

Ambitwistor superstring in the Green–Schwarz formulation

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Abstract In this paper we construct the ambitwistor superstring in the Green–Schwarz formulation. The model is obtained from the related pure spinor version. We show that the spectrum contains only ten-dimensional supergravity and that kappa symmetry in a curved background implies some of the standard constraints.

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1 Introduction

Perturbative superstring theory determines not only physical states but also their interactions. These include supergravity interactions, gauge interactions and a tower of very massive states in ten dimensions. The presence of this infinite set of states complicates calculations of scattering amplitudes but it also provides a control of the quantum aspects of the theory. One could eliminate the massive states in the limit $\alpha' \rightarrow 0$. In this way, the spectrum obtained in perturbation quantization is the massless states only. The world-sheet description of these class of strings has been recently introduced in [1]. This new string can be seen as the $\alpha' \rightarrow 0$ limit of the usual string. Note that they provide the scattering amplitudes proposed by Cachazo, He and Ye (CHY) [2–4] in gauge and gravity theories. A manifest space-time supersymmetric extension was constructed in [5] using the pure

spinor formalism where the CHY scattering amplitudes were generalized to ten-dimensional superspace. Since supergravity theories are not exact quantum theories, it is expected that world-sheet loops contributions to scattering amplitudes should modify them. In our case, the modification should be done on type II supergravity. A one-loop analysis was done in [6] for the NS–NS part of the spectrum. It would be interesting to determine the RR contributions by using our pure spinor formulation.

A natural next step is to consider the string moving in a curved space. In the RNS case it was done in [7] where the superdiffeomorphism algebra is satisfied if the background NS–NS fields satisfy the supergravity equations of motion in ten dimensions. Their result is exact for all orders in world-sheet perturbation theory. The RR contributions can be naturally obtained using the pure spinor formalism. The coupling to a curved background was studied in [8], where the BRST invariance of the particle-like hamiltonian of the system helps to put the background superfields on-shell. It was noted that the system has an additional symmetry whose BRST invariance implies the so-called nilpotency constraints in [9] involving $H = dB$, where B is the super two-form of type II supergravity in ten dimensions. It is natural to consider the generator of this symmetry as part of the BRST charge. This was done in [10] where the nilpotency of the new BRST charge and the BRST invariance of the particle-like hamiltonian determine the type II supergravity constraints in ten dimensions of [9]. In the present paper, we obtain the Green–Schwarz superstring for the pure spinor string of [10].

This paper is organized as follows. In Sect. 2 we define the GS ambitwistor superstring in flat superspace. We show that the action is invariant under space-time supersymmetry transformations as well as under kappa symmetry. In this case, the necessary Virasoro-like constraints are determined. In Sect. 3, we relate the GS superstring to the pure spinor ambitwistor string in flat space. The relation allows one to obtain the particle-like hamiltonian of the pure spinor ambitwistor superstring from the Virasoro-like constraints

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of the GS ambitwistor superstring. In Sect. 4 the spectrum of the GS ambitwistor superstring is computed using the light-cone gauge quantization. The spectrum is described by the linearization of the type II supergravity fields, as expected. It would be interesting to apply the well known light-cone methods to compute scattering amplitudes in this string and see if they match with the ones obtained from the Mason–Skinner string and the pure spinor version. We conclude in Sect. 5 by studying the GS action in a curved superspace. The action turns out to be invariant under kappa symmetries if the constraints of type II supergravity are imposed.

2 Flat superspace formulation

We construct an action for the ambitwistor in Green–Schwarz formalism. We find the κ symmetry of the theory and the Virasoro-like constraints. The world-sheet fields are the conjugate pairs (X^m, P_m) and the super coordinates $(\theta, \hat{\theta})$ which describe the type II superspace in ten dimensions. As it was shown in [10] for the pure spinor case, the supersymmetry transformations for these variables are

$$\begin{aligned} \delta\theta^\alpha &= \varepsilon^\alpha, \quad \delta\hat{\theta}^{\hat{\alpha}} = \hat{\varepsilon}^{\hat{\alpha}}, \quad \delta X^m = -\frac{i}{2}(\varepsilon\gamma^m\theta) - \frac{i}{2}(\hat{\varepsilon}\gamma^m\hat{\theta}), \\ \delta P_m &= \frac{i}{2}\partial(\varepsilon\gamma_m\theta - \hat{\varepsilon}\gamma_m\hat{\theta}). \end{aligned} \tag{2.1}$$

Note that these transformations satisfy the usual super Poincaré algebra, that is, $[\delta_1, \delta_2]$ leaves $(\theta, \hat{\theta}, P)$ invariant and translates X^m in an amount proportional to $\varepsilon_1\gamma^m\varepsilon_2 + \hat{\varepsilon}_1\gamma^m\hat{\varepsilon}_2$.

