_R P_R)]\psi. \quad (7)$$

Writing $\psi = \begin{pmatrix}\phi\\ \chi\end{pmatrix}$ in Eq. (7) and following the standard procedure [40], the Hamiltonian for the large component of the spinor can be obtained:

$$\mathcal{H} = \frac{1}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\boldsymbol{p}) + \frac{\Delta f^0}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{p}) + \frac{f^0}{2} + O(f^2_{L,R})$$

where $f^0 = f^0_L + f^0_R$ and $\Delta f^0 = f^0_L - f^0_R$. In the above equation, we have neglected terms proportional to $G^2_{\rm F}$. In the presence of an external electromagnetic field, the momentum $\boldsymbol{p}$ has to be replaced by $\boldsymbol{p} - e\boldsymbol{A}$. Thus the Hamiltonian for a charged fermion in interacting with an external electromagnetic field and background neutrino is given by

$$\mathcal{H} = \frac{(\boldsymbol{p}-e\boldsymbol{A})^2}{2m} - \boldsymbol{\mu}\cdot\boldsymbol{B} + eA^0 + \frac{\Delta f^0}{2m}\boldsymbol{\sigma}\cdot(\boldsymbol{p}-e\boldsymbol{A}) + \frac{f^0}{2} \quad (8)$$

where $\boldsymbol{\mu} = \frac{eg}{4m}\boldsymbol{\sigma}$ is the electron magnetic moment and g is the Landè g-factor. The first three terms on the right hand side are well known and very well studied in the literature. The fourth and fifth terms are due to the neutrino background. The last term might contribute to the energy of the system, but it will not enter into the equations of motion as the neutrino background is considered to be constant. If the neutrino background varies with space and time, this term would modify the force equation as $\boldsymbol{F}\propto\nabla f = \nabla\psi^*_{\nu}\psi_{\nu}$. This force is called the ponderomotive force. Such a scenario was studied in Ref. [9]; however, in the formalism in this reference the fourth term was not considered.

In order to find the equation of motion for a charged particle in an electromagnetic field and the neutrino background, one can use Eq. (8) and the Heisenberg equation $\dot{\hat{O}} = i[\hat{\mathcal{H}}, \hat{O}]$ and write

$$\boldsymbol{v} = \frac{p-e}{\boldsymbol{A}}m + \frac{\Delta f^0}{2m}\boldsymbol{\sigma} \quad (9)$$

where we wrote $\dot{\boldsymbol{x}} = \boldsymbol{v}$. We have

$$\dot{\boldsymbol{p}} = \frac{e}{m}(\boldsymbol{p}-e\boldsymbol{A})_k\nabla\boldsymbol{A_k} + \frac{eg}{m}\nabla(\boldsymbol{s}\cdot\boldsymbol{B}) - e\nabla A^0 \quad (10)$$

where we have defined $\boldsymbol{s} = \boldsymbol{\sigma}/2$ and

$$\dot{\boldsymbol{s}} = \mu_{\rm B}(\boldsymbol{s}\times\boldsymbol{B}) - \Delta f^0(\boldsymbol{s}\times\boldsymbol{v}). \quad (11)$$

From Eqs. (9–11) we get

$$\ddot{\boldsymbol{x}} = \frac{e}{m}[\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}] + \frac{e\Delta f^0}{2m^2}(\boldsymbol{s}\times\boldsymbol{B}) + \frac{eg}{2m^2}\nabla(\boldsymbol{s}\cdot\boldsymbol{B}) \quad (12)$$

3 The hydrodynamic equations

In this section we follow the methods developed in Ref. [33] to derive the hydrodynamic equations from the quantum Lagrangian for spin half particles. We consider a system of electrons and ions in the presence of a homogeneous neutrino background. The neutrino background is assumed to have a left–right asymmetry. Furthermore we treat electrons as quantum particles and ions as classical particles so that we can neglect the spin dynamics and other quantum effects for ions.

For simplicity, first let us consider Eq. (8) without the neutrino interaction term. We can decompose the wave function as $\psi_{\alpha} = \sqrt{n_{\alpha}}e^{iS_{\alpha}}\chi_{\alpha}$, where n_{α} is the density, S_{α} is the phase and χ_{α} is a two component spinor in which the spin-1/2 information is contained. Inserting this decomposition and considering the real and imaginary parts of the resulting equation we get the continuity and momentum conservation

equation for the "species α":

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \boldsymbol{v}_\alpha) = 0, \tag{13}$$

$$m_\alpha \left(\frac{\partial}{\partial t} + \boldsymbol{v}_\alpha \cdot \nabla\right) \boldsymbol{v}_\alpha = q_\alpha(\boldsymbol{E} + \boldsymbol{v}_\alpha \times \boldsymbol{B}) + 2\mu(\nabla \otimes \boldsymbol{B}) \cdot \boldsymbol{s}_\alpha - \nabla Q_\alpha - \frac{1}{m_\alpha n_\alpha} \nabla \cdot (n_\alpha \boldsymbol{\Sigma}_\alpha). \tag{14}$$

The velocity is defined via $m_\alpha \boldsymbol{v}_\alpha = \boldsymbol{j}_\alpha / \psi^\dagger \psi$, from which we obtain

$$m_\alpha \boldsymbol{v}_\alpha = (\nabla S - i\chi_\alpha^\dagger \nabla \chi_\alpha) - q_\alpha \boldsymbol{A} \tag{15}$$

and

$$\boldsymbol{s}_\alpha = \frac{1}{2} \chi_\alpha^\dagger \boldsymbol{\sigma} \chi_\alpha \tag{16}$$

The quantity Q_α is known as the quantum potential (Bohm potential) defined as

$$Q_\alpha = -\frac{1}{2m_\alpha \sqrt{n}_\alpha} \nabla^2 \sqrt{n}_\alpha \tag{17}$$

and $\boldsymbol{\Sigma}_\alpha$ is the symmetric spin gradient tensor.

$$\boldsymbol{\Sigma}_\alpha = \nabla \boldsymbol{s}_{(\alpha)a} \otimes \nabla \boldsymbol{s}^a_{(\alpha)} \tag{18}$$

where $a = 1, 2, 3$. By contracting the Pauli equation with $\psi^\dagger \boldsymbol{\sigma}$, one can obtain the spin evolution equation as

$$(\frac{\partial}{\partial t} + \boldsymbol{v}_\alpha \cdot \boldsymbol{\nabla}) \boldsymbol{s}_\alpha = 2\mu(\boldsymbol{s}_\alpha \times \boldsymbol{B}) + \frac{\boldsymbol{s}_\alpha \times [\partial_a (n_\alpha \partial^a \boldsymbol{s}_\alpha)]}{m_\alpha n_\alpha}. \tag{19}$$

