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Cosmological attractors and anisotropies in two measure theories, effective EYMH systems, and off-diagonal inflation models

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Abstract Applying the anholonomic frame deformation method, we construct various classes of cosmological solutions for effective Einstein–Yang–Mills–Higgs, and two measure theories. The types of models considered are Freedman–Lemaître–Robertson–Walker, Bianchi, Kasner and models with attractor configurations. The various regimes pertaining to plateau-type inflation, quadratic inflation, Starobinsky type and Higgs type inflation are presented.

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1 Introduction

Over time, the Cosmological Constant Problem (CCP) has evolved from the "Old Cosmological Constant Problem" [1,2], where the concern was on why the observed vacuum energy density of the universe is exactly zero, to the present form pertaining to the evidence establishing the accelerating expansion of the universe [3, For reviews of this subject see for example]. One is therefore faced with the "New Cosmological Constant Problem" [4,5]. In other words, the problem has shifted from the question why the CCP is exactly zero, but to why the vacuum energy density is so small. Various attempts to address the issue range from the conventional to the esoteric. Conventional field theoretic models are based on a single scalar field (quintessence) while the esoteric models involve tachyons, phantoms and K-essence. The latter may also admit multi scalar field configurations. Such models have also been supplemented further to take into consideration the recent observational data from Planck [6,7] and BICEP2 [8]. In all these models the inflationary paradigm [9,10] is the underlying theme; see also an opposite point of view in [11]. However, present data is insufficient to determine precisely what the initial conditions were that drove inflation. In addressing the present situation there are essen-

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tially two main approaches entertained. In one approach it is assumed that there is a basic mechanism driving to zero the vacuum energy but some "residual" interactions survive that slightly shift the vacuum energy density towards the presently observed small non-zero value. In the alternative approach it is assumed that the true vacuum energy will exactly be zero when the final state of the theory is reached and the present state pertaining to the small non-zero vacuum energy density is the result of our universe having not reached that final state yet.

In this work, we will adapt the point of view that the above two scenarios represent equally viable solutions to the CCP and both can be entertained naturally if one considers off-diagonal inhomogeneous cosmological solutions. Alternative constructs are also possible and are discussed in [10,12,13] in a different class of theories. As will be demonstrated, for certain well-defined conditions, the models considered in this work can be treated as effective two measure theories (TMTs) studied in Refs. [14–19]. In these theories, the modified gravitational and matter field equations of TMTs generate effective Einstein–Yang–Mills–Higgs (EYMH) systems which can be solved in analytic form using geometric methods. The underlying principle of the geometric method is based on the anholonomic frame deformation method (AFDM) [20–24]. The main idea of the AFDM is to rewrite equivalently Einstein equations, and various modifications of it, on a (pseudo) Riemannian manifold $\mathbf{V}$ in terms of an "auxiliary" linear connection $\mathbf{D}$. This connection, together with the Levi-Civita (LC) connection ∇, is defined in a metric compatible form by a split metric structure $\mathbf{g} = \{\mathbf{g}_{\alpha\beta} = [g_{ij}, g_{ab}]\}$. In order to establish our notation, we take $\dim \mathbf{V} = 4$, with the conventional splitting of coordinates as $3+1$, and the equivalent splitting as $2+2$, respectively. The signature of the metric on $\mathbf{V}$ is taken to be $(+,+,+,-)$. Indices $i, j, k, \ldots$ take values $1, 2$, while indices $a, b, \ldots$ take values $3, 4$ and the local coordinates are denoted by $u^\alpha = (x^i, y^a)$, or collectively as $u = (x, y)$.[1] Quantities under consideration and with a left label (for instance, ${}^{\mathbf{g}}\mathbf{D}$) emphasize that the geometric object ($\mathbf{D}$) is uniquely determined by $\mathbf{g}$. Unless otherwise stated, Einstein's summation convention is assumed throughout with the caveat that upper and lower labels are omitted if this does not result in ambiguities. We emphasize that $\mathbf{D}$ contains non-trivial anholonomically induced torsion $\mathbf{T}$ relating to the underlying nonholonomic frame structure. Such a torsion field is completely defined by the metric and the nonholonomic (equivalently, anholonomic and/or non-integrable) distortion relations,

$$\mathbf{D} = \nabla + \mathbf{Z}[\mathbf{T}], \tag{1}$$

when both the linear connections and the distortion tensor $\mathbf{Z}[\mathbf{T}]$ are uniquely determined by certain well-defined geometric and/or physical principles. Physical models are constructed following the principle that all geometric constructions are adapted to a nonholonomic splitting with an associated nonlinear connection (N-connection) structure $\mathbf{N} = \{N_i^a(u)\}$ that splits into the Whitney sum consisting of the conventional horizontal (h) and vertical (v) components,

$$\mathbf{N}: \ T\mathbf{V} = {}^h\mathbf{V} \oplus {}^v\mathbf{V} \equiv h\mathbf{V} \oplus v\mathbf{V}, \tag{2}$$

where $T\mathbf{V}$ is the tangent bundle.[2] For such a splitting, all geometric constructions can be carried out equivalently with ∇ using the so-called canonical distinguished connection (d-connection), $\widehat{\mathbf{D}}$. Here $\widehat{\mathbf{D}}$ is distinct from $\mathbf{D}$. This linear connection is N-adapted, i.e., preserves under parallelism the N-connection splitting, and it is uniquely determined (together with ∇) by the constraints

$$\mathbf{g} \rightarrow \begin{cases} \nabla : \nabla \mathbf{g} = 0; \ {}^{\nabla}\mathbf{T} = 0, & \text{the Levi-Civita connection;} \\ \widehat{\mathbf{D}} : \widehat{\mathbf{D}}\, \mathbf{g} = 0; \ h\widehat{\mathbf{T}} = 0, \ v\widehat{\mathbf{T}} = 0, & \text{the canonical d-connection.} \end{cases} \tag{3}$$

It is to be noted that in general, a d-connection $\mathbf{D}$ can equivalently split into the N-adapted horizontal (h) and vertical (v) components, respectively, as $h\mathbf{D}$ and $v\mathbf{D}$, (or equivalently, as $= ({}^h\mathbf{D}, {}^v\mathbf{D})$). But such a splitting may not be compatible, (i.e., $\mathbf{D}\, \mathbf{g} \neq 0,$) as it can carry arbitrary amount of torsion $\mathbf{T}$, and hence is not subject to the aforementioned constraints depicted in (3).

The advantage of the canonical d-connection $\widehat{\mathbf{D}}$ is that in this framework hatted Einstein equations result,

$$\widehat{\mathbf{G}}_{\alpha\beta} := \widehat{R}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\, \widehat{R} = \mathbf{\Upsilon}_{\alpha\beta}(u). \tag{4}$$

Here the hatted Einstein tensor $\widehat{\mathbf{G}}$ and the effective source term $\mathbf{\Upsilon}$ are defined in standard form following geometric methods and N-adapted variational calculus but for quantities $(\mathbf{g}, \widehat{\mathbf{D}})$ instead of the usual $(\mathbf{g}, \nabla)$. The hatted Einstein equations decouple with respect to a class of N-adapted frames

[1] The $2+2$ splitting is convenient for constructing exact cosmological solutions with generic off-diagonal metrics which cannot be diagonalized by coordinate transforms in a finite spacetime region. Nevertheless, realistically, we shall have to consider $3+1$ splitting, for instance, in Sect. 4.3.1 in order to study off-diagonal deformations of FLRW configurations in TMTs, with effective fluid energy-momentum stress tensor.

[2] Boldface symbols will be used in order to emphasize that certain spaces and/or geometric objects are adapted to a N-connection. Here we note that, for instance, ${}^h\mathbf{V}$ is equivalent to $h\mathbf{V}$ (in order to avoid ambiguities, we present both types of notations used in our former work and the references therein). Such a conventional decomposition (equivalently, fibred structure) can always be constructed on any 4-d metric-affine manifold. In general relativity, it is known as the diadic decomposition of tetrads. The most important outcome of our work [20–24] is that we proved that (modified) Einstein equations can be decoupled and solved in very general forms both for a N-adapted $2+2$ splitting and a d-connection $\widehat{\mathbf{D}}$ (this auxiliary connection was not considered in former work with diadic structures).

for various classes of metrics with one-Killing symmetry [22,23]. This allows us to integrate (4) in a very general form by generic off-diagonal metrics, metrics that otherwise cannot be diagonalized in a finite spacetime region by coordinate transformations that are determined via a set of generating and integration functions depending on all spacetime coordinates and various types of commutative or noncommutative parameters.[3] Solutions thus determined describe various geometric and physical models in modified gravity theories with non-trivial nonholonomically induced torsion, $\widehat{\mathbf{T}} \neq 0$, and generalized connections. As special cases, we extract LC-configurations and construct new classes of cosmological solutions in Einstein's gravity if we constrain the set of possible generating and integration functions to satisfy the following conditions:

$$\widehat{\mathbf{T}} = 0, \qquad (5)$$

$$\mathbf{g}_{\alpha\beta} = \mathbf{g}_{\alpha\beta}(t), \qquad (6)$$

where the metric $\mathbf{g}_{\alpha\beta}(t)$ in the unprimed bases can be related to metric in the primed bases via frame transformations, i.e., $\mathbf{g}_{\alpha\beta}(t) = e^{\alpha'}_{\ \alpha} e^{\beta'}_{\ \beta} \mathbf{g}_{\alpha'\beta'}(t)$, where $e^{\alpha'}_{\ \alpha}$ represents the tetrad frame field. For instance, $\mathbf{g}_{\alpha'\beta'}$ can be a Bianchi type metric, or a diagonalized homogeneous Friedmann–Lemaître–Robertson–Walker (FLRW) type metric. In general, $\mathbf{g}_{\alpha'\beta'}$ may not be a solution of any gravitational field equations but we shall always impose the constraint that its nonholonomic deformation $\mathbf{g}_{\alpha\beta}$ always is a solution of the hatted Einstein equation (4).

