

Solar system tests for realistic $f(T)$ models with non-minimal torsion–matter coupling

Rui-Hui Lin, Xiang-Hua Zhai^a, Xin-Zhou Li^b

Shanghai United Center for Astrophysics (SUCA), Shanghai Normal University, 100 Guilin Road, Shanghai 200234, China

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Abstract In the previous paper, we have constructed two $f(T)$ models with non-minimal torsion–matter coupling extension, which are successful in describing the evolution history of the Universe including the radiation-dominated era, the matter-dominated era, and the present accelerating expansion. Meantime, the significant advantage of these models is that they could avoid the cosmological constant problem of Λ CDM. However, the non-minimal coupling between matter and torsion will affect the tests of the Solar system. In this paper, we study the effects of the Solar system in these models, including the gravitation redshift, geodetic effect and perihelion precession. We find that Model I can pass all three of the Solar system tests. For Model II, the parameter is constrained by the uncertainties of the planets' estimated perihelion precessions.

1 Introduction

General relativity (GR), whose centenary was recently celebrated [1, 2], uses the metric tensor as the fundamental dynamical variable, and chooses the torsionless Levi-Civita connection to describe the gravitation field. In the teleparallel equivalent of general relativity (TEGR) [3–5], the curvatureless Weizenböck connection is chosen instead, and the tetrad field plays the role of fundamental variable. TEGR provides the gauge structure of gravitation: the gauge theory for the group of translation on the tangent bundle of the spacetime. In this sense, teleparallelism may be considered as an approach to including gravity into the unification of gauge theories and maybe, further, to quantize the gravity.

Under the framework of TEGR, different modifications of gravity theory have been proposed for better understanding of the currently accelerating Universe [6, 7]. One possible scheme is $f(T)$ theory, in which the torsion scalar T in the

gravitation Lagrangian is replaced by an arbitrary function $f(T)$ [8–14]. Compared to $f(R)$ theory, the field equations are second order instead of fourth one, which is an important advantage of $f(T)$ theory. As a further extension, and an analogy to that of $f(R)$ theories [15–29], non-minimal torsion–matter coupling $f(T)$ gravity has been proposed and studied [30]. In the previous research [31], using the observation data of type Ia supernovae (SNeIa), cosmic microwave background (CMB), and baryon acoustic oscillations (BAO), we established two concrete $f(T)$ models with non-minimal torsion–matter coupling extension, and we found that they are successful in describing the observation of the Universe and its large-scale structure and evolution. The joint fitting led to $\Omega_{m0} = 0.255 \pm 0.010$, $\Omega_{b0}h^2 = 0.0221 \pm 0.0003$ and $H_0 = 68.54 \pm 1.27$ for model I and $\Omega_{m0} = 0.306 \pm 0.010$, $\Omega_{b0}h^2 = 0.0225 \pm 0.0003$ and $H_0 = 60.97 \pm 0.44$ for model II at 1σ confidence level, where H_0 is the Hubble parameter at present, $h = H_0/(100\text{km/s/Mpc})$, and Ω_{m0} and Ω_{b0} are the density parameters of dark matter and baryon matter. In a recent work, we also studied the structure formation of these two models [32].

On the other hand, any modified gravity theory must confront the Solar system tests which have been passed in GR [33–37]. In fact, Solar system effects in some modified gravities including minimal coupling $f(T)$ models have been considered [38–44]. In Ref. [42], the authors have investigated the perihelion precession, light bending, Shapiro time delay and gravitation redshift in minimal coupling $f(T)$ gravity with $f(T) = T + \alpha T^n$, and constrained the parameter α for $n = 2, 3$. However, non-minimal geometry–matter coupling theories predict that the movements of massive particles are no longer geodesic, which will have influences on some of the Solar system effects. Similar cases have been studied in some non-minimal coupling $f(R)$ theories (see e.g. [16, 45, 46]).

In this paper, we consider the Solar system tests including the gravitation redshift, geodetic effect, and perihelion precession in the two models we have constructed in Ref.

^a e-mail: zhaixh@shnu.edu.cn

^b e-mail: kyhz@shnu.edu.cn

[31]. The gravitation redshift is one representative effect that does not involve massive particle. Since the two models we are considering assume that the coupling between radiation and torsion is still minimal for simplicity, this kind of effects is practically the same as the minimal coupling theory. The geodetic effect and perihelion precession, however, involve the movement of massive objects, and thus should be affected by the non-minimal coupling between matter and torsion. Judging from this, it is necessary to combine the cosmological restriction with Solar system effects for the parameters in two models. We have gotten the cosmological restriction using the observation data of SNeIa, CMB and BAO as follows [31]: $A = 0.188 \pm 0, 048$, $B = 0.510 \pm 0.060$ (for Model I); $A = 0.633 \pm 0.012$ and B is a free parameter (for Model II). It is our task in this paper to check whether or not these values of parameters will pass the Solar system tests.