In the presence of a neutrino background, the continuity equation remains unchanged. But momentum conservation and the spin evolution equations are modified in the following way:

$$m_\alpha (\frac{\partial}{\partial t} + \boldsymbol{v}_\alpha \cdot \nabla) \boldsymbol{v}_\alpha = q_\alpha(\boldsymbol{E} + \boldsymbol{v}_\alpha \times \boldsymbol{B}) + 2\mu(\nabla \otimes \boldsymbol{B}) \cdot \boldsymbol{s}_\alpha - \nabla Q_\alpha - \frac{1}{m_\alpha n_\alpha} \nabla \cdot (n_\alpha \boldsymbol{\Sigma}_\alpha) + \frac{\Delta f^0}{2m} \boldsymbol{s}_\alpha \times \boldsymbol{B}, \tag{20}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_\alpha \cdot \nabla\right) \boldsymbol{s}_\alpha = 2\mu(\boldsymbol{s}_\alpha \times \boldsymbol{B}) - \frac{\Delta f^0}{2} \boldsymbol{s}_\alpha \times \boldsymbol{v}_\alpha + \frac{\boldsymbol{s}_\alpha \times [\partial_a (n_\alpha \partial^a \boldsymbol{s}_\alpha)]}{m_\alpha n_\alpha}. \tag{21}$$

Now, in order to define hydrodynamic quantities, we need to specify how to calculate the expectation values. Suppose that we have N wave functions with the same kind of particles with magnetic moment μ charge q and mass m, so that the wave function for the entire system can be factorised as $\psi = \psi_{(1)}\psi(2)\cdots\psi_{(N)}$. Then we can define the total particle density for charge q as $n_q = \sum_\alpha n_\alpha$ and the expectation value of any quantity f as $\langle f \rangle = \sum_\alpha \frac{n_\alpha}{n_q} f$.

Using these arguments we define the total fluid velocity $\boldsymbol{V}_q = \langle \boldsymbol{v}_\alpha \rangle$ and $\boldsymbol{S}_q = \langle \boldsymbol{s}_\alpha \rangle$. In order to simplify further calculations, we redefine these quantities such that $\boldsymbol{w}_\alpha = \boldsymbol{v}_\alpha - \boldsymbol{V}_q$ and $\boldsymbol{\mathcal{S}}_\alpha = \boldsymbol{s}_\alpha - \boldsymbol{S}_q$, satisfying $\langle \boldsymbol{w}_\alpha \rangle = 0$ and $\langle \boldsymbol{\mathcal{S}}_\alpha \rangle = 0$. Now taking the ensemble average of the Eqs. (13), (20) and (21) we get the following expressions:

$$\frac{\partial n_q}{\partial t} + \nabla \cdot (n_q \boldsymbol{V}_q) = 0, \tag{22}$$

$$m_q n_q \left(\frac{\partial}{\partial t} + \boldsymbol{V}_q \cdot \nabla\right) \boldsymbol{V}_q = q n_q (\boldsymbol{E} + \boldsymbol{V}_q \times \boldsymbol{B}) - \nabla \cdot \boldsymbol{\Pi} - \nabla P + \boldsymbol{\mathcal{C}}_{qi} + \boldsymbol{F}_{\mathrm{Q}} + \boldsymbol{F}_{\nu e}, \tag{23}$$

and

$$n_q \left(\frac{\partial}{\partial t} + \boldsymbol{V}_q \cdot \nabla\right) \boldsymbol{S}_q = 2\mu_{\mathrm{B}} n_q \boldsymbol{S}_q \times \boldsymbol{B} - \frac{\Delta f^0}{2} \boldsymbol{S}_q \times \boldsymbol{V}_q + \boldsymbol{\Omega}_s - \nabla \cdot \boldsymbol{K}_q + \boldsymbol{K}_{\nu e}, \tag{24}$$

where $\boldsymbol{\Pi}$ is the traceless anisotropic part of the pressure tensor and P is the homogeneous part. $\mathcal{C}_{qi}$ represents the collision between particle with charge q and ion denoted using the letter i and the quantum force density $\boldsymbol{F}_{\mathrm{Q}}$ and the force due to the interaction with the neutrino back ground $\boldsymbol{F}_{\nu e}$ are defined by

$$\boldsymbol{F}_{\mathrm{Q}} = 2\mu_{\mathrm{B}} n_q (\nabla \otimes \boldsymbol{B}) \cdot \boldsymbol{S}_q - n_q \langle \nabla Q_\alpha \rangle - \frac{1}{m} \nabla \cdot (n_q \boldsymbol{\Sigma}) - \frac{1}{m} \nabla \cdot (n_q \tilde{\Sigma}) - \frac{1}{m} \nabla \cdot [(\nabla \boldsymbol{S}_a) \otimes \langle \nabla \mathcal{S}^a_\alpha \rangle + n_q \langle \nabla \mathcal{S}^a_\alpha \rangle \otimes (\nabla \boldsymbol{S}_a)] \tag{25}$$

and

$$\boldsymbol{F}_{\nu e} = n_e e \frac{\Delta f^0}{2m} \mathbf{S_q} \times \boldsymbol{B}. \tag{26}$$

The quantities $\boldsymbol{\Omega}_{\mathrm{s}}$, $\boldsymbol{\Sigma}$ and $\tilde{\boldsymbol{\Sigma}}$ depend on the spin of the particles and their precise definitions can be found in Ref. [33]. $\boldsymbol{K}_q = \langle \boldsymbol{S}_\alpha \otimes \boldsymbol{w}_\alpha \rangle$ is the spin-thermal coupling and $\boldsymbol{K}_{\nu e} = \epsilon_{ijk} \frac{\Delta f^0}{2} \langle \boldsymbol{S}_{\alpha j} \boldsymbol{w}_{\alpha k} \rangle$ is the spin-thermal coupling induced by neutrino interactions.