In general, gravitational field equations (4) constitute a sophisticated system of nonlinear partial differential equations (PDEs) as opposed to the occurrence of ordinary differential equations (ODEs) in conventional general relativity. The AFDM, on the other hand, allows us to find new classes of solutions by decoupling the PDEs. We emphasize that, in the AFDM approach advocated here, constraints of type (5) and/or (6) are to be imposed after the inhomogeneous $\mathbf{g}_{\alpha\beta}(x^i, y^3, t)$ are constructed in general form. If the aforementioned constraints are imposed from the very beginning in order to transform PDEs into ODEs, a large class of generic off-diagonal and diagonal solutions will be compromised. The specific goal of this work is to apply the AFDM method and explicitly construct solutions in effective TMTs addressing attractors, acceleration, dark energy and dark matter effects in the new cosmological models.

This work is organized as follows. In Sect. 2, we provide a brief introduction to the geometry of nonholonomic deformations in Einstein gravity and modifications that lead to effective TMTs. In such theories we shown how the gravitational and matter field equations can be decoupled and solved in very general off-diagonal forms for the canonical d-connection with constraints for LC-configurations. Section 3 is devoted to off-diagonal and diagonal cosmological solutions with small vacuum density. Also constructed and analysed are the off-diagonal inhomogeneous cosmological solutions with nonholonomically induced torsion. In Sect. 4, we study the equivalence of effective TMTs with sources for nonlinear potentials and EYMH self-dual fields resulting in attractor type behaviour. In Sect. 5, we analyze in explicit form how exact cosmological solutions with locally anisotropic attractor properties can be generated by deforming FLRW type diagonal metrics and off-diagonal Bianchi type cosmological models. Conclusions are presented in Sect. 6.

[3] In general, symmetric metrics of the type $\mathbf{g}_{\alpha\beta}(x^1, x^2, y^3, y^4 = t)$, with t being a time-like coordinate, contain a maximum of six independent variables since four coefficients from the ten components of the metric tensor of a 4-d spacetime can be transformed away via coordinate transforms as a result of the Bianchi identities.

2 Nonholonomic deformations

For clarity, we elaborate upon our notation first. On a (pseudo) Riemannian manifold we prescribe an N-connection with horizontal (h) and vertical (v) decompositions (h and v splitting) (2) as ($\mathbf{V}$, $\mathbf{N}$). To this we associate structures of N-adapted local bases, $\mathbf{e}_\nu = (\mathbf{e}_i, e_a)$, and cobases, $\mathbf{e}^\mu = (e^i, \mathbf{e}^a)$, which are the following N-elongated partial derivatives and differentials:

$$\mathbf{e}_i := \partial/\partial x^i - N_i^a(u)\partial/\partial y^a,\ e_a := \partial_a = \partial/\partial y^a, \qquad (7)$$

$$\text{and } e^i = dx^i,\ \mathbf{e}^a = dy^a + N_i^a(u)dx^i. \qquad (8)$$

The frame basis $\mathbf{e}_\nu = (\mathbf{e}_i, e_a)$, satisfy the nonholonomy relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W^\gamma_{\alpha\beta} \mathbf{e}_\gamma, \qquad (9)$$

with non-trivial nonholonomy coefficients

$$W^b_{ia} = \partial_a N^b_i, \quad W^a_{ji} = \Omega^a_{ij} = \mathbf{e}_j\left(N^a_i\right) - \mathbf{e}_i(N^a_j). \qquad (10)$$

Such a basis is holonomic if and only if $W^\gamma_{\alpha\beta} = 0$. This is trivially satisfied in a coordinate basis if $\mathbf{e}_\alpha = \partial_\alpha$. As holonomic dual basis, we take $\mathbf{e}^\mu = du^\mu$.

The geometric objects on $\mathbf{V}$ are defined with respect to the N-adapted frames (7), (8). These are referred to as distinguished objects or d-objects in short. A vector $Y(u) \in T\mathbf{V}$ is parameterized as a d-vector. Explicitly, $\mathbf{Y} = \mathbf{Y}^\alpha \mathbf{e}_\alpha = \mathbf{Y}^i \mathbf{e}_i + \mathbf{Y}^a e_a$, or $\mathbf{Y} = (hY, vY)$, with $hY = \{\mathbf{Y}^i\}$ and $vY = \{\mathbf{Y}^a\}$. Likewise, in this frame work, the coefficients of d-tensors, N-adapted differential forms, d-connections, and d-spinors are easily accommodated.

Any metric tensor $\mathbf{g}$ on $\mathbf{V}$, defined as a second rank symmetric tensor, takes the following structure with respect to the dual local coordinate basis:

$$\mathbf{g} = \underline{g}_{\alpha\beta} du^{\alpha} \otimes du^{\beta},$$

where

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b g_{ab} & N_j^e g_{ae} \\ N_i^e g_{be} & g_{ab} \end{bmatrix}. \quad (11)$$

Equivalently, $\mathbf{g}$ serves as the d-metric and, in tensor product notation, is taken to be

$$\mathbf{g} = g_{\alpha}(u)\mathbf{e}^{\alpha} \otimes \mathbf{e}^{\beta} = g_i(x)dx^i \otimes dx^i + g_a(x,y)\mathbf{e}^a \otimes \mathbf{e}^a. \quad (12)$$

Linear connections on $\mathbf{V}$ are introduced in N-adapted and N-non-adapted forms in the standard way. By definition, a d-connection $\mathbf{D} = (hD, vD)$ preserves under parallelism the N-connection splitting (2). Any d-connection $\mathbf{D}$ acts as a covariant derivative operator, $\mathbf{D}_{\mathbf{X}}\mathbf{Y}$, for a d-vector $\mathbf{Y}$ in the direction of a d-vector $\mathbf{X}$. With respect to N-adapted frames (7) and (8), we can compute the relevant quantities of interest in N-adapted coefficient form when $\mathbf{D} = \{\Gamma^{\gamma}_{\ \alpha\beta} = (L^i_{\ jk}, L^a_{\ bk}, C^i_{\ jc}, C^a_{\ bc})\}$. The coefficients $\Gamma^{\gamma}_{\ \alpha\beta}$ are computed for the horizontal and vertical components of $\mathbf{D}_{\mathbf{e}_{\alpha}}\mathbf{e}_{\beta} := \mathbf{D}_{\alpha}\mathbf{e}_{\beta}$ by substituting $\mathbf{X}$ for $\mathbf{e}_{\alpha}$ and $\mathbf{Y}$ for $\mathbf{e}_{\beta}$.

We compute the d-torsion $\mathcal{T}$, the d-torsion nonmetricity $\mathcal{Q}$, and the d-curvature $\mathcal{R}$ for any d-connection $\mathbf{D}$ from the following standard formulae:

$$\mathcal{T}(\mathbf{X},\mathbf{Y}) := \mathbf{D}_{\mathbf{X}}\mathbf{Y} - \mathbf{D}_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}], \ \mathcal{Q}(\mathbf{X}) := \mathbf{D}_{\mathbf{X}}\mathbf{g}, \quad (13)$$

$$\mathcal{R}(\mathbf{X},\mathbf{Y}) := \mathbf{D}_{\mathbf{X}}\mathbf{D}_{\mathbf{Y}} - \mathbf{D}_{\mathbf{Y}}\mathbf{D}_{\mathbf{X}} - \mathbf{D}_{[\mathbf{X},\mathbf{Y}]}. \quad (14)$$

The N-adapted coefficients are correspondingly labelled

$$\mathcal{T} = \left\{\mathbf{T}^{\gamma}_{\ \alpha\beta} = \left(T^i_{\ jk}, T^i_{\ ja}, T^a_{\ ji}, T^a_{\ bi}, T^a_{\ bc}\right)\right\}, \ \mathcal{Q} = \{\mathbf{Q}^{\gamma}_{\ \alpha\beta}\},$$
$$\mathcal{R} = \left\{\mathbf{R}^{\alpha}_{\ \beta\gamma\delta} = \left(R^i_{\ hjk}, R^a_{\ bjk}, R^i_{\ hja}, R^c_{\ bja}, R^i_{\ hba}, R^c_{\ bea}\right)\right\}.$$

The Levi-Civita connection ∇ (LC) and the canonical d-connection $\widehat{\mathbf{D}}$ defined by Eq. (3) are also expressed in terms of the local N-adapted form. The coefficients of $\widehat{\mathbf{D}} = \{\widehat{\Gamma}^{\gamma}_{\ \alpha\beta} = (\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc})\}$ depend on $(g_{\alpha\beta}, N^a_i)$ and are computed using the following formulae:

$$\widehat{L}^i_{jk} = \frac{1}{2}g^{ir}\left(\mathbf{e}_k g_{jr} + \mathbf{e}_j g_{kr} - \mathbf{e}_r g_{jk}\right),$$
$$\widehat{C}^a_{bc} = \frac{1}{2}g^{ad}\left(e_c g_{bd} + e_b g_{cd} - e_d g_{bc}\right)$$
$$\widehat{C}^i_{jc} = \frac{1}{2}g^{ik}e_c g_{jk}, \ \widehat{L}^a_{bk} = e_b(N^a_k)$$
$$+\frac{1}{2}g^{ac}\left(\mathbf{e}_k g_{bc} - g_{dc}\, e_b N^d_k - g_{db}\, e_c N^d_k\right). \quad (15)$$

