

Augmented superfield approach to gauge-invariant massive 2-form theory

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Abstract We discuss the complete sets of the off-shell nilpotent (i.e. $s_{(a)b}^2 = 0$) and absolutely anticommuting (i.e. $s_b s_{ab} + s_{ab} s_b = 0$) Becchi–Rouet–Stora–Tyutin (BRST) (s_b) and anti-BRST (s_{ab}) symmetries for the $(3 + 1)$ -dimensional (4D) gauge-invariant massive 2-form theory within the framework of an augmented superfield approach to the BRST formalism. In this formalism, we obtain the coupled (but equivalent) Lagrangian densities which respect both BRST and anti-BRST symmetries on the constrained hypersurface defined by the Curci–Ferrari type conditions. The absolute anticommutativity property of the (anti-) BRST transformations (and corresponding generators) is ensured by the existence of the Curci–Ferrari type conditions which emerge very naturally in this formalism. Furthermore, the gauge-invariant restriction plays a decisive role in deriving the *proper* (anti-) BRST transformations for the Stückelberg-like vector field.

1 Introduction

The antisymmetric 2-form $B^{(2)} = \frac{1}{2!}(dx^\mu \wedge dx^\nu)B_{\mu\nu}$ gauge field $B_{\mu\nu}$ ($= -B_{\nu\mu}$) [1, 2] has paved a great deal of interest of the theoretical physicists during the past few decades because of its relevance in the realm of (super-) string theories [3, 4], (super-) gravity theories [5], dual description of a massless scalar field [6, 7] and modern developments in noncommutative geometry [8]. It has also been quite popular in the mass generation of the 1-form $A^{(1)} = dx^\mu A_\mu$ gauge field A_μ , without taking any help of Higgs mechanism, where 2-form and 1-form gauge fields merged together in a particular fashion through a well-known topological ($B \wedge F$) term [9–14].

The Becchi–Rouet–Stora–Tyutin (BRST) formalism is one of the most elegant and intuitively appealing theoreti-

cal approaches to covariantly quantizing gauge theories [15–18]. The gauge symmetry is always generated by the first-class constraints present in a given theory, in Dirac’s terminology [19, 20]. In the BRST formalism, the classical local gauge symmetry of a given physical theory is traded with two global BRST and anti-BRST symmetries at the quantum level [21, 22]. These symmetries obey two key properties: (1) nilpotency of order two, and (2) absolute anticommutativity. The first property implies that these symmetries are fermionic in nature whereas second property shows that they are linearly independent of each other. In the literature, it has been shown that only the BRST symmetry is not sufficient to yield the ghost decoupling from the physical subspace of the total quantum Hilbert space of states. The addition of a nilpotent anti-BRST symmetry plays an important role in removing the unphysical ghost degeneracy [23]. Thus, the anti-BRST symmetry is not just a decorative part; rather, it is an integral part of this formalism and plays a fundamental role in providing us with the consistent BRST quantization.

The superfield approach to BRST formalism is the theoretical approach that provides the geometrical origin as well as deep understanding about the (anti-) BRST symmetry transformations [24–26]. The Curci–Ferrari condition [21], which is a hallmark of the non-Abelian 1-form gauge theory, emerges very naturally as an off-shoot of the superfield formalism. This condition plays a central role in providing the absolute anticommutativity property of the (anti-) BRST transformations and also responsible for the derivation of the coupled (but equivalent) Lagrangian densities. In recent years, the “augmented” superfield formalism, an extended version of Bonora–Tonin superfield formalism, has been applied to the interacting gauge systems such as (non-)Abelian 1-form gauge theories interacting with Dirac fields [27–31] and complex scalar fields [32, 33], gauge-invariant version of the self-dual chiral boson [34], 4D Freedman–Townsend model [35], 3D Jackiw–Pi model [36], vector Schwinger model in 2D [37] and modified version

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of 2D Proca theory [38]. In this approach, the celebrated horizontality condition and gauge-invariant restrictions are blend together in a physically meaningful manner to derive the proper off-shell nilpotent and absolutely anticommuting (anti-) BRST symmetry transformations.

As far as the quantization of the 4D (non-)Abelian 2-form gauge theories is concerned, the canonical and BRST quantizations have been carried out [39–44]. The 2-form gauge theory is a reducible theory and, thus, requires ghost for ghost in the latter quantization scheme. In the non-Abelian case, a compensating auxiliary vector field is required for the consistent quantization as well as in order to avoid the well-known no-go theorem [45]. In fact, this auxiliary field is needed to close the symmetry algebra and, thus, the theory respects the vector gauge symmetry present in the theory. Furthermore, within the framework of BRST formalism, the free Abelian 2-form gauge theory in (3 + 1) dimensions of spacetime provides a field-theoretic model for the Hodge theory where all the de Rham cohomological operators (d, δ, Δ) and Hodge duality ($*$) operation of differential geometry find their physical realizations in the language of the continuous symmetries and discrete symmetry, respectively [46,47]. In addition, it has also been shown that the free Abelian 2-form gauge theory, within the framework of BRST formalism, provides a new kind of quasi-topological field theory (q-TFT), which captures some features of Witten type TFT and a few aspects of Schwarz type TFT [48].

We have also studied the 4D topologically massive (non-)Abelian 2-form theories where 1-form gauge bosons acquire mass through a topological ($B \wedge F$) term without spoiling the gauge invariance of the theory. With the help of superfield formalism, we have derived the off-shell nilpotent as well as absolutely anticommuting (anti-) BRST transformations and also shown that the topological ($B \wedge F$) term remains unaffected by the presence of the Grassmannian variables when we generalize it on the (4, 2)-dimensional supermanifold [49,50]. In the non-Abelian case, we have found some novel observations. For the sake of brevity, the conserved and nilpotent (anti-) BRST charges do not generate the proper (anti-) BRST transformations for the compensating auxiliary vector field [51,52]. Moreover, in contrast to the Nakanishi–Lautrup fields, the nilpotency and absolute anticommutativity properties of the (anti-) BRST transformations also fail to produce the correct (anti-) BRST symmetry transformations for the compensating auxiliary field.

The contents of our present investigation are organized as follows. In Sect. 2, we briefly discuss the 4D massive 2-form theory and its constraints structure. Section 3 is devoted to the coupled (but equivalent) Lagrangian densities that respect the off-shell nilpotent (anti-) BRST symmetries. We discuss the salient features of the Curci–Ferrari type conditions in this section, too. In Sect. 4, we discuss the conserved charges as the generator of the off-shell nilpotent (anti-) BRST transfor-

mations. The global continuous ghost-scale symmetry and BRST algebra among the symmetry transformations (and corresponding generators) are shown in Sect. 5. Section 6 deals with the derivation of the proper (anti-) BRST symmetry transformations with the help of augmented superfield formalism. We capture the (anti-) BRST invariance of the coupled Lagrangian densities in terms of the superfields and Grassmannian translational generators in Sect. 7. Finally, in Sect. 8, we provide the concluding remarks.

In Appendix A, we show an explicit proof of the anticommutativity of the conserved (anti-) BRST charges.

2 Preliminaries: (3 + 1)-dimensional massive Abelian 2-form theory

We begin with the (3 + 1)-dimensional (4D) massive Abelian 2-form theory which is described by the following Lagrangian density¹:

$$\mathcal{L} = \frac{1}{12} H^{\mu\nu\eta} H_{\mu\nu\eta} - \frac{m^2}{4} B^{\mu\nu} B_{\mu\nu}, \tag{1}$$

where the totally antisymmetric 3-form $H^{(3)} = \frac{1}{3!}(dx^\mu \wedge dx^\nu \wedge dx^\eta)H_{\mu\nu\eta}$ defines the curvature tensor $H_{\mu\nu\eta} = \partial_\mu B_{\nu\eta} + \partial_\nu B_{\eta\mu} + \partial_\eta B_{\mu\nu}$ for the Abelian 2-form $B^{(2)} = \frac{1}{2!}(dx^\mu \wedge dx^\nu)B_{\mu\nu}$ antisymmetric field $B_{\mu\nu}$. The 3-form curvature $H^{(3)} = dB^{(2)}$ owes its origin in the exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$). In the above, m represents a constant mass parameter.

It is evident that due to the existence of mass term, the Lagrangian density does not respects the following gauge symmetry:

$$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu(x) - \partial_\nu \Lambda_\mu(x), \tag{2}$$

where $\Lambda_\mu(x)$ is an infinitesimal local vector gauge parameter. In fact, the above Lagrangian density transforms as $\delta\mathcal{L} = -m^2 B^{\mu\nu}(\partial_\mu \Lambda_\nu)$. The basic reason behind this observation is that the above Lagrangian density is endowed with the second-class constraints, in language of Dirac’s prescription for the classification scheme of constraints [19,20], namely

$$\chi^i = \Pi^{0i} \approx 0, \quad \xi^i = -(2\partial_j \Pi^{ij} + m^2 B^{0i}) \approx 0, \tag{3}$$

where Π^{0i} and Π^{ij} are the canonical conjugate momenta corresponding to the dynamical fields B_{0i} and B_{ij} , respectively. Here, the symbol ‘ \approx ’ defines weak equality in the sense of Dirac. Due to the existence of mass term in the Lagrangian

¹ We adopt the conventions and notations such that the 4D flat Minkowski metric endowed with mostly negative signatures: $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diga}(+1, -1, -1, -1)$. Here, the Greek indices $\mu, \nu, \kappa, \dots = 0, 1, 2, 3$ correspond to the spacetime directions, whereas the Latin indices $i, j, k, \dots = 1, 2, 3$ stand for the space directions only. We also follow the convention $\frac{\delta B_{\mu\nu}}{\delta B_{\kappa\sigma}} = \frac{1}{2!}(\delta_\mu^\kappa \delta_\nu^\sigma - \delta_\nu^\kappa \delta_\mu^\sigma)$.

