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Behaviour of charged collapsing fluids after hydrostatic equilibrium in R^n gravity

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Abstract The purpose of this paper is to study the transport equation and its coupling with the Maxwell equation in the framework of R^n gravity. Using Müller–Israel–Stewart theory for the conduction of dissipative fluids, we analyze the temperature, heat flux, viscosity and thermal conductivity in the scenario of relaxation time. All these thermodynamical variables appear in the form of a single factor whose influence is discussed on the evolution of relativistic model for the heat conducting collapsing star.

1 Introduction

The evolution of gravitational collapse and self-gravitating systems has been widely discussed in General Relativity (GR). This type of study usually is based upon perturbing the system by changing its equilibrium state. The tendency of the evolution of the object is studied as soon as it departs from the equilibrium state. This method of perturbation consists of only linear terms by ignoring the quadratic and higher order terms. In such cases, the evolution processes of the self-gravitating systems may take place on the hydrostatic time scale and a quasi-static approximation could fail. Then it is necessary to study the evolution of the system immediately after its departure from the equilibrium state on a time scale of the order of relaxation times. The relaxation process may change the final outcome of the gravitational collapse drastically. There are particular cases of the collapsing spheres in the literature, where relaxation time may cause the bounce or collapse of the evolving system [1].

The applications of the electromagnetic field in astronomy and astrophysics is an active research domain. A lot of work has been devoted to a discussion of the collective effects of electromagnetic and gravitational fields. For example, the relativistic jets are a natural outcome of some of the most violent and spectacular astrophysical phenomena, such as the core collapse of massive stars in gamma-ray bursts (GRBs) and the accretion onto supermassive black holes in active galactic nuclei (AGN) [2]. It is generally accepted that these jets are powered electromagnetically by the magnetized rotation of a central compact object, i.e., a black hole or neutron star. The main source of power of AGN and GRB jets is the rotational energy of the central black hole [3,4] and its accretion disk. The naturally occurring low mass density and hence high magnetization of black-hole magnetospheres suggests that the relativistic jets originate directly from the black-hole ergosphere. As the plasma is attracted towards the compact object, it is accelerated to relativistic speeds and the in-falling material typically forms an accretion disc around the compact object [5]. Plasma thermalization processes within the accretion disc are thought to accelerate charged particles and launch jets through shocks [6].

The phenomenology of gravitational collapse is of great interest in modified gravity theories. In particular, $f(R)$ gravity is more popular due to its straightforward generalization of GR and its cosmological applications to accommodating the early [7] or late time [8] acceleration of the universe. It is supposed that $f(R)$ gravity can produce some kind of repulsive force similar to that of dark energy in Einstein gravity. Thus, the question arises whether such a kind of repulsive effect in $f(R)$ gravity can hinder the gravitational collapse and the formation of black holes. Different aspects of gravitational collapse and black-hole formation for the spherically symmetric solution in $f(R)$ theory have been explored [9]; however, the dynamical and transport process of forming black holes in $f(R)$ gravity through gravitational collapse has not widely been discussed. This work may contribute to explore such questions.

Many choices for the function $f(R)$ have appeared in the literature which aimed to explain dark energy and accelerating universe [10]. However, in this paper, we restrict

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ourselves to a power-law form of $f(R)$ gravity, that is, $f(R) = R^n$. This model has great physical significance, being determined by the presence of Noether symmetries in the interaction Lagrangian [11]. Earlier, this model was constrained by a solar system test and in the attempt to explain the accelerating universe [12]. However, to obtain analytical or qualitative insight on exact solutions, it is perfectly acceptable model to study $f(R)$ theory as a toy model. Also, the applications of this model are directed to the study of the dark matter by using spherically symmetric solutions via Noether symmetries [13]. Such an approach shows that R^n type models are compatible with the spherically symmetry, which is closely associated with the Birkhoff theorem. The validity of this theorem is directly related to the physical properties of a self-gravitating system, e.g., stability and stationarity etc. [14].

In fact, the relations between the fundamental plane parameters of galaxies and the corrected Newtonian potential, coming from R^n, can be found and justified from a physical point of view to fit the observations [15]. Furthermore, the excellent agreement of the theoretical and observed rotation curves and the values of the stellar mass-to-light ratios with the predictions of population synthesis models make us feel confident that R^n gravity may represent a good candidate to solve both the dark energy problem on cosmological scales and the dark matter one on galactic scales with the same value of the slope n of the higher order gravity Lagrangian [16].

In this paper, we discuss the transport equation of gravitational collapse along with the Maxwell source in R^n gravity. We derive the general transport equation for the $f(R) = R^n$ model and then fix $n = 2$ as well to write some results because the R^2 term could work effectively at the infrared scale. In Sect. 2, the modified Einstein field equations for the R^n gravity combined with the Maxwell source are presented. In Sect. 3, we will formulate the dynamical equations. The central problem, the transport equation, is analyzed in Sect. 4 and results are provided in the last section.