It is useful to define invariant fields under supersymmetry to get invariant objects like the action. We define the combinations

$$\begin{aligned} \Pi^m &= \partial X^m + \frac{i}{2}\theta\gamma^m\partial\theta + \frac{i}{2}\hat{\theta}\gamma^m\partial\hat{\theta}, \\ \bar{\Pi}^m &= \bar{\partial} X^m + \frac{i}{2}\theta\gamma^m\bar{\partial}\theta + \frac{i}{2}\hat{\theta}\gamma^m\bar{\partial}\hat{\theta}, \end{aligned} \tag{2.2}$$

$$P_L^m = P^m - \partial X^m - i\theta\gamma^m\partial\theta, \quad P_R^m = P^m + \partial X^m + i\hat{\theta}\gamma^m\partial\hat{\theta}, \tag{2.3}$$

which are invariant under supersymmetry. The combinations $P_{L,R}$ are not independent, they satisfy $P_R^m - P_L^m = 2\Pi^m$. Note that $(\partial\theta, \bar{\partial}\theta, \partial\hat{\theta}, \bar{\partial}\hat{\theta})$ are also invariant under supersymmetry. The Green–Schwarz action is written in terms of these supersymmetry invariant objects and it is given by

$$\begin{aligned} S_{\text{GS}} &= \int d^2z \frac{1}{2}(P_L^m + P_R^m)\bar{\Pi}_m \\ &\quad + \frac{i}{2}(\theta\gamma_m\partial\theta)\bar{\Pi}^m - \frac{i}{2}(\theta\gamma_m\bar{\partial}\theta)\Pi^m \end{aligned}$$

$$\begin{aligned} & - \frac{i}{2}(\hat{\theta}\gamma_m\partial\hat{\theta})\bar{\Pi}^m + \frac{i}{2}(\hat{\theta}\gamma_m\bar{\partial}\hat{\theta})\Pi^m \\ & + \frac{1}{4}(\theta\gamma^m\partial\theta)(\hat{\theta}\gamma_m\bar{\partial}\hat{\theta}) - \frac{1}{4}(\theta\gamma^m\bar{\partial}\theta)(\hat{\theta}\gamma_m\partial\hat{\theta}). \end{aligned} \tag{2.4}$$

We now verify that this action is invariant under (2.1). Consider first the term involving ε in the variation of the action. This variation becomes proportional to

$$\int d^2z (\varepsilon\gamma^m\partial\theta)(\theta\gamma^m\bar{\partial}\theta) - (\varepsilon\gamma^m\bar{\partial}\theta)(\theta\gamma^m\partial\theta). \tag{2.5}$$

On the one hand, this expression, after using the Fierz identity $\gamma_{\alpha(\beta}^m(\gamma_m)_{\gamma\delta)} = 0$, is equal to

$$\int d^2z (\varepsilon\gamma^m\theta)(\partial\theta\gamma^m\bar{\partial}\theta). \tag{2.6}$$

On the other hand, integrating by parts (2.5) is equal to

$$-2 \int d^2z (\varepsilon\gamma^m\theta)(\partial\theta\gamma^m\bar{\partial}\theta). \tag{2.7}$$

Therefore, (2.5) vanishes (up to a total derivative). A similar implication is obtained for the terms involving $\hat{\varepsilon}$. Then the action (2.4) is invariant under space-time supersymmetry.

The equations of motion derived from the action (2.5) are

$$\bar{\Pi}^m = 0, \quad \bar{\partial} P_m - \frac{1}{2}\partial(\theta\gamma_m\bar{\partial}\theta) + \frac{1}{2}\partial(\hat{\theta}\gamma_m\bar{\partial}\hat{\theta}) = 0, \tag{2.8}$$

$$P_L^m(\gamma_m\bar{\partial}\theta)_\alpha = 0, \quad P_R^m(\gamma_m\bar{\partial}\hat{\theta})_{\hat{\alpha}} = 0. \tag{2.9}$$

These equations are easily solved in the light-cone gauge, as we describe below, and the world-sheet fields are holomorphic.

We now verify that this action is invariant under κ -symmetry and find the Virasoro-like constraints for (2.4). The κ -transformations are defined by

$$\delta\theta^\alpha = P_L^m(\kappa\gamma_m)^\alpha, \quad \delta\hat{\theta}^{\hat{\alpha}} = P_R^m(\hat{\kappa}\gamma_m)^{\hat{\alpha}}, \tag{2.10}$$

$$\delta X^m = \frac{i}{2}\delta\theta\gamma^m\theta + \frac{i}{2}\delta\hat{\theta}\gamma^m\hat{\theta}, \quad \delta P_m = -\frac{i}{2}\partial(\delta\theta\gamma_m\theta - \delta\hat{\theta}\gamma_m\hat{\theta}). \tag{2.11}$$

As usual, the κ transformations for the X field has the opposite sign of the supersymmetry transformation. This property is also true for the P field. We can obtain the κ transformations for P_L, P_R and Π . They are

$$\delta\Pi^m = i\delta\theta\gamma^m\partial\theta + i\delta\hat{\theta}\gamma^m\partial\hat{\theta}, \quad \delta\bar{\Pi}^m = i\delta\theta\gamma^m\bar{\partial}\theta + i\delta\hat{\theta}\gamma^m\bar{\partial}\hat{\theta}, \tag{2.12}$$

$$\delta P_L^m = -2i\delta\theta\gamma^m\partial\theta, \quad \delta P_R^m = 2i\delta\hat{\theta}\gamma^m\partial\hat{\theta}. \tag{2.13}$$

