

Analytic expressions of amplitudes by the cross-ratio identity method

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Abstract In order to obtain the analytic expression of an amplitude from a generic CHY-integrand, a new algorithm based on the so-called cross-ratio identities has been proposed recently. In this paper, we apply this new approach to a variety of theories including the non-linear sigma model, special Galileon theory, pure Yang–Mills theory, pure gravity, Born–Infeld theory, Dirac–Born–Infeld theory and its extension, Yang–Mills-scalar theory, and Einstein–Maxwell and Einstein–Yang–Mills theory. CHY-integrands of these theories which contain higher-order poles can be calculated conveniently by using the cross-ratio identity method, and all results above have been verified numerically.

1 Introduction

In the past a few years, an elegant new formulation of the tree-level S-matrix in arbitrary dimensions for a wide range of field theories has been presented by Cachazo, He and Yuan (CHY) [1–5]. This formulation describes the scattering amplitude for n massless particles as a multidimensional contour integral over the moduli space of punctured Riemann spheres $\mathcal{M}_{0,n}$. It can be unified into a concise expression,

$$\begin{aligned} \mathcal{A}_n &= \int \left(\frac{\prod_{i=1}^n dz_i}{\text{vol } SL(2, \mathbb{C})} \right) \left(\prod' \delta(\mathcal{E}_i) \right) \mathcal{I}_n(k, \epsilon, z) \\ &= \int \left(\begin{matrix} z_{rs} z_{st} z_{tr} & \prod_{i \in \{1, 2, \dots, n\} \setminus \{r, s, t\}} dz_i \end{matrix} \right) \\ &\quad \times \left(\begin{matrix} z_{ab} z_{bc} z_{ca} & \prod_{i \in \{1, 2, \dots, n\} \setminus \{a, b, c\}} \delta(\mathcal{E}_i) \end{matrix} \right) \mathcal{I}_n(k, \epsilon, z), \end{aligned} \quad (1)$$

where z_i is the puncture location in \mathbb{CP}^1 for the i -th particle, and z_{ij} is defined as $z_{ij} \equiv z_i - z_j$. The second line in

(1) is obtained by fixing the gauge redundancy of the Möbius $SL(2, \mathbb{C})$ group. The δ -functions impose the scattering equations

$$\mathcal{E}_i \equiv \sum_{j \in \{1, 2, \dots, n\} \setminus \{i\}} \frac{s_{ij}}{z_{ij}} = 0, \quad (2)$$

where $s_{ij} \equiv (k_i + k_j)^2 = 2k_i \cdot k_j$ are the ordinary Mandelstam variables (in general, we use $s_{ij\dots k} \equiv (k_i + k_j + \dots + k_k)^2$ conventionally). These equations fully localize the integration to a sum over $(n-3)!$ solutions, and no actual integration is required for calculating the n -point amplitude. The Möbius invariant integrand $\mathcal{I}_n(k, \epsilon, z)$ is a rational function of the complex variables z_i , external momenta k_i and polarization vectors ϵ_i . $\mathcal{I}_n(k, \epsilon, z)$ depends on the theory under consideration and carries all the information as regards the wave functions of external particles.

Although conceptually simple and elegant, when applied to practical evaluations, the essential step of finding the full set of analytic solutions becomes a major obstacle, due to the Abel–Ruffini theorem that there is no algebraic solution to the general polynomial equations of degree five or higher with arbitrary coefficients. Moreover, after summing over all those solutions, one often ends up with a remarkably simple result. It is natural to wonder if there were better techniques to produce the analytic expression obtained by summing over all solutions. To overcome this difficulty, many methods have been proposed from various directions [6–18]. Among these approaches, one of the most efficient ways is the integration rules proposed by Baadsgaard, Bjerrum–Bohr, Bourjaily and Damgaard [17, 18]. Inspired by the computation of amplitudes in the field theory limit of string theory, they derived a simple set of combinatorial rules which immediately give the result after integration for any Möbius invariant integrand involving simple poles only. One can get the desired final result after integration by applying these rules directly rather than solving scattering equations. However, the requirement that the CHY-integrand contains only sim-

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ple poles could not be satisfied in general. Logically, there are two alternative issues to bypass this disadvantage. One is to search the integration rules for higher-order poles [19], the other is to decompose an integrand of higher-order poles into that of simple poles [17,20].

Recently, an algorithm of solving this problem has been proposed by Cardona, Feng, Gomez and Huang, based on the so-called cross-ratio identities [21]. These identities reflect the relations between rational functions in terms of z_{ij} with different structures of poles. By applying the cross-ratio identities iteratively, a systematic algorithm for reducing CHY-integrands with higher-order poles has been established. After decomposing the integrand into terms with simple poles only, one can compute the desired analytic result via the integration rules. This is the first systematic way to get the analytic expression of an amplitude from a generic CHY-integrand.

Although this algorithm can be applied to any Möbius invariant integrand in principle, an important question is, can it be terminated within finite steps for any CHY-integrand? In [22], it has been proved that any weight-two rational function of z_{ij} can be decomposed as a sum of PT-factors with kinematic coefficients via the cross-ratio identities within finite steps. Since any term from a known CHY-integrand can be expressed as a product of two weight-two rational functions, one can conclude that all known CHY-integrands can be decomposed into terms of only simple poles by applying the cross-ratio identity method.

To verify its validity, it is worth applying this new method to integrands of various theories and checking the result numerically. In this paper we consider the following theories: non-linear sigma model (NLSM), special Galileon theory (SG), pure Yang–Mills theory (YM), pure gravity (GR), Born–Infeld theory (BI), Dirac–Born–Infeld theory (DBI) and its extension, Yang–Mills-scalar theory (YMS), Einstein–Maxwell theory (EM), Einstein–Yang–Mills theory (EYM). All known CHY-integrands involving higher-order poles are contained in the cases above. In the meanwhile, theories corresponding to CHY-integrands with simple poles only, such as the scalar theory with ϕ^3 or ϕ^4 interaction, will not be considered in this paper. We divide them into three classes according to different building blocks of integrands. The first class includes NLSM, SG, YM, BI as well as GR. Integrands of these theories can be constructed from a $2n \times 2n$ antisymmetric matrix Ψ . The second class includes DBI, EM and a special case of YMS of which integrands depend on antisymmetric matrices $[\Psi]_{a,b;a}$, $[\mathcal{X}]_b$ as well as Ψ . The third class includes the general YMS, the extended DBI and EYM, which contains the mixed traces of the generators of Lie groups. Integrands of these theories are related to a polynomial $\sum_{\{i,j\}} \mathcal{P}_{\{i,j\}}$, or equivalently, an antisymmetric matrix Π . Computation shows that all amplitudes considered in this paper can be calculated efficiently within finite steps.