density, both constraints belong to the category of second-class constraints as one can check that the primary (χ^i) and secondary (ξ_j) constraints lead to a non-vanishing Poisson bracket: $[\chi^i(\mathbf{x}, t), \xi_j(\mathbf{x}', t)] = m^2 \delta_j^i \delta^3(\mathbf{x} - \mathbf{x}')$. Thus, the mass term in the Lagrangian density spoils the gauge invariance. However, on one hand, the gauge invariance can be restored by setting mass parameter equal to zero (i.e. $m = 0$). But this leads to the massless 2-form gauge theory. On other hand, we can restore the gauge invariance by exploiting the power and strength of the well-known Stückelberg technique (see, e.g. [53, 54] for details). Thus, we re-define the field $B_{\mu\nu}$ as

$$B_{\mu\nu} \longrightarrow B_{\mu\nu} = B_{\mu\nu} - \frac{1}{m} \Phi_{\mu\nu}, \tag{4}$$

where $\Phi_{\mu\nu} = (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu)$ and ϕ_μ is the Stückelberg-like vector field. As a consequence, we obtain the following gauge-invariant Stückelberg-like Lagrangian density for the massive 2-form theory [55, 56]:

$$\mathcal{L}_S = \frac{1}{12} H^{\mu\nu\eta} H_{\mu\nu\eta} - \frac{m^2}{4} B^{\mu\nu} B_{\mu\nu} - \frac{1}{4} \Phi^{\mu\nu} \Phi_{\mu\nu} + \frac{m}{2} B^{\mu\nu} \Phi_{\mu\nu}. \tag{5}$$

Here $\Phi_{\mu\nu}$ defines the curvature for the Stückelberg-like vector field ϕ_μ . In the language of differential form, we can write $\Phi^{(2)} = d\phi^{(1)} = \frac{1}{2!} (dx^\mu \wedge dx^\nu) \Phi_{\mu\nu}$. We, interestingly, point out that the above Lagrangian density and the Lagrangian density for the 4D topologically massive ($B \wedge F$) theory have shown to be equivalent by Buscher’s duality procedure [55, 56]. Furthermore, due to the introduction of Stückelberg-like vector field, the second-class constraints get converted into the first-class constraints [19, 20]. These first-class constraints are listed as follows:

$$\Theta = \Pi^0 \approx 0, \quad \Theta^i = \Pi^{0i} \approx 0, \\ \Sigma = \partial_i \Pi^i \approx 0, \quad \Sigma^i = -(2\partial_j \Pi^{ij} + m\Pi^i) \approx 0, \tag{6}$$

where Π^0 and Π^i are canonical conjugate momenta corresponding to the fields ϕ_0 and ϕ_i , respectively. It is elementary to check that the Poisson brackets among all the first-class constraints turn out to be zero. Further, the first-class constraints Σ and Σ^i are not linearly independent. They are related as $\partial_i \Sigma^i + m\Sigma = 0$ which implies that the Lagrangian (5) describes a reducible gauge theory [56]. These first-class constraints are the generators of two independent local and continuous gauge symmetry transformations, namely

$$\delta_1 \phi_\mu = \partial_\mu \Omega, \quad \delta_1 B_{\mu\nu} = 0, \\ \delta_2 B_{\mu\nu} = -(\partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu), \quad \delta_2 \phi_\mu = -m \Lambda_\mu, \tag{7}$$

where the Lorentz scalar $\Omega(x)$ and Lorentz vector $\Lambda_\mu(x)$ are the local gauge parameters. It is straightforward to check that under above the gauge transformations, the Lagrangian density remains invariant (i.e. $\delta_1 \mathcal{L}_S = 0$ and $\delta_2 \mathcal{L}_S = 0$). As

a consequence, the combined gauge symmetry transformations $\delta = (\delta_1 + \delta_2)$ also leave the Lagrangian density (\mathcal{L}_S) invariant.

3 Coupled Lagrangian densities: off-shell nilpotent and absolutely anticommuting (anti-) BRST symmetries

The coupled (but equivalent) Lagrangian densities for the 4D Stückelberg-like massive Abelian 2-form theory incorporate the gauge-fixing and Faddeev–Popov ghost terms within the framework of BRST formalism. In full blaze of glory, these Lagrangian densities (in the Feynman gauge) are given as follows:

$$\mathcal{L}_B = \frac{1}{12} H_{\mu\nu\eta} H^{\mu\nu\eta} - \frac{1}{4} m^2 B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} \Phi_{\mu\nu} \Phi^{\mu\nu} + \frac{1}{2} m B_{\mu\nu} \Phi^{\mu\nu} - B^2 - B(\partial_\mu \phi^\mu + m\varphi) + B_\mu B^\mu - B^\mu (\partial^\nu B_{\nu\mu} - \partial_\mu \varphi + m\phi_\mu) - (\partial_\mu \bar{C} - m\bar{C}_\mu)(\partial^\mu C - mC^\mu) - m^2 \bar{\beta}\beta + (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)(\partial^\mu C^\nu) + \partial_\mu \bar{\beta} \partial^\mu \beta + \left(\partial_\mu \bar{C}^\mu + \frac{1}{2} \rho + m\bar{C} \right) \lambda + \left(\partial_\mu C^\mu - \frac{1}{2} \lambda + mC \right) \rho, \tag{8}$$

$$\mathcal{L}_{\bar{B}} = \frac{1}{12} H_{\mu\nu\eta} H^{\mu\nu\eta} - \frac{1}{4} m^2 B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} \Phi_{\mu\nu} \Phi^{\mu\nu} + \frac{1}{2} m B_{\mu\nu} \Phi^{\mu\nu} - \bar{B}^2 + \bar{B}(\partial_\mu \phi^\mu - m\varphi) + \bar{B}_\mu \bar{B}^\mu + \bar{B}^\mu (\partial^\nu B_{\nu\mu} + \partial_\mu \varphi + m\phi_\mu) - (\partial_\mu \bar{C} - m\bar{C}_\mu)(\partial^\mu C - mC^\mu) - m^2 \bar{\beta}\beta + (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)(\partial^\mu C^\nu) + \partial_\mu \bar{\beta} \partial^\mu \beta + \left(\partial_\mu \bar{C}^\mu + \frac{1}{2} \rho + m\bar{C} \right) \lambda + \left(\partial_\mu C^\mu - \frac{1}{2} \lambda + mC \right) \rho, \tag{9}$$

where the vector fields \bar{B}_μ, B_μ and scalar fields \bar{B}, B are the Nakanishi–Lautrup type auxiliary fields, the vector fields $(\bar{C}_\mu)C_\mu$ and scalar fields $(\bar{C})C$ (with $\bar{C}_\mu \bar{C}^\mu = C_\mu C^\mu = 0, C_\mu \bar{C}_\nu + \bar{C}_\nu C_\mu = 0, C_\mu C_\nu + C_\nu C_\mu = 0, \bar{C}^2 = C^2 = 0, C\bar{C} + \bar{C}C = 0$, etc.) are the fermionic (anti-) ghost fields, $\bar{\beta}, \beta$ are the bosonic ghost-for-ghost fields, $(\rho)\lambda$ are the fermionic auxiliary (anti-) ghost fields. The fermionic (anti-) ghost fields $(\bar{C}_\mu)C_\mu, (\bar{C})C$ and $(\rho)\lambda$ carry ghost number equal to $(-1) + 1$ whereas bosonic (anti-) ghost fields $(\bar{\beta})\beta$ have ghost number equal to $(-2) + 2$. The remaining fields carry zero ghost number. The commuting (anti-) ghost fields $(\bar{\beta})\beta$ and scalar field φ are required for the stage-one reducibility in the theory (see, e.g. [42] for details).

The above Lagrangian densities respect the following off-shell nilpotent (i.e. $s_{(a)b}^2 = 0$) and absolutely anticommuting (i.e. $s_b s_{ab} + s_{ab} s_b = 0$) (anti-) BRST symmetry transformations ($s_{(a)b}$):

$$\begin{aligned} s_b B_{\mu\nu} &= -(\partial_\mu C_\nu - \partial_\nu C_\mu), & s_b C_\mu &= -\partial_\mu \beta, \\ s_b \phi_\mu &= \partial_\mu C - m C_\mu, & s_b \bar{C}_\mu &= B_\mu, & s_b \bar{\beta} &= -\rho, \\ s_b C &= -m\beta, & s_b \bar{C} &= B, & s_b \bar{B} &= -m\lambda, & s_b \varphi &= \lambda, \\ s_b \bar{B}_\mu &= -\partial_\mu \lambda, & s_b [B, \rho, \lambda, \beta, B_\mu, H_{\mu\nu\kappa}] &= 0, \end{aligned} \tag{10}$$

$$\begin{aligned} s_{ab} B_{\mu\nu} &= -(\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), & s_{ab} \bar{C}_\mu &= -\partial_\mu \bar{\beta}, \\ s_{ab} \phi_\mu &= \partial_\mu \bar{C} - m \bar{C}_\mu, & s_{ab} C_\mu &= \bar{B}_\mu, & s_{ab} \beta &= -\lambda, \\ s_{ab} \bar{C} &= -m\bar{\beta}, & s_{ab} C &= \bar{B}, & s_{ab} B &= -m\rho, & s_{ab} \varphi &= \rho, \\ s_{ab} B_\mu &= -\partial_\mu \rho, & s_{ab} [\bar{B}, \rho, \lambda, \bar{\beta}, \bar{B}_\mu, H_{\mu\nu\kappa}] &= 0. \end{aligned} \tag{11}$$

It is straightforward to check that the Lagrangian densities \mathcal{L}_B and $\mathcal{L}_{\bar{B}}$ under the off-shell nilpotent BRST and anti-BRST symmetry transformations transform to the total spacetime derivatives, respectively, as

$$\begin{aligned} s_b \mathcal{L}_B &= -\partial_\mu [B(\partial^\mu C - m C^\mu) - B_\nu(\partial^\mu C^\nu - \partial^\nu C^\mu) \\ &\quad + \rho(\partial^\mu \beta) - \lambda B^\mu], \\ s_{ab} \mathcal{L}_{\bar{B}} &= \partial_\mu [\bar{B}(\partial^\mu \bar{C} + m \bar{C}^\mu) - \bar{B}_\nu(\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \\ &\quad - \lambda(\partial^\mu \bar{\beta}) + \rho \bar{B}^\mu]. \end{aligned} \tag{12}$$

As a consequence, the action integrals remain invariant (i.e. $s_b \int dx^4 \mathcal{L}_B = 0, s_{ab} \int dx^4 \mathcal{L}_{\bar{B}} = 0$) under the nilpotent (anti-) BRST transformations (10) and (11).

