

Interacting diffusive unified dark energy and dark matter from scalar fields

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Abstract Here we generalize ideas of unified dark matter–dark energy in the context of two measure theories and of dynamical space time theories. In two measure theories one uses metric independent volume elements and this allows one to construct unified dark matter–dark energy, where the cosmological constant appears as an integration constant associated with the equation of motion of the measure fields. The dynamical space-time theories generalize the two measure theories by introducing a vector field whose equation of motion guarantees the conservation of a certain Energy Momentum tensor, which may be related, but in general is not the same as the gravitational Energy Momentum tensor. We propose two formulations of this idea: (I) by demanding that this vector field be the gradient of a scalar, (II) by considering the dynamical space field appearing in another part of the action. Then the dynamical space time theory becomes a theory of Diffusive Unified dark energy and dark matter. These generalizations produce non-conserved energy momentum tensors instead of conserved energy momentum tensors which leads at the end to a formulation of interacting DE–DM dust models in the form of a diffusive type interacting Unified dark energy and dark matter scenario. We solved analytically the theories for perturbative solution and asymptotic solution, and we show that the Λ CDM is a fixed point of these theories at large times. Also a preliminary argument as regards the good behavior of the theory at the quantum level is proposed for both theories.

1 Introduction

The best explanation, and fitting with data for the accelerated expansion of our universe, is the Λ CDM model, which tells us that our universe contains 68% of dark energy, and 27% of dark matter. This model present two big questions: The

cosmological constant problem [1–3] – why there is a large disagreement between the vacuum expectation value of the energy momentum tensor which comes from quantum field theory and the observable value of dark energy density – and the coincidence problem [4] – why the observable values of dark energy and dark matter densities in the late universe are of the same order of magnitude.

In order to solve this problem, many approaches emerged [5–8]. One interesting suggestion was a diffusive exchange of energy between dark energy and dark matter made by Calogero [9, 10], Haba et al. [11], and Szydlowski and Stachowski [12], with some solution to cosmic problems. The basic notion is that the diffusion equation (or more exactly, the Fokker–Planck equation [13, 14], which describes the time evolution of the probability density function of the velocity of a particle under the influence of random forces), implies a non-conserved stress energy tensor $T^{\mu\nu}$, which has some current source j^μ :

$$\nabla_\mu T^{\mu\nu} = 3\sigma j^\nu \quad (1)$$

where σ is the diffusion coefficient of the fluid. This generalization is Lorentz invariant and fit for curved space-time. The current j^μ is a time-like covariantly conserved vector field and its conservation tells us that the number of particles in this fluid is constant.

However, in the gravitational equations, the Einstein tensor is proportional to the conserved stress energy tensor $\nabla_\mu T_{(G)}^{\mu\nu} = 0$, which we labeled with “G” [15, 16]. So Calogero came up with what he called ϕ CDM model, which achieves a conserved total energy momentum tensor appearing in the right hand side of Einstein’s equation. But for the dark energy and dust stress tensors there is some source current for those tensors (however, the sum is conserved):

$$-\nabla_\mu T_{(\Lambda)}^{\mu\nu} = \nabla_\mu T_{(\text{Dust})}^{\mu\nu} = J^\nu, \quad \nabla_\mu J^\mu = 0. \quad (2)$$

As Calogero mentioned [9], the diffusion model introduced in his paper lacks an action principle formulation. Therefore

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we develop from a generalization of two measure theories [17–28] a “diffusive energy theory” which can produce on one hand a non-conserved stress energy tensor ($T_{(\chi)}^{\mu\nu}$), as in (1), and on the other hand the conserved stress energy tensor ($T_{(G)}^{\mu\nu}$) that we know from the right hand side of Einstein’s equation. As we will see, this suggested theory is asymptotically different from the ϕ CDM model, and more close in this limit to the standard Λ CDM.

2 Two measure theories and dynamical time theories

There have been theoretical approaches to gravity theories, where a fundamental constraint is implemented; like in two measure theories where one works, in addition to the regular measure of integration in the action $\sqrt{-g}$, with yet another measure, which is also a density and which is also a total derivative. In this case, one can use for constructing this measure four scalar fields φ_a , where $a = 1, 2, 3, 4$. Then we can define the density $\Phi = \varepsilon^{\alpha\beta\gamma\delta} \varepsilon_{abcd} \partial_\alpha \varphi_a \partial_\beta \varphi_b \partial_\gamma \varphi_c \partial_\delta \varphi_d$, and we can write an action that uses both of these densities:

$$S = \int d^4x \Phi \mathcal{L}_1 + \int d^4x \sqrt{-g} \mathcal{L}_2. \tag{3}$$

As a consequence of the variation with respect to the scalar fields φ_a , assuming that \mathcal{L}_1 and \mathcal{L}_2 are independent of the scalar fields φ_a , we obtain

$$A_a^\alpha \partial_\alpha \mathcal{L}_1 = 0 \tag{4}$$

where $A_a^\alpha = \varepsilon^{\alpha\beta\gamma\delta} \varepsilon_{abcd} \partial_\beta \varphi_b \partial_\gamma \varphi_c \partial_\delta \varphi_d$. Since $\det[A_a^\alpha] \sim \Phi^3$ as one easily sees, for $\Phi \neq 0$, (4) implies that $\mathcal{L}_1 = M = \text{Const}$. This result can be expressed as covariant conservation of a stress energy momentum of the form $T_{(\Phi)}^{\mu\nu} = \mathcal{L}_1 g^{\mu\nu}$, and using the second order formalism, where the covariant derivative of $g_{\mu\nu}$ is zero, we obtain $\nabla_\mu T_{(\Phi)}^{\mu\nu} = 0$. This implies $\partial_\alpha \mathcal{L}_1 = 0$. This suggests generalizing the idea of the two measure theory, by imposing the covariant conservation of a more non-trivial kind of the energy momentum tensor, which we denote $T_{(\chi)}^{\mu\nu}$ [29]. Therefore, we consider an action of the form