The paper is organized as follows: In Sect. 2 we briefly review the $f(T)$ theory with power law torsion–matter coupling extension and our two cosmologically fitted models. In Sect. 3, we calculate the weak field limit of the spherically symmetric background metric. In Sects. 4, 5, and 6 we consider the gravitation redshift, geodetic effect and perihelion precession in our two models, respectively. We conclude our research in Sect. 7.

We use the Latin alphabet ($a, b, c, \dots = 0, 1, 2, 3$) to denote the tangent space indices, and Greek alphabet ($\mu, \nu, \rho, \dots = 0, 1, 2, 3$) to denote the spacetime indices. We assume the Lorentz metric of Minkowski spacetime

$$\eta = \eta_{ab} dx^a \otimes dx^b \tag{1}$$

has the form $\eta_{ab} = \text{diag}(+1, -1, -1, -1)$. And we use the unit $c = 8\pi G = 1$ throughout the paper.

2 Brief review of the theory

On the spacetime differentiable manifold \mathcal{M} , one can generally find dual pairs of linearly independent frame fields $\{e_a, e^a\}$, such that $e^a(e_b) = \delta_b^a$. One particular pair would be the gradients $\{\partial_a\}$ of the tangent space coordinates $\{x^a\}$ and their covectors $\{dx^a\}$. One such frame field, also known as the tetrad or vierbein field, forms a base for the vectors on the tangent space $T_p\mathcal{M}$ at each point $p \in \mathcal{M}$. On the overlap area of different patches of \mathcal{M} , the members of one base can be written in terms of members of the other, e.g.

$$e_a = e_a^\mu \partial_\mu \quad \text{or} \quad e^a = e^a_\mu dx^\mu. \tag{2}$$

The spacetime metric g

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu \tag{3}$$

is related to the tangent space metric η by

$$\eta_{ab} = g(e_a, e_b) = g_{\mu\nu} e_a^\mu e_b^\nu, \tag{4}$$

or conversely

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b. \tag{5}$$

And hence the determinant

$$|e| \equiv \det(e_\mu^a) = \sqrt{-g}. \tag{6}$$

TEGR uses these vierbein fields as dynamical variables to define the Weitzenböck connection [47]

$$\tilde{\Gamma}_{\nu\mu}^\lambda \equiv e_a^\lambda \partial_\mu e_\nu^a = -e_\nu^a \partial_\mu e_a^\lambda. \tag{7}$$

The torsion and contorsion tensors are then given by

$$T_{\mu\nu}^\lambda \equiv \tilde{\Gamma}_{\nu\mu}^\lambda - \tilde{\Gamma}_{\mu\nu}^\lambda = e_i^\lambda (\partial_\mu e_\nu^i - \partial_\nu e_\mu^i) \tag{8}$$

and

$$K^{\mu\nu}{}_\rho \equiv -\frac{1}{2} (T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho^{\mu\nu}). \tag{9}$$

Utilizing the tensor

$$S_\rho^{\mu\nu} \equiv \frac{1}{2} (K^{\mu\nu}{}_\rho + \delta_\rho^\mu T^{\lambda\nu}{}_\lambda - \delta_\rho^\nu T^{\lambda\mu}{}_\lambda), \tag{10}$$

one can define the torsion scalar

$$T \equiv T_{\mu\nu}^\rho S_\rho^{\mu\nu}. \tag{11}$$

TEGR [3–5] uses this scalar T as the gravitation Lagrangian density.

For another vierbein field $\{e'_a, e'^a\}$, contracting Eq. (5) with $e'^c{}^\mu e'_d{}^\nu$, one has

$$g_{\mu\nu} e'^c{}^\mu e'_d{}^\nu = \eta_{cd} = \eta_{ab} (e_a^\mu e'_c{}^\mu) (e_b^\nu e'_d{}^\nu). \tag{12}$$

This means $\Lambda_c^a = e_a^\mu e'_c{}^\mu$ is one of the entries of a Lorentz transformation. That is, different vierbein fields satisfying the Lorentz transformation relation

$$e'^c{}_\mu = \Lambda_c^a e_a^\mu \tag{13}$$

will give the same spacetime metric. Though this does not make much difference inTEGR, it does have some significant effects on $f(T)$ theories, one of which is that the $f(T)$ theories are no longer Lorentz invariant theories [48]. The fact that different vierbein fields (related by a Lorentz group) lead to the same metric calls for an appropriate choice of

vierbein for $f(T)$ theories and their extensions, especially when spherical coordinates is considered [49].