Under κ -transformations the action changes as

$$\delta S_{GS} = \int d^2z i P_L^m (\delta\theta\gamma_m\bar{\partial}\theta) + i P_R^m (\delta\hat{\theta}\gamma^m\bar{\partial}\hat{\theta}). \tag{2.14}$$

After using (2.10), the variation of the action will vanish if Virasoro-like constraints are imposed. Consider the first term in (2.14). It is equal to

$$P_L^m P_L^n (\kappa\gamma_n\gamma_m\bar{\partial}\theta) = P_L^m P_{Lm} (\kappa_\alpha\bar{\partial}\theta^\alpha). \tag{2.15}$$

Similarly, the second term in (2.14) is equal to

$$P_R^m P_R^n (\hat{\kappa}\gamma_n\gamma_m\bar{\partial}\hat{\theta}) = P_R^m P_{Rm} (\hat{\kappa}_{\hat{\alpha}}\bar{\partial}\hat{\theta}^{\hat{\alpha}}). \tag{2.16}$$

Plugging these results into (2.14) we obtain the necessity of imposing the constraints

$$H_L = P_L^m P_{Lm} = 0, \quad H_R = P_R^m P_{Rm} = 0, \tag{2.17}$$

which are the Virasoro-like constraints. These constraints can be added to the action through Lagrange multipliers. The action becomes

$$S_{GS} + \int d^2z \frac{1}{4} e_L P_L^m P_{Lm} - \frac{1}{4} e_R P_R^m P_{Rm}, \tag{2.18}$$

and the Lagrange multipliers transform under κ transformations as

$$\delta e_L = -4\kappa_\alpha (\bar{\partial}\theta^\alpha + e_L\partial\theta^\alpha), \quad \delta e_R = 4\hat{\kappa}_{\hat{\alpha}} (\bar{\partial}\hat{\theta}^{\hat{\alpha}} - e_R\partial\hat{\theta}^{\hat{\alpha}}). \tag{2.19}$$

It is interesting to note that both the bosonic and the RNS ambitwistor strings are easily related to their respective point-particle cases in the first order formulation if all fields do not depend on the spatial world-sheet coordinate. However, this is not true when there is space-time supersymmetry. In the pure spinor [8, 10] version and the GS version presented here, the naive point-particle limit does not give the usual Brink–Schwarz super-particle or the pure spinor super-particle. The main difference is that P^m still transforms under supersymmetry in the point-particle limit. Nevertheless, at least in the Brink–Schwarz case, since both actions will be supersymmetric and κ -symmetric, they should be related by a field redefinition. In order to obtain the ambitwistor string from the Brink–Schwarz action, more guess work is necessary to find the correct set of local symmetries.

3 Relation to the pure spinor formalism

It is well known that there is a relation between the Green–Schwarz string and the pure spinor string [11]. The relation begins with the observation that the momentum conjugate

of the fermionic coordinates in (2.4) determines a constraint which is used to define a BRST charge. This is done with the help of the pure spinor variables [12]. The definition of the conjugate variables $p_\alpha, \hat{p}_{\hat{\alpha}}$ for θ and $\hat{\theta}$ respectively, gives the constraints

$$d_\alpha = p_\alpha - \frac{i}{2} (P_m - \partial X_m) (\gamma^m\theta)_\alpha + \frac{1}{4} (\theta\gamma_m\partial\theta) (\gamma^m\theta)_\alpha, \tag{3.1}$$

$$\hat{d}_{\hat{\alpha}} = \hat{p}_{\hat{\alpha}} - \frac{i}{2} (P_m + \partial X_m) (\gamma^m\hat{\theta})_{\hat{\alpha}} - \frac{1}{4} (\hat{\theta}\gamma_m\partial\hat{\theta}) (\gamma^m\hat{\theta})_{\hat{\alpha}}, \tag{3.2}$$

which are the fermionic constraints used in [10] to define the pure spinor BRST charge

$$Q = \oint \lambda^\alpha d_\alpha + \hat{\lambda}^{\hat{\alpha}} \hat{d}_{\hat{\alpha}}, \tag{3.3}$$

where the pure spinors λ and $\hat{\lambda}$ satisfy $\lambda\gamma^m\lambda = \hat{\lambda}\gamma^m\hat{\lambda} = 0$. Clearly, this condition and the algebra of d and \hat{d} imply that $Q^2 = 0$, then this charge is used as the BRST generator in the pure spinor formalism. The BRST transformation of the fermionic coordinates are

$$Q\theta^\alpha = \lambda^\alpha, \quad Q\hat{\theta}^{\hat{\alpha}} = \hat{\lambda}^{\hat{\alpha}}. \tag{3.4}$$

The conjugate variables of the pure spinors are ω and $\hat{\omega}$ are defined up to the gauge transformations $\delta\omega_\alpha = (\lambda\gamma^m)_\alpha\Lambda_m$ and $\delta\hat{\omega}_{\hat{\alpha}} = (\hat{\lambda}\gamma^m)_{\hat{\alpha}}\hat{\Lambda}_{\hat{m}}$. They transform under (3.3) as

$$Q\omega_\alpha = d_\alpha, \quad Q\hat{\omega}_{\hat{\alpha}} = \hat{d}_{\hat{\alpha}}. \tag{3.5}$$

Oda and Tonin related the pure spinor string and the Green–Schwarz such that the difference of the corresponding actions is [11]

$$Q \left(\int d^2z \omega_\alpha \bar{\partial}\theta^\alpha + \hat{\omega}_{\hat{\alpha}} \bar{\partial}\hat{\theta}^{\hat{\alpha}} \right). \tag{3.6}$$

Note that this statement does not imply that the pure spinor and the Green–Schwarz actions are cohomologically equivalent because Q^2 on ω and $\hat{\omega}$ are gauge transformations.