$$S = \mathcal{S}_{(\chi)} + \mathcal{S}_{(R)} = \int d^4x \sqrt{-g} \chi_{\mu;\nu} T_{(\chi)}^{\mu\nu} + \frac{1}{16\pi G} \int d^4x \sqrt{-g} R \tag{5}$$

where $\chi_{\mu;\nu} = \partial_\nu \chi_\mu - \Gamma_{\mu\nu}^\lambda \chi_\lambda$. If we assume $T_{(\chi)}^{\mu\nu}$ to be independent of χ_μ and having $\Gamma_{\mu\nu}^\lambda$ being defined as the Christoffel connection coefficients, then the variation with respect to χ_μ gives a covariant conservation: $\nabla_\mu T_{(\chi)}^{\mu\nu} = 0$.

Notice the fact that the energy density is the canonically conjugated variable to χ^0 , which is what we expect from

a dynamical time (here represented by the dynamical time χ^0). Some cosmological solutions of (5) have been studied in [30], in the context of spatially flat radiation-like solutions, and considering gauge field equations in curved space time.

For a related approach where a set of dynamical space-time coordinates are introduced, not only in the measure of integration, but also in the lagrangian, see [31].

3 Diffusive energy theory from action principle

Let us consider a four dimensional case, where there is a coupling between the scalar field χ and the stress energy momentum tensor $T_{(\chi)}^{\mu\nu}$:

$$S_{(\chi)} = \int d^4x \sqrt{-g} \chi_{\mu;\nu} T_{(\chi)}^{\mu\nu} \tag{6}$$

where, $\mu; \nu$ are covariant derivatives of the scalar field. When $\Gamma_{\mu\nu}^\lambda$ is being defined as the Christoffel connection coefficients, the variation with respect to χ gives a covariant conservation of a current f^μ :

$$\nabla_\mu T_{(\chi)}^{\mu\nu} = f^\nu; \nabla_\nu f^\nu = 0, \tag{7}$$

which is the source of the stress energy momentum tensor. This corresponds to the “dynamical space-time” theory (5), where the dynamical space-time 4-vector χ_μ is replaced by a gradient of a scalar field χ . In the “dynamical space theory” we obtain four equations of motion, by the variation of χ_μ , which correspond to covariant conservation of the energy momentum tensor, $\nabla_\mu T_{(\chi)}^{\mu\nu} = 0$. By changing the four vector to a gradient of a scalar $\partial_\mu \chi$ at the end, what we do is to change the conservation of the energy momentum tensor to the asymptotic conservation of the energy momentum tensor (7) which corresponds to a conservation of a current $\nabla_\nu f^\nu = 0$. In an expanding universe, the current f_μ gets diluted, so then we recover asymptotically a covariant conservation law for $T_{(\chi)}^{\mu\nu}$ again. Equation (7) has a close correspondence with the one obtained in a “diffusion scenario” for DE–DM exchange [9, 10].

This stress energy tensor is substantially different from the stress energy tensor we all know, which is defined as $\frac{8\pi G}{c^4} T_{(G)}^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$. In this case, the stress energy momentum tensor $T_{(\chi)}^{\mu\nu}$ is not conserved (but there is some conserved current f^ν , which is the source to this stress energy momentum tensor non-conservation), here there is some conserved stress energy tensor $T_{(G)}^{\mu\nu}$, which comes from variation of the action according to the metric:

$$T_{(G)}^{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_M)}{\delta g^{\mu\nu}}; \nabla_\mu T_{(G)}^{\mu\nu} = 0. \tag{8}$$

The lagrangian \mathcal{L}_M could be the modified term $\chi_{,\mu;\nu}T_{(\chi)}^{\mu\nu}$, but as we will see, it could be added to more action terms. Using different expressions for $T_{(\chi)}^{\mu\nu}$, which depend on other variables, will give the connection between the scalar field χ and those other variables.

Notice that for the theory the shift symmetry holds, and

$$\chi \rightarrow \chi + C_\chi; T_{(\chi)}^{\mu\nu} \rightarrow T_{(\chi)}^{\mu\nu} + g^{\mu\nu}C_T \tag{9}$$

will not change any equation of motion. When C_χ, C_T are some arbitrary constants. This means that if the matter is coupled through its energy momentum tensor as in (9), the process of redefinition of the energy momentum tensor will not affect the equations of motion. Of course, a redefinition of such a type of energy momentum tensor is exactly what is done in the process of normal ordering in quantum field theory, for example.

4 Diffusive energy theory without high derivatives

Another model that does not involve high derivatives is obtained, by keeping χ_μ as a 4-vector, which is not gradient, but we introduce the vector field χ_μ in another part of the action:

$$S_{(\chi,A)} = \int d^4x \sqrt{-g} \chi_{\mu;\nu} T_{(\chi)}^{\mu\nu} d^4x + \frac{\sigma}{2} \int d^4x \sqrt{-g} (\chi_\mu + \partial_\mu A)^2 d^4x \tag{10}$$

where A is another scalar field. Then from variation with respect to χ_μ we obtain

$$\nabla_\mu T_{(\chi)}^{\mu\nu} = \sigma(\chi^\mu + \partial^\mu A) \tag{11}$$

as (7), where the source is

$$f^\mu = \sigma(\chi^\mu + \partial^\mu A). \tag{12}$$

But in contrast to (7), where f_μ appears as an integration function, here f^μ appears as a function of the dynamical fields. From the variation with respect to A , we indeed see that the current $\chi^\mu + \partial^\mu A$ is conserved, which means again with (7) that $\nabla_\mu \nabla_\nu T_{(\chi)}^{\mu\nu} = 0$, but it does not tell us that all of the equations of motion are the same. Nevertheless, asymptotically, for the late universe, the two theories coincide.