2 Field equations for R^n gravity and Maxwell source

The 4-dimensional ($\mu, \nu = 0, 1, 2, 3$) action in $f(R)$ gravity along with the Maxwell source and matter Lagrangian is defined as

$$
S = \frac{1}{2} \int d^4 x \sqrt{-g} \left(\frac{f(R)}{\kappa} - \frac{F}{2\pi} \right)
$$
$$
+ \int d^4 x L_m(g_{\mu\nu}, \Psi_m), \tag{2.1}
$$

where κ stands for the coupling constant and $F = \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ is the Maxwell invariant and L_m is the Lagrangian for the matter source depending upon the $g_{\mu\nu}$ and the matter field. The Maxwell equations in a gravitational field enhance the

gravity background by the mass–energy relationship. Deriving the field equations by varying the above action with respect to $g_{\mu\nu}$, we get the following set of field equations:

$$
f_R R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu f_R + g_{\mu\nu} \Box f_R
$$
$$
= \kappa (T^m_{\mu\nu} + E_{\mu\nu}), \tag{2.2}
$$

where $T^m_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta L_m}{\delta g^{\mu\nu}}$. The above equations can be written in the standard format as follows:

$$
G_{\mu\nu} = \kappa (T^D_{\mu\nu} + T^m_{\mu\nu} + E_{\mu\nu}), \tag{2.3}
$$

where the quantities on the right hand side follow:

$$
T^{(D)}_{\mu\nu} = \frac{1}{\kappa} \left(\frac{f - R f_{,R}}{2} g_{\mu\nu} + \nabla_\mu \nabla_\nu f_{,R} - g_{\mu\nu} \Box f_{,R} \right), \tag{2.4}
$$

$$
T^m_{\mu\nu} = (\rho + p) u_\mu u_\beta - p g_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu, \tag{2.5}
$$

$$
E_{\mu\nu} = \frac{1}{4\pi} \left(-F^\gamma_\mu F_{\nu\gamma} + \frac{1}{4} F^{\gamma\delta} F_{\gamma\delta} g_{\mu\nu} \right). \tag{2.6}
$$

Here q^u denotes the heat flow vector satisfying $q^\mu u_\mu$, the quantity $F_{\mu\nu} = \Phi_{\nu,\mu} - \Phi_{\mu,\nu}$ is called the field strength tensor and Φ_μ is the electro magnetic tensor. In terms of this strength tensor, the field equations for the Maxwell source can be written as

$$
F^{\mu\nu}_{;\nu} = \mu_0 J^\mu, \tag{2.7}
$$
$$
F_{[\mu\nu;\gamma]} = 0, \tag{2.8}
$$
$$
J^\mu = \rho(t, r) V^\mu. \tag{2.9}
$$

The quantities J^μ, μ_0, V^μ and ρ are the four-current, magnetic permeability, the four-velocity and the charge density, respectively. In this paper, we assume that the charge is at rest and hence the magnetic field is 0, so that

$$
\Phi_\mu = \Phi(t, r) \delta^0_\mu. \tag{2.10}
$$

We consider a spherically symmetric spacetime with general metric components A, B, and C as a function of time and radial coordinates. This interior metric represent the matter source which is undergoing dissipative process causing the gravitational collapse. This interior matter is bounded by a spherical surface Σ and is given by

$$
ds^2_- = A^2 dt^2 - B^2 dr^2 - C^2(t, r)(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.11}
$$

For the metric exterior to the boundary surface, we consider a spacetime represented in the form of a total charge Q and a total mass M of the collapsing matter inside the Σ. This is given by

$$
ds^2_+ = \left(1 - \frac{2M(v)}{r} + \frac{Q^2}{r} \right) dv^2 + 2 dr dv
$$
$$
- r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.12}
$$

For the general interior spacetime, the Maxwell field equations will take the form

$$\frac{\partial^2 \Phi}{\partial r^2} - \left(\frac{B'}{B} + \frac{A'}{A} - \frac{2C'}{C} \right) \frac{\partial \Phi}{\partial r} = 4\pi\rho B^2 A, \qquad (2.13)$$

$$\frac{\partial}{\partial t} \left(\frac{\partial \Phi}{\partial r} \right) - \left(\frac{\dot{B}}{B} + \frac{\dot{A}}{A} - \frac{2\dot{C}}{C} \right) \frac{\partial \Phi}{\partial r} = 0. \qquad (2.14)$$