One of the extensions of $f(T)$ theories is the non-minimal torsion–matter coupling $f(T)$ gravity, proposed by Harko et al. [30]. The action of the non-minimal torsion–matter coupling $f(T)$ gravity can be written as

$$S = -\frac{1}{2} \int |e| [1 + f_1(T)] T d^4x + \int |e| [1 + f_2(T)] \mathcal{L}_m d^4x, \tag{14}$$

where \mathcal{L}_m is the Lagrangian density of matter. With the action principle applied on Eq. (14) with respect to the vierbein, the equation of motion is then given by

$$\begin{aligned} \frac{4}{|e|} f \partial_\beta (|e| S_\sigma^{\alpha\beta} e_a^\sigma) + 4e_a^\sigma S_\sigma^{\alpha\beta} \partial_\beta f \\ + 4f S_\rho^{\alpha\sigma} T_{\sigma\beta}^\rho e_a^\beta + (1 + f_1) T e_a^\alpha \\ = -2(1 + f_2) \mathcal{T}_\beta^\alpha e_a^\beta, \end{aligned} \tag{15}$$

where $f = f(T) \equiv 1 + f_1(T) + f_1'(T)T - 2f_2'(T)\mathcal{L}_m$, and \mathcal{T}_β^α is the energy-momentum tensor of matter given by

$$\frac{\delta(|e|\mathcal{L}_m)}{\delta e^\alpha_a} = -|e|\mathcal{T}_\beta^\alpha e_a^\beta. \tag{16}$$

And it takes the usual form for a perfect fluid,

$$\mathcal{T}_{\mu\nu} = p g_{\mu\nu} - (\varepsilon + p) u_\mu u_\nu, \tag{17}$$

where p and ε are the pressure and energy density of the matter, respectively, and u^μ is the 4-velocity.

The covariant derivative (related to the Levi-Civita connection) of Eq. (15) gives

$$\nabla^\nu \mathcal{T}_{\mu\nu} = -\frac{f_2'(T)\nabla^\nu T}{1 + f_2(T)} (\mathcal{T}_{\mu\nu} + g_{\mu\nu}\mathcal{L}_m). \tag{18}$$

This suggests that the energy-momentum tensor is no longer conservative. However, contracting Eq. (18) with u^μ , we have

$$\begin{aligned} u^\mu \nabla^\nu \mathcal{T}_{\mu\nu} &= \frac{f_2'(T)\nabla^\nu T}{1 + f_2(T)} (\varepsilon - \mathcal{L}_m) \\ &= -u^\mu \nabla_\mu \varepsilon - (\varepsilon + p) \nabla_\mu u^\mu. \end{aligned} \tag{19}$$

If we take the matter Lagrangian density to be $\mathcal{L}_m = \varepsilon$ [30,50–53], then as discussed in Ref. [31], in cosmological cases, i.e. with the Friedmann–Lemaître–Robertson–Walker (FLRW) metric $g_{\mu\nu} = \text{diag}(1, -a, -a, -a)$, Eq. (19) will lead back to the usual conservation law of matter

$$\dot{\varepsilon} + 3H(\varepsilon + p) = 0. \tag{20}$$

In Ref. [31], we have constructed two concrete models with

$$f_1 = \begin{cases} \frac{12BH_0^4}{|T|^2}, & \text{for Model I,} \\ \frac{BH_0}{\sqrt{|T|}}, & \text{for Model II,} \end{cases} \quad f_2 = \frac{2AH_0^2}{\Omega_{M0}|T|}$$

where $\Omega_{M0} = \Omega_{m0} + \Omega_{b0}$ is the current density parameter of matter, fitted with SNIa, CMB, and BAO data. The best-fit values for the parameters A, B are

$$\begin{aligned} \text{Model I : } & A = 0.188 \pm 0.048, \\ & B = 0.510 \pm 0.060; \\ \text{Model II : } & A = 0.633 \pm 0.012, \\ & B \text{ is a free parameter.} \end{aligned} \tag{21}$$

We have found that these two models are successful in describing the observation of the Universe and its large-scale structure and evolution. In order to see whether these two models can pass the Solar system tests, we will consider the gravitation redshift, geodetic effect, and perihelion precession of these two models in the following sections.