To start with, we discuss a toy model in one dimension describing a system that allows the non-conservation of a certain energy function, which increases or decreases linearly with time, while there is another energy which is conserved. It is of interest to compare with a mechanism that produces non-conserved energy momentum tensors which leads to a formulation of interacting DE-DM models; however, there are crucial differences.

5 A mechanical system with a constant power and diffusive properties

In order to see the applications of the ideas, we start with a simple action of one dimensional particle in a potential $V(x)$. We introduce a coupling between the total energy of the particle $\frac{1}{2}m\dot{x}^2 + V(x)$ and the second derivative of some dynamical variable B :

$$S = \int \ddot{B} \left[\frac{1}{2}m\dot{x}^2 + V(x) \right] dt. \tag{13}$$

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$$S = \int \ddot{B} \left[\frac{1}{2}m\dot{x}^2 + V(x) \right] dt. \tag{14}$$

The equation of motion according to the dynamical variable B shows that the second derivative of the total energy is zero. In other words, the total energy of the particle is linear in time:

$$\frac{1}{2}m\dot{x}^2 + V(x) = E(t) = Pt + E_0 \tag{15}$$

where P is a constant power which is given to the particle or taken from it, and E_0 is the total energy of the particle at time equal to zero.

From the equation of motion according to coordinate x we get a close connection between the dynamical variable B and the coordinate of the particle:

$$m\ddot{x} \frac{d^2B}{dt^2} + m\dot{x} \frac{d^3B}{dt^3} = V'(x) \frac{d^2B}{dt^2}, \tag{16}$$

which with Eq. (15) gives

$$\frac{\ddot{B}}{\dot{B}} = \frac{2V'(x)}{\sqrt{2m(E(t) - V(x))}} - \frac{P}{2(E(t) - V(x))}. \tag{17}$$

To get a feeling of these kinds of theories, let us look at the case of a harmonic oscillator $V(x) = \frac{1}{2}kx^2$. First of all, we see from Eq. (5) and the condition that the right hand side be positive; since the left hand side obviously is positive, we see that there is a boundary time $\tau = -\frac{E_0}{P}$, for which for $P > 0$ we get $t > \tau$, and for $P < 0$ there is a maximal time $t < \tau$. Let us consider the case that the power P is positive. The equations of motion in that case will not oscillate, but they will grow exponentially until the “Pt” term present in Eq. (15) dominates, when $\langle x^2 \rangle \propto t$. This is very similar to Brownian motion.

The momenta for this toy model are

$$\pi_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}\ddot{B}, \tag{18}$$

$$\pi_B = \frac{\partial \mathcal{L}}{\partial \dot{B}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \ddot{B}} = -\frac{d}{dt} E(t), \tag{19}$$

$$\Pi_B = \frac{\partial \mathcal{L}}{\partial \ddot{B}} = E(t). \tag{20}$$

Using the Hamiltonian formalism (with second order derivative [32,33]) we see that the hamiltonian of the system is

$$\begin{aligned} \mathcal{H} &= \dot{x}\pi_x + \dot{B}\pi_B + \ddot{B}\Pi_B - \mathcal{L} \\ &= \pi_x \sqrt{\frac{2}{m}(\Pi_B - V(x))} + \dot{B}\pi_B = \text{Const.} \end{aligned} \tag{21}$$

Since the action is not explicitly dependent on time, the hamiltonian is conserved. So even if the total energy of the particle is not conserved, we have the conserved hamiltonian (21). This notion is equivalent to a non-conserved stress energy tensor $T_{(\chi)}^{\mu\nu}$, in addition to the conserved stress energy $T_{(G)}^{\mu\nu}$, which appears in the Einstein equation.

Notice that this hamiltonian is not necessarily bounded from below. However, there are only mild instabilities in the solutions. For example, for the case $V(x) = 0$, we get $\dot{x} \propto \dot{B} \propto t^{\frac{1}{2}}$. In the case of a harmonic oscillator, where $V(x) = \frac{1}{2}kx^2$, there is an even milder behavior at large times: $x \propto t^{\frac{1}{2}}$, which resembles a diffusive behavior, or Brownian motion. This behavior is a mild kind of instability, since no exponential growth appears, only power law growth. The related model in cosmology, as we will see, because of the coupling to an expanding space-time shows damped perturbations, shows a trend towards a fixed point solution, where it coincides with the standard Λ CDM model. This is because whatever potential instabilities the model may have in a flat background, the expanding space (most notably the de Sitter space) has the counter property of red shifting any perturbation; this effect overcomes and cancels these rather soft instabilities (power law instabilities that may exist for the solution in flat space) as we will see in Sect. 7. The exponential expansion is known to counter all kind of unstable behaviors, for example, it goes against the gravitational instability and a big enough cosmological constant can prevent galaxy formation; our case is much simpler than that, but the basic reason is the same. In this context it is important to notice that in an expanding universe a non-covariant conservation of an energy momentum tensor, which may imply that some energy density is increasing in the locally inertial frame, does not mean a corresponding increase of the energy density in the comoving cosmological frame. For example, a non-covariant conservation of the dust component of the universe, in the examples we study, will produce a still decreasing

dust density, although there is a positive flow of energy in the inertial frame. The result of this flow of energy in the local inertial frame is going to be just that the dust energy density decreases a bit slower than the conventional dust in the comoving frame.