In the following sections, we will replace the subscript q with e and i for electrons and ions, respectively. Since we are considering ions as classical particles, we can neglect the contributions from spin and other quantum effects for ions. Thus, the fluid equations for ions read

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \boldsymbol{V}_i) = 0, \tag{27}$$

$$m_i n_i \left(\frac{\partial}{\partial t} + \boldsymbol{V}_i \cdot \nabla\right) \boldsymbol{V}_i = q_i n_i (\boldsymbol{E} + \boldsymbol{V}_i \times \boldsymbol{B}) - \nabla \cdot \boldsymbol{\Pi}_i - \nabla P_i + \boldsymbol{\mathcal{C}}_{iq}. \tag{28}$$

Note that there is no spin evolution equation for ions. Therefore whatever the spin contributions, the dynamics of the system is only due to the spin of the electrons. Now we can construct the single fluid equations from the above equations for electrons and ions. In order to do that we define the total mass density, $\rho = (m_e n_e + m_i n_i)$, the centre of mass velocity of the fluid $\rho \boldsymbol{V} = (m_e n_e \boldsymbol{V}_e + m_i n_i \boldsymbol{V}_i)$ and the current density $\boldsymbol{j} = (-e n_e \boldsymbol{V}_e + Z e n_i \boldsymbol{V}_i)$ and assuming quasi-neutrality $n_e = Z n_i$, one can immediately obtain the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0 \tag{29}$$

and the momentum conservation equation

$$\rho \left(\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla\right) \boldsymbol{V} = \boldsymbol{j} \times \boldsymbol{B} - \nabla \cdot \boldsymbol{\Pi} - \nabla P + \boldsymbol{F}_{\mathrm{Q}} + \boldsymbol{F}_{ve}. \tag{30}$$

Note that, with the assumption of quasi-neutralit,y we can write $n_e = \rho/(m e_e + m_i)$ and $\boldsymbol{V}_e = \boldsymbol{V} - m_i \boldsymbol{j}/Ze\rho$. Therefore we can express the quantum terms in terms of the total density and the centre of mass velocity of the fluid and current. Thus the spin evolution equation becomes

$$\rho \left(\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla\right) \boldsymbol{S} = \frac{m_i}{Ze\rho} \boldsymbol{j} \cdot \nabla \boldsymbol{S} + 2\mu\rho \boldsymbol{S} \times \boldsymbol{B} - (m_e + m_i/Z) \nabla \cdot \boldsymbol{K}_e + (m_e + m_i/Z) \boldsymbol{\Omega}_{\mathrm{s}} - \frac{e\Delta f^0 \rho}{2m_e} \boldsymbol{S} \times \left(\boldsymbol{V} - \frac{m_i \boldsymbol{j}}{Ze\rho}\right) - \frac{e\Delta f^0}{2m_e} \langle \boldsymbol{w}_e \times \boldsymbol{S}_e \rangle; \tag{31}$$

in general, for a magnetised medium with magnetisation density $\boldsymbol{M}$ we can write the free current density $\boldsymbol{j} = \frac{\nabla \times \boldsymbol{B}}{\mu_0} - \boldsymbol{j}_M$, where $\boldsymbol{j}_M = \nabla \times \boldsymbol{M}$ is the magnetisation current density. Note that here we have discarded the displacement current term $\frac{\partial \boldsymbol{E}}{\partial t}$.

In order to simplify the further calculation, we consider only the transverse waves, in which case the Bohm potential i.e. $\langle Q_\alpha \rangle$ term in Eq. (25) can be dropped [39]. Furthermore, all the other terms in Eq. (25) are second order in the spin variable and of order $\hbar^2$. We neglect these terms. However, the $\boldsymbol{F}_{ve}$ term in Eq. (23) and the spin dynamic equation (24) are of order $\hbar$. These terms are retained in the calculation. In such a situation we can write the total force density exerted on the fluid element as

$$F^i = -\partial^i \left(\frac{B_0^2}{2} - \boldsymbol{B} \cdot \boldsymbol{M}\right) + \boldsymbol{B} \cdot \nabla H^i - \partial^i P - \partial^j \Pi^{ij}. \tag{32}$$

For an isotropic plasma, the trace-free part of the pressure tensor Π^{ij} is zero. It is worth noting that the spatial part of the stress tensor takes the form $T^{ij} = -H^i B^j + (B^2/2 - \boldsymbol{B} \cdot \boldsymbol{M})\delta^{ij}$ [41], apart from the pressure terms. Thus the total force density on a magnetised fluid element can be written as $F^i = -\partial^j T^{ij}$. Therefore the momentum conservation equation takes the form

$$\rho \left(\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla\right) \boldsymbol{V} = -\nabla \left(\frac{B^2}{2} - \boldsymbol{B} \cdot \boldsymbol{M}\right) + (\boldsymbol{B} \cdot \nabla) \boldsymbol{H} - \nabla P - \nabla \cdot \boldsymbol{\Pi}. \tag{33}$$

Following the procedure in Ref. [42] we can write

$$\boldsymbol{j} \sim \frac{\sigma m_i}{\rho e} \nabla P + \sigma (\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}) + \frac{\sigma m_i}{\rho e} \boldsymbol{j} \times \boldsymbol{B} + \frac{\sigma m_i}{\rho e} \boldsymbol{F}_{\mathrm{Q}} - n_0 \frac{\sigma m_i}{\rho} \left(\frac{\Delta f^0}{2m_e}\right) \boldsymbol{S} \times \boldsymbol{B}. \tag{34}$$

Taking $\rho \sim n_0 m_i$ the expression for the total current can be written as

$$\boldsymbol{j} \sim \sigma (\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}) - \sigma \left(\frac{\Delta f^0}{2m_e}\right) \boldsymbol{S} \times \boldsymbol{B} + \boldsymbol{j}_M. \tag{35}$$

For the above expression for the hydrodynamic current, the time evolution for the magnetic field $\boldsymbol{B}$ is given by

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\eta \nabla \times (\nabla \times \boldsymbol{B}) + \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) - \left(\frac{\Delta f^0}{2m_e}\right) \nabla \times (\boldsymbol{S} \times \boldsymbol{B}) + \eta \nabla \times \boldsymbol{j}_M \tag{36}$$

where $\eta = 1/\sigma$ is the resistivity.

4 Neutrino induced vorticity, Alfvén wave and normal modes

In this section, we consider a very simple scenario. A background magnetic field $\boldsymbol{B}_0 = B_0 \hat{\boldsymbol{z}}$ is applied to the plasma. As a result, there is a non-zero constant magnetisation in