By using the coefficients of $\nabla = \{\Gamma^{\gamma}_{\ \alpha\beta}\}$, written with respect to (7) and (8), we compute the coefficients of the distortion d-tensor $\widehat{\mathbf{Z}}^{\gamma}_{\ \alpha\beta} = \widehat{\Gamma}^{\gamma}_{\ \alpha\beta} - \Gamma^{\gamma}_{\ \alpha\beta}$, which is the N-adapted coefficient formula for (1). We elaborate upon geometric and physical models in equivalent form by working with two metric compatible connections $\widehat{\mathbf{D}}$ and ∇ because all N-adapted coefficients for $\widehat{\mathbf{Z}}^{\gamma}_{\ \alpha\beta} = \widehat{\Gamma}^{\gamma}_{\ \alpha\beta}$ and $\Gamma^{\gamma}_{\ \alpha\beta}$ are completely defined by the same metric structure $\mathbf{g}$. The non-trivial d-torsions coefficients $\widehat{\mathbf{T}}^{\gamma}_{\ \alpha\beta}$ are computed by setting $\mathbf{D} = \widehat{\mathbf{D}}$ in (13) and determined by the nonholonomy relations,

$$\widehat{T}^i_{\ jk} = \widehat{L}^i_{jk} - \widehat{L}^i_{kj}, \ \ \widehat{T}^i_{\ ja} = \widehat{C}^i_{jb}, \ \ \widehat{T}^a_{\ ji} = -\Omega^a_{\ ji},$$
$$\widehat{T}^c_{aj} = \widehat{L}^c_{aj} - e_a(N^c_j), \ \ \widehat{T}^a_{\ bc} = \widehat{C}^a_{bc} - \widehat{C}^a_{cb}. \quad (16)$$

Any (pseudo) Riemannian geometry is formulated on a nonholonomic manifold $\mathbf{V}$ using two equivalent geometric quantities, $(\mathbf{g}, \nabla)$ or $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$. In the "standard" method we take $\mathbf{D} \to \nabla$ when ${}^{\nabla}T^{\gamma}_{\ \alpha\beta} = 0$, ${}^{\nabla}Q^{\gamma}_{\ \alpha\beta} = 0$, and ${}^{\nabla}R^{\alpha}_{\ \beta\gamma\delta}$ is computed following Eqs. (14). For the "geometric variables" $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$, using similar formulae, we compute $\mathbf{D} = \widehat{\mathbf{D}}$ in standard form respectively the Riemann d-tensor $\widehat{\mathcal{R}}$ and the Ricci d-tensor $\widehat{\mathcal{R}}ic\{= \widehat{\mathbf{R}}_{\ \beta\gamma}\}$. The nonsymmetric d-tensor $\widehat{\mathbf{R}}_{\alpha\beta}$ of $\widehat{\mathbf{D}}$ is characterized by the following four h and v N-adapted coefficients:

$$\widehat{\mathbf{R}}_{\alpha\beta} = \left\{\widehat{R}_{ij} := \widehat{R}^k_{\ ijk}, \ \widehat{R}_{ia} := -\widehat{R}^k_{\ ika},\right.$$
$$\left.\widehat{R}_{ai} := \widehat{R}^b_{\ aib}, \ \widehat{R}_{ab} := \widehat{R}^c_{\ abc}\right\}, \quad (17)$$

and the "alternative" scalar curvature

$$\widehat{R} := \mathbf{g}^{\alpha\beta}\widehat{\mathbf{R}}_{\alpha\beta} = g^{ij}\widehat{R}_{ij} + g^{ab}\widehat{R}_{ab}. \quad (18)$$

The Einstein d-tensor of $\widehat{\mathbf{D}}$ in hatted form is

$$\widehat{\mathbf{G}}_{\alpha\beta} := \widehat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\ \widehat{R} \quad (19)$$

and is a nonholonomic distortion of the standard form, $G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\ R$, which is computed from ∇. We solve the equations resulting from the constraints (5) and get solutions to a system of first order PDE equations

$$\widehat{L}^c_{aj} = e_a(N^c_j), \ \ \widehat{C}^i_{jb} = 0, \Omega^a_{\ ji} = 0. \quad (20)$$

Nonholonomic deformations of fundamental geometric objects on a pseudo-Riemannian manifold $\mathbf{V}$ with N-connection 2+2 splitting are determined by the transforming of the fundamental geometric data $(\mathring{\mathbf{g}}, \mathring{\mathbf{N}}, {}^{\circ}\widehat{\mathbf{D}}) \to (\widehat{\mathbf{g}}, \mathbf{N}, \widehat{\mathbf{D}})$, where the "prime" data $(\mathring{\mathbf{g}}, \mathring{\mathbf{N}}, {}^{\circ}\widehat{\mathbf{D}})$ may or not be a solution of certain gravitational field equations in a (modified) theory of gravity but the "target" data $(\widehat{\mathbf{g}}, \mathbf{N}, \widehat{\mathbf{D}})$ affirmatively define exact solutions of (4) with metrics parameterized in the form (11) and (21).

The prime metric is parameterized as

$$\mathring{\mathbf{g}} = \mathring{g}_{\alpha}(u)\mathring{\mathbf{e}}^{\alpha} \otimes \mathring{\mathbf{e}}^{\beta} = \mathring{g}_i(x)dx^i \otimes dx^i + \mathring{g}_a(x,y)\mathring{\mathbf{e}}^a \otimes \mathring{\mathbf{e}}^a, \text{ for}$$
$$\mathring{\mathbf{e}}^{\alpha} = (dx^i, \mathbf{e}^a = dy^a + \mathring{N}^a_i(u)dx^i),$$
$$\mathring{\mathbf{e}}_{\alpha} = (\mathring{\mathbf{e}}_i = \partial/\partial y^a - \mathring{N}^b_i(u)\partial/\partial y^b, \ e_a = \partial/\partial y^a).$$

As an explicit example, we take $\mathring{\mathbf{g}}$ to be a Friedman–Lemaître–Robertson–Walker (FLRW) type diagonal metric with $\mathring{N}^b_i = 0$. The target off-diagonal metric is of type (12) with $\mathbf{e}^a$

taken as in (8). With additional parameterizations via the so-called gravitational "polarization" functions $\eta_\alpha = (\eta_i, \eta_a)$, the metric $\widehat{\mathbf{g}}$ takes the form

$$\widehat{\mathbf{g}} = g_\alpha(u)\mathbf{e}^\alpha \otimes \mathbf{e}^\beta = g_i(x)\mathrm{d}x^i \otimes \mathrm{d}x^i + g_a(x,y)\mathbf{e}^a \otimes \mathbf{e}^a$$
$$= \eta_i(x^k)\mathring{g}_i \mathrm{d}x^i \otimes \mathrm{d}x^i + \eta_a(x^k, y^b)\mathring{h}_a \mathbf{e}^a \otimes \mathbf{e}^a. \quad (21)$$

In the special case in which $\eta_\alpha \to 1$ and $N_i^a = \mathring{N}_i^a$, we get a trivial nonholonomic transformation (deformation).

For the data $(\widehat{\mathbf{g}}, \widehat{\mathbf{D}})$ the effective source for a scalar field ϕ and a gauge field $\mathbf{F}^{\check{a}}_{\mu\nu}$ in modified gravitational interactions (4) is the energy-momentum tensor ${}^e\mathbf{T}_{\alpha\beta}$ where

$${}^e\mathbf{T}_{\alpha\beta} = \frac{1}{2}\left[\mathbf{e}_\alpha\phi\ \mathbf{e}_\beta\phi + \mathbf{e}_\beta\phi\ \mathbf{e}_\alpha\phi\ - \widehat{\mathbf{g}}_{\alpha\beta}\widehat{\mathbf{g}}^{\mu\nu}\mathbf{e}_\mu\phi\ \mathbf{e}_\nu\phi\right.$$
$$\left. + \widehat{\mathbf{g}}_{\alpha\beta}\ {}^eV(\phi)\right] + \mathbf{F}^{\check{a}}_{\ \alpha\nu}\mathbf{F}^{\check{a}\nu}_{\ \ \beta} - \frac{1}{4}\widehat{\mathbf{g}}_{\alpha\beta}\mathbf{F}^{\check{a}}_{\ \nu\mu}\mathbf{F}^{\check{a}\nu\mu}, \quad (22)$$

where $\check{a}$ is an internal group index. This tensor is constructed with respect to the N-adapted (co) frames (7), (8) following the same procedure as in Refs. [14–19], and $\mathbf{\Upsilon}_{\alpha\beta} = \frac{\kappa}{2}\ {}^e\mathbf{T}_{\alpha\beta}$ where κ is the gravitational constant. The explicit coordinate dependence for $\mathbf{\Upsilon}$ is

$$\mathbf{\Upsilon}^1_{\ 1} = \mathbf{\Upsilon}^2_{\ 2} = \Upsilon(x^k, t); \quad \mathbf{\Upsilon}^3_{\ 3} = \mathbf{\Upsilon}^4_{\ 4} = {}^v\mathbf{\Upsilon}(x^k)]. \quad (23)$$