Independently of this, we will see how it is possible to construct theories with positive Euclidean action that describe diffusive DE–DM unification.

6 Gravity, “k-essence” and diffusive behavior

Our starting point is the following non-conventional gravity-scalar-field action, which will produce a diffusive type of interacting DE–DM theory:

$$\begin{aligned} \mathcal{S} &= \frac{1}{16\pi G} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \mathcal{L}(\phi, X) \\ &\quad + \int d^4x \sqrt{-g} \chi_{,\mu;\nu} T_{(\chi)}^{\mu\nu}, \end{aligned} \tag{22}$$

with the following explanations for the different terms: R is the Ricci scalar which appears in the Einstein–Hilbert action. $\mathcal{L}(\phi, X)$ is the general-coordinate invariant Lagrangian of a single scalar field ϕ , which can be of an arbitrary generic “k-essence” type: some function of a scalar field ϕ and the combination $X = \partial_\mu \phi \partial^\mu \phi$ [34–36]):

$$\mathcal{L}(\phi, X) = \sum_{N=1}^{\infty} A_N(\phi) X^N - V(\phi). \tag{23}$$

As we will see, this last action will produce a diffusive interaction between DE–DM type theory. For the ansatz of $T_{(\chi)}^{\mu\nu}$ we choose to use some tensor which is proportional to the metric, with a proportionality function $\Lambda(\phi, X)$:

$$T_{(\chi)}^{\mu\nu} = g^{\mu\nu} \Lambda(\phi, X) \Rightarrow \mathcal{S}_{(\chi)} = \int d^4x \Lambda \square \chi. \tag{24}$$

From the variation of the scalar field χ we get $\square \Lambda = 0$, whose solution will be interpreted as a dynamically generated cosmological constant with diffusive source.

We take the simple example for this generalized theory, and for the functions \mathcal{L}, Λ we take the first order of the Taylor expansion from (23), or $\mathcal{L} = \Lambda = X$ ($A_1 = 1, A_2 = A_3 = \dots = 0$). From the variation according to the scalar field we get a conserved current $j_{;\mu}^\mu = 0$:

$$j_\alpha = 2(\square \chi + 1)\phi_{,\alpha}. \tag{25}$$

For a cosmological solution we take into account only the change as a function of time $\phi = \phi(t)$. From that we see that the ‘0’ component of the current j_α is non-zero. The last variation we should take is according to the metric (using

the identities in Appendix A), which gives a conserved stress energy tensor:

$$T_{(G)}^{\mu\nu} = g^{\mu\nu}(-\Lambda + \chi^{\cdot\sigma}\Lambda_{,\sigma}) + j^\mu\phi^{\cdot\nu} - \chi^{\cdot\mu}\Lambda^{\cdot\nu} - \chi^{\cdot\nu}\Lambda^{\cdot\mu}. \tag{26}$$

For cosmological solutions the interpretation of the dark energy is by a term proportional to the metric $-\Lambda + \chi^{\cdot\sigma}\Lambda_{,\sigma}$, and dark matter dust by the ‘00’ component of the tensor $j^\mu\phi^{\cdot\nu} - \chi^{\cdot\mu}\Lambda^{\cdot\nu} - \chi^{\cdot\nu}\Lambda^{\cdot\mu}$. Let us see the solution for the Friedman–Robertson–Walker metric:

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right]. \tag{27}$$

The basic combination becomes $\mathcal{L} = \Lambda = X = \partial_\mu\phi\partial^\mu\phi = -\dot{\phi}^2$. Notice that there are high derivative equations, but all such types of equations correspond to conservation laws. For example, we see that the variation of the scalar field (24) will give $\frac{d}{dt}(2\dot{\phi}\ddot{\phi}a^3) = 0$, which can be integrated to

$$2\dot{\phi}\ddot{\phi} = \frac{C_2}{a^3}, \tag{28}$$

which can be integrated again to give

$$\dot{\phi}^2 = C_1 + C_2 \int \frac{dt}{a^3}. \tag{29}$$

The conserved current from Eq. (25) gives the relation

$$2\dot{\phi}(\square\chi + 1) = \frac{C_3}{a^3}, \tag{30}$$

which can also be integrated to give

$$\dot{\chi} = \frac{1}{a^3} \int a^3 dt + \frac{C_4}{a^3} - \frac{C_3}{2a^3} \int \frac{dt}{\dot{\phi}}, \tag{31}$$

which provides the solution for the scalar field χ . From (26) we get the terms for the DE–DM densities:

$$\rho_{de} = \dot{\phi}^2 + 2\dot{\chi}\dot{\phi}\ddot{\phi}, \tag{32}$$

$$\rho_{dm} = \frac{C_3}{a^3}\dot{\phi} - 4\dot{\chi}\dot{\phi}\ddot{\phi}, \tag{33}$$

and the pressure of DE: $p_{de} = -\rho_{de}$ and DM: $p_{dm} = 0$. This leads to the Friedman equations with (32) and (33) as source, and there are a few approximations that we want to discuss. The first one is the asymptotic solution.