At this juncture, the following remarks are in order:

1. The above Lagrangian densities are coupled because the Nakanishi–Lautrup type auxiliary fields B, \bar{B} and B_μ, \bar{B}_μ are related to each other through the celebrated Curci–Ferrari (CF) type of conditions:

$$B + \bar{B} + m\varphi = 0, \quad B_\mu + \bar{B}_\mu + \partial_\mu \varphi = 0. \tag{13}$$

2. It is to be noted that \mathcal{L}_B and $\mathcal{L}_{\bar{B}}$ transform under the continuous anti-BRST and BRST transformations, respectively, as follows:

$$\begin{aligned} s_{ab} \mathcal{L}_B &= \partial_\mu [B_\nu(\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) - B(\partial^\mu \bar{C} - m \bar{C}^\mu) \\ &\quad + \rho(\partial_\nu B^{\nu\mu} + \bar{B}^\mu + m\phi^\mu) - \lambda(\partial^\mu \bar{\beta})] \\ &\quad + m\rho[B + \bar{B} + m\varphi] \\ &\quad - (\partial_\mu \rho)[B^\mu + \bar{B}^\mu + \partial^\mu \varphi] \\ &\quad - m[B_\mu + \bar{B}_\mu + \partial_\mu \varphi](\partial^\mu \bar{C} - m \bar{C}^\mu) \\ &\quad + \partial_\mu [B + \bar{B} + m\varphi](\partial^\mu \bar{C} - m \bar{C}^\mu) \\ &\quad + \partial_\mu [B_\nu + \bar{B}_\nu + \partial_\nu \varphi](\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu), \\ s_b \mathcal{L}_{\bar{B}} &= \partial_\mu [\bar{B}(\partial^\mu C - m C^\mu) \\ &\quad - \bar{B}_\nu(\partial^\mu C^\nu - \partial^\nu C^\mu)] \end{aligned}$$

$$\begin{aligned} & - \lambda(\partial_\nu B^{\nu\mu} - B^\mu + m\phi^\mu) - \rho(\partial^\mu \beta)] \\ & + m\lambda[B + \bar{B} + m\varphi] \\ & - (\partial_\mu \lambda)[B^\mu + \bar{B}^\mu + \partial^\mu \varphi] \\ & + m[B_\mu + \bar{B}_\mu + \partial_\mu \varphi](\partial^\mu C - m C^\mu) \\ & - \partial_\mu [B + \bar{B} + m\varphi](\partial^\mu C - m C^\mu) \\ & + \partial_\mu [B_\nu + \bar{B}_\nu + \partial_\nu \varphi](\partial^\mu C^\nu - \partial^\nu C^\mu), \end{aligned} \tag{14}$$

As a consequence, the coupled Lagrangian densities respect both BRST and anti-BRST symmetries on the 4D constraints hypersurface defined by the CF conditions (13). This reflects the fact that the coupled Lagrangian densities are equivalent on the constrained hypersurface defined by CF type of restrictions.

3. The CF conditions are (anti-) BRST invariant as one can check that

$$\begin{aligned} s_{(a)b}[B + \bar{B} + m\varphi] &= 0, \\ s_{(a)b}[B_\mu + \bar{B}_\mu + \partial_\mu \varphi] &= 0. \end{aligned} \tag{15}$$

Thus, these conditions are physical conditions.

4. Further, the absolute anticommutativity property of the (anti-) BRST symmetry transformations is satisfied due to the existence of the CF conditions. For the sake of brevity, we note that

$$\begin{aligned} \{s_b, s_{ab}\} B_{\mu\nu} &= -\partial_\mu (B_\nu + \bar{B}_\nu) + \partial_\nu (B_\mu + \bar{B}_\mu), \\ \{s_b, s_{ab}\} \phi_\mu &= +\partial_\mu (B + \bar{B}) - m(B_\mu + \bar{B}_\mu). \end{aligned} \tag{16}$$

Now, it is clear from the above that $\{s_b, s_{ab}\} B_{\mu\nu} = 0$ and $\{s_b, s_{ab}\} \phi_\mu = 0$ if and only if the CF conditions are satisfied. For the remaining fields, the anticommutativity property is trivially satisfied.

To sum up the above results, we again emphasize on the fact that the CF conditions play a decisive role in providing the absolute anticommutativity of the (anti-) BRST transformations. These are also responsible for the coupled (but equivalent) Lagrangian densities. Furthermore, the CF type conditions are the physical restrictions (on the theory) in the sense that they are (anti-) BRST invariant conditions. We shall see later on that these CF conditions emerge very naturally within the framework of superfield approach to BRST formalism (cf. Sect. 6, below).

4 Conserved (anti-) BRST charges

According to Noether’s theorem, the invariance of the actions (corresponding to the coupled Lagrangian densities) under the continuous (anti-) BRST symmetries yield the conserved (anti-) BRST currents $J_{(a)b}^\mu$:

$$\begin{aligned}
 J_b^\mu &= -\frac{1}{2}(\partial_\nu C_\eta - \partial_\eta C_\nu)H^{\mu\nu\eta} + B_\nu(\partial^\mu C^\nu - \partial^\nu C^\mu) \\
 &\quad - B(\partial^\mu C - mC^\mu) - (\partial_\nu C - mC_\nu)(\Phi^{\mu\nu} - mB^{\mu\nu}) \\
 &\quad + (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu)(\partial_\nu \beta) - m\beta(\partial^\mu \bar{C} - m\bar{C}^\mu) \\
 &\quad - \rho(\partial^\mu \beta) + \lambda B^\mu, \\
 J_{ab}^\mu &= -\frac{1}{2}(\partial_\nu \bar{C}_\eta - \partial_\eta \bar{C}_\nu)H^{\mu\nu\eta} - \bar{B}_\nu(\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \\
 &\quad + \bar{B}(\partial^\mu \bar{C} - m\bar{C}^\mu) - (\partial_\nu \bar{C} - m\bar{C}_\nu)(\Phi^{\mu\nu} - mB^{\mu\nu}) \\
 &\quad - (\partial^\mu C^\nu - \partial^\nu C^\mu)(\partial_\nu \bar{\beta}) + m\bar{\beta}(\partial^\mu C - mC^\mu) \\
 &\quad - \lambda(\partial^\mu \bar{\beta}) + \rho \bar{B}^\mu.
 \end{aligned} \tag{17}$$

The conservation law ($\partial_\mu J_b^\mu = 0$) for the BRST current J_b^μ can be proven by using the following Euler–Lagrange equations of motion:

$$\begin{aligned}
 \partial_\mu H^{\mu\nu\eta} - (\partial^\nu B^\eta - \partial^\eta B^\nu) - m(\Phi^{\nu\eta} - mB^{\nu\eta}) &= 0, \\
 \partial_\mu \Phi^{\mu\nu} + \partial^\nu B - m(\partial_\mu B^{\mu\nu} + B^\nu) &= 0, \\
 B_\mu &= \frac{1}{2}(\partial^\nu B_{\nu\mu} - \partial_\mu \varphi + m\phi_\mu), \\
 B &= -\frac{1}{2}(\partial_\mu \phi^\mu + m\varphi), \quad \partial_\mu B^\mu + mB = 0, \\
 \square C_\mu - \partial_\mu(\partial_\nu C^\nu) + \partial_\mu \lambda - m(\partial_\mu C - mC_\mu) &= 0, \\
 \square \bar{C}_\mu - \partial_\mu(\partial_\nu \bar{C}^\nu) - \partial_\mu \rho - m(\partial_\mu \bar{C} - m\bar{C}_\mu) &= 0, \\
 \square C - m(\partial_\nu C^\nu) + m\lambda &= 0, \quad (\square + m^2)\beta = 0, \\
 \square \bar{C} - m(\partial_\nu \bar{C}^\nu) - m\rho &= 0, \quad (\square + m^2)\bar{\beta} = 0, \\
 \lambda &= (\partial_\mu C^\mu + mC), \quad \rho = -(\partial_\mu \bar{C}^\mu + m\bar{C}).
 \end{aligned} \tag{18}$$

These equations of motion have been derived from \mathcal{L}_B . Similarly, for the conservation ($\partial_\mu J_{ab}^\mu = 0$) of anti-BRST current J_{ab}^μ , we have used the equations of motion derived from $\mathcal{L}_{\bar{B}}$. We point out that most of the equations of motion are the same for \mathcal{L}_B and $\mathcal{L}_{\bar{B}}$. The Euler–Lagrange equations of motion that are different from (18) and derived from $\mathcal{L}_{\bar{B}}$ are listed as follows:

$$\begin{aligned}
 \partial_\mu H^{\mu\nu\eta} + (\partial^\nu \bar{B}^\eta - \partial^\eta \bar{B}^\nu) - m(\Phi^{\nu\eta} - mB^{\nu\eta}) &= 0, \\
 \partial_\mu \Phi^{\mu\nu} - \partial^\nu \bar{B} - m(\partial_\mu B^{\mu\nu} - \bar{B}^\nu) &= 0, \\
 \bar{B}_\mu &= -\frac{1}{2}(\partial^\nu B_{\nu\mu} + \partial_\mu \varphi + m\phi_\mu), \\
 \bar{B} &= \frac{1}{2}(\partial_\mu \phi^\mu - m\varphi), \quad \partial_\mu \bar{B}^\mu + m\bar{B} = 0.
 \end{aligned} \tag{19}$$

It is interesting to mention that the appropriate equations of motion derived from \mathcal{L}_B and $\mathcal{L}_{\bar{B}}$ [cf. (18) and (19)] produce the CF conditions (13).