The derivatives with respect to time and radius are denoted by a dot and a prime, respectively. Following the conservation law of the four-current, i.e., $J^\mu;_\mu = 0$, we obtain the expression for the charge and the electric field intensity per unit surface area as follows:

$$q(r) = 2\pi \int_0^r \rho B C^2 dr, \qquad (2.15)$$

$$E(t, r) = \frac{q}{4\pi C^2}. \qquad (2.16)$$

We consider the background model $f(R) = R^n$ to formulate the field equations using the interior spacetime metric. The non-zero components of the field equations are as follows:

$$G_{00} = \frac{\kappa}{nR^{n-1}} \left[\rho A^2 + \frac{A^2}{\kappa} \left\{ \frac{(1-n)R^n}{2} \right. \right.$$
$$+ \frac{n(n-1)[(n-2)R^{n-3}R'^2 + R'']}{B^2}$$
$$+ \left(\frac{2\dot{C}}{C} - \frac{\dot{B}}{B} \right) \frac{n(n-1)R^{n-2}\dot{R}}{A^2}$$
$$\left. \left. + \left(\frac{2C'}{C} - \frac{B'}{B} \right) \frac{n(n-1)R^{n-3}R'}{B^2} \right\} + 2\pi E^2 \right], \qquad (2.17)$$

$$G_{01} = \frac{\kappa}{nR^{n-1}} \left[-qAB + \frac{1}{\kappa} \left(n(n-1)[(n-2)R^{n-3}R'\dot{R} \right. \right.$$
$$+ R^{n-2}\dot{R}'] - \frac{A'}{A}[n(n-1)R^{n-2}\dot{R}]$$
$$\left. \left. - \frac{\dot{B}}{B}[n(n-1)R^{n-2}R'] \right) \right], \qquad (2.18)$$

$$G_{11} = \frac{\kappa}{nRn-1} \left[p_r B^2 - 2\pi E^2 - \frac{B^2}{\kappa} \left\{ \frac{(1-n)R^n}{2} \right. \right.$$
$$+ \frac{n(n-1)R^{n-2}\dot{R}}{A^2} \times \left(\frac{\dot{A}}{A} + \frac{2\dot{C}}{C} \right)$$
$$- \{n(n-1)[(n-2)Rn - 3\dot{R} + R^{n-2}\ddot{R}]\}A^2$$
$$\left. \left. + \left(\frac{A'}{A} + \frac{2C'}{C} \right) \frac{n(n-1)R^{n-2}R'}{B^2} \right\} - 2\pi E^2 \right], \qquad (2.19)$$

$$G_{22} = \frac{\kappa}{nR^{n-1}} \left[p_\perp C^2 - \frac{C^2}{\kappa} \left\{ \frac{(1-n)R^n}{2} \right. \right.$$
$$- \frac{n(n-1)[(n-2)R^{n-3}\dot{R} + R^{n-2}\ddot{R}]}{A^2}$$

$$+ \frac{n(n-1)[(n-2)R^{n-3}R'^2 + R^{n-2}R'']}{B^2}$$
$$+ \frac{nR^{n-1}}{A^2} \left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right)$$
$$\left. \left. + \left(\frac{A'}{A} - \frac{B'}{B} + \frac{C'}{C} \right) \frac{n(n-1)R^{n-2}R'}{B^2} \right\} + 2\pi E^2 \right]. \qquad (2.20)$$

To discuss the collapsing matter inside the star, the proper time, proper radial derivatives and the collapsing velocity of the dissipative fluid can be written

$$D_T = \frac{1}{A}\frac{\partial}{\partial t}, \quad D_C = \frac{1}{C'}\frac{\partial}{\partial r} \quad U = D_T C = \frac{\dot{C}}{A}, \qquad (2.21)$$

where the velocity is always considered negative to represent collapse. The time derivative of the collapsing velocity, the acceleration, $D_T U$, can be calculated using Eqs. (2.19) and (2.21) as follows:

$$D_T U = \frac{A'}{AB}\tilde{E} - \frac{\kappa}{2nR^{n-1}}$$
$$\times \left[p - \frac{1}{\kappa} \left\{ -\frac{n(n-1)[(n-2)R^{n-3}\dot{R} + R^{n-2}\ddot{R}]}{A^2} \right. \right.$$
$$+ \frac{n(n-1)R^{n-2}\dot{R}}{A^2} \left(\frac{\dot{A}}{A} + \frac{2\dot{C}}{C} \right)$$
$$\left. \left. + \frac{n(n-1)R^{n-2}R'}{B^2} \left(\frac{A'}{A} + \frac{2C'}{C} \right) + \frac{(1-n)R^n}{2} \right\} \right]. \qquad (2.22)$$