Elements with $\alpha \neq \beta$ are all taken to be zero. The effective nonlinear scalar potential ${}^eV$ is determined by two scalar potentials $V(\phi)$ and $U(\phi)$ as

$${}^eV = (V+M)^2/4U, \quad (24)$$

where M is a constant. The Einstein d-tensor $\widehat{\mathbf{G}}_{\alpha\beta}$ is given in N-adapted form by Eq. (19). The resulting nonlinear system of PDEs can be integrated in explicit form for arbitrary parameterizations of type $\mathbf{\Upsilon}^\beta_{\ \delta} = diag[\mathbf{\Upsilon}_\alpha]$.[4]

As a specific example, we take the TMT effective action

$$S = \frac{1}{\kappa}\int \mathrm{d}^4u\sqrt{|\widehat{\mathbf{g}}_{\alpha\beta}|}\left[\widehat{R} + {}^m\widehat{L}\right] \quad (25)$$

studied in [14,15,17–19] for ${}^m\widehat{L}$ resulting in the energy-momentum tensor (22) and where $\widehat{R}$ is the scalar curvature. Modified Einstein equations are derived in the light of the LC-conditions (20). The energy-momentum tensor follows from the variation in N-adapted form using the N-elongated partial derivatives and differentials:

$${}^e\mathbf{T}_{\mu\nu} := -\frac{2}{\sqrt{|\widehat{\mathbf{g}}_{\alpha\beta}|}}\frac{\delta(\sqrt{|\widehat{\mathbf{g}}_{\alpha\beta}|}\ {}^m\widehat{L})}{\delta\ \widehat{\mathbf{g}}^{\mu\nu}}.$$

We consider a new 'scaled' d-metric $\mathbf{g}_{\alpha\beta}$ where

$$\widehat{\mathbf{g}}_{\alpha\beta} = e^{-2\widehat{\sigma}(u)}\mathbf{g}_{\alpha\beta}, \text{ and}$$
$$e^{-2\widehat{\sigma}(u)} = 2U/(V+M) = \Phi/\sqrt{|\mathbf{g}_{\alpha\beta}|}, \quad (26)$$

where $e^{-2\widehat{\sigma}}$ is the scale factor determined in terms of the constant and potentials used in the effective potential ${}^eV$ (24). The function

$$\Phi = \varepsilon^{\mu\nu\alpha\beta}\mathbf{e}_\mu\mathbf{A}_{\nu\alpha\beta} = \varepsilon^{\mu\nu\alpha\beta}\varepsilon^{\underline{abcd}}\mathbf{e}_\mu\varphi_{\underline{a}}\ \mathbf{e}_\nu\varphi_{\underline{b}}\ \mathbf{e}_\alpha\varphi_{\underline{c}}\ \mathbf{e}_\beta\varphi_{\underline{d}},$$

with four scalar fields $\varphi_{\underline{a}}$, $(\underline{a} = 1,2,3,4)$, defines the second measure in TMTs. The effective gravitational theory (25) with the source ${}^e\mathbf{T}_{\alpha\beta}$ (18) and rescaling properties (26) is equivalent to the theory given by the following action:

$$S = \int {}^1L\Phi\mathrm{d}^4u + \int {}^2L\sqrt{|\mathbf{g}_{\alpha\beta}|}\mathrm{d}^4x$$
$$+ \int N\phi\varepsilon^{\mu\nu\alpha\beta}F^{\check{a}}_{\mu\nu}F^{\check{a}}_{\alpha\beta}\mathrm{d}^4u, \quad (27)$$

where

$${}^1L = -\frac{1}{\kappa}\widehat{R}(\mathbf{g}) + \frac{1}{2}\mathbf{g}^{\mu\nu}\mathbf{e}_\mu\phi\ \mathbf{e}_\nu\phi - V(\phi) \text{ and}$$
$${}^2L = U(\phi) - \frac{1}{4}F^{\check{a}}_{\mu\nu}F^{\check{a}\mu\nu}. \quad (28)$$

In the above, the N term.[5] It is a CP violating parameter and is determined to be very small from constraints from phenomenology. The non-Riemannian configuration is determined from the canonical d-connection $\widehat{\Gamma}^\alpha_{\beta\gamma}$ for $\mathbf{g}_{\alpha\beta}$.

Identifying the scalar indices as interior indices ("overline check") and varying (27) with respect to $\varphi_{\check{a}}$ in N-adapted form, we obtain the equation

$$\mathbf{A}^\mu_{\check{a}}\ \mathbf{e}_\mu\ {}^1L = 0. \quad (29)$$

The solution of this equation is $\mathbf{e}_\mu\ {}^1L = 0$, or ${}^1L = M = const$. Thus for any $M \neq 0$, we obtain a spontaneous breaking of global scale invariance of the theory. This follows from the mismatch between the left hand side and the right hand side of the equation. If we fix M as an integration constant for the right hand side, the left hand side has a non-trivial transformation. In terms of the metric $\widehat{\mathbf{g}}_{\alpha\beta}$, the equation for the scalar field becomes

$$\mathbf{e}_\mu(\sqrt{|\mathbf{g}_{\alpha\beta}|}\widehat{\mathbf{g}}^{\mu\nu}\phi) + \sqrt{|\mathbf{g}_{\alpha\beta}|}\frac{d\ {}^eV(\phi)}{d\phi} + N\varepsilon^{\mu\nu\alpha\beta}F^{\check{a}}_{\mu\nu}F^{\check{a}}_{\alpha\beta} = 0. \quad (30)$$

Not considering effective gauge interactions, i.e. for $N = 0$, we define the vacuum states for $V + M = 0$, where ${}^eV = 0$ and $\mathrm{d}\ {}^eV/\mathrm{d}\phi = 0$ (it is also considered that $\mathrm{d}\ {}^eV/\mathrm{d}\phi$ is finite and $U \neq 0$). We conclude that the basic feature of TMTs

[4] We can consider other distributions which do not allow for the construction of solutions in explicit form. Our geometric approach will be applied to such N-connection splitting and frame/ coordinate transforms that parameterize the effective sources in some form and will admit the decoupling of the (modified) Einstein equations.

[5] Such a "non-boldface" symbol should not be confused with the N-connection $\mathbf{N} = \{N^a_I\}$; we maintain standard notations in gravity theories with N-connections (boldface symbols). In TMT models N has a completely different meaning as introduced in [14–19].

do not depend on the type of nonholonomic distributions on spacetimes if we work with metric compatible canonical d-connections or the LC connections. For both cases, we solve the "old" cosmological constant problem, implying that the vacuum state with zero cosmological constant is achieved for different types of linear connections and without resort to fine tuning. Independently of whether we change the value of constant M or add a constant to V, we still satisfy the conditions ${}^{e}V=0$ and $\mathrm{d}\,{}^{e}V/\mathrm{d}\phi=0$ if $V+M=0$. Here we also note that if we consider $N\neq 0$, it implies that an external source drives the scalar field away from such vacuum points and can be addressed in terms of instanton effects.

N-adapted variations with respect to $\mathbf{g}^{\mu\nu}$ result in the equation

$$\Phi\left[-\frac{1}{\kappa}\widehat{R}(\mathbf{g})+\frac{1}{4}(\mathbf{e}_{\mu}\phi\ \mathbf{e}_{\nu}\phi+\ \mathbf{e}_{\nu}\phi\ \mathbf{e}_{\mu}\phi)\right] -\frac{1}{2}\sqrt{|\mathbf{g}_{\alpha\beta}|}U(\phi)\mathbf{g}_{\mu\nu}+\sqrt{|g|}\left[\mathbf{F}^{a}_{\mu\alpha}\mathbf{F}^{a\ \alpha}_{\beta}-\frac{1}{4}\mathbf{g}_{\mu\nu}\mathbf{F}^{a}_{\alpha\beta}\mathbf{F}^{a\alpha\beta}\right]=0, \quad (31)$$

where $\underline{a}=\check{a}$ for this class of TMT theories. Additional constraints for LC-configurations when Eq. (20) for the data $(\mathbf{g},\widehat{\mathbf{D}}[\mathbf{g}])$ are satisfied transform (31) into the system (17) in [16]. A small vacuum density determined by instantons was analyzed for LC-configurations of (30). It is a cumbersome task to find cosmological solutions of the system defined by Eqs. (29)–(31). Nevertheless, it is possible to construct generic off-diagonal cosmological solutions for the systems of modified commutative and noncommutative Einstein–Yang–Mills–Higgs fields using the AFDM [20,23,26]. Our strategy is to find solutions for the theory (25) resulting in modified Einstein equation (4) with effective stress-energy tensor (22) and effective source (23). Metrics such as $\widehat{\mathbf{g}}_{\alpha\beta}$, in general, transform into $\mathbf{g}_{\alpha\beta}$ for the theory (27) using N-adapted conformal transforms of type (26).