the system even in the absence of any perturbations, which also implies that $\boldsymbol{S}\times\boldsymbol{B}=0$ for the plasma at equilibrium. In this case the spin of the electrons align anti-parallel to the magnetic field to reduce the energy and therefore we can assume the equilibrium magnetisation density $\boldsymbol{M}_0$ to take the form $\boldsymbol{M}_0=-\mu_{\rm B}n_e\boldsymbol{S}_0=\mu_{\rm B}n_e\xi\left(\frac{\mu_{\rm B}B_0}{T_e}\right)\hat{\boldsymbol{z}}$, where $\xi(x)=tanh(x)$ is the Brillouin function. For the following discussions we make the approximation $\xi\left(\frac{\mu_{\rm B}B_0}{T_e}\right)\sim\left(\frac{\mu_{\rm B}B_0}{T_e}\right)$ so that $\boldsymbol{S}_0\sim-\frac{1}{2}\frac{\mu_{\rm B}B_0}{T}\hat{\boldsymbol{z}}$. Furthermore we assume that there are no electromagnetic perturbations in the system and the fluid velocity enters into the governing equations as perturbation. That is, $\boldsymbol{E}=0$, $\boldsymbol{B}=B_0\hat{\boldsymbol{z}}$, $\boldsymbol{V}=\delta\boldsymbol{V}$ and $\boldsymbol{S}=\boldsymbol{S}_0+\boldsymbol{\delta S}$. With these assumptions, up to linear order in the perturbations, we use the hydrodynamic equations in the following form:

$$\frac{\partial\delta\boldsymbol{S}}{\partial t}=2\mu_{\rm B}\delta\boldsymbol{S}\times\boldsymbol{B}_0-\left(\frac{\Delta f^0}{2m_e}\right)\boldsymbol{S}_0\times\delta\boldsymbol{V},\tag{37}$$

$$\rho_0\frac{\partial\delta\boldsymbol{V}}{\partial t}=-\frac{\mu_{\rm B}n_e}{T}\Delta f^0\nabla(\boldsymbol{B}_0\cdot\delta\boldsymbol{V})+\frac{\mu_{\rm B}n_e}{T}\Delta f^0(\boldsymbol{B}_0\cdot\nabla)\delta\boldsymbol{V}.\tag{38}$$

From Eq. (37) we get $\frac{\partial\delta\boldsymbol{S}\cdot\boldsymbol{B}_0}{\partial t}=0$. In order to satisfy these conditions, we choose $\delta\boldsymbol{S}\cdot\boldsymbol{B}_0=0$. We also take the space-time dependence of the perturbations to be of the following form:

$$\delta\boldsymbol{V}(t,\boldsymbol{x})=\delta\boldsymbol{V}_{\omega,\boldsymbol{k}}e^{-i(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})},\quad\delta\boldsymbol{S}(t,\boldsymbol{x})=\delta\boldsymbol{S}_{\omega,\boldsymbol{k}}e^{-i(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})}.\tag{39}$$

With these assumptions we get

$$\delta\boldsymbol{S}_{\omega,\boldsymbol{k}}=-\left(\frac{\Delta f^0}{2\Omega_e}\right)|\boldsymbol{S}_0|\delta\boldsymbol{V}_{\omega,\boldsymbol{k}}\tag{40}$$

where $\Omega_e=\frac{eB_0}{m_e}$. To obtain the above expression we have assumed that $\frac{\omega^2}{\Omega_e^2}\ll1$. Thus, from Eqs. (38) and (40), we get

$$-i\omega\delta\boldsymbol{V}_{\omega,\boldsymbol{k}}=\left(\frac{\mu_{\rm B}n_e}{T}\Delta f^0\right)\boldsymbol{\Omega}_{\omega,\boldsymbol{k}}\times\boldsymbol{B}_0\tag{41}$$

where $\boldsymbol{\Omega}_{\omega,\boldsymbol{k}}=i\boldsymbol{k}\times\delta\boldsymbol{V}_{\omega,\boldsymbol{k}}$ is the vorticity in the Fourier space. Note that, in the above expressions, we have kept only terms up to linear order in Δf^0. From the above expression we can see that the vorticity term will not contribute to the fluid dynamics if $\Delta f^0=0$. Therefore we conclude that $\boldsymbol{\Omega}$ is induced via the electron neutrino interaction. From Eq. (41) we can obtain the dispersion relations. For the case $\boldsymbol{k}||\boldsymbol{B}_0$,

$$\omega=-\left(\frac{\mu_{\rm B}n_eB_0}{\rho_0}\right)\left(\frac{\Delta f^0}{T}\right)k.\tag{42}$$

The group velocity of this new mode is given by

$$v_{\rm g}=\left|\frac{{\rm d}\omega}{{\rm d}k}\right|=\left|\left(\frac{\mu_{\rm B}n_eB_0}{\rho_0}\right)\left(\frac{\Delta f^0}{T}\right)\right|\tag{43}$$

$$\sim2\sqrt{2}\left(\frac{\mu_{\rm B}n_eB_0}{\rho_0}\right)\left(\frac{G_{\rm F}}{T}\right)|n_{\nu e}-n_{\bar{\nu}e}|\,.\tag{44}$$

Equation (42) corresponds to a new type of transverse mode propagating in the direction parallel to the background magnetic field, *induced* by the asymmetry in the neutrino background. The wave velocity not just depends on the strength of magnetic field but also on the neutrino asymmetry. This new mode is similar to the one found in the very high energy plasma with the chiral-anomaly [30]. In contrast to Ref. [30], in our work the electrons are not considered to be chirally polarised. However, the parity violating interaction in our work arises due to a neutrino–electron interaction. Furthermore, the effect of dissipation can easily be introduced by incorporating the contribution of the finite shear viscosity $-ik^2\eta_{\rm vis}$ and the resistivity $-i\sigma_1B_0^2$ into the dispersion relation (42), where $\eta_{\rm vis}$ is the kinematic viscosity and $\sigma_1=\sigma/\rho_0$ with σ being the resistivity.

Next, we consider the effect of electromagnetic perturbations. That is, we take the perturbations in the following form:

$$\boldsymbol{V}=\delta\boldsymbol{V},\quad\boldsymbol{B}=B_0\hat{\boldsymbol{z}}+\delta\boldsymbol{B},\quad\boldsymbol{E}=\delta\boldsymbol{E}.\tag{45}$$

For this case, the linearised hydrodynamic equations, Eqs. (29) and (33), take the form

$$\frac{\partial\rho}{\partial t}+\rho_0\nabla\cdot\delta\boldsymbol{V}=0,\tag{46}$$

$$\rho_0\frac{\partial\delta\boldsymbol{V}}{\partial t}=-\nabla\left(\boldsymbol{B}_0\cdot\delta\boldsymbol{B}-\boldsymbol{M}_0\cdot\delta\boldsymbol{B}-\delta\boldsymbol{M}\cdot\boldsymbol{B}_0\right)+\boldsymbol{B}_0\cdot\nabla\delta\boldsymbol{H}-\nabla P.\tag{47}$$