$\tilde{E}$ has been given in terms of the Misner and Sharp mass function m as

$$\tilde{E} = \left[1 + U^2 + \frac{2m}{C} \right]^{1/2}. \qquad (2.23)$$

3 Transport equation

To study the transport equation, first we need to formulate the dynamical equations of the collapsing fluid by using contracted Bianchi identities achieved by taking the covariant derivative with respect to the four-velocity V_α and four vector $\chi_\alpha = B^{-1}\delta_1^\alpha$ along the radial direction, $[T^{\alpha\beta}\overset{(m)}{} + T^{\alpha\beta}\overset{(D)}{} + E^{\alpha\beta}];_\beta V_\alpha = 0$ and $[T^{\alpha\beta}\overset{(m)}{} + T^{\alpha\beta}\overset{(D)}{} + T^{\alpha\beta}\overset{(m)}{}];_\beta \chi_\alpha = 0$, respectively. The resulting lengthy equations obtained from these identities are given in the appendix. Extracting the term $\frac{A'}{AB}(\rho + p)$ from Eq. (2.22), we get

$$\frac{A'}{AB}(\rho + p) = \frac{(\rho + p)}{\tilde{E}} D_T U + \frac{(\rho + p)\kappa}{2nR^{n-1}\tilde{E}}$$
$$\times \left[p - \frac{1}{\kappa} \left\{ \frac{n(n-1)R^{n-2}\dot{R}}{A^2} \right. \right.$$

$$\times \left(\frac{\dot{A}}{A} + \frac{2\dot{C}}{C} \right) + \frac{(1-n)R^n}{2}$$
$$+ \frac{n(n-1)R^{n-2}R'}{B^2} \left(\frac{A'}{A} + \frac{2C'}{C} \right)$$
$$- \frac{n(n-1)[(n-2)R^{n-3}\dot{R} + R^{n-2}\ddot{R}]}{A^2} \Bigg\} \Bigg].$$
(3.1)

Replacing this term in Eq. (5.7), we get the following expression for the acceleration of the collapsing fluid:

$$(\rho + p)D_T U r = -(\rho + p) \Bigg[\frac{\kappa p_r}{2nR^{n-1}}$$
$$- \frac{1}{2nR^{n-1}} \Bigg\{ \frac{(1-n)R^n}{2} - D_T \left(\frac{n(n-1)R^{n-2}\dot{R}}{A} \right)$$
$$+ \frac{nR^{n-1}D_T C}{C} + \frac{\tilde{E}}{B} \left(\frac{A'}{A} + \frac{2C'}{C} \right) nR^{n-1}D_C R \Bigg\} \Bigg]$$
$$- \tilde{E} \Bigg[\left(\frac{D_T B}{B} - \frac{D_T C}{C} \right) \times 2q - D_T q \Bigg]$$
$$- \tilde{E}^2 D_C p + \frac{\tilde{E}}{\kappa B} S_{R^n},$$

where we denote by S_{R^n} the term as appearing purely due to R^n gravity. Explicitly, it is written in the Appendix.

To derive the equation for the heat flux, we use the Müller–Israel–Stewart theory which helps us to write the thermal conductivity in terms of a linear combination of various fluxes, e.g., the four-velocity, heat flux, etc. This theory has been conceived in a series of papers by Israel and Stewart [17–19] followed by work of Müller [20]. The study of the transport equation obtained from this theory provides the information as regards the transfer of mass, heat and momentum during the collapse of the matter. The equation is given by

$$\tau h^{\alpha\beta} u^{\gamma} q_{\beta;\gamma} + q^{\alpha}$$
$$= -\eta h^{\alpha\beta}(T_{,\beta} + a_\beta T) - \frac{1}{2}\eta T^2 \left(\frac{\tau u^\beta}{\eta T^2} \right)_{;\beta} q^\alpha.$$
(3.2)

Here $h^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ is the projection tensor whereas the notation η, τ, T and $a_\beta T$ denotes the thermal conductivity, the relaxation time, the temperature and the Tolman inertial term with $a_\alpha = u_{\alpha;\beta}u^\beta$ being the acceleration, respectively. The non-zero and independent component of the above equation is given by

$$\tau \dot{q} = -qA - \frac{1}{2}\eta q T^2 \left(\frac{\tau}{\eta T^2} \right)^{\cdot}$$
$$- \frac{1}{2}\tau q \left(\frac{\dot{B}}{B} + 2\frac{\dot{C}}{C} \right) + \frac{\eta A^2}{B} \left(\frac{T}{A} \right)'.$$
(3.3)