7 Asymptotic solution and stability of the theory

We can solve asymptotically and by the way show the basic stability of the theory (which should eliminate any concerns related to the formal unboundedness of the action). First we solve for $\dot{\chi}$; see Eq. (31). We see that the leading term is the fraction $\frac{1}{a^3} \int a^3 dt$. For an asymptotically de Sitter space, where $a(t) \approx a_0 \exp(H_0 t)$, we see that there is a unique asymptotic value:

$$\lim_{t \rightarrow \infty} \dot{\chi} = \frac{1}{3H_0}. \tag{34}$$

This is in accordance with our expectations that the expansion of the universe will stabilize the solutions, indeed (30) is basically equivalent to the equation of a particle rolling down a linear potential plus additional negligible terms as $a(t)$ goes to infinity; the fixed point solution is of course that of constant velocity, when friction \times velocity = force = 1; since friction = 3H, we obtain Eq. (34).

With this information we can check what the asymptotic value of DE is, from (28), (29) and (32). We see that in this limit, the non-constant part of $\dot{\phi}^2$ is canceled by $2\dot{\chi}\dot{\phi}\ddot{\phi}$, and then asymptotically

$$\rho_{de} = C_1 + O\left(\frac{1}{a^6}\right); \tag{35}$$

with the same analysis for DM density we obtain

$$\rho_{dm} = \left(C_3\sqrt{C_1} - \frac{2C_2}{3H_0}\right) \frac{1}{a^3} + O\left(\frac{1}{a^6}\right). \tag{36}$$

The Friedman equation provides a relation between C_1 and H_0 (the asymptotic value of Hubble constant) which is $H_0^2 = \frac{8\pi G}{3}C_1$. For negative C_2 we have decaying dark energy, the last term of the contribution for dark energy density is positive (and the opposite). This behavior, where $C_2 < 0$, has a chance of explaining the coincidence problem, because unlike the standard Λ CDM model, where the dark energy is exactly constant, and the dark matter decreases like a^{-3} , in our case, dark energy can slowly decrease, instead of being constant, and dark matter also decreases, but not as fast as a^{-3} .

As suggested, this behavior can be understood by the observation that in an expanding universe a non-covariant conservation of an energy momentum tensor, which may imply that some energy density is increasing in the locally inertial frame, does not mean a corresponding increase of the energy density in the comoving cosmological frame, here in particular the non-covariant conservation of the dust component of the universe will produce a still decreasing dust density, although, for $C_2 < 0$, there will be a positive flow of energy in the inertial frame to the dust component, but the

result of this flow of energy in the local inertial frame will be just that the dust energy density will decrease a bit more slowly than conventional dust (but still it decreases).

This is yet another example where potential instabilities are softened or in this case eliminated by the expansion of the universe. It is well known in the case of the Jeans gravitational instability, which is much softer in the expanding universe and also in other situations [37].

Another application of this mechanism could be to use it to explain the particle production, “taking vacuum energy and converting it into particles” as expected from the inflation reheating epoch. Maybe this can be combined with a mechanism that creates standard model particles.

As we see, the expansion of the universe stabilizes the solutions, such that for large times all of them become indistinguishable from Λ CDM, which appears as an attractor fixed point of our theory, showing a basic stability of the solutions at large times. Choosing C_1 as positive is necessary, because of the demand that the terms with $\sqrt{C_1}$ will not be imaginary. But for the other constants of integration, there is only the condition $C_3\sqrt{C_1} > \frac{2C_2}{3H_0}$, which gives a positive dust density at large times.

8 $C_2 = 0$ solution

Another special case is when $C_2 = 0$. That means that the dark energy of this universe is constant: we have $\dot{\phi}^2 = C_1$ and $\ddot{\phi} = 0$. The equation of motions for the dark energy and dust (32) and (33) are independent on the scalar field χ , and therefore the density of dust in that universe behaves as $\frac{C_3\sqrt{C_1}}{a^3}$. This solution says there is no interaction between dark energy and dark matter. This is precisely the solution of the two measure theory [38–40], with the action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R + \int d^4x (\Phi + \sqrt{-g}) \mathcal{L}(X, \phi), \tag{37}$$

which provides a unified picture of DE–DM. For more about two measure theory and related models and solutions for DE–DM, see the discussion in Appendix C. The FRWM for both theories gives the solution

$$\rho_{DE} = \dot{\phi}^2 = C_1, \tag{38}$$

$$\rho_{Dust} = \frac{\sqrt{C_1}C_3}{a^3}. \tag{39}$$

For this trivial case, $C_2 = 0$, there is no diffusion effect between dark matter and dark energy. The current f^μ , which is the source of the stress energy tensor $T_{(\chi)}^{\mu\nu}$ (see (7)) is zero, and both stress energy tensors are conserved. This is equivalent to Λ CDM. The exact solution of the case of constant

dark energy and dust, using (38) and (39) is [41]

$$a_0(t) = \left(\frac{C_3}{\sqrt{C_1}} \right)^{\frac{1}{3}} \sinh^{2/3}(\alpha t) \tag{40}$$

where $\alpha = \frac{3}{2}\sqrt{C_1}$. From comparing to the Λ CDM solution, we can see how the observables values are related to the constant of integration that come from the solution of the theory:

$$\Omega_\Lambda = \frac{C_1}{H}; \quad \Omega_m = \frac{C_1\sqrt{C_3}}{H}, \tag{41}$$

where H is the Hubble constant for the late universe. For exploring the non-trivial diffusive effect for $C_2 \neq 0$, we use perturbation theory.