We use this property to find the pure spinor version of (2.18). The action in the pure spinor formalism is (2.18) plus

$$Q \left(\int d^2z \omega_\alpha \bar{\partial}\theta^\alpha + \hat{\omega}_{\hat{\alpha}} \bar{\partial}\hat{\theta}^{\hat{\alpha}} - e_L\omega_\alpha\partial\theta^\alpha - e_R\hat{\omega}_{\hat{\alpha}}\partial\hat{\theta}^{\hat{\alpha}} \right). \tag{3.7}$$

Assuming the Lagrange multipliers e_L and e_R are BRST invariants, we finally find that the action is

$$S = \int d^2z P_m \bar{\partial} X^m + p_\alpha \bar{\partial} \theta^\alpha + \hat{p}_{\hat{\alpha}} \bar{\partial} \hat{\theta}^{\hat{\alpha}} + \omega_\alpha \bar{\partial} \lambda^\alpha + \hat{\omega}_{\hat{\alpha}} \bar{\partial} \hat{\lambda}^{\hat{\alpha}} + e_L T_L + e_R T_R, \tag{3.8}$$

where

$$T_L = \frac{1}{4} P_L^m P_{Lm} - d_\alpha \partial \theta^\alpha - \omega_\alpha \partial \lambda^\alpha, \\ T_R = -\frac{1}{4} P_R^m P_{Rm} - \hat{d}_{\hat{\alpha}} \partial \hat{\theta}^{\hat{\alpha}} - \hat{\omega}_{\hat{\alpha}} \partial \hat{\lambda}^{\hat{\alpha}}, \tag{3.9}$$

which correspond to the pure spinor generalization of the constraints H_L, H_R in the Green–Schwarz case. Note that T_L and T_R are stress-like tensors which were used in [13] to find b -like ghosts. That is, there exist b_L and b_R that satisfy $Q_L b_L = T_L$ and $Q_R b_R = T_R$. Note also that $T_L + T_R$ is the stress-energy tensor of (3.8) and that

$$\mathcal{H} = -T_L + T_R, \tag{3.10}$$

is a symmetry of (3.8) discovered in [10] and that helps to find the supergravity constraints when the model of (3.8) is coupled to a curved background. Note that the last two terms in (3.8) are BRST trivial so they can be removed from the action. The reason for this is that both T_L and T_R are BRST exact [13]. Note also that this is not the case in the bosonic string case [1]. Here the corresponding Lagrange multipliers are gauge fixed and lead to bc ghosts in the quantization of the model.

4 Light-cone gauge quantization and spectrum

The action (2.4) has a large set of local symmetries that allows us to fix some of its dynamical variables. The kappa-symmetry parameters, see (2.10), can be used to fix the fermionic coordinates to satisfy

$$\gamma^+ \theta = 0, \quad \gamma^+ \hat{\theta} = 0. \tag{4.1}$$

The only component of $\theta \gamma^m \partial \theta$ that does not vanish is $\theta \gamma^- \partial \theta = \delta_{ab} \theta^a \partial \theta^b$. We will suppress δ_{ab} from now on. The same simplification happens for the other derivative and for the other spinor variable.

In this gauge the action simplifies to

$$S_{lc} = \int d^2z \left(P^i \bar{\partial} X^i + P^+ \bar{\partial} X^- + P^- \bar{\partial} X^+ + \frac{i}{2} P_L^+ \theta^a \bar{\partial} \theta^a + \frac{i}{2} P_R^+ \hat{\theta}^{\hat{a}} \bar{\partial} \hat{\theta}^{\hat{a}} \right). \tag{4.2}$$

Off shell this is as far as we can go. The equations of motion coming from (4.2) imply all world-sheet variables are holomorphic. We now use the Virasoro-like constraints to fix

$$P_L^+ = P^+ - \partial X^+ = k_L^+, \quad P_R^+ = P^+ + \partial X^+ = k_R^+. \tag{4.3}$$

This means that

$$X^+ = x_0^+ + \frac{1}{2} (k_R^+ - k_L^+) (\tau - \sigma), \quad P^+ = \frac{1}{2} (k_R^+ + k_L^+). \tag{4.4}$$

Periodicity in σ in the closed string implies $k_L^+ = k_R^+ = k^+$.