Eliminating the expression for the quantity, $\frac{A'}{A}$, from Eq. (3.1) and then substituting it in Eq. (2.23), we obtain

$$D_T q = -\frac{\eta T^2 q}{2\tau} D_T \left(\frac{\tau}{\eta T^2} \right) - \frac{q}{2} \left(\frac{D_T B}{B} + \frac{2D_T C}{C} + \frac{1}{\tau} \right)$$
$$+ \frac{\eta \tilde{E}}{\tau} \left(D_C T - \frac{D_T U}{\tau \tilde{E}^2} \right) - \frac{\eta T}{\tau \tilde{E}} \Bigg[\frac{\kappa p_r}{2nR^{n-1}}$$
$$- \frac{1}{2nR^{n-1}} \Bigg\{ - D_T \left(\frac{n(n-1)R^{n-2}\dot{R}}{A} \right)$$
$$+ \frac{nR^{n-1}D_T C}{C} + \frac{(1-n)R^n}{2}$$
$$+ \frac{\tilde{E}}{B} \left(\frac{A'}{A} + \frac{2C'}{C} \right) nR^{n-1}D_C R \Bigg\} \Bigg].$$
(3.4)

To see the effects of the heat flux on the dissipative process of the collapsing fluid, we use this version of the transport equation in Eq. (5.8) and fix $n = 2$ to obtain

$$(\rho + p) \left[1 - \frac{\eta T}{\tau(\rho + p)} \right] D_T U$$
$$= -\left[1 - \frac{\eta T}{\tau(\rho + p)} \right] \left(\frac{\rho + p}{4R} \right) \Bigg[\frac{\kappa p}{4R}$$
$$+ \frac{1}{4R} \Bigg\{ D_T \left(\frac{2\dot{R}}{A} \right) + \frac{R^2}{2} - \frac{2R D_T C}{C}$$
$$- \frac{2\tilde{R}E}{B} \left(\frac{A'}{A} + \frac{2C'}{C} \right) D_C R \Bigg\} \Bigg]$$
$$+ \tilde{E} \Bigg[\frac{\eta T^2 q}{2\tau} D_T \left(\frac{\tau}{\eta T^2} \right) - \tilde{E}^2 D_C p \frac{q}{2} \left(\frac{D_T B}{B} \right.$$
$$\left. + \frac{2D_T C}{C} + \frac{1}{\tau} \right) + \frac{\eta \tilde{E}}{\tau} D_C T \Bigg] + \frac{\tilde{E}}{\kappa B} S_{R^2}.$$
(3.5)

This equation yields the energy transport in a star. There are three ways of energy transfer from hot to cold layers of the star, i.e., conduction, radiation and convection. Usually, photons carry energy from the hot interior core of a collapsing star to the outer cold space. If photons/radiation are unable to transfer the total energy of the hot interior star to the surface of the star, then the method of convection is used to process energy transfer. In the method of convection, hotter gases rise to the upper levels of the star surfaces to radiate their energy and meanwhile cooler gases sink towards the hot interior to collect energy. The third way of the transport energy is the conduction method in which each atom transfers its energy to its neighbouring atoms; however, this method is usually ignored due to its low efficiency.

4 Discussion and results

In this paper, we have discussed the dynamics of the dissipative fluid after its departure from the hydrostatic equilibrium by using the transport equation of a radiating charged

fluid. We have adopted the power-law version of $f(R)$ theory which could fit well the observations and encourages further investigation on R^n gravity from both the theoretical and observational points of view [21]. The field equations have been derived for R^n theory and Maxwell source. The conservation equations for the matter yielded two types of dynamical equations. These dynamical equations give the information as regards the dynamics of the collapsing fluid and also are further used in the transport equation for Müller–Israel–Stewart theory of dissipative fluids to get the hydrostatic equilibrium evolution equation.

It is found that the resulting evolution equation (4.1) critically depends on a factor composed of thermodynamic variables. On the left hand side of this equation, the term $D_T U$ is the acceleration, whereas the product term $(\rho + p)$ is the inertial mass density. Thus by Newton's law, the right hand side term represents the gravitational force term along with the repulsive term S_{R^2}. To interpret this, we assume that the gravitational force term overcomes the effect of repulsive term; then we see that both sides of the equation are affected by the factor $1 - \frac{\eta T}{\tau(\rho+p)}$. Also, the same factor appears on the right hand side of the equation and hence represents the consistency of the equivalence principle. If we denote this factor by β, then we may have the following possibilities:

(i) If $0 < \beta < 1$, then the inertia of heat causes a decrease in the inertial and gravitational mass densities due to a fractional factor. If the evolution proceeds in such a way that $\beta \to 1$, then the effective inertial mass density of the fluid element approaches 0. For a very small value of the relaxation time at present time, we may speculate that β may increase substantially in a pre-supernova event. In fact, at the last stages of a massive star evolution, the decrease of inertial densities would prevent the propagation of photons and neutrinos [22].