This paper is organized as follows: In Sect. 2 we give a brief review of the cross-ratio identity method. Based on this approach, calculations of amplitudes of theories in the three classes above are given in Sects. 3, 4 and 5 respectively. Finally, we give a brief summary in Sect. 6.

2 Brief review of the cross-ratio identity method

For the reader’s convenience, we will give a brief introduction to the cross-ratio identity method in this section [21], then we will discuss its validity for general CHY-integrands.

2.1 The systematic decomposition algorithm

Firstly, we need to define the order of poles of an integrand. A generic n -point CHY-integrand consists of terms as rational functions of z_{ij} in the form

$$\mathcal{I} = \frac{1}{\prod_{i,j \in \{1,2,\dots,n\}, i < j} z_{ij}^{\beta_{ij}}}, \tag{3}$$

with the integer power β_{ij} under the constraint of the Möbius invariance: $\sum_j \beta_{ij} + \sum_j \beta_{ji} = 4$ for arbitrary $i \in \{1, 2, \dots, n\}$. For a subset $\Lambda = \{i_1, i_2, \dots, i_{|\Lambda|}\} \subset \{1, 2, \dots, n\}$, the pole index χ_Λ is defined as

$$\chi_\Lambda = \left(\sum_{i',j' \in \Lambda} \beta_{i'j'} \right) - 2(|\Lambda| - 1), \tag{4}$$

where $|\Lambda|$ denotes the length of the set Λ . If $\chi_\Lambda \geq 0$, a pole $\frac{1}{s_\Lambda^{\chi_\Lambda+1}}$ will arise in the result. It is straightforward to verify $\chi_\Lambda = \chi_{\bar{\Lambda}}$, which reflects the momentum conservation constraint $s_\Lambda = s_{\bar{\Lambda}}$, where the subset $\bar{\Lambda} = \{1, 2, \dots, n\} \setminus \Lambda$ is the complement of Λ . Thus, it is necessary to choose independent Λ . If a CHY-integrand has m independent subsets $\Lambda_1, \Lambda_2, \dots, \Lambda_m$ with $\chi_{\Lambda_i} \geq 0$, the order of the poles of the integrand is defined as

$$\Upsilon[\mathcal{I}] = \sum_{i=1}^m \chi_{\Lambda_i}. \tag{5}$$

Then an integrand which results in simple poles only must satisfy $\Upsilon[\mathcal{I}] = 0$.

In order to apply the integration rules, it is necessary to decompose an integrand with $\Upsilon[\mathcal{I}] > 0$ into terms with $\Upsilon[\mathcal{I}'] = 0$. This can be achieved by multiplying the cross-ratio identities to the integrand iteratively. The cross-ratio identity for the set Λ is given by

$$1 = - \sum_{i \in \Lambda \setminus \{j\}} \sum_{b \in \bar{\Lambda} \setminus \{p\}} \frac{s_{ib} z_{bp} z_{ij}}{s_\Lambda z_{ib} z_{jp}} \equiv \mathbb{I}_n[\Lambda, j, p], \tag{6}$$

where j and p are selected manually. This identity holds on the support of the scattering equations and the momentum conservation constraint. One can expand the original \mathcal{I} into $(|\Lambda| - 1)(n - |\Lambda| - 1)$ terms via the operation $\mathcal{I} = \mathbb{I}_n[\Lambda, j, p]\mathcal{I}$. This operation will not break the manifest Möbius invariance, since the cross-ratio identity is of weight zero under Möbius transformations for any node i with $i \in \{1, 2, \dots, n\}$. Obviously, such an operation decreases χ_Λ by 1; thus the order of the pole $\frac{1}{s_\Lambda^{x+1}}$ has been reduced if $\chi_\Lambda > 0$.

The systematic reduction algorithm is presented in the following:

1. Count the order of poles $\Upsilon[\mathcal{I}]$ of the integrand \mathcal{I} . If $\Upsilon[\mathcal{I}] > 0$, find the full set of independent subsets with $\chi_{\tilde{\Lambda}_i} > 0$ (say, there are m subsets):

$$\tilde{\Lambda}_1, \tilde{\Lambda}_2, \dots, \tilde{\Lambda}_m. \tag{7}$$

2. Step 1: start from the first set $\tilde{\Lambda}_1$, collect all $|\tilde{\Lambda}_1|(n - |\tilde{\Lambda}_1|)$ cross-ratio identities of $\tilde{\Lambda}_1$ with different choices of j and b as

$$\begin{aligned} &\mathbb{I}_n[\tilde{\Lambda}_1, j, p] \quad \text{where } j \in \tilde{\Lambda}_1, \\ &p \in \{1, 2, \dots, n\} \setminus \tilde{\Lambda}_1. \end{aligned} \tag{8}$$

3. Step 2: decompose the CHY-integrand \mathcal{I} by applying the first cross-ratio identity

$$\mathcal{I} = \mathbb{I}_n[\tilde{\Lambda}_1, j, p]\mathcal{I} = \sum_\ell c_\ell \mathcal{I}'_\ell, \tag{9}$$

where the c_ℓ are rational functions of the Mandelstam variables.

4. Count all $\Upsilon[\mathcal{I}'_\ell]$.
 - If all $\Upsilon[\mathcal{I}'_\ell] < \Upsilon[\mathcal{I}]$, return $\sum_\ell c_\ell \mathcal{I}'_\ell$.
 - If there exists any $\Upsilon[\mathcal{I}'] \geq \Upsilon[\mathcal{I}]$, test the second cross-ratio identity in step 2 and so on, until we find a cross-ratio identity satisfying all $\Upsilon[\mathcal{I}'_\ell] < \Upsilon[\mathcal{I}]$.
 - If after running over all cross-ratio identities of the set Λ_1 , there is still no such an identity satisfying all $\Upsilon[\mathcal{I}'_\ell] < \Upsilon[\mathcal{I}]$, then take the first identity in step 2 again but now we stop till an identity satisfies all $\Upsilon[\mathcal{I}'_\ell] \leq \Upsilon[\mathcal{I}]$, and return $\sum_\ell c_\ell \mathcal{I}'_\ell$.
5. Perform the procedure above for each \mathcal{I}'_ℓ , and repeat the same operation iteratively, then end with the expression such that the order of poles of each term is zero.
6. If, after some steps, there always exist terms with the order of poles no less than $\Upsilon[\mathcal{I}]$, restart the algorithm by starting from $\tilde{\Lambda}_2$, etc.