3 Spherically symmetric weak field solution

Firstly, we calculate the background metric. Since we are considering the spherically symmetric vacuum solution, Eq. (15) can be rewritten as

$$\begin{aligned} \frac{4}{|e|} f \partial_\beta (|e| S_\sigma^{\alpha\beta} e_a^\sigma) + 4e_a^\sigma S_\sigma^{\alpha\beta} \partial_\beta f \\ + 4f S_\rho^{\alpha\sigma} T_{\sigma\beta}^\rho e_a^\beta + (1 + f_1) T e_a^\alpha = 0, \end{aligned} \tag{22}$$

with $f = f(T) = 1 + f_1(T) + f_1'(T)T$. Therefore, we are actually dealing with the minimal coupling $f(T)$ theory since \mathcal{L}_m vanishes at outside of the central mass. As mentioned before, one should be careful when it comes to choosing vierbein fields in spherical coordinates for $f(T)$ theories. The general spherically symmetric metric can be written as

$$g_{\mu\nu} = \text{diag}(e^{\xi(r,t)}, -e^{\zeta(r,t)}, -r^2, -r^2 \sin^2 \theta). \tag{23}$$

It is found [49] that the vierbein field

$$e^a_\mu = \begin{pmatrix} e^{\frac{\xi}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{\zeta}{2}} \sin \theta \cos \phi & e^{\frac{\zeta}{2}} \sin \theta \sin \phi & e^{\frac{\zeta}{2}} \cos \theta \\ 0 & -r \cos \theta \cos \phi & -r \cos \theta \sin \phi & r \sin \theta \\ 0 & r \sin \theta \sin \phi & -r \sin \theta \cos \phi & 0 \end{pmatrix} \tag{24}$$

is a viable choice of vierbein for $f(T)$ theories in that it does not impose extra constraints on the form of $f(T)$ and it preserves the Birkhoff theorem, that is,

$$\xi = \xi(r), \quad \zeta = \zeta(r). \tag{25}$$

The torsion scalar is then given by

$$T(r) = \frac{2}{r^2} e^{-\zeta} (1 + e^{\frac{\zeta}{2}}) (1 + e^{\frac{\zeta}{2}} + r\xi'), \tag{26}$$

where the prime indicates a derivative with respect to r .

Thus, Eq. (22) can reduce to

$$f(T)[\xi'(r) + \zeta'(r)] = 2[1 + e^{\frac{\zeta(r)}{2}}]f'(T)T'(r),$$

$$e^{\zeta(r)}r^2T^2f_1'(T) = 2f(T)[e^{\zeta(r)} - 1 - r\xi'(r)]. \tag{27}$$

In the weak field limit $\xi, \zeta \ll 1$, these equations can be solved up to linear approximation as

$$\xi_I(r) = -\frac{2M}{r} + \frac{3BH_0^4r^4}{20},$$

$$\zeta_I(r) = \frac{2M}{r} - \frac{9BH_0^4r^4}{10}, \tag{28}$$

for Model I, and

$$\xi_{II}(r) = -\frac{2M}{r} + \frac{BH_0r}{\sqrt{2}},$$

$$\zeta_{II}(r) = \frac{2M}{r}, \tag{29}$$

for Model II, where the integral constant is chosen to be the central spherically symmetric mass M as in the Schwarzschild solution. These results are consistent with the general weak field solution for minimal coupling power law $f(T)$ theory given in Refs. [41, 54].

With the (vacuum) background vierbein field solved, we are ready to calculate the Solar system effects.

4 Gravitation redshift

The gravitation redshift for a radiation signal is given by

$$\frac{\nu}{\nu_0} = \frac{e_0^0(r)}{e_0^0(r_0)}, \tag{30}$$

where ν, ν_0 are the frequencies of the signal measured at the position of r, r_0 , respectively. Since this effect does not involve any massive test particle, the background solutions (28) and (29) can be used directly. Thus, up to linear approximation, the gravitation redshift is

$$\left(\frac{\nu}{\nu_0}\right)_I \simeq 1 - \frac{M}{r} + \frac{M}{r_0} + \frac{3BH_0^4}{40}(r^4 - r_0^4) \tag{31}$$

for Model I and

$$\left(\frac{\nu}{\nu_0}\right)_{II} \simeq 1 - \frac{M}{r} + \frac{M}{r_0} + \frac{BH_0}{2\sqrt{2}}(r - r_0) \tag{32}$$

for Model II. The common terms in each case are the prediction of GR, and the last terms are the modifications from the models.

The Hydrogen-Maser in Gravity Probe A redshift experiment [55] launched a spacecraft nearly vertically upward to 10,000 km to measure this effect. Using the cosmologically best-fit values (21), we have the gravitation redshift modifications $\delta(\frac{\nu}{\nu_0})_I \leq 9.045 \times 10^{-78}$ for Model I and $\delta(\frac{\nu}{\nu_0})_{II} = 2.331B \times 10^{-20}$ for Model II. The redshift experiment reaches a $\delta_{rs} = 10^{-14}$ accuracy [55]. It is obvious that the modifications are within the error for both models if $B < 10^6$ for Model II.