We can now use canonical quantization to find the spectrum of the model. The commutators for the spinors are

$$\{\theta^a(\tau, \sigma), \theta^b(\tau, \sigma')\} = \frac{\pi}{k^+} \delta^{ab} \delta(\sigma - \sigma'), \\ \{\hat{\theta}^{\hat{a}}(\tau, \sigma), \hat{\theta}^{\hat{b}}(\tau, \sigma')\} = \frac{\pi}{k^+} \delta^{\hat{a}\hat{b}} \delta(\sigma - \sigma'). \tag{4.5}$$

These relations imply that the modes in the expansion

$$\theta^a = \frac{1}{\sqrt{2k^+}} \sum_{-\infty}^{\infty} \theta_n^a e^{-in(\tau-\sigma)}, \quad \hat{\theta}^{\hat{a}} = \frac{1}{\sqrt{2k^+}} \sum_{-\infty}^{\infty} \hat{\theta}_n^{\hat{a}} e^{-in(\tau-\sigma)} \tag{4.6}$$

satisfy

$$\{\theta_n^a, \theta_m^b\} = \delta^{ab} \delta_{n+m}, \quad \{\hat{\theta}_n^{\hat{a}}, \hat{\theta}_m^{\hat{b}}\} = \delta^{\hat{a}\hat{b}} \delta_{n+m}. \tag{4.7}$$

As usual, the commutation relation of the zero modes imply that the vacuum is degenerate,

$$|\dot{a}\hat{a}\rangle_0, \quad |aj\rangle_0, \quad |i\hat{a}\rangle_0, \quad |ij\rangle_0, \tag{4.8}$$

where the spinor zero modes act as

$$\theta_0^a |i\hat{a}\rangle = \sigma_i^{a\hat{a}} |i\hat{a}\rangle, \quad \theta^a |i\hat{a}\rangle = \sigma_i^{a\hat{a}} |i\hat{a}\rangle \tag{4.9}$$

and similarly with $\hat{\theta}^{\hat{a}}$.

Now we turn to the bosonic coordinates and their conjugate momenta. The gauge fixing (4.3) mean we can use the Virasoro-like constraints to find P_R^- and P_L^- in terms of all other variables. The quantization of the remaining coordinates do not follow the usual one from the bosonic string, they are in fact more closely related to the RNS ghosts β and γ . The mode expansions are given by

$$X^i = \sum_{-\infty}^{\infty} x_n^i e^{-in(\tau-\sigma)}, \quad P^i = \sum_{-\infty}^{\infty} p_n^i e^{-in(\tau-\sigma)}, \tag{4.10}$$

and the commutation relations are given by

$$[x_n^i, p_m^j] = i\delta^{ij}\delta_{n+m}. \tag{4.11}$$

This follows from

$$[X^i(\tau, \sigma), P^j(\tau, \sigma')] = \pi i\delta^{ij}\delta(\sigma - \sigma'). \tag{4.12}$$

Reality of X and P implies that

$$(x_n^i)^\dagger = x_{-n}^i, \quad (p_n^j)^\dagger = p_{-n}^j. \tag{4.13}$$

It remains to find P^- and X^- . They they are found using the Virasoro-like constraints,

$$\begin{aligned} P^- - \partial X^- &= -\frac{1}{2k^+} P_L^i P_L^i + \theta^a \partial \theta^a, \\ P^- + \partial X^- &= -\frac{1}{2k^+} P_R^i P_R^i - \hat{\theta}^a \partial \hat{\theta}^a. \end{aligned} \tag{4.14}$$

Note that the zero mode of the left hand side of the two expressions above is the same,

$$p_0^- = \frac{1}{2\pi} \oint d\sigma (P^- - \partial X^-) = \frac{1}{2\pi} \oint d\sigma (P^- + \partial X^-). \tag{4.15}$$

Physical states $|\Psi\rangle$ have to satisfy the condition that they are eigen states of p_0^- :

$$p_0^- |\Psi\rangle = k^- |\Psi\rangle. \tag{4.16}$$

This condition fixes the mass of the state $|\Psi\rangle$ using

$$\begin{aligned} p_0^- &= -\frac{1}{2k^+} \left(p_0^i p_0^i + \sum_{n=1}^{\infty} (p_{-n}^i p_n^i + n^2 x_{-n}^i x_n^i \right. \\ &\quad \left. + in(x_{-n}^i p_n^i - p_{-n}^i x_n^i) + n\theta_{-n}^a \theta_n^a) \right), \end{aligned} \tag{4.17}$$

$$\begin{aligned} p_0^- &= -\frac{1}{2k^+} \left(p_0^i p_0^i + \sum_{n=1}^{\infty} (p_{-n}^i p_n^i + n^2 x_{-n}^i x_n^i \right. \\ &\quad \left. - in(x_{-n}^i p_n^i - p_{-n}^i x_n^i) - n\hat{\theta}_{-n}^a \hat{\theta}_n^a) \right). \end{aligned} \tag{4.18}$$

Note that x_0^i , θ_0^a and $\hat{\theta}_0^a$ do not appear in the expressions above and the normal ordering constant from x_n^i and p_n^i cancels the one from θ_n^a and $\hat{\theta}_n^a$.