The entire algorithm can be implemented in MATHEMATICA. For a given CHY-integrand, if this algorithm can be terminated within finite steps, one can obtain an expression with terms that have simple poles only and finally get the analytic result by applying the integration rules.

2.2 The feasibility of this method

Given the decomposition algorithm, it is natural to ask whether all Möbius invariant integrands can be computed in this way. It is obvious that the algorithm can be performed for any integrand. The question is, can it be terminated within finite steps? An ideal situation is, we can always find cross-ratio identities such that all terms satisfy $\Upsilon[\mathcal{I}'_\ell] < \Upsilon[\mathcal{I}]$ at each step, then the decomposition procedure can be terminated after $\Upsilon[\mathcal{I}]$ steps at most. However, this happens for some particular integrands rather than for all. Thus, we need to consider if it is possible that at every step there are terms carrying the structure of poles such that $\Upsilon[\mathcal{I}'_\ell] = \Upsilon[\mathcal{I}]$, or even $\Upsilon[\mathcal{I}'_\ell] > \Upsilon[\mathcal{I}]$, for all choices of $\tilde{\Lambda}_i, j$ and p . If this happens, the corresponding integrand cannot be calculated by the method introduced in the previous subsection. In order to fully understand the cross-ratio identity method, one needs to prove that the situation above can be excluded in general, or clarify when such a situation might arise.

Actually, the sum of the χ_Λ for length- t subsets of $\{1, 2, \dots, n\}$ is fully determined by the condition $\sum_{j \in \{1, 2, \dots, n\}} \beta_{ij} = 4$:

$$\chi_t \equiv \sum_{|\Lambda_i|=t} \chi_{\Lambda_i} = -2(t - 1)C_n^2 + 2n C_{n-2}^{n-t}. \tag{10}$$

Thus, the sum of all χ_Λ , $\chi_{\text{total}} \equiv \sum_\Lambda \chi_\Lambda = \sum_t \chi_t$, is invariant under any action which maintains the Möbius invariance. If χ_{total} is positive for some integer n , it is impossible to decompose the corresponding integrand into terms with simple poles only. Fortunately, a little algebra leads to the conclusion that $\chi_t > 0$ if and only if $n < t + 1$, thus χ_{total} can never be positive.

On the other hand, it has been proved that a weight-two rational function of z_{ij} can always be transformed to the sum of PT-factors $\frac{1}{z_{i_1 i_2} z_{i_2 i_3} \dots z_{i_n i_1}}$ with kinematic coefficients via the cross-ratio identities within finite steps [22]. Any term of a known CHY-integrand in the literature can be expressed as a product of two weight-two rational functions. Hence, although it is not clear whether the CHY-integrand for any physical theory can be decomposed as products of weight-two functions, one can use the cross-ratio identities to decompose any known CHY-integrand into terms which contain simple poles only.

In this paper, we will choose the original algorithm rather than the one which decomposes a weight-two function into PT-factors, since the former is more convenient to be imple-

mented in MATHEMATICA. Indeed, the feasibility of this algorithm has not been proved, since the procedure of decomposing a weight-two function into PT-factors cannot ensure $\Upsilon[\mathcal{I}'_\ell] \leq \Upsilon[\mathcal{I}]$ at each step. However, as can be seen in the subsequent sections, all known CHY-integrands can be computed by the original algorithm efficiently, i.e., the condition $\Upsilon[\mathcal{I}'_\ell] \leq \Upsilon[\mathcal{I}]$ can always be satisfied, at least for all known CHY-integrands.

3 Amplitudes of theories in the first class

For theories in this class, the most important object in the construction of the integrand \mathcal{I}_n is the $2n \times 2n$ antisymmetric matrix

$$\Psi = \left(\begin{array}{c|c} A & -C^T \\ \hline C & B \end{array} \right) \tag{11}$$

where A, B and C are the $n \times n$ matrices given by

$$A_{ij} = \begin{cases} \frac{k_i \cdot k_j}{z_{ij}} & i \neq j, \\ 0 & i = j, \end{cases} \quad B_{ij} = \begin{cases} \frac{\epsilon_i \cdot \epsilon_j}{z_{ij}} & i \neq j, \\ 0 & i = j, \end{cases} \tag{12}$$

and

$$C_{ij} = \begin{cases} \frac{\epsilon_i \cdot k_j}{z_{ij}} & i \neq j, \\ -\sum_{l=1, l \neq i}^n \frac{\epsilon_i \cdot k_l}{z_{il}} & i = j. \end{cases} \tag{13}$$

One also needs to introduce the reduced Pfaffian $\text{Pf}'\Psi = \frac{(-)^{i+j}}{z_{ij}} \text{Pf}\Psi_{ij}^{ij}$ where Ψ_{ij}^{ij} denotes the minor obtained by deleting rows and columns labeled by i and j , with $i, j \in \{1, 2, \dots, n\}$. On the support of scattering equations, the reduced Pfaffian $\text{Pf}'\Psi$ is invariant with respect to the permutation of particle labels. In addition, a useful factor is defined as

$$\mathcal{C}_{\{i_1, i_2, \dots, i_s\}} = \sum_{\sigma \in S_s / \mathbb{Z}_s} \frac{\text{Tr}(T^{I_{\sigma(i_1)}} T^{I_{\sigma(i_2)}} \dots T^{I_{\sigma(i_s)}})}{z_{\sigma(i_1)\sigma(i_2)} z_{\sigma(i_2)\sigma(i_3)} \dots z_{\sigma(i_s)\sigma(i_1)}}, \tag{14}$$

where T^I are generators of the Lie group under consideration.

The diagonal terms of the matrix C will break the manifest Möbius invariance, since they are not of a uniform weight under Möbius transformations. To keep the validity of the integration rules, they need to be rewritten as

$$\begin{aligned} C_{ii} &= \sum_{l \neq i} \left(\frac{\epsilon_i \cdot k_l}{z_{ai}} + \frac{(\epsilon_i \cdot k_l) z_{la}}{z_{ai} z_{il}} \right) \\ &\Rightarrow \sum_{l \neq i, a} \frac{(\epsilon_i \cdot k_l) z_{la}}{z_{ai} z_{il}}, \quad a \neq i, \end{aligned} \tag{15}$$

where momentum conservation and the gauge invariant condition $\epsilon_i \cdot k_i = 0$ have been used. The new formula of C_{ii} gives the weight two for node i and weight zero for other nodes, then the term-wise Möbius invariance is guaranteed. Throughout this paper, we choose

$$C_{ii} = \begin{cases} \sum_{l=3}^n \frac{(\epsilon_1 \cdot k_l) z_{l2}}{z_{21} z_{1l}} & i = 1, \\ \sum_{l \neq 1, i} \frac{(\epsilon_i \cdot k_l) z_{l1}}{z_{1i} z_{il}} & i > 1. \end{cases} \tag{16}$$

With these ingredients, we can now investigate theories in the first class one by one.