5 Geodetic effect

Consider a gyroscope described by a spin 4-vector s^μ orbiting around the central mass. Its 4-velocity is given by $u^\mu = (u^t, 0, 0, u^\phi)$ where u^ϕ is the orbiting angular velocity, and $u_\mu u^\mu = 1$. s^μ is spacelike such that $u_\mu s^\mu = 0$. It is found that as long as the direct effects of tidal forces can be neglected, s^μ is Fermi–Walker transported along u^μ [37, 56], that is,

$$u^\nu \nabla_\nu s^\mu = u^\mu (u^\lambda \nabla_\lambda u^\nu) s_\nu. \tag{33}$$

Contracting Eq. (18) with the induced metric $h^{\mu\lambda} = g^{\mu\lambda} - u^\mu u^\nu$, we have

$$h^{\mu\lambda} \nabla^\nu \mathcal{T}_{\mu\nu} = h^{\mu\lambda} \nabla_\mu p - (\varepsilon + p) u^\mu \nabla_\mu u^\lambda$$

$$= -h^{\mu\lambda} \frac{f_2'(T) \nabla_\mu T}{1 + f_2(T)} (p + \mathcal{L}_m). \tag{34}$$

Combining Eqs. (33) and (34), and suppressing the hydrodynamical term $\nabla_\nu p$, which is not related to f_2 , we have

$$u^\nu \nabla_\nu s^\mu = \frac{ds^\mu}{dt} + \Gamma_{\nu\lambda}^\mu s^\lambda u^\nu = s^\lambda u^\mu \frac{f_2'(T) \partial_\lambda T}{1 + f_2(T)}, \tag{35}$$

where $\Gamma_{\nu\lambda}^\mu$ is the Levi-Civita connection and we have taken $\mathcal{L}_m = \varepsilon$ [30, 50–52]. The r, θ, ϕ components of Eq. (35) are

$$\frac{ds^r}{dt} + \frac{1}{2} \xi'(r) e^{\xi-\zeta} s^t u^t - r \sin^2 \theta e^{-\zeta} s^\phi u^\phi = 0,$$

$$\frac{ds^\phi}{dt} + \frac{1}{r} s^r u^\phi + \cot \theta s^\theta u^\phi = u^\phi s^r \frac{f_2'(T) T'(r)}{1 + f_2(T)},$$

$$\frac{ds^\theta}{dt} - \sin \theta \cos \theta s^\phi u^\phi = 0. \tag{36}$$

The first and third equations vanish because $u^r, u^\theta = 0$. Since $s^\mu u_\mu = 0$, let $\theta = \pi/2$, we can put

$$s^t u^t = e^{-\xi} r^2 s^\phi u^\phi \tag{37}$$

into Eq. (36) and then we have

$$\begin{aligned} \frac{ds^r}{dt} + \left[\frac{1}{2}\xi'(r)r^2 - r \right] e^{-\zeta} s^\phi u^\phi &= 0, \\ \frac{ds^\phi}{dt} + \left[\frac{1}{r} - \frac{f_2'(T)T'(r)}{1+f_2(T)} \right] s^r u^\phi &= 0, \\ \frac{ds^\theta}{dt} &= 0. \end{aligned} \tag{38}$$

The general solution of Eq. (38) can be written as

$$\begin{aligned} s^r &= s_0^r \sin \Omega t, \\ s^\phi &= s_0^\phi \cos \Omega t, \\ s^\theta &= s_0^\theta = \text{const.}, \end{aligned} \tag{39}$$

where the angular frequency Ω is given by

$$\Omega^2 = (u^\phi)^2 \left[r - \frac{1}{2}\xi'(r)r^2 \right] \left[\frac{1}{r} - \frac{f_2'(T)T'(r)}{1+f_2(T)} \right] e^{-\zeta}. \tag{40}$$

On the other hand, from Eq. (34), we have

$$u^\nu \nabla_\nu u^\mu = \frac{du^\mu}{dt} + \Gamma_{\nu\lambda}^\mu u^\nu u^\lambda = h^{\mu\nu} \frac{f_2'(T)\nabla_\nu T(r)}{1+f_2(T)}. \tag{41}$$

Using $u_\mu u^\mu = 1$, we have

$$(u^\phi)^2 = \frac{\frac{1}{2}\xi'(r) - \frac{f_2'(T)T'(r)}{1+f_2(T)}}{r(1 - \frac{r}{2}\xi'(r))}. \tag{42}$$