We integrate in explicit form Eq. (4) with a source (23) for the N-adapted coefficients of a metric $\widehat{\mathbf{g}}$ (21) parameterized in the form

$$g_i=e^{\psi(x^k)},\quad g_a=\omega(x^k,y^b)h_a(x^k,t),$$
$$N_i^3=n_i(x^k,t),\quad N_i^4=w_i(x^k,t) \quad (32)$$

and supplementing with frame/coordinate transformations that satisfy the conditions $h_a^{\diamond}\neq 0$, $\mathbf{\Upsilon}_{2,4}\neq 0$.[6] For convenience, the partial derivatives $\partial_\alpha=\partial/\partial u^\alpha$ are labelled

$$\partial_1 s=s^{\bullet}=\partial s/\partial x^1,\ \partial_2 s=s'=\partial s/\partial x^2,$$
$$\partial_3 s=\partial s/\partial y^3,\ \partial_4 s=\partial s/\partial t=\partial_t s,\ \partial^2 s/\partial t^2=\partial^2_{tt}s.$$

The non-trivial components of the Ricci and Einstein d-tensors are computed using the N-adapted coefficients of the canonical d-connection (15) for the metric ansatz (21) with data (32) for $\omega=1$ introduced, respectively, in (17), (18) and (19). Eventually, we arrive at the following system of nonlinear PDEs:

$$\widehat{R}^1_1=\widehat{R}^2_2=\frac{1}{2g_1g_2}\left[\frac{g_1^{\bullet}g_2^{\bullet}}{2g_1}+\frac{(g_2^{\bullet})^2}{2g_2}-g_2^{\bullet\bullet}+\frac{g_1'g_2'}{2g_2}+\frac{(g_1')^2}{2g_1}-g_1''\right]=-\,{}^{v}\mathbf{\Upsilon}, \quad (33)$$

$$\widehat{R}^3_3=\widehat{R}^4_4=\frac{1}{2h_3h_4}\left[\frac{(\partial_t h_3)^2}{2h_3}+\frac{\partial_t h_3\ \partial_t h_4}{2h_4}-\partial^2_{tt}h_3\right]=-\mathbf{\Upsilon} \quad (34)$$

$$\widehat{R}_{3k}=\frac{h_3}{2h_4}\partial^2_{tt}n_k+\left(\frac{h_3}{h_4}\partial_t h_4-\frac{3}{2}\partial_t h_3\right)\frac{\partial_t n_k}{2h_4}=0, \quad (35)$$

$$\widehat{R}_{4k}=\frac{w_k}{2h_3}\left[\partial^2_{tt}h_3-\frac{(\partial_t h_3)^2}{2h_3}-\frac{\partial_t h_3\ \partial_t h_4}{2h_4}\right]+\frac{\partial_t h_3}{4h_3}\left(\frac{\partial_k h_3}{h_3}+\frac{\partial_k h_4}{h_4}\right)-\frac{\partial_k\partial_t h_3}{2h_3}=0. \quad (36)$$

The torsionless (Levi-Civita, LC) conditions (5), (20) transform into

$$\partial_t w_i=(\partial_i-w_i\partial_t)\ln\sqrt{|h_4|},\ (\partial_i-w_i\partial_t)\ln\sqrt{|h_3|}=0,$$
$$\partial_k w_i=\partial_i w_k,\ \partial_t n_i=0,\ \partial_i n_k=\partial_k n_i. \quad (37)$$

The system of nonlinear PDE (33)–(36) posses an important decoupling property which admits step by step integration of such equations. To achieve this, first we introduce the coefficients

$$\alpha_i=(\partial_t h_3)\ (\partial_i\varpi),\ \ \beta=(\partial_t h_3)\ (\partial_t\varpi),$$
$$\gamma=\partial_t\left(\ln|h_3|^{3/2}/|h_4|\right), \quad (38)$$

where

$$\varpi=\ln|\partial_t h_3/\sqrt{|h_3h_4|}|. \quad (39)$$

The coefficients serve as generating functions. For $\partial_t h_a\neq 0$ and $\partial_t\varpi\neq 0$,[7] we rewrite the equations in the form

$$\psi^{\bullet\bullet}+\psi''=2\,{}^{v}\mathbf{\Upsilon} \quad (40)$$
$$\partial_t\varpi\ \partial_t h_3=2h_3h_4\mathbf{\Upsilon} \quad (41)$$
$$\partial^2_{tt}n_i+\gamma\partial_t n_i=0, \quad (42)$$
$$\beta w_i-\alpha_i=0, \quad (43)$$
$$\partial_i\omega-n_i\partial_3\omega-(\partial_i\varpi/\partial_t\varpi)\partial_t\omega=0. \quad (44)$$

The functions $\psi(x^k)$ are found by solving a two dimensional Poisson equation (40) for any prescribed source ${}^{v}\mathbf{\Upsilon}(x^k)$. Equations (39) and (41) convert any two functions to two others from a set of four, h_a, ϖ and $\mathbf{\Upsilon}$. In one explicit form, h_3 and h_4 are determined for any prescribed $\varpi(x^k,t)$ and

[6] For simplicity, we shall omit "hats" on coefficients of type g_i, g_a, n_i, w_i etc. related to $\widehat{\mathbf{g}}$ if it will not lead to ambiguities.

[7] Non-trivial solutions result if such conditions are not satisfied; in such cases, we need to consider other special methods for generating solutions.

$\Upsilon(x^k,t)$. Once h_a are determined, we integrate twice w.r.t. t in (42) and find $n_i(x^k,t)$. In the final step we solve for $w_i(x^i,y^a)$ by solving a system of linear algebraic equations (43). Equation (44) is necessary to accommodate a non-trivial conformal (in the vertical "subspace") factor $\omega(x^i,y^a)$ that depends on all four coordinates. For convenience, we shall use $\Psi := e^{\varpi}$ as our redefined generating function.

We have shown that TMT theories as determined by actions of type (27) can be formulated in nonholonomic variables as effective EYMH systems with modified Einstein field equation (4). This allows one to apply the AFDM and decouple such systems of nonlinear PDEs in very general form and write them equivalently as systems of type (40)–(44). This procedure and the resulting equations provide important results for mathematical cosmology. For instance, by considering the coordinate $y^4 = t$ to be time-like, one can show that TMT theories and other modified gravity models can be integrated in general forms.

3 Off-diagonal cosmological solutions with small vacuum density

In this section we provide a series of examples of new classes of exact solutions of modified Einstein equations with (non) homogeneous cosmological configurations constructed by applying the AFDM. We emphasize that all solutions generated in this section will be for a TMT theory with sources (22) parameterized in the form (23), when the effective nonlinear scalar potential is taken in the form (24). In a similar form, we can construct solutions with effective sources for other types of modified gravity theories like in [25,47–49].

For any $\partial_t\varpi \neq 0$, $\partial_t h_a \neq 0$ and $\Upsilon \neq 0$, we write (41) and (39) as

$$h_3 h_4 = (\partial_t\varpi)(\partial_t h_3)/2\Upsilon \quad \text{and} \quad |h_3 h_4| = (\partial_t h_3)^2 e^{-2\varpi}. \tag{45}$$

Using $\Psi := e^{\varpi}$ and introducing the first equation into the second in (45), we obtain the relation $|\partial_t h_3| = \partial_t[\Psi^2]/4|\Upsilon|$. Integrating with respect to t, we get

$$h_3[\Psi,\Upsilon] = {}^0h_3(x^k) - \frac{1}{4}\int dt \frac{\partial_t(\Psi^2)}{\Upsilon}, \tag{46}$$

where ${}^0h_3 = {}^0h_3(x^k)$ is an integration function. We use the first equation in (45) and compute

$$h_4[\Psi,\Upsilon] = \frac{1}{2}\frac{\partial_t\Psi}{\Upsilon}\frac{\partial_t h_3}{\Psi}\frac{}{h_3}. \tag{47}$$

Formulae for h_a are expressed in a more convenient form by considering an effective cosmological constant $\Lambda_0 = const \neq 0$ and a redefined generating function, $\Psi \to \tilde{\Psi}$, subject to the condition

$$\frac{\partial_t[\Psi^2]}{\Upsilon} = \frac{\partial_t[\tilde{\Psi}^2]}{\Lambda_0},$$

where the integration function ${}^0h_3(x^k)$ from (46) is formally introduced either in $\widetilde{\Psi}$ or equivalently in Υ.

Our final results are

$$h_3[\widetilde{\Psi},\Lambda_0] = \frac{\widetilde{\Psi}^2}{4\Lambda_0} \quad \text{and} \quad h_4[\widetilde{\Psi},\Lambda_0,\Xi] = \frac{(\partial_t\widetilde{\Psi})^2}{\Xi} \tag{48}$$

and hold for an effective cosmological constant $\Lambda_0 \neq 0$ so that redefinition of the generating functions, $\Psi \longleftrightarrow \widetilde{\Psi}$, are unambiguous where

$$\Psi^2 = \Lambda_0^{-1}\int dt\Upsilon\partial_t(\widetilde{\Psi}^2) \quad \text{and} \quad \widetilde{\Psi}^2 = \Lambda_0\int dt\Upsilon^{-1}\partial_t(\Psi^2). \tag{49}$$

The functional

$$\Xi[\Upsilon,\widetilde{\Psi}] = \int dt\Upsilon\partial_t(\widetilde{\Psi}^2)$$

in the formula for h_4 in (48) is interpreted as a redefined source $\Upsilon \to \Xi$ for a prescribed generating function $\widetilde{\Psi}$ when $\Upsilon = \partial_t\Xi/\partial_t(\widetilde{\Psi}^2)$. Such effective sources contain information on effective matter field contributions in modified gravity theories. We work with the generating quantities, $(\Psi, {}^v\Lambda)$ and $[\widetilde{\Psi},\Lambda_0,\Xi]$ related via Eqs. (49) in terms of the prescribed effective cosmological constant Λ_0. The numerical value of Λ_0 is fixed to meet present day constraints from cosmology.