3.1 Non-linear sigma model

We begin with the simplest case, the NLSM, whose standard Lagrangian in Cayley parametrization is

$$\mathcal{L}^{\text{NLSM}} = \frac{1}{8\lambda^2} \text{Tr}(\partial_\mu U^\dagger \partial^\mu U), \tag{17}$$

where

$$U = (\mathbb{I} + \lambda\Phi)(\mathbb{I} - \lambda\Phi)^{-1}, \quad \Phi = \phi_I T^I. \tag{18}$$

Here \mathbb{I} is the identity matrix and the T^I are generators of $U(N)$. The CHY-integrand of NLSM is given by [5]

$$\mathcal{I}^{\text{NLSM}} = \lambda^{n-2} C_n (\text{Pf}'A(k, z))^2. \tag{19}$$

For this case, it is sufficient to calculate the color-ordered partial amplitude $\mathcal{A}^{\text{NLSM}}(1, 2, \dots, n)$ defined via

$$\begin{aligned} \mathcal{A}_n^{\text{NLSM}} &= \sum_{\sigma \in S_n / \mathbb{Z}_n} \text{Tr}(T^{I_{\sigma(1)}} T^{I_{\sigma(2)}} \dots T^{I_{\sigma(n)}}) \mathcal{A}^{\text{NLSM}} \\ &\times (\sigma(1), \sigma(2), \dots, \sigma(n)). \end{aligned} \tag{20}$$

In other words, we focus on the color-ordered partial integrand

$$\mathcal{I}^{\text{NLSM}}(1, 2, \dots, n) = \frac{(\text{Pf}'A(k, z))^2}{z_{12} z_{23} \dots z_{n1}}. \tag{21}$$

Here the coupling constant has been omitted.

We start from the 6-point amplitude $\mathcal{A}_6^{\text{NLSM}}$. By definition, the corresponding color-ordered partial integrand is

$$\begin{aligned} \mathcal{I}^{\text{NLSM}}(1, 2, \dots, 6) &= \frac{(k_1 \cdot k_2)^2 (k_3 \cdot k_4)^2}{z_{12}^3 z_{23}^3 z_{34}^3 z_{45}^3 z_{56}^3 z_{61}} \\ &+ \frac{2(k_1 \cdot k_2)(k_2 \cdot k_4)(k_2 \cdot k_3)(k_3 \cdot k_4)}{z_{12}^2 z_{14} z_{23}^2 z_{34}^2 z_{45}^3 z_{56}^3 z_{61}} \\ &- \frac{2(k_1 \cdot k_2)(k_1 \cdot k_3)(k_2 \cdot k_4)(k_3 \cdot k_4)}{z_{12}^2 z_{13} z_{23} z_{24} z_{34}^2 z_{45}^3 z_{56}^3 z_{61}} \\ &+ \frac{(k_1 \cdot k_4)^2 (k_2 \cdot k_3)^2}{z_{12} z_{14}^2 z_{23}^3 z_{34} z_{45}^3 z_{56}^3 z_{61}} \end{aligned}$$

$$\begin{aligned}
 &+ \frac{(k_1 \cdot k_3)^2(k_2 \cdot k_4)^2}{z_{12}z_{13}^2z_{23}z_{24}^2z_{34}z_{45}z_{56}^3z_{61}} \\
 &- \frac{2(k_1 \cdot k_3)(k_1 \cdot k_4)(k_2 \cdot k_3)(k_2 \cdot k_4)}{z_{12}z_{13}z_{14}z_{23}^2z_{24}z_{34}z_{45}z_{56}^3z_{61}}. \tag{22}
 \end{aligned}$$

The pole structure of (22) is listed as follows:

	1st term	2nd term	3rd term	4th term	5th term	6th term
Pole	$\frac{1}{s_{12}^2s_{23}^2s_{34}^2s_{45}^2s_{56}^2s_{61}^2s_{561}}$	$\frac{1}{s_{56}^2s_{12}^2s_{23}^2s_{34}^2s_{123}^2s_{561}}$	$\frac{1}{s_{56}^2s_{12}^2s_{34}^2s_{123}^2s_{561}}$	$\frac{1}{s_{23}^2s_{56}^2s_{14}^2s_{123}^2s_{561}}$	$\frac{1}{s_{56}^2s_{13}^2s_{24}^2s_{123}^2s_{561}}$	$\frac{1}{s_{56}^2s_{23}^2s_{123}^2s_{561}}$

It can be seen from the table that every term contains higher-order poles which need to be decomposed. Via the cross-ratio identity method, One can accomplish the decomposition within three steps. Below is the table with #[ALL], the number of resulting terms and #[H], the number of terms of higher-order poles in each Round of decomposition:

	Round 1	Round 2	Round 3
#[ALL]	18	30	38
#[H]	6	4	0

Integrations of these terms can be bypassed by applying the integration rules. Summing all terms from the final result, we obtain

$$\begin{aligned}
 \mathcal{A}^{\text{NLSM}}(1, 2, \dots, 6) &= \frac{(s_{12} + s_{23})(s_{45} + s_{56})}{s_{123}} \\
 &+ \frac{(s_{23} + s_{34})(s_{56} + s_{61})}{s_{234}} + \frac{(s_{34} + s_{45})(s_{61} + s_{12})}{s_{345}} \\
 &- (s_{12} + s_{23} + s_{34} + s_{45} + s_{56} + s_{61}). \tag{23}
 \end{aligned}$$

For this simple example, the full computation takes less than a minute in MATHEMATICA. One can see the manifest cyclic symmetry in (23), which is the characteristic of the color-ordered partial amplitude. This analytic result is confirmed numerically against the one obtained from solving scattering equations numerically.