Let us now find what are the possible eigenstates of the two operators on the left hand side of (4.17) and (4.18). We will use $|\Psi\rangle_0$ to represent any of the states in (4.8). A state with transverse momentum \vec{k} is given by

$$|\Psi\rangle_{\vec{k}} = e^{ik^i x_0^i} |\Psi\rangle_0. \tag{4.19}$$

We have to define how the whole set of oscillators act on $|\Psi\rangle_{\vec{k}}$. In order to do that, let us define the linear combinations

$$\begin{aligned} \frac{1}{\sqrt{2}}(P^i - \partial X^i) &= \frac{1}{\sqrt{2}} \sum_{-\infty}^{\infty} (p_n^i + inx_n^i) e^{-in(\tau-\sigma)} \\ &= \sum_{-\infty}^{\infty} a_n^i e^{-in(\tau-\sigma)}, \end{aligned} \tag{4.20}$$

$$\begin{aligned} \frac{1}{\sqrt{2}}(P^i + \partial X^i) &= \frac{1}{\sqrt{2}} \sum_{-\infty}^{\infty} (p_n^i - inx_n^i) e^{-in(\tau-\sigma)} \\ &= \sum_{-\infty}^{\infty} \bar{a}_n^i e^{-in(\tau-\sigma)}. \end{aligned} \tag{4.21}$$

Using the reality condition we see that

$$(a_n^i)^\dagger = a_{-n}^i, \quad (\bar{a}_n^i)^\dagger = \bar{a}_{-n}^i. \tag{4.22}$$

The commutation relation (4.11) implies

$$[a_n^i, \bar{a}_m^j] = 0, \quad [a_n^i, a_m^j] = -n\delta^{ij}\delta_{n+m}, \quad [\bar{a}_n^i, \bar{a}_m^j] = n\delta^{ij}\delta_{n+m}. \tag{4.23}$$

The Hamiltonian of the system is

$$\begin{aligned} H &= \frac{1}{4\pi} \oint d\sigma (: H_R : - : H_L :) \\ &= \sum_{n=1}^{\infty} (-a_{-n}^i a_n^i + \bar{a}_{-n}^i \bar{a}_n^i + n\theta_{-n}^a \theta_n^a + n\hat{\theta}_{-n}^a \hat{\theta}_n^a). \end{aligned} \tag{4.24}$$

The normal ordering is defined by moving all positive modes to the right. The constant resulting from this operation cancels. The action of the oscillators on $|\Psi\rangle_{\vec{k}}$ will be defined such that H vanishes on it,

$$a_n^i |\Psi\rangle_{\vec{k}} = \bar{a}_n^i |\Psi\rangle_{\vec{k}} = \theta_n^a |\Psi\rangle_{\vec{k}} = \hat{\theta}_n^a |\Psi\rangle_{\vec{k}} = 0 \quad \text{for } n \geq 1. \tag{4.25}$$

This corresponds to imposing the requirement that all positive frequency modes kill the vacuum, as usual. We see that modes with negative n create an excitation with energy n .

Now it is possible to see that the only state that satisfies (4.16) with both (4.17) and (4.18) is $|\Psi\rangle_{\vec{k}}$ with vanishing mass. This is the expected answer since we already know the spectrum of both RNS and pure spinor versions of the ambitwistor string. Although we did not compute it, it is likely that the super Poincaré algebra only closes when $d = 10$, just like the usual GS string.

The zero mode of the operator $k \cdot P$ is not identically zero, but it acts as zero on the physical states. This is very similar to what happens with the $\beta\gamma$ ghosts in RNS, where the vacuum is annihilated by $\gamma_{\frac{1}{2}}$, the zero mode of $\gamma(z)$ with NS boundary conditions. In terms of a local operator, the vacuum corresponds to $\delta(\gamma(z))$. In the same way, if a covariant quantization was possible, the construction of the operator corresponding to $|\Psi\rangle_{\vec{k}}$ would contain $\delta(k \cdot P(z))$. This

explains the delta function present in the vertex operator in the Mason–Skinner string.

5 Curved background

The action in a curved background is given by

$$S = \int d^2z \mathcal{P}_a \bar{\Pi}^a - \Pi^A \bar{\Pi}^B B_{BA}, \tag{5.1}$$

where $\Pi^A = \partial Z^M E_M^A$, $\bar{\Pi}^A = \bar{\partial} Z^M E_M^A$ with Z^M being the superspace coordinates and E_M^A being the supervielbein. The flat space limit of this action becomes (2.4).

The κ -transformations are

$$\begin{aligned} \delta Z^M E_M^a &= 0, \quad \delta Z^M E_M^\alpha = (\mathcal{P}_a - \Pi_a)(\gamma^a \kappa)^\alpha, \\ \delta Z^M E_M^{\hat{a}} &= (\mathcal{P}_a + \Pi_a)(\gamma^a \hat{\kappa})^\alpha, \end{aligned} \tag{5.2}$$

and we determine the transformation of \mathcal{P} by demanding the invariance of the action (5.1) under κ -transformations. Let us define $\sigma^\alpha = \delta Z^M E_M^\alpha$ and $\sigma^{\hat{\alpha}} = \delta Z^M E_M^{\hat{\alpha}}$. The variation of the action is