The spin evolution equation becomes

$$\left(\frac{\partial\boldsymbol{S}}{\partial t}\right)=2\mu_{\rm B}\boldsymbol{S}\times\boldsymbol{B}-\frac{\Delta f^0}{2}\boldsymbol{S}\times\delta\boldsymbol{V}\tag{48}$$

where $\boldsymbol{S}=\boldsymbol{S}_0+\delta\boldsymbol{S}$. For a perfectly conducting medium ($\eta\to0$), Eq. (36) becomes

$$\frac{\partial\delta\boldsymbol{B}}{\partial t}=\nabla\times(\delta\boldsymbol{V}\times\boldsymbol{B}_0)-\left(\frac{\Delta f}{2m_e}\right)\nabla\times(\delta\boldsymbol{S}\times\boldsymbol{B}_0+\boldsymbol{S}_0\times\delta\boldsymbol{B})\,.\tag{49}$$

Following the same procedure in the last section with the same assumptions, we get the expression for $\delta\boldsymbol{S}$,

$$\delta\boldsymbol{S}_{\omega,\boldsymbol{k}}=\frac{\mu_{\rm B}}{T}\delta\boldsymbol{B}_{\omega,\boldsymbol{k}}-\frac{\Delta f^0}{T}\delta\boldsymbol{V}_{\omega,\boldsymbol{k}}\tag{50}$$

where ω_p^2 is the plasma frequency. Using Eqs. (46), (47) and (50) and using the approximation $\boldsymbol{M}_0 = -\mu_B n_e \boldsymbol{S}_0 = \mu_B n_e \eta \left(\frac{\mu_B B_0}{T_e}\right)\hat{z}$ we get

$$-i\omega\rho_0\delta\boldsymbol{V}_{\omega,\boldsymbol{k}} = i\left[\frac{\omega_p^2}{m_e T}\right]\boldsymbol{B}_0 \times \left(\boldsymbol{k}\times\delta\boldsymbol{B}_{\boldsymbol{k},\omega}\right) - \left[\frac{\mu_B n_e}{T}\Delta f^0\right]\boldsymbol{B}_0\times\boldsymbol{\Omega}_{\boldsymbol{k},\omega} \tag{51}$$

where $\boldsymbol{\Omega}_{\boldsymbol{k},\omega} = i\boldsymbol{k}\times\delta\boldsymbol{V}$ is the vorticity in the Fourier space. We can see that the last term in Eq. (51) is proportional to the neutrino asymmetry of the background expression for the velocity in the Fourier space; thus,

$$\delta\boldsymbol{V}_{\boldsymbol{k},\omega} = \left(\frac{\boldsymbol{B}_0\cdot\boldsymbol{k}}{\rho_0\omega^2}\left[\frac{\omega_p^2}{m_e T}-1\right]+\frac{1}{\omega}\frac{\mu_B n_e}{\rho_0 T}\Delta f^0\right) \times\left((\boldsymbol{B}_0\cdot\delta\boldsymbol{V}_{\omega,\boldsymbol{k}})\boldsymbol{k}-(\boldsymbol{B}_0\cdot\boldsymbol{k})\delta\boldsymbol{V}_{\omega,\boldsymbol{k}}\right). \tag{52}$$

Note that we have neglected the contributions from the pressure terms in the above expression. Taking $\boldsymbol{k}$ in the direction of background magnetic field and assuming $\boldsymbol{B}_0\cdot\delta\boldsymbol{V} = 0$, we get the following dispersion relation:

$$\omega = -\frac{\tilde{v}_A}{\sqrt{\rho_0\alpha}}\frac{\mu_B n_e}{2T}\Delta f^0 k \pm \tilde{v}_A k \tag{53}$$

where $\alpha = (1-\frac{\omega_p^2}{m_e T})$ and $\tilde{v}_A = v_A\alpha^{1/2}$ is the spin-modified Alfvén velocity [33]. Here we note that the quantity α describes the spin corrections and in the absence of spin dynamics $\alpha = 1$. It is clear from Eq. (53) that the group velocity v_g can have the two values given by

$$v_g^{\pm} = \tilde{v}_A\left|1\pm\frac{1}{\sqrt{\rho_0\alpha}}\frac{\mu_B n_e}{2T}\Delta f^0\right|, \tag{54}$$

which is impossible in the absence of any neutrino asymmetry ($\Delta f^0 = 0$). Thus we can have two different group velocities for the Alfvén waves propagating parallel or anti-parallel to $\boldsymbol{B}_0$.

For finite values of the conductivity, we have to take into account of the first and last terms of Eq. (36) and the dispersion relation can be obtained from

$$\omega^2+\omega\left[i\alpha\eta k^2+\frac{\mu_B n_e\Delta f^0}{T\sqrt{\rho_0\alpha}}\tilde{v}_A k\right]-\tilde{v}_A^2 k^2 = 0. \tag{55}$$

Solving for ω we get

$$\omega = -\frac{1}{2}\left[i\alpha\eta k^2+\frac{\mu_B n_e\tilde{v}_A}{T\sqrt{\rho_0\alpha}}\Delta f^0 k\right] \pm\frac{1}{2}\sqrt{\left[i\alpha\eta k^2+\frac{\mu_B n_e\tilde{v}_A}{T\sqrt{\rho_0\alpha}}\Delta f^0 k\right]^2+4\tilde{v}_A^2 k^2}. \tag{56}$$

We can see that in the absence of any neutrino asymmetry and η, Eq. (56) reduces to $\omega^2 = \tilde{v}_A^2 k^2$, which is the same as in magnetohydrodynamics with spin corrections as obtained in [33].

5 Neutrino asymmetry and the pulsar kick

We use our formalism for a qualitative calculation of the observed pulsar kick [43–45]. There have been several attempts to explain the reason for the kick, see e.g. Refs. [46–49]. Recently there have been attempts to explain the pulsar kick using anomalous hydrodynamic theories (see e.g. Ref. [50]), but the exact reason for the pulsar kick is not yet resolved.