Then we turn to the 8-point amplitude $\mathcal{A}_8^{\text{NLSM}}$. The integrand has 120 terms and all terms contain higher-order poles. Performing the cross-ratio identity method, this integrand can be decomposed into 4340 terms with simple poles only within 6 steps. The table of #[ALL] and #[H] in each Round of decomposition is given as follows:

	Round 1	Round 2	Round 3	Round 4	Round 5	Round 6
#[ALL]	600	1128	1904	2924	4084	4340
#[H]	124	142	169	150	32	0

From these terms with simple poles, we get the desired analytic expression of the amplitude via the integration rules. The final result can be simplified into the form

$$\mathcal{A}^{\text{NLSM}}(1, 2, \dots, 8) = \text{Part1} - \text{Part2} + \text{Part3}, \tag{24}$$

with

$$\begin{aligned}
 \text{Part1} &= \frac{(s_{12} + s_{23})(s_{45} + s_{56})(s_{78} + s_{8123})}{s_{123}s_{456}} \\
 &+ \frac{(s_{23} + s_{34})(s_{56} + s_{67})(s_{81} + s_{1234})}{s_{234}s_{567}} \\
 &+ \frac{(s_{34} + s_{45})(s_{67} + s_{78})(s_{12} + s_{2345})}{s_{345}s_{678}} \\
 &+ \frac{(s_{45} + s_{56})(s_{78} + s_{81})(s_{23} + s_{3456})}{s_{456}s_{781}} \\
 &+ \frac{(s_{56} + s_{67})(s_{81} + s_{12})(s_{34} + s_{4567})}{s_{567}s_{812}} \\
 &+ \frac{(s_{67} + s_{78})(s_{12} + s_{23})(s_{45} + s_{5678})}{s_{678}s_{123}} \\
 &+ \frac{(s_{78} + s_{81})(s_{23} + s_{34})(s_{56} + s_{6789})}{s_{781}s_{234}} \\
 &+ \frac{(s_{81} + s_{12})(s_{34} + s_{45})(s_{67} + s_{7812})}{s_{812}s_{345}} \\
 &+ \frac{(s_{12} + s_{23})(s_{56} + s_{67})(s_{1234} + s_{4567})}{s_{123}s_{567}} \\
 &+ \frac{(s_{23} + s_{34})(s_{67} + s_{78})(s_{2345} + s_{5678})}{s_{234}s_{678}} \\
 &+ \frac{(s_{34} + s_{45})(s_{78} + s_{81})(s_{3456} + s_{6781})}{s_{345}s_{781}} \\
 &+ \frac{(s_{45} + s_{56})(s_{81} + s_{12})(s_{4567} + s_{7812})}{s_{456}s_{812}}, \tag{25}
 \end{aligned}$$

$$\begin{aligned}
 \text{Part2} &= \frac{(s_{12} + s_{23})(s_{45} + s_{56} + s_{67} + s_{78} + s_{8123} + s_{1234})}{s_{123}} \\
 &+ \frac{(s_{23} + s_{34})(s_{56} + s_{67} + s_{78} + s_{81} + s_{1234} + s_{2345})}{s_{234}} \\
 &+ \frac{(s_{34} + s_{45})(s_{67} + s_{78} + s_{81} + s_{12} + s_{2345} + s_{3456})}{s_{345}} \\
 &+ \frac{(s_{45} + s_{56})(s_{78} + s_{81} + s_{12} + s_{23} + s_{3456} + s_{4567})}{s_{456}} \\
 &+ \frac{(s_{56} + s_{67})(s_{81} + s_{12} + s_{23} + s_{34} + s_{4567} + s_{5678})}{s_{567}} \\
 &+ \frac{(s_{67} + s_{78})(s_{12} + s_{23} + s_{34} + s_{45} + s_{5678} + s_{6781})}{s_{678}} \\
 &+ \frac{(s_{78} + s_{81})(s_{23} + s_{34} + s_{45} + s_{56} + s_{6781} + s_{7812})}{s_{781}}
 \end{aligned}$$

$$+ \frac{(s_{81} + s_{12})(s_{34} + s_{45} + s_{56} + s_{67} + s_{7812} + s_{8123})}{s_{812}}, \tag{26}$$

$$\text{Part3} = 2(s_{12} + s_{23} + s_{34} + s_{45} + s_{56} + s_{67} + s_{78} + s_{81}) + s_{1234} + s_{2345} + s_{3456} + s_{4567}. \tag{27}$$

This result has been verified numerically.

3.2 Special Galileon theory

The next theory under consideration is SG. The general pure Galileon Lagrangian is

$$\mathcal{L}^{\text{SG}} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \sum_{m=3}^{\infty} g_m \mathcal{L}_m, \tag{28}$$

with

$$\mathcal{L}_m = \phi \det \{ \partial^{\mu_i} \partial_{\nu_j} \phi \}_{i,j=1}^{m-1}. \tag{29}$$

We restrict ourselves on the special situation in which there exist constraints on the coupling constants g_m such that all amplitudes with an odd number of external particles vanish. Then the CHY-integrand \mathcal{I}_n of this theory is given by [5]

$$\mathcal{I}^{\text{SG}} = (\text{Pf}' A(k, z))^4, \tag{30}$$

where the coupling constants have been omitted.

With this setup, we choose the 6-point amplitude $\mathcal{A}_6^{\text{SG}}$ as an example. The integrand has 15 terms, and all terms contain higher-order poles. One can divide it into 3169 terms with simple poles only within 10 steps. The table of #[ALL] and #[H] in each *Round* of decomposition is given by

	Round 1	Round 2	Round 3	Round 4	Round 5
#[ALL]	45	108	234	468	873
#[H]	45	72	162	180	198

	Round 6	Round 7	Round 8	Round 9	Round 10
#[ALL]	1215	1944	3068	3151	3169
#[H]	261	530	53	12	0

Although the final result is too lengthy to be presented, it has been confirmed numerically.

3.3 Yang–Mills theory

Then we turn to the pure YM case. The CHY-integrand of YM is [5]

$$\mathcal{I}^{\text{YM}} = \mathcal{C}_n \text{Pf}' \Psi(k, \epsilon, z). \tag{31}$$

Similar to the case of NLSM, it is sufficient to consider the color-ordered partial integrand

$$\mathcal{I}^{\text{YM}}(1, 2, \dots, n) = \frac{\text{Pf}' \Psi(k, \epsilon, z)}{z_{12} z_{23} \dots z_{n1}}. \tag{32}$$

Let us take the 6-point color-ordered amplitude $\mathcal{A}^{\text{YM}}(1, 2, \dots, 6)$ as an example. The partial integrand $\mathcal{I}^{\text{YM}}(1, 2, \dots, 6)$ has 3420 terms and 1120 of them contain higher-order poles. The decomposition procedure can be terminated within 5 steps via the cross-ratio identity method, and the analytic expression of $\mathcal{A}^{\text{YM}}(1, 2, \dots, 6)$ is verified numerically. The table of #[ALL] and #[H] in each *Round* of decomposition is given by

	Round 1	Round 2	Round 3	Round 4	Round 5
#[ALL]	6174	8624	10459	11167	11252
#[H]	834	533	162	28	0

It is worth noticing that, when checking the result numerically, the values of external momenta must satisfy the momentum conservation constraint, which is necessary for the derivation of the cross-ratio identities. However, those of polarization vectors can be chosen arbitrarily since they are irrelevant to the cross-ratio identities and the integration. We have verified the result with polarization vectors $\epsilon_i \cdot k_i \neq 0$ as well as $\epsilon_i \cdot k_i = 0$, and we find that the analytic expression reproduces the value obtained from solving the scattering equations numerically.