$$\begin{aligned} \delta S &= \int d^2z [\delta \mathcal{P}_a \bar{\Pi}^a + \mathcal{P}_a (-\sigma^\alpha (\bar{\Pi}^A T_{A\alpha}{}^a + \bar{\Pi}^b \Omega_{\alpha b}{}^a) \\ &\quad - \sigma^{\hat{\alpha}} (\bar{\Pi}^A T_{A\hat{\alpha}}{}^a + \bar{\Pi}^b \Omega_{\hat{\alpha} b}{}^a)) \\ &\quad - \sigma^\alpha \Pi^A \bar{\Pi}^B H_{BA\alpha} - \sigma^{\hat{\alpha}} \Pi^A \bar{\Pi}^B H_{BA\hat{\alpha}}], \end{aligned} \tag{5.3}$$

where Ω is the Lorentz connection, T is the torsion and $H = dB$. The terms involving $\Pi^A \bar{\Pi}^B$ with A and B taking spinor values vanish, implying that

$$H_{\alpha\beta\gamma} = H_{\alpha\hat{\beta}[\gamma]} = H_{\alpha\hat{\beta}\hat{\gamma}} = H_{\hat{\alpha}\hat{\beta}[\gamma]} = 0. \tag{5.4}$$

Similarly, the terms involving $\sigma^\alpha \Pi^A \bar{\Pi}^{\hat{\beta}}$, $\sigma^{\hat{\alpha}} \Pi^A \bar{\Pi}^{\hat{\beta}}$ and $\sigma^\alpha \mathcal{P}_a \bar{\Pi}^{\hat{\beta}}$, $\sigma^{\hat{\alpha}} \mathcal{P}_a \bar{\Pi}^{\hat{\beta}}$ vanish too. This implies that

$$T_{\alpha\hat{\beta}}{}^a = H_{\alpha\hat{\beta}a} = 0. \tag{5.5}$$

Using these results, the variation of the action becomes

$$\begin{aligned} \delta S &= \int d^2z -\sigma^\alpha (\mathcal{P}^a T_{\alpha\beta a} - \Pi^a H_{\alpha\beta a}) \bar{\Pi}^\beta \\ &\quad - \sigma^{\hat{\alpha}} (\mathcal{P}^a T_{\hat{\alpha}\hat{\beta}a} - \Pi^a H_{\hat{\alpha}\hat{\beta}a}) \bar{\Pi}^{\hat{\beta}} \\ &\quad + (\delta \mathcal{P}_a + \sigma^\alpha (-H_{\alpha\beta a} \Pi^\beta + (T_{\alpha a}{}^b - \Omega_{\alpha a}{}^b) \mathcal{P}_b - H_{\alpha ab} \Pi^b) \\ &\quad + \sigma^{\hat{\alpha}} (-H_{\hat{\alpha}\hat{\beta}a} \bar{\Pi}^{\hat{\beta}} + (T_{\hat{\alpha} a}{}^b - \Omega_{\hat{\alpha} a}{}^b) \mathcal{P}_b - H_{\hat{\alpha} ab} \bar{\Pi}^b)) \bar{\Pi}^a. \end{aligned} \tag{5.6}$$

We would like to obtain a result similar to (2.14) of flat space. That is, the Virasoro-like constraints should appear. In curved

space they are proportional to $(\mathcal{P}_a \pm \Pi_a)^2$. Consider the first term in (5.6). Because σ^α involves $(\mathcal{P}_a - \Pi_a)$, the Virasoro-like constraint $(\mathcal{P}_a - \Pi_a)^2$ will appear if $H_{\alpha\beta a} = T_{\alpha\beta a} = (\gamma_a)_{\alpha\beta}$. Similarly, for the second in (5.6), the Virasoro-like constraint $(\mathcal{P}_a + \Pi_a)^2$ will appear if $H_{\hat{\alpha}\hat{\beta}a} = -T_{\hat{\alpha}\hat{\beta}a} = (\gamma_a)_{\hat{\alpha}\hat{\beta}}$. Then we have the constraints for the background superfields,

$$T_{\alpha\beta a} - H_{\alpha\beta a} = 0, \quad T_{\hat{\alpha}\hat{\beta}a} + H_{\hat{\alpha}\hat{\beta}a} = 0. \tag{5.7}$$

We also need to cancel the second term in (5.6) which gives the transformation of \mathcal{P} , that is,

$$\begin{aligned} \delta \mathcal{P}_a &= \sigma^\alpha ((\gamma_a)_{\alpha\beta} \Pi^\beta - (T_{\alpha a}{}^b - \Omega_{\alpha a}{}^b) \mathcal{P}_b + H_{\alpha ab} \Pi^b) \\ &\quad - \sigma^{\hat{\alpha}} ((\gamma_a)_{\hat{\alpha}\hat{\beta}} \bar{\Pi}^{\hat{\beta}} - (T_{\hat{\alpha} a}{}^b - \Omega_{\hat{\alpha} a}{}^b) \mathcal{P}_b - H_{\hat{\alpha} ab} \bar{\Pi}^b). \end{aligned} \tag{5.8}$$