We note that the energy flux associated with the wave is equal to the energy density in the wave times the group velocity [51], which is the Poynting vector $\boldsymbol{P} = \boldsymbol{E}\times\boldsymbol{B}$ in our case [51,52]. The Poynting vector can be expressed in the form

$$\boldsymbol{P} = (\omega A^2)\boldsymbol{k} \tag{57}$$

where A is the magnitude of the vector potential $\boldsymbol{A}_{\omega,\boldsymbol{k}}$. Using Eq. (53) we write

$$|\boldsymbol{P}| = k^2A^2\tilde{v}_A\left(1\pm\frac{1}{\sqrt{\rho_0\alpha}}\frac{\mu_B n_e}{2T}\Delta f^0\right) \tag{58}$$

$$= (k^2A^2)v_g. \tag{59}$$

From Eq. (59) we infer that the energy density associated with the wave is k^2A^2. Further we note from Eq. (58) that the energy transported in the direction of the background field $\boldsymbol{B}_0$ and the energy going opposite to $\boldsymbol{B}_0$ are different due to the parity violation within the system. An excess amount of energy is transported in the direction of the magnetic field. This excess amount of energy transported per unit area per unit time is given by

$$\Delta P = (k^2A^2\tilde{v}_A)\left(\frac{\Delta f^0}{T}\frac{\mu_B n_e}{\sqrt{\rho_0\alpha}}\right). \tag{60}$$

This is essentially the momentum carried by the *excess* photons leaving the pulsar per unit area per unit time. Therefore the change in velocity experienced by the pulsar can be expressed as

$$\Delta V_{NS} = \frac{\Delta P}{M_{NS}}\times\Delta t\times(\text{area}) \tag{61}$$

where $M_{NS}\sim 10^{30}$ kg is the mass of the neutron star and Δt is the time span we assume for the kick to last, which

is approximately 10 s. The radius of the neutron star $R_{\rm NS}$ is approximately 10 km. Taking $k \sim A \sim T$, $\Delta n_{ve} \sim 1.6 \times 10^8$ (MeV)^3, $T \sim 10^{12}$ K and $B_0 \sim (10^{15}-10^{16})$ Gauss [53], we get $\Delta V_{\rm NS} \sim (10^2-10^3)$ km/s, which is within the order of magnitude of the observed pulsar kicks.

6 Discussion and conclusion

In conclusion we have developed spin magnetohydrodynamic equations in the presence of asymmetric background neutrinos and analysed the normal modes of the plasma in the presence of a constant magnetic field. We have shown that a new kind of wave (Alfvén) is generated as described by Eq. (42) and speed of this wave depends on the neutrino asymmetry. Such a wave can be generated in a dense astrophysical plasma such as a magnetar. For example for B_0^{15} Gauss, $T \sim 10MeV$ and $\Delta n_{ve} \sim 1.6 \times 10^8$ (MeV)^3, one can estimate the velocity of the wave (in units of speed of light) around 10^{-5}. We have shown that the background neutrino asymmetry can modify the wave velocity in directions parallel and anti-parallel to the external magnetic field (as shown in Eq. (54)). We have used our formalism to calculate the kick received by a pulsar during its birth. An order of magnitude calculation matches with the observations $\Delta V_{\rm NS} \simeq (10^2-10^3)$ km/s. In the appendix we have derived the relativistic hydrodynamical equation for the electrons using the Dirac equation. For the case when the electrons are relativistic the estimate given here for the Alfvén velocity can be suppressed by a factor $1/\sqrt{2}$.

Funded by SCOAP^3.

Appendix A

In many astrophysical situations it is necessary to consider the system temperature to be greater than its rest-mass and therefore we discuss a relativistic generalisation of the electron fluid. For such a generalisation in the context of a quantum plasma one needs to start with the Dirac equation. Work by Pauli [54], Harish Chandra [55] and Takabayasi [56] has shown that the Dirac equation can be cast into hydrodynamical form. Here we use the methodology similar to that given in [56] (see also [57]) to describe the fluid equations for relativistic electrons in the presence of the asymmetric neutrino background. In the standard MHD approximation the electron contributes in defining the current whereas the ion provides the inertia and therefore significantly contributes to the fluid velocity [33,42]. In this appendix we first derive electron fluid equations from the Dirac equation and then carry out the MHD approximations with the non-relativistic ion fluid and obtain an expression for the relativistic corrections to the MHD current; and finally we discuss the changes this bring about our (non-relativistic) results on the Alfvén waves. The subsequent derivation is rather lengthy and involved; we would like to refer the reader to Ref. [56] for further details.

Following [56] we start with the bilinear covariants with hydrodynamic variables and establish the relations among them using the properties of the gamma matrices. Also we establish their evolution equations from the moments of the corresponding Dirac equation. We choose the following bilinear covariants:

$$\Omega = \bar{\psi}\psi \tag{A1}$$
$$\bar{\Omega} = i\bar{\psi}\gamma^5\psi \tag{A2}$$
$$S^\mu = \bar{\psi}\gamma^\mu\psi, \tag{A3}$$
$$\bar{S}^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi, \tag{A4}$$
$$M^{\mu\nu} = \bar{\psi}\sigma^{\mu\nu}\psi, \tag{A5}$$
$$\bar{M}^{\mu\nu} = i\bar{\psi}\gamma^5\sigma^{\mu\nu}\psi, \tag{A6}$$

where $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$. One can obtain the equations of motion for ψ and $\bar{\psi}$ from Eq. (1) and using these equations of motion one can write the following two generic equations involving the dynamics of the above bilinear forms:

$$i(\bar{\psi}\gamma^A\gamma^\mu\partial_\mu\psi + \partial_\mu\bar{\psi}\gamma^\mu\gamma^A\psi) - eA_\mu\bar{\psi}[\gamma^A, \gamma^\mu]\psi - \frac{\Delta f_\mu}{2}\bar{\psi}[\gamma^A, \gamma^\mu\gamma^5]\psi = 0, \tag{A7}$$

$$i(\bar{\psi}\gamma^A\gamma^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^\mu\gamma^A\psi) - eA_\mu\bar{\psi}\{\gamma^A, \gamma^\mu\}\psi - \frac{\Delta f_\mu}{2}\bar{\psi}\{\gamma^A, \gamma^\mu\gamma^5\}\psi - 2m\bar{\psi}\gamma^A\psi = 0. \tag{A8}$$

Using the definition for the covariant differential operator $\delta^*_\mu(\bar{\psi}\gamma^A\psi) = i(\bar{\psi}\gamma^A\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^A\psi) - 2eA_\mu\bar{\psi}\gamma^A\psi$ we define

$$j_\mu = (1/2m)\delta^*_\mu\Omega, \tag{A9}$$
$$\bar{j}_\mu = (1/2m)\delta^*_\mu\bar{\Omega}, \tag{A10}$$
$$T^\nu_\mu = (1/2m)\delta^*_\mu S^\nu, \tag{A11}$$
$$\bar{T}^\nu_\mu = (1/2m)\delta^*_\mu\bar{S}^\nu, \tag{A12}$$
$$N^{\mu\nu}_\alpha = (1/2m)\delta^*_\alpha M^{\mu\nu}, \tag{A13}$$
$$\bar{N}^{\mu\nu}_\alpha = (1/2m)\delta^*_\alpha\bar{M}^{\mu\nu}. \tag{A14}$$