(ii) If $\beta \to 0$, then there is no effect on the inertial mass density and gravitational force. In addition, this case may lead to the fact that $\eta T \to 0$. If this happens, then the core becomes degenerate, starts to cool and the star must become a white dwarf. This case may be fitted to small bodies, such as Saturn, which is stable against the gravitational collapse. If we gave Saturn a slight squeeze, both the gravitational force and the pressure within its core would increase. The gravitational force would rise simply as the inverse-square of the radius, but the force of the pressure would rise faster than the inverse-square of the radius. This imbalance of forces would cause Saturn to expand back to its equilibrium radius regardless of how cold Saturn grows.

(iii) If $\beta > 1$, then it changes sign and hence the gravitational force term becomes positive implying a reversal of collapse. Consequently, this case may stop the collapse and make the star explode. If this does not happen,

the collapse would lead to a region of instability. This mechanism is assumed to cause type II supernovae.

(iv) The case when $\beta = 1$: we get the critical point during gravitational collapse. In this case, the force terms on the right hand side of the evolution equation will also be 0 and we are left with a constraint equation,

$$\frac{\eta T^2 q}{2\tau} D_T \left(\frac{\tau}{\eta T^2}\right) - \tilde{E}^2 D_C p \frac{q}{2}$$
$$\times \left(\frac{D_T B}{B} + \frac{2 D_T C}{C} + \frac{1}{\tau}\right)$$
$$+ \frac{\eta \tilde{E}}{\tau} D_C T = \frac{\tilde{E}}{\kappa B} S_{R^2}. \tag{4.1}$$

This equation represents the dissipative regime of the collapsing sphere immediately after its departure from the equilibrium state on a time scale of the relaxation time τ. If we suppose that, before hydrostatic equilibrium, there is no dissipation, then all terms on the left hand side will vanish due to the vanishing of q and η. At the moment of hydrostatic equilibrium, the relaxation time influences the evolution process. Some particular values of the relaxation time may cause the bounce or collapse of the sphere [23,24].

If the inertial mass density $(\rho + p)$ will be 0 for the perfect fluid case, then the discussion will be the same as for the factor $\frac{\eta T}{\tau(\rho+p)}$ when it approaches 1. It is mentioned that we are evaluating the system immediately after its leaving the equilibrium state, hence the thermodynamical variables are hard to interpret numerically, however, as a guess, the values may be $[\eta] \approx 10^{37}$, $[T] \approx 10^{13}$, $[\tau] \approx 10^{-4}$, $[\rho] \approx 10^{12}$ [22]. In general, the obtained results represent a general self-gravitating dissipative fluid model.

5 Appendix

For the model $f(R) = R^n$, the derivatives of $\frac{df}{dR} = f_R$ used in the field equation are given by

$$\dot{f}_R = n(n-1)R^{n-2}\dot{R} \tag{5.1}$$
$$f'_R = n(n-1)R^{n-2}R' \tag{5.2}$$
$$\ddot{f}_R = n(n-1)[(n-2)R^{n-3}\dot{R} + R^{n-2}\ddot{R}] \tag{5.3}$$
$$\dot{f'}_R = n(n-1)[(n-2)R^{n-3}R'\dot{R} + R(n-2)\dot{R}'] \tag{5.4}$$
$$f''_R = n(n-1)[(n-2)R^{n-3}R'^2 + R^{n-2}R'']. \tag{5.5}$$

The first dynamical equation, $[\overset{(m)}{T^{\alpha\beta}} + \overset{(D)}{T^{\alpha\beta}} + E^{\alpha\beta}]_{;\beta} V_\alpha = 0$:

$$\frac{\dot\rho}{A} + \frac{q'}{B} + \frac{q}{B}\left(\frac{A'}{A} + \frac{2C'}{C}\right) + \frac{(\rho+p)}{A}\left(\frac{\dot B}{B} + \frac{2\dot C}{C}\right)$$

$$+ \frac{A}{\kappa}\left[\frac{1}{A^2 B^2} \times \left\{ n(n-1)(n-2)R^{n-3}R'\dot R \right.\right.$$

$$+ R^{n-2}\dot R' - \frac{A'}{A}n(n-1)R^{n-2}\dot R$$

$$\left.\left. - \frac{\dot B}{B}n(n-1)R^{n-2}R' \right\}\right]_{,1}$$

$$+ \frac{A}{\kappa}\left\{\frac{n(n-1)}{A^2 B^2}(n-2)R^{n-3}R'^2 \right.$$

$$- \frac{n(n-1)R^{n-2}\dot R}{A^2}\left(\frac{\dot B}{B} - \frac{2\dot C}{C}\right)$$