Thus, the precessing rate of the gyroscope $\omega = u^\phi - \Omega$. For the two models being considered, up to linear approximation, it is

$$\begin{aligned} \omega_I &= \sqrt{\frac{M}{r^3}} \left(\frac{3M}{2r} - \frac{AH_0^2 r^2}{8\Omega_{M0}} - \frac{3BH_0^4 r^4}{40} \right), \\ \omega_{II} &= \sqrt{\frac{M}{r^3}} \left(\frac{3M}{2r} - \frac{AH_0^2 r^2}{8\Omega_{M0}} + \frac{5BH_0 r}{8\sqrt{2}} \right). \end{aligned} \tag{43}$$

The first term $3M/2r$ is the prediction of GR, the rest terms are the modifications from the models. Thus the relative modifications are

$$\begin{aligned} \frac{\Delta\omega_I}{\omega_{GR}} &= -\frac{AH_0^2 r^3}{12\Omega_{M0}M} - \frac{BH_0^4 r^5}{20M}, \\ \frac{\Delta\omega_{II}}{\omega_{GR}} &= -\frac{AH_0^2 r^3}{12\Omega_{M0}M} + \frac{5BH_0 r^2}{12\sqrt{2}M}. \end{aligned} \tag{44}$$

The Gravity Probe B mission [57] tested this effect¹ with four gyroscopes at the typical altitude of 642 km. Analysis of the data resulted in a relative error of geodetic precessing rate less than 0.4%. With the cosmologically best-fit value

¹ It measured also the gravitomagnetic spin precession induced in GR by the Earth's rotation; see, e.g., [58].

(21) and the mass of Earth ($GM_\oplus \simeq 22487.9 \text{ eV}^{-1}$, with the gravitational constant G recovered), the relative modifications of our models are $|\Delta\omega_I/\omega_{GR}| \leq 3.005 \times 10^{-31}$ and $\Delta\omega_{II}/\omega_{GR} = 2.152B \times 10^{-11}$. It is obvious that the modifications from both models are within the experiment error if $B < 10^8$ for Model II.

6 Perihelion precession

For a massive test particle orbiting around the central mass, the equation of motion can be obtained from Eq. (41):

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = h^{\mu\nu} \frac{f_2'(T)\nabla_\nu T(r)}{1+f_2(T)}. \tag{45}$$

Assume that the orbit lies in the $\theta = \pi/2$ plane and $d\theta = 0$. The t and ϕ components of the equation of motion are

$$\begin{aligned} \frac{d^2 t}{d\tau^2} &= -\left[\xi'(r) + \frac{f_2'(T)T'(r)}{1+f_2(T)} \right] \frac{dr}{d\tau} \frac{dt}{d\tau}, \\ \frac{d^2 \phi}{d\tau^2} &= -\left[\frac{2}{r} + \frac{f_2'(T)T'(r)}{1+f_2(T)} \right] \frac{dr}{d\tau} \frac{d\phi}{d\tau}, \end{aligned} \tag{46}$$

which can be integrated:

$$\frac{dt}{d\tau} = \frac{ke^{-\xi}}{1+f_2}, \quad \frac{d\phi}{d\tau} = \frac{l}{r^2(1+f_2)}, \tag{47}$$

where k, h are integral constants.

Now from the line element

$$(d\tau)^2 = e^\xi (dt)^2 - e^\zeta (dr)^2 - r^2 (d\phi)^2, \tag{48}$$

we have

$$\begin{aligned} \left(\frac{dr}{d\tau} \right)^2 &= e^{\xi-\zeta} \left(\frac{dt}{d\tau} \right)^2 - r^2 e^{-\zeta} \left(\frac{d\phi}{d\tau} \right)^2 \\ &= \frac{k^2 e^{-\xi-\zeta}}{(1+f_2)^2} - \frac{l^2 e^{-\zeta}}{r^2(1+f_2)^2}. \end{aligned} \tag{49}$$

Up to linear approximation, Eq. (49) can be expressed as

$$\begin{aligned} \left(\frac{dr}{d\tau} \right)_I^2 &= k^2 - 1 + \frac{2M}{r} + \frac{l^2}{r^2} \left(\frac{2M}{r} - 1 \right) \\ &\quad + \frac{3BH_0^4}{4} \left(k^2 - \frac{6}{5} \right) r^4 - \frac{9BH_0^4 l^2 r^2}{10} \\ &\quad - \frac{AH_0^2 k^2 r^2}{2\Omega_{M0}} + \frac{AH_0^2 l^2}{2\Omega_{M0}} \end{aligned} \tag{50}$$