9 Perturbative solution

The conclusion from this correspondence is that the diffusion between dark energy and dark matter dust at the late universe is very small, since that is the effect of the C_2 term, and therefore we can estimate the solution by perturbation theory. So we obtain two dimensionless terms, which are dependent on time and scale factor, and they tell us the “diffusion rate”:

$$\lambda_1(t, t_0) = \frac{C_2}{C_1} \int_{t_0}^t \frac{dt}{a^3}, \tag{42}$$

$$\lambda_2(t, t_0) = \frac{C_2}{\sqrt{C_1}C_3} \dot{\chi}(t, t_0), \tag{43}$$

where the integration is between two close times t_0 and t . For $C_2 = 0$, both λ_1 and λ_2 are equal to zero, and there is no dissipative effect, which as we saw, gives us the Λ CDM model. For any non-zero $\lambda_{1,2} \ll 1$, the stress energy tensor $T_{(\chi)}^{\mu\nu}$ is not conserved, and there is a little diffusion effect.

The use of defining these two dimensionless terms is evident when C_2 is small enough for using perturbation theory. By using λ_1 we can write the scalar field term as $\dot{\phi}^2 = C_1(1 + \lambda)$. The definition for λ_2 is from the assumption that the leading term in (33), whose scale $\sqrt{C_1}C_3$, is much bigger than the other term $\dot{\chi}\dot{\phi}\dot{\phi}$ (with the $\dot{\chi}C_2$ component, using (28)). The total contributions for the densities, in the context of perturbation theory at the first order, are

$$\rho_{de} = C_1 \left(1 + \lambda_1 + \frac{C_3}{\sqrt{C_1}}\lambda_2 \right) + O_2(\lambda_1, \lambda_2), \tag{44}$$

$$\rho_{dm} = \frac{\sqrt{C_1}C_3}{a^3} \left(1 - \frac{1}{2}(\lambda_1 + \lambda_2) \right) + O_2(\lambda_1, \lambda_2). \tag{45}$$

We can see from those terms that in the deviation from the unperturbed standard solution, the behavior of dark energy and dust are opposite – for increasing dark energy (for example the components are $C_2 < 0$; $C_1, C_3, C_4 > 0$), the dark matter amount ($a^3 \rho_{dm}$) gets lower. Or in the case of decreasing dark energy, the amounts of dark matter increases (and $C_1, C_2, C_3, C_4 > 0$).

10 Equation of motion and solutions for diffusive energy without higher derivatives

For the second class of theories we proposed in (10)–(12), we can write the diffusive energy action, without high derivatives:

$$S = \frac{1}{16\pi G} \int \sqrt{-g} R + \int \sqrt{-g} \Lambda + \int \sqrt{-g} \chi_{\mu;\nu} T_{(\chi)}^{\mu\nu} + \frac{\sigma}{2} \int \sqrt{-g} (\chi_\mu + \partial_\mu A)^2 \tag{46}$$

and, as before, the stress energy tensor $T_{(\chi)}^{\mu\nu} = g^{\mu\nu} \Lambda$. From the variation with respect to the vector field χ_μ :

$$\nabla_\mu \Lambda = f_\mu = \sigma (\chi_\mu + \partial_\mu A). \tag{47}$$

The variations with respect to the scalars A and ϕ :

$$f_{;\mu}^\mu = 0, \tag{48}$$

$$j_\alpha = 2(\chi_{;\lambda}^\lambda + 1)\phi_{,\alpha}; J_{;\alpha}^\alpha = 0, \tag{49}$$

as (7) and (25). Finally for the stress energy tensor, which comes from variation with respect to the metric we obtain

$$T_{(G)}^{\mu\nu} = g^{\mu\nu} \left(-\Lambda + \chi^{\cdot\lambda} \Lambda_{,\lambda} - \frac{1}{2\sigma} \Lambda^{\cdot\lambda} \Lambda_{,\lambda} \right) + j^\mu \phi^{,\nu} - \chi^{\cdot\mu} \Lambda^{,\nu} - \chi^{\cdot\nu} \Lambda^{,\mu} + \frac{1}{\sigma} \Lambda^{\cdot\mu} \Lambda^{,\mu}. \tag{50}$$

Both theories, (22) and (46), give rise to similar final equations of motion, besides the variation according to the metric, which asymptotically for large times behave in the same way. The new terms $-\frac{1}{2\sigma} \Lambda^{\cdot\mu} \Lambda_{,\mu}$ and $\frac{1}{\sigma} \Lambda^{\cdot\mu} \Lambda^{,\mu}$ are negligible at the late universe, since they go as $\frac{1}{a^6}$. For the early universe those terms may be very important, which we will study in future publications.

The modified model of diffusion gives rise to the simpler model when σ goes to infinity. Since, in this case, the extra $\frac{\sigma}{2} \int \sqrt{-g} (\chi_\mu + \partial_\mu A)^2$ term forces $\chi_\mu = -\partial_\mu A$ (because $(\chi_\mu + \partial_\mu A)^2 = 0$ and we are also disregarding light like solutions for $(\chi_\mu + \partial_\mu A)$, which do not appear relevant to cosmology), i.e. the χ_μ is a gradient of a scalar. Therefore the theory of dynamical time (5) with a source (46) becomes a diffusive action with high derivatives (6) and (22).