The temporal components of the conserved currents (i.e. $Q_{(a)b} = \int d^3x J_{(a)b}^0$) lead to the following charges $Q_{(a)b}$:

$$\begin{aligned}
 Q_b &= \int d^3x \left[-\frac{1}{2}(\partial_i C_j - \partial_j C_i)H^{0ij} \right. \\
 &\quad + B_i(\partial^0 C^i - \partial^i C^0) - B(\partial^0 C - mC^0) \\
 &\quad \left. - (\partial_i C - mC_i)(\Phi^{0i} - mB^{0i}) - m\beta(\partial^0 \bar{C} - m\bar{C}^0) \right. \\
 &\quad \left. + (\partial^0 \bar{C}^i - \partial^i \bar{C}^0)(\partial_i \beta) - \rho(\partial^0 \beta) + \lambda B^0 \right], \\
 Q_{ab} &= \int d^3x \left[-\frac{1}{2}(\partial_i \bar{C}_j - \partial_j \bar{C}_i)H^{0ij} \right. \\
 &\quad - \bar{B}_i(\partial^0 \bar{C}^i - \partial^i \bar{C}^0) + \bar{B}(\partial^0 \bar{C} - m\bar{C}^0) \\
 &\quad - (\partial_i \bar{C} - m\bar{C}_i)(\Phi^{0i} - mB^{0i}) + m\bar{\beta}(\partial^0 C - mC^0) \\
 &\quad \left. - (\partial^0 C^i - \partial^i C^0)(\partial_i \bar{\beta}) - \lambda(\partial^0 \bar{\beta}) + \rho \bar{B}^0 \right].
 \end{aligned} \tag{20}$$

It turns out that these conserved charges are the generators of the corresponding symmetry transformations. For instance, one can check that

$$\begin{aligned}
 s_{(a)b}\Psi &= -i[\Psi, Q_{(a)b}]_\pm, \\
 \Psi &= B_{\mu\nu}, \phi_\mu, C_\mu, \bar{C}_\mu, \beta, \bar{\beta}, C, \bar{C}, \varphi,
 \end{aligned} \tag{21}$$

where (\pm) signs, as the subscript, on the square bracket correspond to the (anti)commutator depending on the generic field Ψ being (fermionic) bosonic in nature. We, further, point out that the conserved (anti-) BRST charges do not produce the proper (anti-) BRST symmetry transformations for the Nakanishi–Lautrup type auxiliary fields $B, \bar{B}, B_\mu, \bar{B}_\mu$ and the auxiliary (anti-) ghost fields (ρ) λ . The transformations of these auxiliary fields can be derived from the requirements of the nilpotency and absolute anticommutativity properties of the (anti-) BRST symmetry transformations.

The (anti-) BRST charges are nilpotent and anticommuting in nature. These properties can be shown in a straightforward manner by exploiting the definition of a generator. For the nilpotency of the (anti-) BRST charges, the following relations are true:

$$\begin{aligned}
 s_b Q_b &= -i\{Q_b, Q_b\} = 0 \Rightarrow Q_b^2 = 0, \\
 s_{ab} Q_{ab} &= -i\{Q_{ab}, Q_{ab}\} = 0 \Rightarrow Q_{ab}^2 = 0.
 \end{aligned} \tag{22}$$

In a similar fashion, one can also show the anticommutativity of the (anti-) BRST charges as

$$\begin{aligned}
 s_b Q_{ab} &= -i\{Q_{ab}, Q_b\} = 0 \Rightarrow Q_b Q_{ab} + Q_{ab} Q_b = 0, \\
 s_{ab} Q_b &= -i\{Q_b, Q_{ab}\} = 0 \Rightarrow Q_b Q_{ab} + Q_{ab} Q_b = 0.
 \end{aligned} \tag{23}$$

The above computations are more algebraically involved. For the sake of completeness, in our Appendix A, we shall provide a complete proof of the first relation that appear in (23) in a simpler way.

Before we wrap up this section, we dwell a bit on the constraint structure of the gauge-invariant Lagrangian density (5) within the framework of BRST formalism. We define a physical state ($|\text{phys}\rangle$) in the quantum Hilbert space of states which respects the (anti-) BRST symmetries. The physicality criteria $Q_{(a)b}|\text{phys}\rangle = 0$ state that the physical state $|\text{phys}\rangle$ must be annihilated by the conserved and nilpotent (anti-) BRST charges $Q_{(a)b}$. In other words, we can say that Faddeev–

Popov ghosts are decoupled from the physical states of the theory. Thus, the physicality criterion $Q_b|\text{phys}\rangle = 0$ produces the following constraint conditions:

$$\begin{aligned} -B|\text{phys}\rangle &= 0, \\ \partial_i(\Phi^{0i} - mB^{0i})|\text{phys}\rangle &= 0, \\ B^i|\text{phys}\rangle &= 0, \\ -(\partial_j H^{0ij} + m(\Phi^{0i} - mB^{0i}))|\text{phys}\rangle &= 0, \end{aligned} \tag{24}$$

which, finally, imply the familiar constraint conditions on the physical state: $\Pi^0|\text{phys}\rangle = 0, \partial_i \Pi^i|\text{phys}\rangle = 0, \Pi^{0i}|\text{phys}\rangle = 0, -(2\partial_j \Pi^{ij} + m\Pi^i)|\text{phys}\rangle = 0$, where $\Pi^0, \Pi^i, \Pi^{0i}, \Pi^{ij}$ are the canonical conjugate momenta with respect to the dynamical fields $\phi_0, \phi_i, B_{0i}, B_{ij}$, respectively. These momenta have been derived from the Lagrangian density (8). The very similar constraint conditions also emerge when we exploit the physicality criterion $Q_{ab}|\text{phys}\rangle = 0$. These constraint conditions are consistent with gauge-invariant Lagrangian (5). As a consequence, the BRST quantization is consistent with the requirements of the Dirac quantization scheme for the constrained systems.

5 Ghost-scale symmetry and BRST algebra

The coupled Lagrangian densities, in addition to the (anti-)BRST symmetries, also respect the following continuous ghost-scale symmetry transformations:

$$\begin{aligned} C_\mu &\rightarrow e^{+\vartheta} C_\mu, \quad \bar{C}_\mu \rightarrow e^{-\vartheta} \bar{C}_\mu, \\ C &\rightarrow e^{+\vartheta} C, \quad \bar{C} \rightarrow e^{-\vartheta} \bar{C}, \quad \rho \rightarrow e^{-\vartheta} \rho, \\ \beta &\rightarrow e^{+2\vartheta} \beta, \quad \bar{\beta} \rightarrow e^{-2\vartheta} \bar{\beta}, \quad \lambda \rightarrow e^{+\vartheta} \lambda, \\ (B_{\mu\nu}, \phi_\mu, B_\mu, \bar{B}_\mu, B, \bar{B}, \varphi) & \\ \rightarrow e^0(B_{\mu\nu}, \phi_\mu, B_\mu, \bar{B}_\mu, B, \bar{B}, \varphi), & \end{aligned} \tag{25}$$

where ϑ is a (spacetime independent) global scale parameter. The numerical factors in the exponentials (i.e. 0, $\pm 1, \pm 2$) define the ghost number of the various fields present in the theory. The infinitesimal version of the above ghost-scale symmetry (with $\vartheta = 1$) leads to the following symmetry transformations (s_g):

$$\begin{aligned} s_g C_\mu &= +C_\mu, \quad s_g \bar{C}_\mu = -\bar{C}_\mu, \quad s_g \rho = -\rho, \\ s_g C &= +C, \quad s_g \bar{C} = -\bar{C}, \quad s_g \lambda = +\lambda, \\ s_g \beta &= +2\beta, \quad s_g \bar{\beta} = -2\bar{\beta}, \\ s_g [B_{\mu\nu}, \phi_\mu, B_\mu, \bar{B}_\mu, B, \bar{B}, \varphi] &= 0, \end{aligned} \tag{26}$$

under which the (coupled) Lagrangian densities remain invariant (i.e. $s_g \mathcal{L}_B = s_g \mathcal{L}_{\bar{B}} = 0$).

According to the Noether theorem, the continuous ghost-scale symmetry yields the conserved current J_g^μ and corresponding charge Q_g , namely

$$\begin{aligned} J_g^\mu &= (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) C_\nu + (\partial^\mu C^\nu - \partial^\nu C^\mu) \bar{C}_\nu \\ &\quad - (\partial^\mu \bar{C} - m\bar{C}^\mu) C - (\partial^\mu C - mC^\mu) \bar{C} \\ &\quad + 2\beta(\partial^\mu \bar{\beta}) - 2\bar{\beta}(\partial^\mu \beta) - \rho C^\mu + \lambda \bar{C}^\mu, \\ Q_g &= \int d^3 [(\partial^0 \bar{C}^i - \partial^i \bar{C}^0) C_i + (\partial^0 C^i - \partial^i C^0) \bar{C}_i \\ &\quad - (\partial^0 \bar{C} - m\bar{C}^0) C - (\partial^0 C - mC^0) \bar{C} \\ &\quad + 2\beta(\partial^0 \bar{\beta}) - 2\bar{\beta}(\partial^0 \beta) - \rho C^0 + \lambda \bar{C}^0]. \end{aligned} \tag{27}$$

It is evident that the above charge is the generator of the corresponding ghost-scale symmetry transformations as one can check that

$$s_g \Psi = \pm i[\Psi, Q_g], \tag{28}$$

where Ψ is the generic field of the theory. The (\pm) signs in front of the commutator are used for the generic field Ψ being (fermionic) bosonic in nature.

At this moment, the following remarks are in order:

1. The conserved ghost charge Q_g does not produce the proper transformations for the auxiliary fields ρ and λ . These transformations can be obtained from other considerations (see (29) below).
2. The continuous symmetry transformations (in their operator form) obey the following algebra:

$$\begin{aligned} s_b^2 &= 0, \quad s_{ab}^2 = 0, \quad \{s_b, s_{ab}\} = 0, \quad [s_g, s_g] = 0, \\ [s_g, s_b] &= +s_b, \quad [s_g, s_{ab}] = -s_{ab}. \end{aligned} \tag{29}$$

3. By exploiting the last two relations of the above equation, we can obtain the proper transformations for ρ and λ . For instance, one can check that

$$[s_g, s_b]\beta = +s_b \bar{\beta} \Rightarrow s_g \rho = -\rho. \tag{30}$$

Similarly, the transformation for the auxiliary field λ can be derived, too.