3.4 Born–Infeld theory

Now we consider BI whose Lagrangian is given by

$$\mathcal{L}^{\text{BI}} = \ell^{-2} \left(\sqrt{-\det(\eta_{\mu\nu} - \ell^2 F_{\mu\nu})} - 1 \right). \tag{33}$$

The CHY-integrand of BI is [5]

$$\mathcal{I}^{\text{BI}} = \ell^{n-2} \text{Pf}' \Psi(k, \tilde{\epsilon}, z) \text{Pf}' A(k, z)^2. \tag{34}$$

For simplicity, we calculate the 6-point amplitude $\mathcal{A}_6^{\text{BI}}$. The integrand contains 20400 terms and 18744 of them involve higher-order poles. Using the cross-ratio identities, one can reduce it to terms with simple poles within 10 steps. The table of #[ALL] and #[H] in each *Round* of decomposition is given by

	Round 1	Round 2	Round 3	Round 4	Round 5
#[ALL]	61200	123267	202067	269132	324740
#[H]	28616	32813	24206	15026	6420

	Round 1	Round 2	Round 3	Round 4	Round 5
#[ALL]	352236	375220	397206	399183	399552
#[H]	3173	3135	304	62	0

This is the most complicated example in this paper; it takes more than a day in MATHEMATICA. The analytic expression of the amplitude is confirmed by numerical verification.

3.5 Gravity

The final theory under consideration in this section is GR. The CHY-integrand of this theory is the product of two inde-

$$([\Psi]_{a,b;a})_{i,j} = \begin{cases} A_{ij} & i, j \in \{1, 2, \dots, n\}, \\ C_{(i-n)j} & i \in \{n+1, n+2, \dots, n+n_a\}, j \in \{1, 2, \dots, n\}, \\ (-C^T)_{i(j-n)} & i \in \{1, 2, \dots, n\}, j \in \{n+1, n+2, \dots, n+n_a\}, \\ B_{(i-n)(j-n)} & i, j \in \{n+1, n+2, \dots, n+n_a\}. \end{cases} \tag{37}$$

pendent copies of the one for YM, each of which has its own gauge choice for polarization vectors [5]

$$\mathcal{I}^{GR} = \text{Pf}'\Psi(k, \epsilon, z) \text{Pf}'\Psi(k, \tilde{\epsilon}, z). \tag{35}$$

The polarization tensor of a graviton is given by $\zeta_{\mu\nu} = \epsilon_\mu \tilde{\epsilon}_\nu$. This integrand leads to amplitudes of gravitons coupled to dilatons and B-fields.

We take the 4-point amplitude \mathcal{A}_4^{GR} as an example. The integrand contains 484 terms, with 228 terms involving higher-order poles. It can be decomposed into terms with simple poles within 2 steps, as shown in the following table:

	Round 1	Round 2
#[ALL]	484	484
#[H]	36	0

Physically, polarization tensors of gravitons are traceless, i.e., they satisfy $\epsilon^\mu \tilde{\epsilon}_\mu = 0$. However, as discussed before, their values can be chosen without imposing any physical constraint when performing the numerical verification.

4 Amplitudes of theories in the second class

In this section we move on to theories in the second class. CHY-integrands of these theories require two new matrices $[\mathcal{X}]_b$ and $[\Psi]_{a,b;a}$ as basic ingredients. a and b are two sets of external particles, whose numbers are denoted by n_a and n_b ,

respectively, and $n = n_a + n_b$ is the total number of particles. $[\mathcal{X}]_b$ is an $n_b \times n_b$ matrix defined as

$$([\mathcal{X}]_b)_{i,j} = \begin{cases} \frac{\delta^{I_i, I_j}}{z_{ij}} & i \neq j, \\ 0 & i = j, \end{cases} \tag{36}$$

where $I^i \in \{1, \dots, M\}$ denotes the i -th $U(1)$ charge of the $U(1)^M$ group. $[\Psi]_{a,b;a}$ is an $(n + n_a) \times (n + n_a)$ matrix obtained from Ψ by deleting rows and columns labeled by $n + i$ for all $i \in [n_a + 1, n]$. More explicitly, its elements are given by

Among n external particles, only particles in subset a contribute their polarization vectors to the matrix $[\Psi]_{a,b;a}$. This is the reason why the integrand including $\text{Pf}'[\Psi]_{a,b;a}$ can describe interactions between bosons with different spins.

It is worth emphasizing that all terms in $\text{Pf}[\mathcal{X}]_b$ are manifestly invariant under the Möbius transformations. From the formula of diagonal terms of the matrix C in (16), terms in the expansion of $\text{Pf}'[\Psi]_{a,b;a}$ also have the manifest Möbius invariance, which ensures the feasibility of the integration rules.

4.1 Special Yang–Mills–scalar theory

The first theory under consideration in this section is the special YMS case which describes the low energy effective action of N coincident D-branes. The Lagrangian of this theory is

$$\mathcal{L}^{YMS} = -\text{Tr} \left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} D^\mu \phi^I D_\mu \phi^I - \frac{g^2}{4} \sum_{I \neq J} [\phi^I, \phi^J]^2 \right), \tag{38}$$

where the gauge group is $U(N)$, and the scalars carry a flavor index I with $I \in \{1, \dots, M\}$ from the M -dimensional space transverse to the D-brane. The corresponding CHY-integrand is [5]

$$\mathcal{I}^{YMS}(g, s) = C_n \text{Pf}[\mathcal{X}]_s(z) \text{Pf}'[\Psi]_{g,s;g}(k, \tilde{\epsilon}, z), \tag{39}$$

where g and s denote the sets of gluons and scalars, respectively. Gluons have polarization vectors ϵ^μ , while scalars do not; thus their kinematical information can be combined into the matrix $[\Psi]_{g,s;g}$. Again, we consider the color-ordered partial amplitude $\mathcal{A}^{YMS}(1, 2, \dots, n)$.