11 Some preliminary ideas on quantization and the boundedness of the Euclidean action

Let us take the action (46); by integration by parts of $\chi_{\mu;\nu} T_{(\chi)}^{\mu\nu}$, and throwing away total derivatives, we obtain the action

$$S = \frac{1}{16\pi G} \int \sqrt{-g} R + \int \sqrt{-g} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \int \sqrt{-g} \chi_\mu \nabla_\nu T_{(\chi)}^{\mu\nu} + \frac{\sigma}{2} \int \sqrt{-g} (\chi_\mu + \partial_\mu A)^2. \tag{51}$$

We notice that there are no derivatives acting on the χ_μ field at this action, and therefore χ_μ is a Lagrange multiplier. It is legitimate to solve χ_μ from its equation of motions, and insert the result back into the action. The equation of motion according to the χ_μ variation is

$$0 = -\nabla_\nu T_{(\chi)}^{\mu\nu} + \sigma (\chi_\mu + \partial_\mu A). \tag{52}$$

Solving for χ_μ and inserting back into the action gives

$$S = \frac{1}{16\pi G} \int \sqrt{-g} R + \int \sqrt{-g} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2\sigma} \int \sqrt{-g} (\nabla_\nu T_{(\chi)}^{\mu\nu})^2 + \int \sqrt{-g} \partial_\nu A \nabla_\alpha T_{(\chi)}^{\nu\alpha}. \tag{53}$$

Considering the functional integral quantization for this theory will give a few integrations over the field variables. The functional integral over the scalar A gives rise to a delta function that enforces the covariant conservation of the current $\nabla_\mu T_{(\chi)}^{\mu\nu} = f^\nu$. The Euclidean functional integral will be

$$\mathcal{Z} = \int \mathcal{D}\phi \delta(\nabla_\nu f^\nu) \exp \left[\frac{1}{2\sigma} \int d^4x \sqrt{g} f_\mu f^\mu - \int d^4x \sqrt{g} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \right]. \tag{54}$$

This partition function excludes the Hilbert–Einstein action term, which has its own problems, which are not particularly of interest to this paper. In the full theory we need to include the integration over all the Euclidean geometries.

We can see that the integration measure is positive definite, and the argument of the integrals in the exponents are negative definite in a Euclidean signature space-time $\text{sign}[+, +, +, +]$, following the Hawking approach [42]. The terms $f_\mu f^\mu$ and $\phi_{,\mu} \phi^{,\mu}$ are positive definite, and by choosing the proper sign of σ , the action is positive definite, and the partition function is convergent. The original theory we formulated in (22) is equivalent to (46) when σ goes to minus infinity.

Therefore, this proof is valid for both theories. However, the simple model (22) has to be regularized by first taking

finite and negative σ , and then letting the σ go to minus infinity. This is a preliminary approach, because in the quantum theory, there are many issues concerning how one goes from the Hamiltonian formulation to the path integral formulation, etc. But we see that the quantum theory has a chance to be well defined.

12 Diffusive dark energy and dust by Calogero

The solution for the Calogero suggestion we presented at the beginning, see (1) and (2), leads to the following interdependence between the densities of dark matter and dark energy and the scale parameter:

$$\rho_{de} = C_1 + C_2 \int \frac{dt}{a^3}, \quad (55)$$

$$\rho_{dm} = \frac{C_3}{a^3} - \frac{C_2 t}{a^3}. \quad (56)$$

A complete set of solutions of these differential equations (in the form of Friedman equations) is very complicated, but one phenomenological solution for this theory predicts a DE–DM ratio similar to the observed one [11]. Both approaches (which are described in this paper and in Calogero's theory) become very similar when the time derivative of the scalar field is low $\dot{\chi} C_2 \ll 1$. In that case, the dark energy density (32) becomes

$$\rho_{de} = C_1 + C_2 \int \frac{dt}{a^3}. \quad (57)$$

The dark matter dust will reduce to the term (33):

$$\rho_{dm} = \frac{C_3}{a^3} \dot{\phi} \quad (58)$$

and for those equations it implies diffusion between dark energy and dark matter dust, like Calogero has found. In this model they assumed that the dark energy and the dust are not separately conserved.

We can see that our asymptotic solution does not fit with Calogero's model, for general C_2 . As opposed to Eq. (2), in our asymptotic solution (35)–(36) the dark energy density becomes constant, providing a behavior much closer to the standard Λ CDM model. The main reason for this nonequivalence of those theories is the role of the χ field, which has the effect of making the exchange between dark matter and dark energy less symmetric than in the ϕ CMD model. In our case, χ makes the decay of DE much lower than in ϕ CMD, and it keeps the DM evolution still decreasing as Λ CDM (a^{-3}).

13 Discussion, conclusions and prospects

In this paper we have generalized the TMT and the dynamical space time theory, which imposes the covariant conservation of an energy momentum tensor. We demand that the dynamical space-time 4-vector χ_μ , which appears in the dynamical space-time theory, be a gradient $\partial_\mu \chi$. We do not obtain the covariant conservation of the energy momentum tensor that is introduced in the action. Instead we obtain current conservation. The current is the divergence of this energy momentum tensor. This current, which drives the non-conservation of the energy momentum tensor, is dissipated in the case of an expanding universe. So we get asymptotic conservation of this energy momentum tensor. Because the four-divergence of the covariant divergence of both the dark matter and dark energy is zero, we can make contact with the dissipative models of [9, 10]. This might give a deeper motivation for these models and allow for the construction of new models.

This energy tensor is not the gravitational energy tensor which appears in the right hand side of the Einstein tensor, in the gravity equations, but the non-covariant conservation of the energy momentum tensor that appears in the action induces an energy momentum transfer between the dark energy and dark matter components, of the gravitational energy momentum tensor, in a way that resembles the ideas in [11]. But one did not provide any action principle to support their ideas. Although the mechanism is similar, our formulation and theirs are not equivalent.