$$- \frac{n(n-1)R^{n-2}R'}{B^2}\left(\frac{B'}{B} - \frac{2C'}{C}\right)$$

$$\left. + \frac{(1-n)R^n}{2A^2}\right\}_{,0} + \frac{\dot A}{\kappa A^2}\left\{\frac{(1-n)R^n}{2A^2}\right.$$

$$+ \frac{n(n-1)}{A^2 B^2}(n-2)R^{n-3}R'^2 - \frac{n(n-1)R^{n-2}\dot R}{A^2}$$

$$\left. \times \left(\frac{\dot B}{B} - \frac{2\dot C}{C}\right) - \frac{n(n-1)R^{n-2}R'}{B^2}\left(\frac{B'}{B} - \frac{2C'}{C}\right)\right\}$$

$$+ \frac{\dot B}{\kappa A B}\left\{\frac{n(n-1)}{B^2}[(n-2)R^{n-3}R'^2 + R^{n-2}R''] \right.$$

$$- \frac{n(n-1)R^{n-2}\dot R}{A^2} \times \left(\frac{\dot A}{A} + \frac{\dot B}{B}\right) + \frac{n(n-1)}{A^2}$$

$$\times [(n-2)R^{n-3}\dot R + R^{n-2}\ddot R] - \left(\frac{A'}{A} + \frac{B'}{B}\right)$$

$$\left. \times \frac{n(n-1)R^{n-2}R'}{B^2}\right\} + \frac{2\dot C}{\kappa A C}\left\{\frac{n(n-1)}{A^2}[(n-2)\right.$$

$$\times R^{n-3}\dot R + R^{n-2}\ddot R] + \frac{n(n-1)R^{n-2}\dot R}{A^2}\left(\frac{\dot C}{C} - \frac{\dot A}{A}\right)$$

$$\left. - \frac{n(n-1)R^{n-2}R'}{B^2}\left(\frac{A'}{A} - \frac{C'}{C}\right)\right\}$$

$$+ \frac{1}{\kappa A B^2}\left(n(n-1)[(n-2)R^{n-3}R'\dot R + R^{n-2}\dot R'] \right.$$

$$\left. - n(n-1)R^{n-2}\dot R\frac{A'}{A} - n(n-1)R^{n-2}R'\frac{\dot B}{B}\right)$$

$$\times \left(\frac{2A'}{A} + \frac{B'}{B} + \frac{C'}{C}\right) = 0. \tag{5.6}$$

The second dynamical equation, $[\overset{(m)}{T^{\alpha\beta}} + \overset{(D)}{T^{\alpha\beta}} + T^{\alpha\beta}]_{;\beta} \chi_\alpha = 0$:

$$\frac{p'}{B} + \frac{\dot q}{A} + \frac{2q}{A}\left(\frac{\dot B}{B} + \frac{\dot C}{C}\right) + \frac{A'}{AB}(\rho+p)$$

$$- \frac{B}{\kappa}\left[\left\{\frac{n(n-1)}{A^2 B^2} \times [(n-2)R^{n-3}R'\dot R + R^{n-2}\dot R'] \right.\right.$$

$$\left. - n(n-1)R^{n-2}\left(\frac{A'}{A}\dot R + \frac{\dot B}{B}R'\right)\right\}_{,0}$$

$$+ \left\{\frac{(1-n)R^n}{2B^2} - \frac{n(n-1)}{A^2}[(n-2)R^{n-3}\dot R + R^{n-2}\ddot R]\right.$$

$$+ \left(\frac{\dot A}{A} + \frac{2\dot C}{C}\right) \times \frac{n(n-1)R^{n-2}\dot R}{A^2} + \frac{B^2}{n(n-1)R^{n-2}R'}$$

$$\left. \times \left(\frac{A'}{A} + \frac{2C'}{C}\right)\right\}_{,1} + \frac{A'}{AB^2} \times \left\{\frac{n(n-1)}{A^2}\right.$$

$$\times [(n-2)R^{n-3}\dot R + R^{n-2}\ddot R] - \frac{n(n-1)R^{n-2}\dot R}{A^2}$$

$$+ \left(\frac{\dot A}{A} + \frac{\dot B}{B}\right) \times \frac{n(n-1)}{B^2}[(n-2)R^{n-3}R'^2 + R^{n-2}R'']$$

$$\left. - \frac{n(n-1)R^{n-2}R'}{B^2}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right\}$$

$$+ \frac{2B'}{B^3}\left(\frac{A'}{A} - \frac{C'}{C}\right)\left\{\frac{(1-n)R^n}{2A^2}\right.$$

$$+ \frac{n(n-1)[(n-2)R^{n-3}\dot R + R^{n-2}\ddot R]}{A^2}$$

$$+ \frac{A'}{A}n(n-1)R^{n-2}\dot R - \frac{\dot B}{B}n(n-1)R^{n-2}R'$$

$$- \frac{n(n-1)R^{n-2}\dot R}{A^2}\left(\frac{\dot A}{A} + \frac{\dot B}{B}\right)$$