The quantities $M^{\mu\nu}$ and $\bar{M}^{\mu\nu}$ can be expressed as $\rho^2 M^{\mu\nu} = -\bar{\Omega}(S^\mu \bar{S}^\nu - S^\nu \bar{S}^\mu) + \Omega\epsilon^{\mu\nu\kappa\lambda} S_\kappa \bar{S}_\lambda$ and $\bar{M}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\kappa\lambda} M_{\kappa\lambda}$, where $\rho = \sqrt{\Omega^2 + \bar{\Omega}^2}$ has the interpretation of a density. From Eqs. (A8) and (A8) we obtain the evolution equation of the above defined quantities. We have

$$\partial_\mu S^\mu = 0, \tag{A15}$$

$$\partial_\mu \bar{S}^\mu = -2m\bar{\Omega} \tag{A16}$$

$$(1/2m)\partial_\nu M^{\mu\nu} + j^\mu - S^\mu + \frac{\Delta f_\nu}{2m}\bar{M}^{\mu\nu} = 0, \tag{A17}$$

$$(1/2m)\partial_\nu \bar{M}^{\mu\nu} + \bar{j}^\mu - \frac{\Delta f_\nu}{2m} M^{\mu\nu} = 0. \tag{A18}$$

Next, one defines [57] the four-velocity $v_\mu = S_\mu/\rho$ and four-spin $w_\mu = \bar{S}_\mu/\rho$ in such a way that they satisfy the following constraints: $v^\mu v_\mu = 1$, $w^\mu w_\mu = -1$ and $v^\mu W_\mu = 0$. From the last constraint, it is clear that $w_0 = \mathbf{v}\cdot\mathbf{w}/u^0$ and thus in the rest-frame the zeroth component of the four-spin $w_0 = 0$. By taking the divergence of $\bar{T}^{\mu\nu}$ one obtains the following equation:

$$\partial_\nu \bar{T}^{\mu\nu} = \frac{e}{m} w_\nu F^{\mu\nu} - \bar{j}^\mu_{st} + \delta f_\nu[-\rho\sin\theta\,(v^\mu w^\nu - v^\nu w^\mu) + \rho\cos\theta\,\epsilon^{\mu\nu\kappa\lambda} v_\kappa w_\lambda], \tag{A19}$$

where we have used $M^{\mu\nu} = [-\rho\sin\theta\,(v^\mu w^\nu - v^\nu w^\mu) + \rho\cos\theta\,\epsilon^{\mu\nu\kappa\lambda} v_\kappa w_\lambda]$ following [56]; and $\bar{j}^\mu_{st}$ has the same standard expression as given in Refs. [56,57]. Besides we have used the new definitions $\cos\theta = \Omega/\rho$ and $\sin\theta = \bar{\Omega}/\rho$. Here we note that the f^μ term for the neutrino current does not appear in the above equation. Equation (A19) is at the single body particle–antiparticle state level and one is required to take the fluid average for a collection of N such states. This N particle spinor must be written as a $4^N \times 4^N$ Slater determinant of N one-particle states; this procedure had been developed in Ref. [57] and we follow it for our calculation. We find the following equation, in thermal equilibrium, for the spatial part of the spin dynamics:

$$\gamma(\partial_t + \mathbf{v}\cdot\nabla)\mathbf{S} = \frac{e/m}{\langle\cos\theta\rangle}(W^0\mathbf{E}/2 + \mathbf{S}\times\mathbf{B}) - \gamma\Delta f^0 \mathbf{S}\times\mathbf{v} + \gamma\Delta f^0 \frac{\langle\sin\theta\rangle}{\langle\cos\theta\rangle}\mathbf{S} \tag{A20}$$

where $\gamma = \frac{1}{\sqrt{1-v^2}}$ and $W^0 = \mathbf{S}\cdot\mathbf{v}$. Here we note that as we have assumed before we have dropped the spin-thermal coupling and the non-linear spin terms. The last two terms on the right hand side give an additional contribution to the spin dynamics of the electron fluid dynamics given in Ref. [57]. This additional term solely depends on the neutrino background, as it should. Following the electron relativistic hydrodynamical model in Ref. [57], we regard θ as a constant parameter which is zero for the non-relativistic quantum case; and for an extreme relativistic quantum case $\theta = \pi/4$. For the non-relativistic spin dynamics (Eq. (24)) earlier results can be reproduced when we take $\theta = 0$. When the electrons are at a relativistic temperature one can replace mn by $(\epsilon + p)$, i.e. by the enthalpy density [58].

Now one can define the total mass density $\rho = (\epsilon + p) + m_i n_i$ where $(\epsilon + p)$ represents the enthalpy density of the electrons. Since in the magnetohydrodynamic equations the inertia of the fluid is dominated by ions, the momentum of the fluid is dominated by the ion momenta [42] (see also [33]). This remains true for the relativistic electron case also provided $\rho \sim m_i n_i$ and Eq. (33) remains valid for us. Next we derive the analog of Eq. (42) when the electrons are relativistic. For this purpose, consider an external magnetic field B_0 in the z-direction, there being no streaming of fluid $\mathbf{v}_0 = 0$. Also there is no electromagnetic perturbations i.e. $\delta\mathbf{E}$, $\delta\mathbf{B}$=0. The background spin vector is anti-parallel to the external magnetic field and given by $S_0 = -\mu_B B_0/2T$ as considered before. Next one can eliminate the electron velocity in the spin equation $\mathbf{v}_e = \mathbf{v} - m_i\mathbf{j}/(Ze\rho)$. Since there is no electromagnetic perturbation for this case $\mathbf{j} = 0$ and one can use Eqs. (33) and (A20) one obtains the following dispersion relation:

$$\omega = -\left(\frac{\mu_B n_e}{\sqrt{\rho_0}}\right)\left(\frac{\Delta f^0}{2T}\langle\cos\theta\rangle\right) k v_A \tag{A21}$$

where $v_A = \frac{B_0}{\sqrt{\rho_0}}$. Here we note that the last term in Eq. (A20) does not contribute significantly to the dispersion relation.

Similarly for the electromagnetic perturbation for the standard Alfvén waves one obtains the following dispersion relation:

$$\omega = -\frac{\tilde{v}_A}{\sqrt{\rho_0\alpha}}\frac{\mu_B n_e}{2T}\Delta f^0\langle\cos\theta\rangle k \pm k\tilde{v}_A. \tag{A22}$$

Here we note that both the new Alfvén waves (Eq. (A21)) and the regular Alfvén waves (Eq. (A22)), in the ideal MHD limit, gets a correction due to the relativistic effect which is characterised by the $\langle\cos\theta\rangle$ factors. Now for the ultrarelativistic limit if one takes $\theta = \pi/4$ following Ref. [57], one gets a $1/\sqrt{2}$ factor suppression in the speed of the new Alfvén wave compared to the non-relativistic case (with $\theta = 0$) case.