From the asymptotic solution we see that when $C_2 < 0$, unlike the standard Λ CDM model, where the dark energy is exactly constant, and the dark matter decreases like a^{-3} , in our case, dark energy can slowly decrease, instead of being constant, and dark matter also decreases, but not as fast as a^{-3} . This special property is different in the ϕ CMD model, where the exchange between DE and DM is much stronger in the asymptotic limit.

This behavior, where $C_2 < 0$, has a chance of explaining the coincidence problem, because unlike the standard Λ CMD model, where the dark energy is exactly constant, and the dark matter decreases like a^{-3} , in our case, dark energy can slowly decrease, instead of being constant, and dark matter also decreases, but not as fast as a^{-3} . This behavior can be understood by the observation that in an expanding universe a non-covariant conservation of an energy momentum tensor, which may imply that some energy density is increasing in the locally inertial frame, does not mean a corresponding increase of the energy density in the comoving cosmological frame. Here in particular the non-covariant conservation of the dust component of the universe will produce a still decreasing dust density, although, for $C_2 < 0$, there will be a positive flow of energy in the inertial frame to the dust component, but the result of this flow of energy in the local inertial frame will be just that the dust energy density will

decrease a bit more slowly than the conventional dust (but it still decreases).

We have seen that in perturbation theory, the behavior of dark energy and dust are different – for increasing dark energy (for example the components are $C_2 < 0; C_1, C_3, C_4 > 0$), the dark matter amount ($a^3 \rho_{dm}$) grows lower. Or in the case of decreasing dark energy, the amounts of dark matter increase (and all the constants of integration are positive).

For another suggestion for diffusive energy action, which does not produce high derivative equations, we have kept the χ_μ field as a 4-vector (not the gradient of a scalar), but now χ_μ appears in another term at the action, in addition to a scalar field A . The equations of motion produce again a diffusive energy equation, but with the additional contribution of two terms that are negligible for the late universe.

A preliminary argument about the good behavior of the theory at the quantum level is also proposed for both theories. Some additional investigations concerning the quantum theory could be developed by using the WDW equation, in the mini-super space approximation.

Also in the future we will study not only the asymptotic behavior, but the full numerical solution of the dark energy and dark matter components, starting from the early universe, for all the theories we suggested.

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14 Appendix A: Identities

We have

$$\frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}} = -\frac{1}{2}(g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu}),$$

$$\frac{\partial \Gamma_{\lambda\sigma}^\tau}{\partial g_{\mu\nu}} = -\frac{1}{2}(g^{\mu\tau} \Gamma_{\lambda\sigma}^\nu + g^{\nu\tau} \Gamma_{\lambda\sigma}^\mu),$$

$$\begin{aligned} \frac{\partial \Gamma_{\lambda\alpha}^\tau}{\partial g_{\mu\nu,\sigma}} &= \frac{1}{4}[g^{\mu\tau}(\delta_\alpha^\nu \delta_\lambda^\sigma + \delta_\lambda^\nu \delta_\alpha^\sigma) + g^{\tau\nu}(\delta_\alpha^\mu \delta_\lambda^\sigma + \delta_\lambda^\mu \delta_\alpha^\sigma) \\ &\quad - g^{\tau\sigma}(\delta_\alpha^\mu \delta_\lambda^\nu + \delta_\lambda^\mu \delta_\alpha^\nu)], \end{aligned}$$

$$T_{(G)}^{\alpha\beta} = \frac{-2}{\sqrt{-g}} \frac{\partial (\sqrt{-g} \chi_{\mu;\nu} T_{(\chi)}^{\mu\nu})}{\partial g_{\alpha\beta}}$$

$$+ \frac{2}{\sqrt{-g}} \frac{\partial}{\partial x^\sigma} \frac{\partial (\sqrt{-g} \chi_{\mu;\nu} T_{(\chi)}^{\mu\nu})}{\partial g_{\alpha\beta,\sigma}}.$$

15 Appendix B

An equivalent expression for (7), when $T_{(\chi)}^{\mu\nu}$ is formulated as a perfect fluid in FRWM space, is

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = \frac{C_2}{a^3};$$

when $C_2 = 0$, the stress energy tensor is conserved, and there is no diffusive effect. For late times, where the scale parameter goes to infinity, we see that the diffusive effect vanishes.

16 Appendix C

TMTs also have many points of similarity with ‘Lagrange multiplier gravity (LMG)’ [43,44]. The Lagrange multiplier field in LMG enforces the condition that a certain function be zero. In the TMT this is equivalent to the constraint that requires some lagrangian to be constant. The two measure models presented here are different from the LMG models of [43,44], and they provide us with an arbitrary constant of integration for the value of a given lagrangian, this constant of integration, if non-zero, can generate spontaneous symmetry breaking of scale invariance, which is present in the theory for example. Recently a lot of interest has been attracted by the so-called ‘mimetic’ dark matter model proposed in [45,46]. The latter employs a special covariant isolation of the conformal degree of freedom in Einstein gravity, whose dynamics mimics cold dark matter as a pressure-less ‘dust’. Important questions concerning the stability of ‘mimetic’ gravity are studied in Refs. [47,48] where also one formulates a generalized tensor–vector–scalar ‘mimetic’ gravity, which avoids those problems. In [49] the idea is applied to inflationary scenarios.

Most versions of the mimetic gravity, except for [47] appear to be equivalent to a special kind of Lagrange multiplier theory or TMT models that were known before, where we have the simple constraint that the kinetic term of a scalar field be constant. This of course gives identical results to the very special TMT where the lagrangian that couples to the new measure is the kinetic term of this scalar field.

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