4. The operator form of the conserved (anti-)BRST charges together with the ghost charge obeys the following graded algebra:

$$\begin{aligned} Q_b^2 &= 0, \quad Q_{ab}^2 = 0, \quad \{Q_b, Q_{ab}\}, \quad [Q_g, Q_g] = 0, \\ [Q_g, Q_b] &= -iQ_b, \quad [Q_g, Q_{ab}] = +iQ_{ab}, \end{aligned} \tag{31}$$

which is also known as standard BRST algebra.

As a consequences of the above algebra, we define an eigenstate $|\zeta\rangle_n$ (in the quantum Hilbert space of states) with respect to the operator iQ_g such that $iQ_g|\zeta\rangle_n = n|\zeta\rangle_n$. Here n defines the ghost number as the eigenvalue of the operator iQ_g . Using the above algebra among the conserved charges, it is straightforward to check that the following relationships are true:

$$\begin{aligned}
 i Q_g Q_b |\zeta\rangle_n &= (n + 1) Q_b |\zeta\rangle_n, \\
 i Q_g Q_{ab} |\zeta\rangle_n &= (n - 1) Q_{ab} |\zeta\rangle_n,
 \end{aligned}
 \tag{32}$$

which imply that the eigenstates $Q_b |\zeta\rangle_n$ and $Q_{ab} |\zeta\rangle_n$ have the eigenvalues $(n + 1)$ and $(n - 1)$, respectively. In other words, The conserved (anti-) BRST charges $Q_{(a)b}$ (decrease)increase the ghost number of the eigenstate $i Q_g |\zeta\rangle_n$ by one unit. Also, we can say that the (anti-) BRST charges $Q_{(a)b}$ carry ghost number equal to $(-1) + 1$ while ghost charge Q_g does not carry any ghost number. These observations also reflect from the expressions of the conserved charges if we look carefully for the ghost number of the various fields that appear in the charges.

6 Augmented superfield approach to BRST formalism

In this section, we shall derive the *proper* off-shell nilpotent and absolutely anticommuting (anti-) BRST symmetry transformations with the help of an extended version of Bonora–Tonin superfield formalism [24,25] where the horizontality condition and gauge-invariant restriction are used in a physically meaningful manner. In this formalism, we generalize our ordinary 4D spacetime to (4, 2)D superspace parameterized by an additional pair of the Grassmannian variables $(\theta, \bar{\theta})$ as

$$x^\mu \rightarrow Z^M = (x^\mu, \theta, \bar{\theta}), \quad \partial_\mu \rightarrow \partial_M = (\partial_\mu, \partial_\theta, \partial_{\bar{\theta}}), \tag{33}$$

where x^μ ($\mu = 0, 1, 2, 3$) are the (bosonic) spacetime coordinates. The super coordinates $Z^M = (x^\mu, \theta, \bar{\theta})$ parametrize the (4, 2)D superspace (with $\theta^2 = 0, \bar{\theta}^2 = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0$) and $\partial_\theta = \partial/\partial\theta, \partial_{\bar{\theta}} = \partial/\partial\bar{\theta}$ are the Grassmannian translational generators along the Grassmannian directions $\theta, \bar{\theta}$. We generalize the exterior derivative d and 2-form $B^{(2)}$ to the super exterior derivative \tilde{d} and super 2-form $\tilde{B}^{(2)}$ on the (4, 2)D supermanifold as follows:

$$\begin{aligned}
 d \rightarrow \tilde{d} &= dZ^M \partial_M \\
 &\equiv dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}}, \quad \tilde{d}^2 = 0,
 \end{aligned}
 \tag{34}$$

$$\begin{aligned}
 B^{(2)} \rightarrow \tilde{B}^{(2)} &= \frac{1}{2!} (dZ^M \wedge dZ^N) \tilde{B}_{MN} \\
 &\equiv \frac{1}{2!} (dx^\mu \wedge dx^\nu) \tilde{B}_{\mu\nu}(x, \theta, \bar{\theta}) \\
 &\quad + (dx^\mu \wedge d\theta) \tilde{\mathcal{F}}_\mu(x, \theta, \bar{\theta}) \\
 &\quad + (dx^\mu \wedge d\bar{\theta}) \tilde{\mathcal{F}}_\mu(x, \theta, \bar{\theta}) \\
 &\quad + (d\theta \wedge d\theta) \tilde{\beta}(x, \theta, \bar{\theta}) \\
 &\quad + (d\bar{\theta} \wedge d\bar{\theta}) \tilde{\beta}(x, \theta, \bar{\theta}) \\
 &\quad + (d\theta \wedge d\bar{\theta}) \tilde{\Phi}(x, \theta, \bar{\theta}).
 \end{aligned}
 \tag{35}$$

The super multiplets as the components of the super 2-form can be expanded along the Grassmannian directions $(\theta$ and $\bar{\theta})$ as follows:

$$\begin{aligned}
 \tilde{B}_{\mu\nu}(x, \theta, \bar{\theta}) &= B_{\mu\nu}(x) + \theta \bar{R}_{\mu\nu}(x) + \bar{\theta} R_{\mu\nu}(x) \\
 &\quad + i\theta\bar{\theta} S_{\mu\nu}(x), \\
 \tilde{\mathcal{F}}_\mu(x, \theta, \bar{\theta}) &= C_\mu(x) + \theta \bar{B}_\mu^{(1)}(x) + \bar{\theta} B_\mu^{(1)}(x) \\
 &\quad + i\theta\bar{\theta} f_\mu(x), \\
 \tilde{\bar{\mathcal{F}}}_\mu(x, \theta, \bar{\theta}) &= \bar{C}_\mu(x) + \theta \bar{B}_\mu^{(2)}(x) + \bar{\theta} B_\mu^{(2)}(x), \\
 &\quad + i\theta\bar{\theta} \bar{f}_\mu(x), \\
 \tilde{\beta}(x, \theta, \bar{\theta}) &= \beta(x) + \theta \bar{f}_1(x) + \bar{\theta} f_1(x) \\
 &\quad + i\theta\bar{\theta} b_1(x), \\
 \tilde{\bar{\beta}}(x, \theta, \bar{\theta}) &= \bar{\beta}(x) + \theta \bar{f}_2(x) + \bar{\theta} f_2(x) \\
 &\quad + i\theta\bar{\theta} b_2(x), \\
 \tilde{\Phi}(x, \theta, \bar{\theta}) &= \varphi(x) + \theta \bar{f}_3(x) + \bar{\theta} f_3(x) \\
 &\quad + i\theta\bar{\theta} b_3(x),
 \end{aligned}
 \tag{36}$$

where the secondary fields $R_{\mu\nu}, \bar{R}_{\mu\nu}, f_\mu, \bar{f}_\mu, f_1, \bar{f}_1, f_2, \bar{f}_2, f_3, \bar{f}_3$ are fermionic in nature and $S_{\mu\nu}, B_\mu^{(1)}, \bar{B}_\mu^{(1)}, B_\mu^{(2)}, \bar{B}_\mu^{(2)}, b_1, b_2, b_3$ are bosonic secondary fields. We shall determine the precise value of these secondary fields with the help of superfield formalism.

It is to be noted that the horizontality condition (HC),

$$dB^{(2)} = \tilde{d}\tilde{B}^{(2)} \iff H^{(3)} = \tilde{\mathcal{H}}^{(3)}, \tag{37}$$

determines the value of all secondary fields in terms of the basic and auxiliary fields of the theory. This HC implies that the l.h.s. is independent of the Grassmannian variables θ and $\bar{\theta}$ when we generalize it on the (4, 2)D supermanifold. The r.h.s. of (37), in its full blaze of glory, can be written as

$$\begin{aligned}
 \tilde{\mathcal{H}}^{(3)} &= \tilde{d}\tilde{B}^{(2)} \\
 &= \frac{1}{3!} (dx^\mu \wedge dx^\nu \wedge dx^\kappa) (\partial_\mu \tilde{B}_{\nu\kappa} + \partial_\nu \tilde{B}_{\kappa\mu} + \partial_\kappa \tilde{B}_{\mu\nu}) \\
 &\quad + \frac{1}{2!} (dx^\mu \wedge dx^\nu \wedge d\theta) [\partial_\theta \tilde{B}_{\mu\nu} + \partial_\mu \tilde{\mathcal{F}}_\nu - \partial_\nu \tilde{\mathcal{F}}_\mu] \\
 &\quad + \frac{1}{2!} (dx^\mu \wedge dx^\nu \wedge d\bar{\theta}) [\partial_{\bar{\theta}} \tilde{B}_{\mu\nu} + \partial_\mu \tilde{\mathcal{F}}_\nu - \partial_\nu \tilde{\mathcal{F}}_\mu] \\
 &\quad + (d\theta \wedge d\theta \wedge d\theta) (\partial_\theta \tilde{\beta}) + (d\bar{\theta} \wedge d\bar{\theta} \wedge d\bar{\theta}) (\partial_{\bar{\theta}} \tilde{\beta}) \\
 &\quad + (dx^\mu \wedge d\theta \wedge d\bar{\theta}) [\partial_\mu \tilde{\Phi} + \partial_\theta \tilde{\mathcal{F}}_\mu + \partial_{\bar{\theta}} \tilde{\mathcal{F}}_\mu] \\
 &\quad + (dx^\mu \wedge d\theta \wedge d\theta) [\partial_\theta \tilde{\mathcal{F}}_\mu + \partial_\mu \tilde{\beta}] \\
 &\quad + (dx^\mu \wedge d\bar{\theta} \wedge d\bar{\theta}) [\partial_{\bar{\theta}} \tilde{\mathcal{F}}_\mu + \partial_\mu \tilde{\beta}] \\
 &\quad + (d\theta \wedge d\bar{\theta} \wedge d\bar{\theta}) [\partial_{\bar{\theta}} \tilde{\Phi} + \partial_\theta \tilde{\beta}] \\
 &\quad + (d\bar{\theta} \wedge d\theta \wedge d\theta) [\partial_\theta \tilde{\Phi} + \partial_{\bar{\theta}} \tilde{\beta}].
 \end{aligned}
 \tag{38}$$