Using formulae h_a (48), we compute the coefficients α_i, β and γ from (38). This allows us to find solutions to Eqs. (42) by integrating two times with respect to t, and (43), solving a system of linear algebraic equations for w_i. As a result, the N-coefficients are expressed recurrently as functionals (an example of which is $[\widetilde{\Psi},\Lambda_0,\Xi]$) and are as follows:

$$\begin{aligned} n_k &= {}_1n_k + {}_2n_k\int dt h_4/(\sqrt{|h_3|})^3 = {}_1n_k \\ &\quad + {}_2\widetilde{n}_k\int dt(\partial_t\widetilde{\Psi})^2/\widetilde{\Psi}^3\Xi, \quad \text{and} \\ w_i &= \partial_i\varpi/\partial_t\varpi = \partial_i\Psi/\partial_t\Psi = \partial_i\Psi^2/\partial_t\Psi^2 \\ &= \int dt\partial_i[\Upsilon\partial_t(\widetilde{\Psi}^2)]/\Upsilon\partial_t(\widetilde{\Psi}^2) = \partial_i\Xi/\partial_t\Xi, \end{aligned} \tag{50}$$

where ${}_1n_k(x^i)$ and ${}_2n_k(x^i)$, or ${}_2\widetilde{n}_k(x^i)$, are integration functions with possible redefinitions by coordinate transforms.

After a tedious calculation for $g_a = \omega^2(x^k,y^a)h_a$ that involves the vertical conformal factor $\omega(u^\alpha)$ depending on all spacetime coordinates, the vertical metric h_a (48) and the N-coefficients N_i^a (50) reveals the fact that the formulae for the Ricci d-tensor $\widehat{\mathbf{R}}_{\alpha\beta}$ (17) are invariant if the first order PDE (44) are satisfied. For non-trivial ω, the solutions to the modified gravitational equation (4), parameterized as a d-metric (21), do not posses in general any Killing symmetries and contain dependencies of ω on $[\psi, h_a, n_i, w_i]$ with as many as six independent variables for $\mathbf{g}_{\alpha\beta}$.

Putting together the solutions for the 2-d Poisson equation (40) and the formulae for the coefficients (48), (50) we conclude as our final result that the system of nonlinear PDEs (40)–(43) for non-vacuum 4-d configurations for the data $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$, and with Killing symmetry on ∂_3 when $\omega = 1$, integrates to the line element

$$ds^2 = g_{\alpha\beta}(x^k, t)du^\alpha du^\beta = e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2] + \omega^2 \frac{\widetilde{\Psi}^2}{4\Lambda_0}\left[dy^3 + \left({}_1n_k + {}_2\widetilde{n}_k \int dt \frac{(\partial_t \widetilde{\Psi})^2}{\widetilde{\Psi}^3\ \Xi}\right) dx^k\right]^2 + \omega^2 \frac{(\partial_t \widetilde{\Psi})^2}{\Xi}\left[dt + \frac{\partial_i \Xi}{\partial_t \Xi} dx^i\right]^2. \quad (51)$$

Such inhomogeneous cosmological solutions with nonholonomically induced torsion are determined by $\psi(x^k)$, $\widetilde{\Psi}(x^k, t)$, $\omega(x^k, y^3, t)$, $\Xi(x^k, t)$ that depend on the effective cosmological constant Λ_0 and integration functions ${}_1n_k$, ${}_2\widetilde{n}_k$. Straightforward computations reveal that, in general, the nonholonomy coefficients $W^\gamma_{\alpha\beta}$ (10) are non-vanishing. Therefore the class of solutions (51) cannot be diagonalized in N-adapted form unless supplemented with additional assumptions on generating/ integration functions and constants. The non-trivial coefficients of the canonical d-torsion (13) are also non-vanishing. They are determined by introducing the coefficients of the d-metric into the N-adapted Eqs. (15) and then into $\widehat{\mathbf{T}}^\gamma_{\alpha\beta}$ (16).

Let us prove that the zero d-torsion conditions (37) for LC-configurations can be solved in explicit form by imposing additional constraints on the d-metrics (51). For the n -coefficients, such conditions are satisfied if ${}_2n_k(x^i) = 0$ and $\partial_i\ {}_1n_j(x^k) = \partial_j\ {}_1n_i(x^k)$. In N-adapted form, such coefficients do not depend on generating functions and sources but only on a corresponding class of integration functions, e.g., ${}_1n_j(x^k) = \partial_i n(x^k)$, for any $n(x^k)$. It is a more difficult task to find explicit solutions for the LC-conditions (37) involving variables $w_i(x^k)$. Such nonholonomic constraints cannot be solved in explicit form for arbitrary data $(\Psi, \boldsymbol{\Upsilon})$, or arbitrary $(\tilde{\Psi}, \Xi, \Lambda_0)$. We first use the property that $\mathbf{e}_i \Psi = (\partial_i - w_i \partial_t)\Psi \equiv 0$ for any Ψ if $w_i = \partial_i \Psi / \partial_t \Psi$ (it follows from Eq. (50)). This results in the expression

$$\mathbf{e}_i H = (\partial_i - w_i \partial_t) H = \frac{\partial H}{\partial \Psi}(\partial_i - w_i \partial_t)\Psi \equiv 0$$

for any functional $H[\Psi]$. The second step is to restrict our construction to a subclass of variables when $H = \tilde{\Psi}[\Psi]$ is a functional which allows us to generate LC-configurations in explicit form. By taking $h_3[\tilde{\Psi}] = \tilde{\Psi}^2/4\Lambda_0$ (48) as a necessary type of functional $H = \tilde{\Psi} = \ln\sqrt{|\ h_3|}$, we satisfy the condition $\mathbf{e}_i \ln\sqrt{|\ h_3|} = 0$ in (37).

Next, we solve for the constraint on h_4. The derivative ∂_4 of $w_i = \partial_i \Psi / \partial_t \Psi$ (50) results in

$$\partial_t w_i = \frac{(\partial_t \partial_i \Psi)(\partial_t \Psi) - (\partial_i \Psi)\partial_t^2 \Psi}{(\partial_t \Psi)^2} = \frac{\partial_t \partial_i \Psi}{\partial_t \Psi} - \frac{\partial_i \Psi}{\partial_t \Psi}\frac{\partial_t^2 \Psi}{\partial_t \Psi}.$$

Substituting in this formula the generating function $\Psi = \check{\Psi}$ gives

$$\partial_t \partial_i \check{\Psi} = \partial_i \partial_t \check{\Psi}, \quad (52)$$

and we deduce that $\partial_t w_i = \mathbf{e}_i \ln|\partial_t \check{\Psi}|$. By extracting $h_4[\check{\Psi}, {}^v\Lambda]$ from (47) with $\check{\Psi}$, we arrive at

$$\mathbf{e}_i \ln\sqrt{|\ h_4|} = \mathbf{e}_i[\ln|\partial_t \check{\Psi}| - \ln\sqrt{|\boldsymbol{\Upsilon}|}].$$

In order to prove this formula we have used (52) and $\mathbf{e}_i \check{\Psi} = 0$. From the last two formulae, we obtain $\partial_t w_i = \mathbf{e}_i \ln\sqrt{|\ h_4|}$ if

$$\mathbf{e}_i \ln\sqrt{|\boldsymbol{\Upsilon}|} = 0.$$

This is possible for either $\boldsymbol{\Upsilon} = const$, or if $\boldsymbol{\Upsilon}$ can be expressed as a functional $\boldsymbol{\Upsilon}(x^i, t) = \check{\boldsymbol{\Upsilon}}[\check{\Psi}]$. If such conditions are not satisfied, we can rescale the generating function $\check{\Psi} \longleftrightarrow \widetilde{\Psi}$, where

$$\check{\Psi}^2 = \Lambda_0^{-1} \int dt \check{\boldsymbol{\Upsilon}} \partial_t(\widehat{\Psi}^2) \quad \text{and} \quad \widehat{\Psi}^2 = \Lambda_0 \int dt \check{\boldsymbol{\Upsilon}}^{-1} \partial_t(\check{\Psi}]^2),$$

when

$$\partial_t \partial_i \widehat{\Psi} = \partial_i \partial_t \widehat{\Psi}. \quad (53)$$

We consider a functional

$$\widehat{\Xi}[\ \check{\boldsymbol{\Upsilon}}, \widehat{\Psi}] = \int dt\ \check{\boldsymbol{\Upsilon}} \partial_t(\widetilde{\Psi}^2)$$

in the formula for h_4 (48) (as a redefined source, $\check{\boldsymbol{\Upsilon}} \to \widehat{\Xi}$), for a prescribed generating function $\widehat{\Psi}$, when $\check{\boldsymbol{\Upsilon}} = \partial_t \widehat{\Xi}/\partial_t(\check{\Psi}^2)$ for any effective cosmological constant Λ_0 in order to satisfy such conditions.

If we introduce a function $\check{A} = \check{A}(x^k, t)$ for which

$$w_i = \check{w}_i = \partial_i \check{\Psi}/\partial_t \check{\Psi} = \partial_i \widehat{\Xi}/\partial_t \widehat{\Xi} = \partial_i \check{A},$$

then $\partial_i w_j = \partial_j w_i$ in (37).

Summarizing the results, we conclude that we have the linear quadratic line element

$$ds^2 = g_{\alpha\beta}(x^k, t)du^\alpha du^\beta = e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2] + \omega^2 \frac{\widehat{\Psi}^2}{4\Lambda_0}[dy^3 + \partial_i n(x^k)dx^i]^2 + \omega^2 \frac{(\partial_t \widehat{\Psi})^2}{\widehat{\Xi}}\ [dt + \partial_i \check{A}\ dx^i]^2, \quad (54)$$

where ω is a solution of

$$\partial_i \omega - \partial_i n\ \partial_3 \omega - (\partial_i \widehat{\Xi}/\partial_t \widehat{\Xi})\ \partial_t \omega = 0,$$

and this defines generic off-diagonal cosmological solutions with zero nonholonomically induced torsion. Such inhomogeneous cosmological solutions are determined by the generating functions and effective sources $\psi(x^k)$, $\widehat{\Psi}(x^k, t)$, $\omega(x^k, y^3, t)$, $\widehat{\Xi}(x^k, t)$, the parameter Λ_0, and the integration functions ${}_1n_i = \partial_i n(x^k)$, respectively. The main result of this section is the demonstration that TMT theories admit generic off-diagonal cosmological solutions of type (51), with nontrivial nonholonomically induced torson, or of type (54), for LC-configurations. Another fundamental physical result is the emergence of a nonlinear symmetry for generating functions, see Eq. (49), for cosmological solutions of such nonlinear systems which allows one to transform arbitrary effective and matter fields sources into an effective cosmological constant Λ_0 treated as an integration parameter. The value of the integration parameter can be fixed by getting compatibility with observational cosmological data.