Using this, the variation of the action is, as promised, similar to flat space case and it is equal to

$$\begin{aligned} \delta S &= \int d^2z -(\mathcal{P}_a - \Pi_a)(\gamma^a \kappa)^\alpha (\mathcal{P}_b - \Pi_b) \gamma_{\alpha\beta}^b \bar{\Pi}^\beta \\ &\quad - (\mathcal{P}_a + \Pi_a)(\gamma^a \hat{\kappa})^{\hat{\alpha}} (\mathcal{P}_b + \Pi_b) \gamma_{\hat{\alpha}\hat{\beta}}^b \bar{\Pi}^{\hat{\beta}}, \end{aligned} \tag{5.9}$$

therefore if the Virasoro-like constraints are

$$\begin{aligned} \mathcal{H}_L &= (\mathcal{P}_a - \Pi_a)(\mathcal{P}^a - \Pi^a) = 0, \\ \mathcal{H}_R &= (\mathcal{P}_a + \Pi_a)(\mathcal{P}^a + \Pi^a) = 0, \end{aligned} \tag{5.10}$$

and the action is invariant under the κ -transformations of (5.2) and (5.8). Note that these Virasoro-like constraints becomes the Virasoro-like constraints of (2.17) because, as shown in [10], the field \mathcal{P} becomes $\frac{1}{2}P_L + P_R$.

As in the flat space case, the action can include the Virasoro-like constraints through the use of Lagrange multipliers so it is equal to

$$S + \int d^2z \frac{1}{4} e_L \mathcal{H}_L - \frac{1}{4} e_R \mathcal{H}_R. \tag{5.11}$$

To obtain the κ -transformations of the Virasoro-like constraints, we need to determine the corresponding variation of $\mathcal{P}_a \pm \Pi_a$. It turns out that these combinations transform as

$$\begin{aligned} \delta(\mathcal{P}_a - \Pi_a) &= 2\delta Z^M E_M^\alpha (\gamma_a)_{\alpha\beta} \Pi^\beta \\ &\quad + (T_{\alpha ab} - \Omega_{\alpha ab}) \delta Z^M E_M^\alpha (\mathcal{P}^b - \Pi^b) \\ &\quad + (T_{\hat{\alpha} ab} - \Omega_{\hat{\alpha} ab}) \delta Z^M E_M^{\hat{\alpha}} (\mathcal{P}^b - \Pi^b), \end{aligned} \tag{5.12}$$

$$\begin{aligned} \delta(\mathcal{P}_a + \Pi_a) = & -2\delta Z^M E_M^{\hat{\alpha}}(\gamma_a)_{\hat{\alpha}\hat{\beta}}\Pi^{\hat{\beta}} \\ & + (T_{\alpha ab} - \Omega_{\alpha ab})\delta Z^M E_M^{\alpha}(\mathcal{P}^b + \Pi^b) \\ & + (T_{\hat{\alpha} ab} - \Omega_{\hat{\alpha} ab})\delta Z^M E_M^{\hat{\alpha}}(\mathcal{P}^b + \Pi^b), \end{aligned} \tag{5.13}$$

if we impose the further constraints

$$H_{\alpha ab} = H_{\hat{\alpha} ab} = 0. \tag{5.14}$$

Note that the term involving the connection does not contribute to the variation of \mathcal{H}_L and \mathcal{H}_R because it is anti-symmetric in the vector indices. Similarly, the torsion component has to be anti-symmetric in the vector indices. That is,

$$T_{\alpha(ab)} = T_{\hat{\alpha}(ab)} = 0. \tag{5.15}$$

The variations of the Virasoro-like constraints are

$$\delta\mathcal{H}_L = 4(\kappa_{\alpha}\Pi^{\alpha})\mathcal{H}_L, \quad \delta\mathcal{H}_R = -4(\hat{\kappa}_{\hat{\alpha}}\Pi^{\hat{\alpha}})\mathcal{H}_R, \tag{5.16}$$

and the action is invariant under κ -transformations if the Lagrange multipliers transform as

$$\delta e_L = 4\kappa_{\alpha}(\bar{\Pi}^{\alpha} - e_L\Pi^{\alpha}), \quad \delta e_R = -4\hat{\kappa}_{\hat{\alpha}}(\bar{\Pi}^{\hat{\alpha}} + e_R\Pi^{\hat{\alpha}}), \tag{5.17}$$

which imply the transformations of the Lagrange multipliers (2.19) in the flat space limit.

In summary, in order to have an action with κ -symmetry in curved background, it is necessary to impose certain constraints on the background fields. They are

$$\begin{aligned} H_{\alpha\beta\gamma} = H_{\hat{\alpha}\hat{\beta}[\gamma]} = H_{\Lambda\alpha\hat{\beta}} = T_{\alpha\hat{\beta}}^a = T_{\alpha\beta a} - H_{\alpha\beta a} \\ = T_{\hat{\alpha}\hat{\beta} a} + H_{\hat{\alpha}\hat{\beta} a} = T_{\alpha(ab)} = T_{\hat{\alpha}(ab)} = 0, \end{aligned} \tag{5.18}$$

and it is also necessary $T_{\alpha\beta}^a = \gamma_{\alpha\beta}^a$ and $T_{\hat{\alpha}\hat{\beta}}^a = \gamma_{\hat{\alpha}\hat{\beta}}^a$. Note that (5.18) is part of the supergravity constraints and is not enough to put the background on-shell.

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