References

1. C. Volpe, arXiv:1411.6533
2. E. Komatsu et al., Astrophys. J. Suppl. Ser. **192**(2), 18 (2010)
3. S. Bashinsky, U. Seljak, Phys. Rev. D **69**, 083002 (2004)
4. K. Subramaniam, J.D. Barrow, Phys. Rev. D **58**, 083502 (1998)
5. K. Jademzik, V. Katalinic, A. Olinto, Phys. Rev. D **57**, 3264 (1997)
6. J.R. Shaw, A. Lewis, Phys. Rev. D **81**, 043517 (2010)
7. R. Bingham, J.M. Dawson, J.J. Su, H.A. Bethe, Phys. Lett. A **193**, 279 (1994)

8. P.K. Shukla, L. Stenflo, R. Bingham, H.A. Bethe, J.M. Dawson, J.T. Mendonça, Phys. Lett. A **233**, 181 (1997)
9. R. Bingham, H.A.B. Bethe, J.M. Dawson, P.K. Shukla, J.J. Su, Phys. Lett. A **220**, 107 (1996)
10. V.N. Tsytovich, R. Bingham, J.M. Dawson, H.A. Bethe, Astropart. Phys. **8**, 297 (1998)
11. L.O. Silva, R. Bingham, J.M. Dawson, W.B. Mori, Phys. Rev. E **59**, 2273 (1999)
12. B. Luís, Phys. Rev. D **61**, 013004 (1999)
13. Salvatore Esposito, Mod. Phys. Lett. A **14**, 1763 (1999)
14. A.J. Brizard, H. Murayama, J.S. Wurtele, Phys. Rev. E **61**, 4410 (2000)
15. B. Luís, Phys. Rev. D **63**, 077302 (2001)
16. A. Serbeto, J.T. Mendonça, P.K. Shukla, L.O. Silva, Phys. Lett. A **305**, 190 (2002)
17. A. Serbeto, Phys. Lett. A **296**, 217 (2002)
18. L.O. Silva, R. Bingham, J.M. Dawson, J.T. Mendoncça, Phys. Rev. Lett. **83**, 2703 (1999)
19. L.O. Silva, R. Bingham, JCAP **05**, 011 (2006)
20. L. Bento, arXiv:hep-ph/9912533v1
21. P.B. Pal, J.F. Nieves, Phys. Rev. D **39**, 652 (1989)
22. J.F. Nieves, Phys. Rev. D **61**, 113008 (2000)
23. J.F. Nieves, S. Sahu, Phys. Rev. D **71**, 073006 (2015)
24. A. Boyarsky, O. Ruchayskiy, M. Shaposhnikov, Phys. Rev. Lett. **109**, 111602 (2012)
25. A.D. Dolgov, D. Grasso, Phys. Rev. Lett **88**, 011301 (2002)
26. S.I. Vainshtein, Ya B. Zeldovich, Usp. Fiz. Nauk **106**, 431 (1972)
27. M. Dvornikov, V.B. Semikoz, JCAP **05**, 002 (2015)
28. J.R. Bhatt, M. George, Int. J. Mod. Phys. D **26**, 1750052 (2017)
29. J.S. Díaz, F.R. Klinkhamer, Phys. Rev. D **93**, 053004 (2016)
30. N. Yamamoto, Phys. Rev. D **93**, 065017 (2016)
31. M. Dvornikov, JCAP **05**, 037 (2015)
32. A.G. Suvorov, A. Mastrano, A. Melatos, Mon. Not. R. Astronom. Soc. **456**, 731 (2016)
33. G. Brodin, M. Marklund, N. J. Phys. **9**, 277 (2007)
34. C. Giunti, C.W. Kim, *Fundamentals of Neutrino Physics and Astrophysics* (Oxford University Press, Oxford, 2007)
35. R.M. Kulsrud, H.P. Furth, E.J. Valeo, M. Goldhaber, Phys. Rev. Lett. **49**, 1248 (1982)
36. S.C. Cowley, R.M. Kulsrud, E. Valeo, Phys. Fluids **49**, 430 (1986)
37. M.W. Walser, D.J. Urbach, K.Z. Hatsagortsyan, S. Hu, C.H. Keitel, Phys. Rev. A **65**, 043410 (2002)
38. S.M. Mahajan, F.A. Asenjo, Phys. Rev. Lett. **107**, 195003 (2011)
39. F. Haas, *Quantum Plasmas: An Hydrodynamic Approach* (Springer, Berlin, 2011)
40. W. Greiner, *Relativistic Quantum Mechanics*, vol. 3 (Springer, Berlin, 1990)
41. M.P. Robinson et al., Phys. Rev. Lett. **85**, 4466 (2000)
42. T.J.M. Boyd, J.J. Anderson, *Plasma Dynamics* (Thomas Nelson and Sons Ltd., Nashville, 1969)
43. R. Minkowski, PASP **82**, 470 (1970)
44. A.G. Lyne, B. Anderson, M.J. Salter, MNRAS **291**, 503 (1982)
45. B.M.S. Hansen, E.S. Phinney, MNRAS **291**, 569 (1997)
46. J.R. Gott, J.E. Gunn, J.P. Ostriker, ApJ **160**, L91 (1970)
47. I. Iben, A.V. Tutukov, ApJ **456**, 738 (1996)
48. A. Kusenko, G. Segré, Phys. Rev. D **59**, 061302 (1999)
49. A. Kusenko, Int. J. Mod. Phys. D **13**, 2065 (2004)
50. M. Kaminiski, C.F. Uhlemann, M. Bleicher, J. Schaffner-Bielich, Phys. Lett. B **760**, 170 (2016)
51. T.H. Stix, *Waves in Plasmas* (Springer Science & Business Media, Berlin, 1992)
52. N.F. Cramer, *The Physics of Alfvén Waves* (Wiley, New York, 2011)
53. A. Reisenegger, arXiv:1305.2542
54. W. Pauli, Ann. Inst. Poincaré **6**, 109 (1936)
55. Harish-Chandra Research Institute, Proc. Indian Acad. Sci. (Math. Sci.) **22**, 30 (1945)
56. T. Takabayasi, Progr. Theor. Phys. Suppl. **4**, 1 (1957)
57. F.A. Asenjo, V. Munoz, J.A. Valdivia, S.M. Mahajan, Phys. Plasmas **18**, 012107 (2011)
58. L.D. Landau, E.M. Lifshitz, *Fluid Mechanics*, 2nd edn. (Pergamon Press, Oxford, 1989)