for Model I and

$$\begin{aligned} \left(\frac{dr}{d\tau} \right)_{II}^2 &= k^2 - 1 + \frac{2M}{r} + \frac{l^2}{r^2} \left(\frac{2M}{r} - 1 \right) \\ &\quad - \frac{BH_0 k^2 r}{\sqrt{2}} - \frac{AH_0^2 k^2 r^2}{2\Omega_{M0}} + \frac{AH_0^2 l^2}{2\Omega_{M0}} \end{aligned} \tag{51}$$

for Model II. The first line of each model represents the GR case. Eq. (51) can be compared to the classical equation of motion

$$\frac{1}{2}m_0\dot{r}^2 + V + \frac{L}{2m_0r^2} = E, \tag{52}$$

where V, E are the potential and total energy, and L is the angular momentum, and m_0 is the mass of the test particle. In our cases, both L and E are constants. Thus in the first lines of Eqs. (50) and (51), we can identify $V_{\text{Newt.}} = -\frac{M}{r}$ as the Newtonian potential, $\mathcal{E} = \frac{k^2-1}{2}$ as the conserved energy per unit mass, l as the conserved angular momentum per unit mass, and $V_{\text{GR}} = -\frac{l^2 M}{r^3}$ as the potential from GR. The rest lines of Eqs. (50) and (51) can be seen as additional potentials from the modifications of the models. Thus, the equation of motion can be re-expressed as

$$\frac{1}{2} \left(\frac{dr}{d\tau} \right)^2 = \mathcal{E} + \frac{l^2}{2r^2} - V_{\text{Newt.}} - V_{\text{GR}} - V_{\text{mod.}}, \tag{53}$$

with $V_{\text{mod.}} = V_1 + V_2 + V_3 + V_4$ and

$$\begin{aligned} V_1(r) &= (2\mathcal{E} + 1) \frac{AH_0^2 r^2}{4\Omega_{M0}}, \\ V_2 &= -\frac{AH_0^2 l^2}{4\Omega_{M0}} \end{aligned} \tag{54}$$

for both models and

$$\begin{aligned} V_{3,I}(r) &= -\frac{3BH_0^4}{8} \left(2\mathcal{E} - \frac{1}{5} \right) r^4, \\ V_{4,I}(r) &= \frac{9BH_0^4 l^2 r^2}{20} \end{aligned} \tag{55}$$

for Model I and

$$\begin{aligned} V_{3,II}(r) &= \frac{BH_0(2\mathcal{E} + 1)r}{2\sqrt{2}}, \\ V_{4,II}(r) &= 0 \end{aligned} \tag{56}$$

for Model II.

V_2 and $V_{4,II}$ are constants, which do not contribute to the precession. All the other potentials are power law functions of r . Perihelion precession for such kind of potentials were calculated in, e.g., [59–61]. For a potential $V(r)$ in the form $V(r) = \gamma r^\lambda$, it is found that the extra perihelion precession per orbital period is

$$\delta\phi = \frac{\lambda\gamma(1 - e_{\text{ecc.}}^2)^{\lambda+1} r_a^{\lambda+1}}{e_{\text{ecc.}} M} I(e_{\text{ecc.}}, \lambda), \tag{57}$$

where $e_{\text{ecc.}}$ and r_a are the eccentricity and the semi major axis of the orbit, respectively, and

$$I(e_{\text{ecc.}}, \lambda) = \int_0^{2\pi} \frac{\cos\theta d\theta}{(1 + e_{\text{ecc.}} \cos\theta)^{\lambda+1}}. \tag{58}$$

Table 1 Extra perihelion precessions of some planets in the Solar system, estimated from the recent ephemerides INPOP10a and EPM2011, in miliarcseconds per year (mas/year)

Planet	EPM2011	INPOP10a
Mercury	-0.020 ± 0.030	0.004 ± 0.006
Venus	0.026 ± 0.016	0.002 ± 0.015
Earth	0.0019 ± 0.0019	-0.0020 ± 0.0090
Mars	-0.00020 ± 0.00037	-0.00040 ± 0.00150
Jupiter	0.587 ± 0.283	-0.410 ± 0.420
Saturn	-0.0032 ± 0.0047	0.0015 ± 0.0065