$$\left. - \frac{n(n-1)R^{n-2}R'}{B^2}\left(\frac{A'}{A} + \frac{B'}{B}\right)\right\}$$

$$+ \frac{2\dot C}{CA^2}\left\{\frac{n(n-1)}{A^2}[(n-2)R^{n-3}\dot R\right.$$

$$+ R^{n-2}\ddot R] + \frac{n(n-1)R^{n-2}\dot R}{A^2}\left(\frac{\dot C}{C} - \frac{\dot A}{A}\right)$$

$$\left. - \frac{n(n-1)R^{n-2}R'}{B^2}\right\} + \frac{1}{A^2 B^2}$$

$$\times [n(n-1)(n-2)R^{n-3}R'\dot R + R^{n-2}\dot R']$$

$$\left. - \left(\frac{\dot A}{A} + \frac{3\dot B}{B} + \frac{2\dot C}{C}\right)\right] = 0, \tag{5.7}$$

$$S_{R^n} = \left[\left(\frac{D_T A}{A} - \frac{D_T B}{B} - \frac{2D_T C}{C}\right)\left\{n(n-1)\right.\right.$$

$$\times D_T R^{n-2}R' - \frac{A'}{A}n R^{n-1}D_T$$

$$\left. \times \left(\frac{n(n-1)R^{n-2}R'D_T B}{B}\right)\right\}$$

$$+ \left\{2nR^{n-1} \times \frac{D_T C}{C} - D_T\left(\frac{n(n-1)R^{n-2}\dot R}{A}\right)\right.$$

$$\left.\left. + \frac{\tilde E}{B}\left(\frac{A'}{A} + \frac{2C'}{C}\right)nR^{n-1}D_C R\right\}\right]_{,1}$$

$$+\frac{A'}{A}\left\{D_T\left(\frac{n(n-1)R^{n-2}\dot{R}}{A}\right)\right.$$

$$+\frac{n(n-1)}{B^2}[(n-2)R^{n-3}R'^2+R^{n-2}R'']$$

$$\left.-\frac{n(n-1)R^{n-2}R'}{B^2}\left(\frac{A'}{A}+\frac{B'}{B}\right)-\frac{nR^{n-1}D_TBD_T}{B}\right\}$$

$$+\frac{2C'}{C}\left\{\frac{n(n-1)}{B^2}\times[(n-2)R^{n-3}R'^2+R^{n-2}R'']\right.$$

$$-nR^{n-1}D_T\left(\frac{D_TB}{B}+\frac{D_TC}{C}\right)-\frac{n(n-1)}{B^2}$$

$$\left.-R^{n-2}R'\left(\frac{B'}{B}+\frac{C'}{C}\right)\right\}$$

$$+\frac{1}{A^2}\left\{n(n-1)[(n-2)R^{n-3}R'\dot{R}+R^{n-2}\dot{R}']\right.$$

$$\left.\left.-n(n-1)R^{n-2}\left(\dot{R}\frac{A'}{A}-R'\frac{\dot{B}}{B}\right)\right\}_{,0}\right]. \tag{5.8}$$

$$S_{R^2}=\left[\left(\frac{D_TA}{A}-\frac{D_TB}{B}-\frac{2D_TC}{C}\right)\right.$$

$$\left\{2D_TR'-\frac{A'}{A}2RD_T\left(\frac{2R'D_TB}{B}\right)\right\}$$

$$+\left\{4R\frac{D_TC}{C}+2RD_CR\frac{\tilde{E}}{B}\left(\frac{A'}{A}+\frac{2C'}{C}\right)\right.$$

$$\left.-D_T\left(\frac{2\dot{R}}{A}\right)\right\}_{,1}+\frac{A'}{A}\left\{D_T\left(\frac{2\dot{R}}{A}\right)\right.$$

$$+\frac{2R''}{B^2}-\frac{2R'}{B^2}\left(\frac{A'}{A}+\frac{B'}{B}\right)-\frac{2RD_TBD_T}{B}\right\}$$

$$-\frac{2C'}{C}\left\{\frac{2R''}{B^2}-2D_T\left(\frac{D_TB}{B}+\frac{D_TC}{C}\right)\right.$$

$$\left.-R'\left(\frac{B'}{B}+\frac{C'}{C}\right)\right\}$$

$$\left.-\frac{2}{A^2}\left\{\left(\dot{R}\frac{A'}{A}-R'\frac{\dot{B}}{B}\right)+\frac{\dot{R}'}{A^2}\right\}_{,0}\right]. \tag{5.9}$$

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