The first example is the 6-point partial amplitude \mathcal{A}^{YMS} ($1g, 2g, 3g, 4g, 5s, 6s$), where external particles $1g, 2g, 3g$ and $4g$ are gluons, while $5s, 6s$ are scalars of the same flavor. The partial integrand has 222 terms and 68 of them contain higher-order poles. The decomposition procedure is shown in the following table:

	Round 1	Round 2	Round 3
#[ALL]	376	514	592
#[H]	51	22	0

The second example is the 6-point amplitude \mathcal{A}^{YMS} ($1g, 2g, 3s_{I_1}, 4s_{I_1}, 5s_{I_2}, 6s_{I_2}$), where $1g$ and $2g$ are gluons, $3s_{I_1}, 4s_{I_1}$ are scalars of one flavor and $5s_{I_2}, 6s_{I_2}$ are scalars of another. The partial integrand contains 15 terms and 7 of them contain higher-order poles. The decomposition can be done within 3 steps as shown in the following table:

	Round 1	Round 2	Round 3
#[ALL]	29	35	43
#[H]	3	4	0

Analytic expressions of these two examples are verified numerically.

4.2 Dirac–Born–Infeld theory

We proceed to consider DBI whose Lagrangian is

$$\mathcal{L}^{\text{DBI}} = \ell^{-2} \left(\sqrt{-\det(\eta_{\mu\nu} - \ell^2 \partial_\mu \phi^I \partial_\nu \phi^I - \ell F_{\mu\nu})} - 1 \right), \tag{40}$$

where I again labels the flavor of scalars. The CHY-integrand of DBI is [5]

$$\mathcal{I}^{\text{DBI}}(\gamma, s) = \ell^{n-2} \text{Pf}[\mathcal{X}]_s(z) \text{Pf}'[\Psi]_{\gamma, s; \gamma}(k, \tilde{\epsilon}, z) (\text{Pf}' A(k, z))^2, \tag{41}$$

where γ denotes the set of photons and s the set of scalars, respectively.

Let us calculate the 6-point amplitude $\mathcal{A}_{2\gamma 2s_{I_1} 2s_{I_2}}^{\text{DBI}}$, which contains two photons and four scalars carrying two flavor indices. The integrand has 82 terms and all of them contain higher-order poles. One can accomplish the decomposition procedure via the cross-ratio identities within 10 steps, as shown in the following table:

	Round 1	Round 2	Round 3	Round 4	Round 5
#[ALL]	246	440	674	904	1169
#[H]	106	120	130	104	67

	Round 6	Round 7	Round 8	Round 9	Round 10
#[ALL]	1285	1528	1908	1937	1943
#[H]	87	174	17	4	0

Again, this result is verified numerically.

4.3 Einstein–Maxwell theory

The final theory in this section is EM which describes gravitons coupled to photons. The CHY-integrand of this theory is given by [5]

$$\mathcal{I}^{\text{EM}} = \text{Pf}[\mathcal{X}]_\gamma(z) \text{Pf}'[\Psi]_{h, \gamma; h}(k, \epsilon, z) \text{Pf}'\Psi(k, \tilde{\epsilon}, z). \tag{42}$$

Here the set of gravitons is denoted by h , and that of photons is denoted by γ . Equation (42) allows the photons to carry more than one flavor in general. The polarization tensor of a graviton is $\zeta^{\mu\nu} = \epsilon^\mu \tilde{\epsilon}^\nu$, and the polarization vector of a photon is $\tilde{\epsilon}^\nu$. The matrix $\Psi(k, \tilde{\epsilon}, z)$ contains $\tilde{\epsilon}^\nu$ for both gravitons and photons, and the matrix $[\Psi]_{h, \gamma; h}(k, \epsilon, z)$ contains the remaining ϵ^μ for gravitons.

Our example is the 5-point amplitude $\mathcal{A}_{3h 2\gamma}^{\text{EM}}$ whose external particles are three gravitons and two photons carrying the same flavor index. The integrand has 5013 terms and 1171 of them contain higher-order poles. The decomposition procedure can be done within 4 steps, as shown in the following table:

	Round 1	Round 2	Round 3	Round 4
#[ALL]	6748	7577	7775	7799
#[H]	488	80	10	0

Again, we have verified this result numerically.

5 Amplitudes of theories in the third class

Theories in the final class correspond to multi-trace mixed amplitudes. More precisely, an amplitude in this section contains external bosons which belong to the set $a \cup b_{\text{Tr}_1} \cup b_{\text{Tr}_2} \cdots \cup b_{\text{Tr}_m}$ with bosons in set a of spin S_a and that in the set b_{Tr_i} of spin $S_b = S_a - 1$. This structure leads to the mixed color factor $\text{Tr}_1 \cup \text{Tr}_2 \cdots \cup \text{Tr}_m$ in the amplitude. The kinematic part of the integrand for these theories can be constructed in two equivalent ways; one is to introduce the polynomial $\sum_{\{i, j\}} \mathcal{P}'_{\{i, j\}}$, the other is to define the matrix Π . Let us assume the amplitude contains m mixed traces, then

$\sum_{\{i,j\}} \mathcal{P}'_{\{i,j\}}$ is a sum over the perfect matching $\{i, j\}$,

$$\sum_{\{i,j\}} \mathcal{P}'_{\{i,j\}} = \sum_{\substack{i_{m-1} < j_{m-1} \in \text{Tr}_{m-1} \\ i_1 < j_1 \in \text{Tr}_1}} \text{sgn}(\{i, j\}) z_{i_1 j_1} \cdots z_{i_{m-1} j_{m-1}} \times \text{Pf}[\Psi]_{a,i_1,j_1,\dots,i_{m-1},j_{m-1};h}, \tag{43}$$

where i_a and j_a are labels of two external particles which belong to b_{Tr_a} . This sum can be recognized as the reduced Pfaffian of the matrix Π . The matrix Π can be constructed from Ψ by performing the so-called squeezing operation iteratively. Terms in the expansion of $\sum_{\{i,j\}} \mathcal{P}'_{\{i,j\}}$ respect the Möbius invariance automatically, while terms in the expansion of $\text{Pf}'\Pi$ break the manifest Möbius invariance thus are forbidden for the integration rules. Hence, we will use $\sum_{\{i,j\}} \mathcal{P}'_{\{i,j\}}$ to express the integrands throughout this section.