The above HC implies the following interesting relationships amongst the superfields:

$$\begin{aligned}
 \partial_\theta \tilde{\mathcal{B}}_{\mu\nu} + \partial_\mu \tilde{\mathcal{F}}_\nu - \partial_\nu \tilde{\mathcal{F}}_\mu &= 0, & \partial_\theta \tilde{\beta} &= 0, \\
 \partial_{\bar{\theta}} \tilde{\mathcal{B}}_{\mu\nu} + \partial_\mu \tilde{\mathcal{F}}_\nu - \partial_\nu \tilde{\mathcal{F}}_\mu &= 0, & \partial_{\bar{\theta}} \tilde{\beta} &= 0, \\
 \partial_\mu \tilde{\Phi} + \partial_\theta \tilde{\mathcal{F}}_\mu + \partial_{\bar{\theta}} \tilde{\mathcal{F}}_\mu &= 0, \\
 \partial_\mu \tilde{\beta} + \partial_\theta \tilde{\mathcal{F}}_\mu &= 0, & \partial_\mu \tilde{\beta} + \partial_{\bar{\theta}} \tilde{\mathcal{F}}_\mu &= 0, \\
 \partial_\theta \tilde{\Phi} + \partial_{\bar{\theta}} \tilde{\beta} &= 0, & \partial_{\bar{\theta}} \tilde{\Phi} + \partial_\theta \tilde{\beta} &= 0.
 \end{aligned} \tag{39}$$

Exploiting the above expressions for the superfields given in (36), we obtain the value of all secondary fields,

$$\begin{aligned}
 R_{\mu\nu} &= -(\partial_\mu C_\nu - \partial_\nu C_\mu), & B^{(1)} &= -\partial_\mu \beta, \\
 \bar{R}_{\mu\nu} &= -(\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), & \bar{B}^{(2)} &= -\partial_\mu \bar{\beta}, \\
 S_{\mu\nu} &= i(\partial_\mu B_\nu^{(2)} - \partial_\nu B_\mu^{(2)}) \equiv -i(\partial_\mu \bar{B}_\nu^{(1)} - \partial_\nu \bar{B}_\mu^{(1)}), \\
 f_\mu &= i\partial_\mu f_3 \equiv -i\partial_\mu \bar{f}_1, & \bar{f}_\mu &= -i\partial_\mu \bar{f}_3 \equiv i\partial_\mu f_2, \\
 B_\mu^{(2)} + \bar{B}_\mu^{(1)} + \partial_\mu \varphi &= 0, & f_1 &= 0, & \bar{f}_2 &= 0, \\
 f_2 + \bar{f}_3 &= 0, & \bar{f}_2 + f_3 &= 0, \\
 b_1 &= 0, & b_2 &= 0, & b_3 &= 0.
 \end{aligned} \tag{40}$$

Substituting these values in the expression of the superfields (36), we obtain the following super-expansions:

$$\begin{aligned}
 \tilde{B}_{\mu\nu}^{(h)}(x, \theta, \bar{\theta}) &= B_{\mu\nu}(x) - \theta(\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)(x) \\
 &\quad - \bar{\theta}(\partial_\mu C_\nu - \partial_\nu C_\mu)(x) \\
 &\quad + \theta\bar{\theta}(\partial_\mu B_\nu - \partial_\nu B_\mu)(x), \\
 \tilde{C}_\mu^{(h)}(x, \theta, \bar{\theta}) &= C_\mu(x) + \theta\bar{B}_\mu(x) - \bar{\theta}(\partial_\mu \beta)(x) \\
 &\quad - \theta\bar{\theta}(\partial_\mu \lambda)(x), \\
 \tilde{\mathcal{F}}_\mu^{(h)}(x, \theta, \bar{\theta}) &= \bar{C}_\mu(x) \\
 &\quad - \theta(\partial_\mu \bar{\beta})(x) - \bar{\theta}B_\mu(x) \\
 &\quad + \theta\bar{\theta}(\partial_\mu \rho)(x), \\
 \tilde{\beta}^{(h)}(x, \theta, \bar{\theta}) &= \beta(x) - \theta\lambda(x), \\
 \tilde{\bar{\beta}}^{(h)}(x, \theta, \bar{\theta}) &= \bar{\beta}(x) - \bar{\theta}\rho(x), \\
 \tilde{\Phi}^{(h)}(x, \theta, \bar{\theta}) &= \varphi(x) + \theta\rho(x) + \bar{\theta}\lambda(x).
 \end{aligned} \tag{41}$$

The superscript (h) on the superfields denotes the expansion of the superfields obtained after the application of HC. In the above super-expansions, we have chosen $\bar{f}_3 = \rho = -f_2$, $f_3 = \lambda = -\bar{f}_1$, $\bar{B}_\mu^{(1)} = \bar{B}_\mu$, $B_\mu^{(2)} = B_\mu$ where B_μ and \bar{B}_μ play the role of Nakanishi–Lautrup type auxiliary fields. These auxiliary fields are required for the linearization of the gauge-fixing terms as well as for the off-shell nilpotency of the (anti-) BRST symmetry transformations.

It is clear from the above super-expansions of the superfields that the coefficients of $\bar{\theta}$ are the BRST transformations whereas the coefficients of θ are the anti-BRST transformations. To be more precise, the BRST transformation (s_b) for any generic field $\Psi(x)$ is equivalent to the translation of the corresponding superfield $\tilde{\Psi}^{(h)}(x, \theta, \bar{\theta})$ along the $\bar{\theta}$ -direction while keeping θ -direction fixed. Similarly, the anti-BRST transformation (s_{ab}) can be obtained by taking the transla-

tion of the superfield along the θ -direction while $\bar{\theta}$ -direction remains intact. Mathematically, these statements can be corroborated in the following fashion:

$$\begin{aligned}
 s_b \Psi(x) &= \left. \frac{\partial}{\partial \bar{\theta}} \tilde{\Psi}^{(h)}(x, \theta, \bar{\theta}) \right|_{\bar{\theta}=0}, \\
 s_{ab} \Psi(x) &= \left. \frac{\partial}{\partial \theta} \tilde{\Psi}^{(h)}(x, \theta, \bar{\theta}) \right|_{\bar{\theta}=0}, \\
 s_b s_{ab} \Psi(x) &= \frac{\partial}{\partial \bar{\theta}} \frac{\partial}{\partial \theta} \tilde{\Psi}^{(h)}(x, \theta, \bar{\theta}).
 \end{aligned} \tag{42}$$

It is worthwhile to mention that the (anti-) BRST transformations of the fermionic auxiliary fields ρ, λ and Nakanishi–Lautrup type fields B_μ, \bar{B}_μ have been derived from the requirements of the nilpotency and absolute anticommutativity properties of the (anti-) BRST symmetry transformations.

So far, we have obtained the off-shell nilpotent and absolutely anticommuting (anti-) BRST symmetries for the 2-form field $B_{\mu\nu}$ and corresponding (anti-) ghost fields. But the (anti-) BRST symmetry transformations of the Stückelberg vector field ϕ_μ and corresponding (anti-) ghost fields are still unknown. For this purpose, it is to be noted that the following quantity remains invariant under the gauge transformations $(\delta = \delta_1 + \delta_2)$:

$$\delta \left[B_{\mu\nu} - \frac{1}{m}(\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) \right] = 0. \tag{43}$$

This is a physical quantity in the sense that it is gauge-invariant. Thus, it remains independent of the Grassmannian variables when we generalize it on the $(4, 2)D$ supermanifold. This gauge-invariant quantity will serve our purpose in deriving the proper (anti-) BRST transformations for the Stückelberg-like vector field ϕ_μ and corresponding (anti)ghost fields $(\bar{C})C$. In terms of the differential forms, we generalize this gauge-invariant restriction on the $(4, 2)D$ supermanifold as

$$B^{(2)} - \frac{1}{m}d\phi^{(1)} = \tilde{B}^{(2)} - \frac{1}{m}d\tilde{\Phi}^{(1)}, \tag{44}$$

where the super 1-form is defined as

$$\begin{aligned}
 \tilde{\Phi}^{(1)} &= dZ^M \Phi_M \\
 &= dx^\mu \tilde{\Phi}_\mu(x, \theta, \bar{\theta}) + d\theta \tilde{\mathcal{F}}(x, \theta, \bar{\theta}) \\
 &\quad + d\bar{\theta} \tilde{\mathcal{F}}(x, \theta, \bar{\theta}).
 \end{aligned} \tag{45}$$

The multiplets of super 1-form, one can express, along the Grassmannian directions as

$$\begin{aligned}
 \tilde{\Phi}_\mu(x, \theta, \bar{\theta}) &= \phi_\mu(x) + \theta \bar{R}_\mu(x) + \bar{\theta} R_\mu(x) + i\theta\bar{\theta} S_\mu(x), \\
 \tilde{\mathcal{F}}(x, \theta, \bar{\theta}) &= C(x) + \theta \bar{B}_1(x) + \bar{\theta} B_1(x) + i\theta\bar{\theta} s(x), \\
 \tilde{\bar{\mathcal{F}}}(x, \theta, \bar{\theta}) &= \bar{C}(x) + \theta \bar{B}_2(x) + \bar{\theta} B_2(x) + i\theta\bar{\theta} \bar{s}(x),
 \end{aligned} \tag{46}$$

where $R_\mu, \bar{R}_\mu, s, \bar{s}$ and $S_\mu, B_1, \bar{B}_1, B_2, \bar{B}_2$ are fermionic and bosonic secondary fields, respectively.