For a small eccentricity $e_{\text{ecc.}}$, $I(e_{\text{ecc.}}, \lambda) \simeq -e_{\text{ecc.}}\pi(1 + \lambda)$. Hence, the prediction of the extra perihelion precession from GR is

$$\delta\phi_{\text{GR}} \simeq \frac{6l^2\pi}{(1 - e_{\text{ecc.}}^2)^2 r_a^2}, \tag{59}$$

which is the well-known result. The extra modifications of perihelion precession are

$$\begin{aligned} \delta\phi_I &= \left[\frac{AH_0^2}{\Omega_{M0}} (1 + 2\mathcal{E}) + \frac{9BH_0^4 l^2}{5} \right] \frac{3(e_{\text{ecc.}}^2 - 1)^3 \pi r_a^3}{2M} \\ &\quad - \frac{3BH_0^4 (e_{\text{ecc.}}^2 - 1)^5 (10\mathcal{E} - 1) \pi r_a^5}{2M} \end{aligned} \tag{60}$$

for Model I and

$$\begin{aligned} \delta\phi_{II} &= \frac{3AH_0^2 (e_{\text{ecc.}}^2 - 1)^3 (1 + 2\mathcal{E}) \pi r_a^3}{2M\Omega_{M0}} \\ &\quad - \frac{BH_0 (e_{\text{ecc.}}^2 - 1)^2 (2\mathcal{E} + 1) \pi r_a^2}{\sqrt{2}M} \end{aligned} \tag{61}$$

for Model II.

The extra perihelion precessions of the planets (from Mercury to Saturn) in the Solar system estimated from the recent ephemerides INPOP10a [62] and EPM2011 [63] (see also [36]) are listed in Table 1.

While it is not very conclusive whether or not some of these planets have anomalous perihelion precessions [36], the preliminary bounds on the two models should be obtained by comparing Eqs. (60) and (61) with the uncertainties of the perihelion precessions in Table 1. Note that Eqs. (60) and (61) give the theoretical predictions of extra changes of the orbit after one revolution, while Table 1 gives the extra precession rates of the planets estimated from the observation. Divided by the orbital periods of the planets, the theoretical predictions from Eqs. (60) and (61) can be compared with observation. Using the semi major axes r_a , eccentricities $e_{\text{ecc.}}$ and the orbital periods T_0 of the planets' orbits, as well as the Solar mass ($GM_\odot \simeq 7.483 \times 10^9 \text{ eV}^{-1}$, with the gravitational constant recovered), we can calculate the extra

Table 2 Extra perihelion precessions of the planets calculated from the models using the cosmological best-fit parameters, in miliarcseconds per year (mas/year)

Planet	Model I	Model II
Mercury	-1.59×10^{-14}	$-26.10B$
Venus	-4.60×10^{-14}	$-38.75B$
Earth	-7.52×10^{-14}	$-45.71B$
Mars	-1.42×10^{-13}	$-56.55B$
Jupiter	-8.86×10^{-13}	$-103.8B$
Saturn	-2.07×10^{-12}	$-134.9B$

precession rates of Models I and II with the cosmologically best-fit values of A , B (21), which are listed in Table 2.

One can see that with the best-fit parameters, the extra perihelion precessions from Model I are less than the uncertainties in Table 1 by about 10 orders of magnitude. Thus, if the unproved anomalies are overlooked it can be concluded that Model I with the best-fit parameters can pass the test of perihelion precession. In the same spirit, if we require that for Model II the extra precessions from the model $|\delta\phi_{II}/T_0|$ listed in Table 2 are less than 10^{-4} mas/year, we can obtain the upper limit of the parameter $B < 7.41 \times 10^{-7}$.

7 Conclusion and discussion

In this paper, we have studied the gravitation redshift, geodesic effect and perihelion precession in the two concrete models of $f(T)$ theory with non-minimal torsion–matter coupling extension, which have been previously constructed and investigated for cosmology. It is found that Model I can pass all three Solar system tests. For Model II, if it is required that the theoretical extra perihelion precessions are indistinguishable under the current observation accuracy and less than 10^{-4} mas/year, the free parameter B should be $< 7.41 \times 10^{-7}$, otherwise the theoretical values will surpass the bounds of the uncertainties of the planets' perihelion precessions.

Neither the gravitation redshift nor Shapiro time delay involves massive particle. Since the coupling between radiation and torsion is still minimal in Models I and II, these effects are practically the same as the minimal coupling theory [38, 39, 42]. Furthermore, the test of Shapiro time delay has been omitted in this paper because its constraint is looser than that of the gravitation redshift for the parameters A and B in the Models I and II.

In a nutshell, we have shown that Models I and II successfully describe not only the observation of the Universe and its large-scale structure and evolution, but also the Solar system effects of gravitation. Therefore, these models are realistic ones. However, we find no acceptable H_0 and Ω_{m0} when $f_1(T)$ and $f_2(T)$ are both proportional to T^2 . Further

research is needed as to whether or not there is a better $f(T)$ model with non-minimal torsion–matter coupling extension. For instance, one can consider the case of $f_1(T) \propto T^m$ and $f_2(T) \propto T^n$ and take optimum of (m, n) using observation at the large and small scales. We will consider this question in the future work.

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