5.1 General Yang–Mills–scalar theory

Let us consider the general YMS with the Lagrangian

$$\mathcal{L}^{\text{gen.YMS}} = -\text{Tr}\left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} D^\mu \phi^I D_\mu \phi^I - \frac{g^2}{4} \sum_{I \neq J} [\phi^I, \phi^J]^2\right) + \frac{\lambda}{3!} f_{IJK} f_{\bar{I}\bar{J}\bar{K}} \phi^{I\bar{I}} \phi^{J\bar{J}} \phi^{K\bar{K}}, \tag{44}$$

which involves the general flavor group and a cubic scalar self-interaction. The trace is for the gauge group, and $f_{\bar{I}\bar{J}\bar{K}}$ and f_{IJK} are the structure constants of gauge and flavor groups respectively. The amplitudes of this theory can only contain a single trace of the gauge group, and multi-traces for the flavor group, as can be seen from the general CHY-integrand [5]

$$\mathcal{I}^{\text{gen.YMS}}(s_{\text{Tr}_1} \cup \cdots \cup s_{\text{Tr}_m}, g) = \mathcal{C}_n \mathcal{C}_{\text{Tr}_1} \cdots \mathcal{C}_{\text{Tr}_m} \times \sum_{\{i,j\}} \mathcal{P}'_{\{i,j\}}(s_{\text{Tr}_1} \cup \cdots \cup s_{\text{Tr}_m}, g), \tag{45}$$

where g denotes the set of gluons and s_{Tr_i} denotes the set of scalars with the trace Tr_i .

Obviously, the simplest example is that the scalars belong to two traces and each trace contains two scalars. However, these amplitudes correspond to the special case of YMS with $\text{Tr}(T^{I_1} T^{I_2})$ replaced by δ^{I_1, I_2} . In other words, the kinematic part of these amplitudes is included in the situations we have calculated in the previous section. Thus, we choose to compute a non-trivial case, the 7-point color-ordered partial amplitude $\mathcal{A}^{\text{gen.YMS}}(1g, (2s, 3s, 4s)_{\text{Tr}_1}, (5s, 6s, 7s)_{\text{Tr}_2})$, which contains one gluon and six scalars with three scalars carrying Tr_1 and the rest three carrying Tr_2 . The integrand has 21 terms and all of them contain higher-order poles. The decomposition procedure can be done within 2 steps, as shown in the following table:

	Round 1	Round 2
#[ALL]	126	216
#[H]	18	0

Again, this analytic result is confirmed by the numerical verification.

5.2 Extended Dirac–Born–Infeld theory

The second theory under consideration is the extended DBI, which is described by the Lagrangian

$$\mathcal{L}^{\text{ext.DBI}} = \ell^{-2} \left(\sqrt{-\det(\eta_{\mu\nu} - \frac{\ell^2}{4\lambda^2} \text{Tr}(\partial_\mu U^\dagger \partial_\nu U) - \ell^2 W_{\mu\nu} - \ell F_{\mu\nu})} - 1 \right), \tag{46}$$

where the matrix $U(\Phi)$ is defined in (18), and

$$W_{\mu\nu} = \sum_{m=1}^{\infty} \sum_{k=0}^{m-1} \frac{2(m-k)}{2m+1} \lambda^{2m+1} \times \text{Tr}(\partial_{[\mu} \Phi \Phi^{2k} \partial_{\nu]} \Phi \Phi^{2(m-k)-1}). \tag{47}$$

The corresponding CHY-integrand is given by [5]

$$\mathcal{I}^{\text{ext.DBI}}(s_{\text{Tr}_1} \cup \cdots \cup s_{\text{Tr}_m}, \gamma) = \mathcal{C}_{\text{Tr}_1} \cdots \mathcal{C}_{\text{Tr}_m} \times \sum_{\{i,j\}} \mathcal{P}'_{\{i,j\}}(s_{\text{Tr}_1} \cup \cdots \cup s_{\text{Tr}_m}, \gamma) (\text{Pf}'A)^2, \tag{48}$$

where γ denotes the set of photons and s_{Tr_i} denotes the set of scalars with the trace Tr_i .

Our example is the 6-point partial amplitude $\mathcal{A}^{\text{ext.DBI}}(1\gamma, (2s, 3s)_{\text{Tr}_1} (4s, 5s, 6s)_{\text{Tr}_2})$, which involves one photon and five scalars, where two scalars carry Tr_1 and three carry Tr_2 . The integrand has 36 terms and 8 of them contain higher-order poles. The decomposition procedure can be done within 3 steps as shown in the table

	Round 1	Round 2	Round 3
#[ALL]	52	88	142
#[H]	18	18	0

This result has been verified numerically.

5.3 Einstein–Yang–Mills theory

The final theory in this section is the Einstein–Yang–Mills theory, which describes the interaction between gravitons and gauge bosons. The general CHY-integrand involving the mixed traces is [5]

$$\mathcal{I}^{\text{EYM}}(g_{\text{Tr}_1} \cup \dots \cup g_{\text{Tr}_m}, h) = \mathcal{C}_{\text{Tr}_1} \cdots \mathcal{C}_{\text{Tr}_m} \times \sum'_{\{i,j\}} \mathcal{P}_{\{i,j\}}(g_{\text{Tr}_1} \cup \dots \cup g_{\text{Tr}_m}, h) \text{Pf}' \Psi, \quad (49)$$

where the set of gravitons is denoted by h and the set of gluons with the trace Tr_i is denoted by g_{Tr_i} .

We consider the 5-point partial amplitude $\mathcal{A}^{\text{EYM}}((1g, 2g)_{\text{Tr}_1}, (3g, 4g, 5g)_{\text{Tr}_2})$, of which all five external particles are gluons with two of them carrying Tr_1 and three of them carrying Tr_2 . The original integrand has 239 terms and 189 of them contain higher-order poles. The decomposition procedure can be done within 3 steps, as shown in the following table:

	Round 1	Round 2	Round 3
#[ALL]	434	531	557
#[H]	115	26	0

As all other examples, this analytic result is confirmed numerically.

6 Conclusion

In this paper, we have applied the cross-ratio identity method to CHY-integrands of various theories including: the non-linear sigma model, special Galileon theory, pure Yang–Mills theory, pure gravity, Born–Infeld theory, Dirac–Born–Infeld theory and its extension, Yang–Mills–scalar theory, and Einstein–Maxwell and Einstein–Yang–Mills theory. All the integrands under consideration are computed conveniently in this way, the decomposition procedures expend 10 steps at most. All the analytic results are verified numerically, thus this method is confirmed for all examples of this paper. Consequently, the cross-ratio identity method is valid and effective for a wide range of CHY-integrands. An interesting observation is that the condition $\Upsilon[\mathcal{I}'_\ell] \leq \Upsilon[\mathcal{I}]$ can always be satisfied at each step, although its rigorous proof is still absent.

In this paper, the most complicated example takes more than a day in MATHEMATICA. The reason of this low efficiency is, the choices of $\tilde{\Lambda}_i$, j and p are tested by brute force in the algorithm. Appropriate choices of the cross-ratio identities at each step can minimize the number of steps of the decomposition, which is crucial for practical calculations. Thus, how to optimize these choices to improve the efficiency is an important future project.

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