The explicit expression of the r.h.s. of (44) can be written in the following fashion:

$$\begin{aligned} \tilde{B}^{(2)} - \frac{1}{m} \tilde{d}\tilde{\Phi}^{(1)} &= \frac{1}{2!} (dx^\mu \wedge dx^\nu) [\tilde{B}_{\mu\nu}^{(h)} \\ &\quad - \frac{1}{m} (\partial_\mu \tilde{\Phi}_\nu - \partial_\nu \tilde{\Phi}_\mu)] \\ &\quad + (dx^\mu \wedge d\theta) [\tilde{\mathcal{F}}_\mu^{(h)} - \frac{1}{m} (\partial_\mu \tilde{\mathcal{F}} - \partial_\theta \tilde{\Phi}_\mu)] \\ &\quad + (dx^\mu \wedge d\bar{\theta}) \left[\tilde{\mathcal{F}}_\mu^{(h)} - \frac{1}{m} (\partial_\mu \tilde{\mathcal{F}} - \partial_{\bar{\theta}} \tilde{\Phi}_\mu) \right] \\ &\quad + (d\theta \wedge d\bar{\theta}) \left[\tilde{\Phi}^{(h)} + \frac{1}{m} (\partial_\theta \tilde{\mathcal{F}} + \partial_{\bar{\theta}} \tilde{\mathcal{F}}) \right] \\ &\quad + (d\theta \wedge d\theta) \left[\tilde{\beta}^{(h)} + \frac{1}{m} \partial_\theta \tilde{\mathcal{F}} \right] \\ &\quad + (d\bar{\theta} \wedge d\bar{\theta}) \left[\tilde{\beta}^{(h)} + \frac{1}{m} \partial_{\bar{\theta}} \tilde{\mathcal{F}} \right]. \end{aligned} \tag{47}$$

Using (44) and setting all the coefficients of the Grassmannian differentials to zero, we obtain the following interesting relationships:

$$\begin{aligned} \tilde{\mathcal{F}}_\mu^{(h)} - \frac{1}{m} (\partial_\mu \tilde{\mathcal{F}} - \partial_\theta \tilde{\Phi}_\mu) &= 0, \\ \tilde{\mathcal{F}}_\mu^{(h)} - \frac{1}{m} (\partial_\mu \tilde{\mathcal{F}} - \partial_{\bar{\theta}} \tilde{\Phi}_\mu) &= 0, \\ \tilde{\beta}^{(h)} + \frac{1}{m} \partial_\theta \tilde{\mathcal{F}} = 0, \quad \tilde{\beta}^{(h)} + \frac{1}{m} \partial_{\bar{\theta}} \tilde{\mathcal{F}} &= 0, \\ \tilde{\Phi}^{(h)} + \frac{1}{m} (\partial_\theta \tilde{\mathcal{F}} + \partial_{\bar{\theta}} \tilde{\mathcal{F}}) &= 0. \end{aligned} \tag{48}$$

Exploiting the above equations together with (41) for the super-expansions given in (46), we obtain the precise value of the secondary fields in terms of the basic and auxiliary fields, namely;

$$\begin{aligned} R_\mu &= \partial_\mu C - mC_\mu, \quad \bar{R}_\mu = \partial_\mu \bar{C} - m\bar{C}_\mu, \\ B_1 &= -m\beta, \quad \bar{B}_2 = -m\beta, \quad s = im\lambda, \\ B_2 + \bar{B}_1 + m\varphi &= 0, \quad \bar{s} = -im\rho, \\ S_\mu &= -i(\partial_\mu B_2 - mB_\mu) \equiv +i(\partial_\mu \bar{B}_1 - m\bar{B}_\mu). \end{aligned} \tag{49}$$

Putting the above relationships into the super-expansions of the superfields (46), we obtain the following explicit expressions for the superfields (46), in terms of the basic and auxiliary fields:

$$\begin{aligned} \tilde{\Phi}_\mu^{(h,g)}(x, \theta, \bar{\theta}) &= \phi_\mu(x) + \theta(\partial_\mu \bar{C} - m\bar{C}_\mu)(x) \\ &\quad + \bar{\theta}(\partial_\mu C - mC_\mu)(x) \\ &\quad + \theta\bar{\theta}(\partial_\mu B - mB_\mu)(x), \\ \tilde{\mathcal{F}}^{(h,g)}(x, \theta, \bar{\theta}) &= C(x) + \theta\bar{B}(x) - \bar{\theta}(m\beta)(x) \\ &\quad - \theta\bar{\theta}(m\lambda)(x), \end{aligned}$$

$$\begin{aligned} \tilde{\mathcal{F}}^{(h,g)}(x, \theta, \bar{\theta}) &= \bar{C}(x) - \theta(m\bar{\beta})(x) + \bar{\theta}B(x) \\ &\quad + \theta\bar{\theta}(m\rho)(x). \end{aligned} \tag{50}$$

Here the superscript (h, g) on the superfields denotes the super-expansions obtained after the application of gauge-invariant restriction (44). In the above, we have made the choices $B_2 = B$ and $\bar{B}_1 = \bar{B}$ for the additional Nakanishi–Lautrup type fields B and \bar{B} . These fields are also required for the off-shell nilpotency of the (anti-) BRST symmetry transformations and linearization of the gauge-fixing term for the Stückelberg vector field ϕ_μ . Again, the (anti-) BRST transformations for the auxiliary fields B and \bar{B} have been derived from the requirements of the nilpotency and absolute anticommutativity of the (anti-) BRST transformations. Thus, one can easily read-off all the (anti-) BRST transformations for the vector field ϕ_μ and corresponding (anti-) ghost fields $(\bar{C})C$ [cf. (10) and (11)].

Before we wrap up this section, we point out that the CF conditions (13) which play the crucial role (cf. Sect. 3) emerge very naturally in this formalism. The *first* CF condition $B_\mu + \bar{B}_\mu + \partial_\mu\varphi = 0$ arises from the HC (28). In particular, the relation $\partial_\mu \tilde{\Phi} + \partial_\theta \tilde{\mathcal{F}}_\mu + \partial_{\bar{\theta}} \tilde{\mathcal{F}}_\mu = 0$, which is a coefficient of the wedge product $(dx^\mu \wedge d\theta \wedge d\bar{\theta})$, leads to the first CF condition. Similarly, the *second* CF condition $B + \bar{B} + m\varphi = 0$ emerges from (48) when we set the coefficient of the wedge product $(d\theta \wedge d\bar{\theta})$ equal to zero. In fact, the last relation of Eq. (48) produces the second CF condition. Furthermore, it is interesting to note that Eq. (44) imposes its own integrability condition [42]. Thus, if we operate super exterior derivative $\tilde{d} = d + d\theta\partial_\theta + d\bar{\theta}\partial_{\bar{\theta}}$ on (44) from left, we obtain

$$\tilde{d} \left(B^{(2)} - \frac{1}{m} d\phi^{(1)} \right) = \tilde{d} \left(\tilde{B}^{(2)} - \frac{1}{m} \tilde{d}\tilde{\Phi}^{(1)} \right). \tag{51}$$

In the above, $B^{(2)}$ and $\phi^{(1)}$ are independent of the Grassmannian variables $(\theta, \bar{\theta})$ and $d^2 = \tilde{d}^2 = 0$. As a result, the above equation turns into the horizontality condition (37).

7 (Anti-) BRST invariance of the coupled Lagrangian densities: superfield approach

In this section, we shall provide the (anti-) BRST invariance of the coupled Lagrangian densities in the context of superfield formalism. To accomplish this goal, we note that the coupled Lagrangian densities, in terms of the off-shell nilpotent and absolutely anticommuting (anti-) BRST symmetries, can be written as

$$\begin{aligned} \mathcal{L}_B &= \mathcal{L}_s + s_b s_{ab} \left[\frac{1}{2} \phi_\mu \phi^\mu - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \bar{C}_\mu C^\mu \right. \\ &\quad \left. + \frac{1}{2} \varphi\varphi + 2\bar{\beta}\beta + C\bar{C} \right], \end{aligned} \tag{52}$$

$$\mathcal{L}_{\bar{B}} = \mathcal{L}_s - s_{ab}s_b \left[\frac{1}{2} \phi_\mu \phi^\mu - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \bar{C}_\mu C^\mu + \frac{1}{2} \varphi \varphi + 2\bar{\beta} \beta + C\bar{C} \right]. \tag{53}$$

For our present 4D model, all terms in square brackets are chosen in such a way that each term carries mass dimension equal to $[M]^2$ in natural units ($\hbar = c = 1$). In fact, the dynamical fields $B_{\mu\nu}, \phi_\mu, C_\mu, \bar{C}_\mu, \beta, \bar{\beta}, \varphi, C, \bar{C}$ have mass dimension equal to $[M]$. The operation of s_b and s_{ab} on any generic field increases the mass dimension by one unit. In other words, s_b and s_{ab} carry mass dimension one. Furthermore, s_b increases the ghost number by one unit when it operates on any generic field whereas s_{ab} decreases the ghost number by one unit when it acts on any field. As a consequence, the above coupled Lagrangian densities are consistent with mass dimension and ghost number considerations. The constant numerals in front of each term are chosen for our algebraic convenience. In full blaze of glory, the above Lagrangian densities (52) and (53) lead to (8) and (9), respectively, modulo the total spacetime derivatives.

In our earlier section (cf. Sect. 2), we have already mentioned that \mathcal{L}_s is gauge-invariant and, thus, it remains invariant under the (anti-) BRST symmetries. As a consequence, both \mathcal{L}_B and $\mathcal{L}_{\bar{B}}$ given in (52) and (53) remain invariant under the operation of $s_{(a)b}$ due to the nilpotency property (i.e. $s_b^2 = 0, s_{ab}^2 = 0$) of $s_{(a)b}$. In terms of the superfields (41), (50) and Grassmannian translational generators, we can generalize the 4D Lagrangian densities to super Lagrangian densities on the (4, 2)D supermanifold as

$$\begin{aligned} \tilde{\mathcal{L}}_B = \tilde{\mathcal{L}}_s + \frac{\partial}{\partial\theta} \frac{\partial}{\partial\bar{\theta}} \left[\frac{1}{2} \tilde{\Phi}^{(h,g)}_\mu \tilde{\Phi}^{\mu(h,g)} - \frac{1}{4} \tilde{B}^{(h)}_{\mu\nu} \tilde{B}^{\mu\nu(h)} + \tilde{\mathcal{F}}^{(h)}_\mu \tilde{\mathcal{F}}^{\mu(h)} + \frac{1}{2} \tilde{\Phi}^{(h)} \tilde{\Phi}^{(h)} + 2\tilde{\beta}^{(h)} \tilde{\beta}^{(h)} + \tilde{\mathcal{F}}^{(h,g)} \tilde{\mathcal{F}}^{(h,g)} \right], \end{aligned} \tag{54}$$