4 Time-like parameterized off-diagonal cosmological solutions

In this section we consider a subclass of solutions pertaining to $g_{\alpha\beta}(x^k, y^3, t)$ extracted from either (51), or (54) which, via frame transformations $g_{\alpha\beta}(u) = e^{\alpha'}_{\alpha}(u)e^{\beta'}_{\beta}(u)g_{\alpha'\beta'}(t)$, result in metrics $g_{\alpha'\beta'}(t)$ that depend only on time-like coordinate t. For applications in modern cosmology, we consider $g_{\alpha'\beta'}(t)$ as certain off-diagonal deformations of the FLRW, or the Bianchi type universes [22,25]. In explicit form, we construct physical models with $\acute{\mathbf{g}} = \{g_{\alpha'\beta'}(t)\} \to \mathring{\mathbf{g}} = \{\mathring{g}_i, \mathring{h}_a\}$ for $\eta_\alpha \to 1$ and $\mathbf{e}^\alpha \to du^\alpha = (dx^i, dy^a)$ in (21). The strategy is first to construct solutions for a class of generating functions and sources with spacetime dependent coordinates and then to restrict the integral varieties to configurations with dependencies only on the time-like coordinate. This procedure requires that $\widetilde{\Psi}(x^k, t) \to \acute{\widetilde{\Psi}}(t), \widehat{\Psi}(x^k, t) \to \acute{\widehat{\Psi}}(t)$; $\Upsilon(x^k, t) \to \acute{\Upsilon}(t)$ with $\Xi[\Upsilon, \widetilde{\Psi}] = \int dt \Upsilon \partial_t(\widetilde{\Psi}^2) \to \acute{\Xi}(t) = \acute{\Xi}[\acute{\Upsilon}(t), \acute{\widetilde{\Psi}}(t)]$; $\widehat{\Xi}[\Upsilon, \widehat{\Psi}] = \int dt \Upsilon \partial_t(\widehat{\Psi}^2) \to \acute{\widehat{\Xi}}(t) = \acute{\widehat{\Xi}}[\acute{\Upsilon}(t), \acute{\widehat{\Psi}}(t)]$; $\partial_i \acute{\Xi} \to \acute{\mathcal{F}}_i(t)$, $\partial_i \acute{\widehat{\Xi}} \to \acute{\widehat{\mathcal{F}}}$ and with $\omega \to 1$. The integration functions ${}_1n_k(x^i)$ and ${}_2\widetilde{n}_k(x^i)$ are considered to be constants of integration, implying $\partial_i n(x^k) \to const.$ and $\partial_i \check{A}(x^k, t) \to \check{\mathcal{F}}_i(t)$.

4.1 Cosmological solutions for the effective EYMH systems and TMT

The effective gravitational theory (25) with source ${}^e\mathbf{T}_{\alpha\beta}$ (18) in TMTs describes a nonlinear parametrical interacting EYMH system where we interpret ϕ as a Higgs field that can carry internal indices and acquire vacuum expectation $\phi_{[0]}$, and couple to the gauge field $\mathbf{A} = \mathbf{A}_\mu e^\mu$ with values in non-Abelian Lie algebra. On the premises defined by the nonholonomic $\mathbf{V}$, the d-operator $\widehat{\mathbf{D}}_\mu$ is elongated additionally to accommodate the gauge potentials in the form $\widehat{\mathcal{D}}_\mu = \widehat{\mathbf{D}}_\mu + ie[\mathbf{A}_\mu,]$, where the commutator $[.,.]$ signifies the non-Abelian structure. The gauge coupling is e and $i^2 = -1$. The gauge field $\mathbf{A}_\mu$ enters the covariant derivative $D_\mu = \mathbf{e}_\mu + ie[\mathbf{A}_\mu,]$ and the "curvature"

$$\mathbf{F}_{\beta\mu} = \mathbf{e}_\beta \mathbf{A}_\mu - \mathbf{e}_\mu \mathbf{A}_\beta + ie[\mathbf{A}_\beta, \mathbf{A}_\mu], \quad (55)$$

where the boldface $\mathbf{F}_{\beta\mu}$ is used for N-adapted constructions.[8]

With respect to N-adapted frames the nonholonomic EYMH equations, postulated either by following geometric principles, or "derived" following an N-adapted variational calculus from (25), are the following:

$$\widehat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2}\widehat{\mathbf{g}}_{\alpha\beta}\ \widehat{R} = \frac{\kappa}{2}\left({}^\phi T_{\beta\delta} + {}^F T_{\beta\delta}\right), \quad (56)$$

$$(\sqrt{|\widehat{\mathbf{g}}|})^{-1}\ D_\mu(\sqrt{|\widehat{\mathbf{g}}|}F^{\mu\nu}) = \frac{1}{2}ie[\phi, D^\nu\phi], \quad (57)$$

$$(\sqrt{|\widehat{\mathbf{g}}|})^{-1}\ D_\mu(\sqrt{|\widehat{\mathbf{g}}|}\phi) = \lambda(\ \phi^2_{[0]} - \phi^2)\phi, \quad (58)$$

where the source (23) is determined by the stress-energy tensor

$$\begin{aligned} {}^\phi T_{\beta\delta} = Tr\Bigg[&\frac{1}{4}(D_\delta\phi\ D_\beta\phi + D_\beta\phi\ D_\delta\phi) \\ &-\frac{1}{4}\widehat{\mathbf{g}}_{\beta\delta} D_\alpha\phi\ D^\alpha\phi\Bigg] - \widehat{\mathbf{g}}_{\beta\delta}\ {}^e V(\phi), \end{aligned} \quad (59)$$

$${}^F T_{\beta\delta} = 2Tr\left(\widehat{\mathbf{g}}^{\mu\nu}\mathbf{F}_{\beta\mu}\mathbf{F}_{\delta\nu} - \frac{1}{4}\widehat{\mathbf{g}}_{\beta\delta}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right). \quad (60)$$

The nonlinear potential ${}^e V(\phi)$ is as in (59) for a TMT if it is taken in the form (24).

The system of nonlinear PDEs (56)–(58) posses a similar decoupling property as in (4) if plausible assumptions are made for gravitational and matter field interactions. To see this and construct new classes of modified EYMH equations we take the "prime" solution to be given by data for a diagonal d-metric $^\circ\mathbf{g} = [^\circ g_i(x^1), {}^\circ h_a(x^k), {}^\circ N^a_i = 0]$ with matter fields ${}^\circ A_\mu(x^1)$ and ${}^\circ\Phi(x^1)$. For $SU(2)$ gauge field configurations, the diagonal ansatz for generating solutions can be written in the form

$$\begin{aligned} {}^\circ\mathbf{g} &= {}^\circ g_i(x^1)dx^i \otimes dx^i + {}^\circ h_a(x^1, x^2)dy^a \otimes dy^a \\ &= q^{-1}(r)dr \otimes dr + r^2 d\theta \otimes d\theta \\ &\quad + r^2 \sin^2\theta d\varphi \otimes d\varphi - \sigma^2(r)q(r)dt \otimes dt, \end{aligned} \quad (61)$$

where the coordinates and metric coefficients are parameterized, respectively, as $u^\alpha = (x^1 = r, x^2 = \theta, y^3 = \varphi, y^4 = t)$ and ${}^\circ g_1 = q^{-1}(r)$, ${}^\circ g_2 = r^2$, ${}^\circ h_3 = r^2\sin^2\theta$, ${}^\circ h_4 = -\sigma^2(r)q(r)$, for $q(r) = 1 - 2m(r)/r - \Lambda r^2/3$, and Λ is a cosmological constant. The function $m(r)$ is interpreted as

[8] For standard gauge field models but on nonholonomic manifolds we can follow a variational principle for a gravitating non-Abelian SU(2) gauge field $\mathbf{A} = \mathbf{A}_\mu \mathbf{e}^\mu$ coupled to a triplet Higgs field ϕ. In such cases, the value $\phi_{[0]}$ is the vacuum expectation of the Higgs field which determines the mass ${}^H M = \sqrt{\lambda}\eta$, when λ is the constant of scalar field self-interaction with potential $\mathcal{V}(\phi) = \frac{1}{4}\lambda Tr(\phi^2_{[0]} - \phi^2)^2$, where the trace Tr is taken on internal indices. In EYMH theory, the gravitational constant $G, \kappa = 16\pi G$, defines the Planck mass $M_{Pl} = 1/\sqrt{G}$ and it is also the mass of gauge boson, ${}^W M = ev$. In the literature, various versions of modified gravity and TMTs are elaborated upon with different types of nonlinear scalar and gauge fields.

the total mass within the radius r for which $m(r) = 0$ defines an empty de Sitter space written in a static coordinate system with a cosmological horizon at $r = r_c = \sqrt{3/\Lambda}$. The solution of (56) associated to the quadratic metric line element (61) is defined by a single magnetic potential $\omega(r)$,

$$^{\circ}A = {}^{\circ}A_2 dx^2 + {}^{\circ}A_3 dy^3 = \frac{1}{2e}\left[\omega(r)\tau_1 d\theta + (\cos\theta\ \tau_3 + \omega(r)\tau_2 \sin\theta)\ d\varphi\right], \quad (62)$$

where τ_1, τ_2, τ_3 are the Pauli matrices. The corresponding solution of (58) is given by

$$\Phi = {}^{\circ}\Phi = \varpi(r)\tau_3. \quad (63)$$

Explicit values for the functions $\sigma(r), q(r), \omega(r), \varpi(r)$ have been found in Ref. [27] for ansatz (61), (62) and (63) when $\left[{}^{\circ}\mathbf{g}(r),\ {}^{\circ}A(r),\ {}^{\circ}\Phi(r)\right]$ define physical solutions with diagonal metrics depending only on the radial coordinate. A typical example is the well-known diagonal Schwarzschild–de Sitter solution (56)–(58) that is given by

$$\omega(r) = \pm 1, \sigma(r) = 1, \quad \phi(r) = 0,$$
$$q(r) = 1 - 2M/r - \Lambda r^2/3$$

and defines a black hole configuration inside a cosmological horizon because $q(r) = 0$ has two positive solutions and $M < 1/3\sqrt{\Lambda}$.