$$\begin{aligned} \tilde{\mathcal{L}}_{\bar{B}} = \tilde{\mathcal{L}}_s - \frac{\partial}{\partial\theta} \frac{\partial}{\partial\bar{\theta}} \left[\frac{1}{2} \tilde{\Phi}^{(h,g)}_\mu \tilde{\Phi}^{\mu(h,g)} - \frac{1}{4} \tilde{B}^{(h)}_{\mu\nu} \tilde{B}^{\mu\nu(h)} + \tilde{\mathcal{F}}^{(h)}_\mu \tilde{\mathcal{F}}^{\mu(h)} + \frac{1}{2} \tilde{\Phi}^{(h)} \tilde{\Phi}^{(h)} + 2\tilde{\beta}^{(h)} \tilde{\beta}^{(h)} + \tilde{\mathcal{F}}^{(h,g)} \tilde{\mathcal{F}}^{(h,g)} \right], \end{aligned} \tag{55}$$

where the super Lagrangian density $\tilde{\mathcal{L}}_s$ is the generalization of the gauge-invariant Lagrangian density \mathcal{L}_s on the (4, 2)D superspace. The former Lagrangian density is given as follows:

$$\begin{aligned} \tilde{\mathcal{L}}_s = \frac{1}{12} \tilde{\mathcal{H}}^{(h)}_{\mu\nu\eta} \tilde{\mathcal{H}}^{\mu\nu\eta(h)} - \frac{m^2}{4} \tilde{B}^{(h)}_{\mu\nu} \tilde{B}^{\mu\nu(h)} - \frac{1}{4} \tilde{\Phi}^{\mu\nu(h,g)} \tilde{\Phi}^{(h,g)}_{\mu\nu} + \frac{m}{2} \tilde{B}^{(h)}_{\mu\nu} \tilde{\Phi}^{\mu\nu(h,g)}. \end{aligned} \tag{56}$$

A straightforward computation shows that $\tilde{\mathcal{L}}_s$ is independent of the Grassmannian variables θ and $\bar{\theta}$ (i.e. $\tilde{\mathcal{L}}_s = \mathcal{L}_s$) which shows that \mathcal{L}_s is gauge-invariant as well as (anti-) BRST invariant Lagrangian density. Mathematically, latter can be expressed in terms of the translational generators as

$$\begin{aligned} \frac{\partial}{\partial\theta} \tilde{\mathcal{L}}_s = 0 &\Rightarrow s_b \mathcal{L}_s = 0, \\ \frac{\partial}{\partial\bar{\theta}} \tilde{\mathcal{L}}_s = 0 &\Rightarrow s_{ab} \mathcal{L}_s = 0. \end{aligned} \tag{57}$$

It is clear from (54) and (55) together with (57), the followings are true:

$$\begin{aligned} \frac{\partial}{\partial\theta} \tilde{\mathcal{L}}_B = 0 &\Rightarrow s_b \mathcal{L}_B = 0, \\ \frac{\partial}{\partial\bar{\theta}} \tilde{\mathcal{L}}_{\bar{B}} = 0 &\Rightarrow s_{ab} \mathcal{L}_{\bar{B}} = 0, \end{aligned} \tag{58}$$

which clearly show the (anti-) BRST invariance of the coupled Lagrangian densities within the framework superfield formalism. The above equation is true due to the validity of the nilpotency (i.e. $\partial_\theta^2 = 0, \partial_{\bar{\theta}}^2 = 0$) of the translational generators ∂_θ and $\partial_{\bar{\theta}}$.

8 Conclusions

In our present investigation, we have studied the 4D gauge-invariant massive Abelian 2-form theory within the framework of BRST formalism where the local gauge symmetries given in (7) are traded with two linearly independent global BRST and anti-BRST symmetries [cf. (10) and (11)]. In this formalism, we have obtained the coupled (but equivalent) Lagrangian densities [cf. (8) and (9)] which respect the off-shell nilpotent and absolutely anticommuting BRST and anti-BRST symmetry transformations on the constrained hypersurface defined by the CF type conditions (13). These CF conditions are (anti-) BRST invariant as well as they also play a pivotal role in the proof of the absolute anticommutativity of the (anti-) BRST transformations and the derivation of the coupled Lagrangian densities. The anticommutativity property for the dynamical fields $B_{\mu\nu}$ and ϕ_μ is satisfied due to the existence of the Curci–Ferrari type conditions [cf. (16)].

The continuous and off-shell nilpotent (anti-) BRST symmetries lead to the derivation of the corresponding conserved (anti-) BRST charges. In addition to these symmetries, the coupled Lagrangian densities also respect the global ghost-scale symmetry which leads to the conserved ghost charge. The operator form of the continuous symmetry transformations and corresponding generators obeys the standard graded BRST algebra [cf. (29) and (30)]. We lay emphasis on the fact that the physicality criterion $Q_{(a)b}|phys\rangle = 0$ produces the first-class constraints, as the physical conditions

(24) on the theory, which are present in the gauge-invariant Lagrangian density (5). Thus, the BRST quantization is consistent with the Dirac quantization of the system having first-class constraints.

It is worthwhile to point out that the (anti-) BRST charges which are the generators of the corresponding symmetry transformations are unable to produce the proper (anti-) BRST symmetry transformations for the Nakanishi–Lautrup fields $B, \bar{B}, B_\mu, \bar{B}_\mu$ and other fermionic auxiliary fields ρ, λ . The transformations of these fields have been derived from the requirements of the nilpotency and absolute anticommutativity properties of the (anti-) BRST transformations. Similarly, the ghost charge is also incapable to generate the proper transformations for the auxiliary ghost fields ρ and λ . We have derived these symmetries from other considerations [cf. (30)] where we have used the appropriate relations that appear in the algebra (29).

Furthermore, we have exploited the augmented version of superfield approach to BRST formalism to derive the off-shell nilpotent and absolutely anticommuting (anti-) BRST symmetries for the 4D dimensional Stückelberg-like massive Abelian 2-form gauge theory. In this approach, besides the horizontality condition (37), we have used the gauge-invariant restriction (44) for the derivation of the complete sets of the BRST and anti-BRST symmetry transformations. The gauge-invariant restriction is required for the derivation of the proper (anti-) BRST transformations for the Stückelberg-like vector field ϕ_μ . One of the spectacular observations, we point out that the horizontality condition, which produces the (anti-) BRST transformations for the 2-form field and corresponding (anti-) ghost fields, can also be obtained from the integrability of (44) [42]. The CF conditions, which are required for the absolute anticommutativity of the (anti-) BRST symmetries, emerge very naturally in the superfield formalism. These (anti-) BRST invariant CF conditions are conserved quantities. Thus, it would be an interesting piece of work to show that these CF conditions commute with the Hamiltonian within the framework of BRST formalism (see, e.g. [57, 58]).

Using the basic tenets of BRST formalism, we have written the coupled (but equivalent) Lagrangian densities in terms of (anti-) BRST symmetries where the mass dimension and ghost number of the dynamical fields have been taken into account. Within the framework of superfield, we have provided the geometrical origin of the (anti-) BRST symmetries in terms of the Grassmannian translational generators. Also, one can capture the basic properties of the (anti-) BRST transformations in the language of the translational generators. Thus, we have been able to write the coupled Lagrangian densities in terms of the superfields and Grassmannian derivatives. As a result, the (anti-) BRST invariance of the super Lagrangian densities become quite simpler and straightforward

due to the nilpotency property of the Grassmannian derivatives.

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Appendix A: Anticommutativity of the BRST and anti-BRST charges

In this appendix, we provide an explicit proof of the anticommutativity of the BRST and anti-BRST charges in a simpler way. The BRST and anti-BRST charges given in (20) can also be simplified by using the equations of motion (18) and (19), respectively, as

$$Q_b = \int d^3x [B_i(\partial^0 C^i - \partial^i C^0) - (\partial^0 B^i - \partial^i B^0)C_i - B(\partial^0 C - mC^0) + (\partial^0 B - mB^0)C - \rho(\partial^0 \beta) + \beta(\partial^0 \rho) + \lambda B^0], \tag{A.1}$$

$$Q_{ab} = \int d^3x [-\bar{B}_i(\partial^0 \bar{C}^i - \partial^i \bar{C}^0) + (\partial^0 \bar{B}^i - \partial^i \bar{B}^0)\bar{C}_i + \bar{B}(\partial^0 \bar{C} - m\bar{C}^0) - (\partial^0 \bar{B} - m\bar{B}^0)\bar{C} - \lambda(\partial^0 \bar{\beta}) + \bar{\beta}(\partial^0 \lambda) + \rho \bar{B}^0]. \tag{A.2}$$

Applying s_{ab} on Q_b and using the equation of motion for the ghost field C_0 [cf. (18)], we obtain

$$s_{ab}Q_b = \int d^3x [B_i(\partial^0 \bar{B}^i - \partial^i \bar{B}^0) - \bar{B}_i(\partial^0 B^i - \partial^i B^0) - B(\partial^0 \bar{B} - m\bar{B}^0) + \bar{B}(\partial^0 B - mB^0)]. \tag{A.3}$$

Now eliminating \bar{B}^i, \bar{B}^0 and \bar{B} by using the CF conditions, the above expression further simplifies as

$$s_{ab}Q_b = \int d^3x [\varphi(\partial_i \partial^i B^0 + m^2 B^0) - \varphi \partial^0 (\partial_i B^i + mB)]. \tag{A.4}$$

Exploiting the equation of motion $\partial_\mu B^\mu + mB = 0$ and an off-shoot $(\square + m^2)B_\mu = 0$ of the Euler–Lagrange equations of motion (18), we obtain

$$s_{ab}Q_b = -i\{Q_b, Q_{ab}\} = 0. \tag{A.5}$$

Similarly, operating BRST transformations s_b on (A.2) and exploiting the equation of motion for the anti-ghost field \bar{C}_0

[cf. (18)], we finally obtain the r.h.s. of (A.3). In fact, we yield

$$s_b Q_{ab} = s_{ab} Q_b. \quad (\text{A.6})$$

As a result of the above equations, the relation $s_b Q_{ab} = s_{ab} Q_b = -i\{Q_b, Q_{ab}\} = 0$ implies the anticommutativity (i.e. $Q_b Q_{ab} + Q_{ab} Q_b = 0$) of the (anti-) BRST charges $Q_{(a)b}$. Here, we again lay emphasis on the fact that the CF conditions play a crucial role in the anticommutativity of the (anti-) BRST charges (and corresponding symmetry transformations).

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