The conditions for nonholonomic deformations of (61) are as follows. The "target" d-metric ${}^{\eta}\mathbf{g}$ with non-trivial N-coefficients, for ${}^{\circ}\mathbf{g} \rightarrow \widehat{\mathbf{g}}$ is parameterized as in (21). The gauge fields are deformed as

$$A_\mu(x^i, y^3) = {}^{\circ}A_\mu(x^1) + {}^{\eta}A_\mu(x^i, y^a), \quad (64)$$

where ${}^{\circ}A_\mu(x^1)$ is of the type (62) and ${}^{\eta}A_\mu(x^i, y^a)$ are functions for which

$$\mathbf{F}_{\beta\mu} = {}^{\circ}F_{\beta\mu}(x^1) + {}^{\eta}\mathbf{F}_{\beta\mu}(x^i, y^a) = s\sqrt{|g|}\varepsilon_{\beta\mu}, \quad (65)$$

where s is a constant and $\varepsilon_{\beta\mu}$ is the absolute antisymmetric tensor. The gauge field curvatures $F_{\beta\mu}$, ${}^{\circ}F_{\beta\mu}$ and ${}^{\eta}\mathbf{F}_{\beta\mu}$ are computed by substituting (62) and (64) into (55). Any antisymmetric $\mathbf{F}_{\beta\mu}$ (65) is a solution of $D_\mu(\sqrt{|g|}F^{\mu\nu}) = 0$, i.e. determines ${}^{\eta}F_{\beta\mu}$, ${}^{\eta}A_\mu$, for any given ${}^{\circ}A_\mu$, ${}^{\circ}F_{\beta\mu}$. For nonholonomic modifications of scalar fields, we take ${}^{\circ}\phi(x^1) \rightarrow \phi(x^i, y^a) = {}^{\phi}\eta(x^i, y^a)\,{}^{\circ}\phi(x^1)$. It is supplemented with a polarization ${}^{\phi}\eta$ for which

$$D_\mu\phi = 0 \quad \text{and} \quad \phi(x^i, y^a) = \pm\phi_{[0]}. \quad (66)$$

This nonholonomic configuration of the nonlinear scalar field is non-trivial even with respect to N-adapted frames ${}^{e}V(\phi) = 0$ and ${}^{F}T_{\beta\delta} = 0$, (59). For ansatz (21), Eqs. (66) are

$$(\partial/\partial x^i - A_i)\phi = n_i\partial_3\phi + w_i\partial_t\phi,\ (\partial_3 - A_3)\,\phi = 0,$$
$$(\partial_4 - A_4)\,\phi = 0.$$

A nonholonomically deformed scalar (Higgs field depending in non-explicit form on two variables because of constraint (66)) modifies indirectly the off-diagonal components of the metric via n_i, w_i and the above conditions for ${}^{\eta}A_\mu$.

The effective gauge field $\mathbf{F}_{\beta\mu}$ (65) with the potential A_μ (64) modified nonholonomically by ϕ and subject to the conditions (66) determine exact solutions of the system (31) if the spacetime metric is chosen to be in the form (21). The energy-momentum tensor is determined to be ${}^{F}T^{\alpha}_{\beta} = -4s^2\delta^{\alpha}_{\beta}$ [28]. Interacting gauge and Higgs fields, with respect to N-adapted frames, result in an effective cosmological constant ${}^{s}\Lambda = 8\pi s^2$, which should be added to the respective source (23).

To conclude, a generic off-diagonal ansatz $\widehat{\mathbf{g}} = [\eta_i\,{}^{\circ}g_i, \eta_a\,{}^{\circ}h_a; w_i, n_i]$ (21) and (effective) gauge-scalar configurations (A, ϕ) subject to conditions mentioned above define a decoupling of the nonlinear PDEs (56)–(58) if the sources (23) are transformed in the form

$$\mathbf{\Upsilon}^{\beta}_{\ \delta} = diag[\mathbf{\Upsilon}_\alpha] \rightarrow \mathbf{\Upsilon}^{\beta}_{\ \delta} + {}^{F}T^{\beta}_{\ \delta} = diag[\mathbf{\Upsilon}_\alpha - 4s^2\delta^{\alpha}_{\beta}]. \quad (67)$$

This is in sharp contrast to the situation where with respect to coordinate frames, such systems of equations describe a very complex, nonlinearly coupled gravitational and gauge–scalar interactions.

4.2 Effective vacuum EYMH configurations in TMTs

The effects of off-diagonal gravitational, scalar and gauge matter fields result in driving the vacuum energy density to zero even when the effective source $\mathbf{\Upsilon}_\alpha$ and cosmological constant Λ_0 are non-trivial. This is possible due to the contributions of effective self-dual gauge fields. Such an effect is discussed in [16] for instantons. If $\mathbf{\Upsilon}^{\beta}_{\ \delta} = 0$ in (67), one imposes further nonholonomic rescaling $\Upsilon \rightarrow \Lambda_0$ when $\Lambda_0 - 4s^2 = 0$. We can generate a very large class of solutions in TMTs with effective EMYH interactions into nonholonomic vacuum configurations of modified Einstein gravity. In this section, we analyze a subclass of generic off-diagonal EYMH interactions which can be encoded as effective vacuum Einstein manifolds of various class and lead to solutions with non-trivial cosmological constant $\Lambda_0 = 4s^2$. In general, such solutions depend parametrically on $\Lambda_0 - 4s^2$ and do not have a smooth limit from non-vacuum to vacuum models. Effects of this type exist both in commutative, noncommutative gauge gravity theories [26], Einstein gravity and its various modifications [20,23], and TMTs. Examples are provided in the following sections.

The Einstein equations (40)–(44) corresponding to a system of nonlinear PDEs (56)–(58) with source $\Upsilon^{\beta}_{\ \delta}$ (67) are

$$\psi^{\bullet\bullet} + \psi'' = 2(\Upsilon - 4s^2), \quad (68)$$
$$\partial_t\varpi\ \partial_t h_3 = 2h_3h_4(\Upsilon - 4s^2), \quad (69)$$

$$\partial^2_{tt} n_i + \gamma \partial_t n_i = 0, \tag{70}$$
$$\beta w_i - \alpha_i = 0, \tag{71}$$
$$\partial_i \omega - n_i \partial_3 \omega - (\partial_i \varpi / \partial_t \varpi) \partial_t \omega = 0. \tag{72}$$

To derive self-consistent solutions of this system for $\Upsilon - 4s^2 = 0$ we consider off-diagonal ansatz depending on all spacetime coordinates,

$$\begin{aligned}\widehat{\mathbf{g}} &= e^{\psi(x^k)}[\mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^2] + h_3(x^k, t) \\ &\quad \underline{h}_3(x^k, y^3)\mathbf{e}^3 \otimes \mathbf{e}^3 + h_4(x^k, t)\mathbf{e}^4 \otimes \mathbf{e}^4, \\ \mathbf{e}^3 &= \mathrm{d}y^3 + n_i(x^k)\mathrm{d}x^i, \mathbf{e}^4 = \mathrm{d}t + w_i(x^k, t)\mathrm{d}x^i,\end{aligned} \tag{73}$$

where the coefficients of this target metric are defined by solutions of the following equations:

$$\ddot{\psi} + \psi'' = 0, \tag{74}$$
$$(\partial_t \varpi)\ \partial_t h_3 = 0, \tag{75}$$
$$\beta w_i - \alpha_i = 0. \tag{76}$$

The coefficients β and α_i are computed following Eqs. (38) for nonzero $\partial_t \varpi$ and $\partial_t h_3$. The coefficients h_a, $\underline{h}_3$ and w_i are additionally subject to the zero-torsion conditions (5), (6) as in the form (20) where, for simplicity, we fix n_i equal to a constant as a trivial solutions of (70).

For Eq. (74), we can take $\psi = 0$, or consider a trivial 2-d Laplace equation with space-like coordinates x^k. There are two possibilities to satisfy the condition (75) and derive the corresponding off-diagonal solutions. In the first case we take $h_3 = h_3(x^k)$, when $\partial_t h_3 = 0$. This implies that Eq. (75) has solutions with zero source for arbitrary function $h_4(x^k, t)$ and arbitrary N-coefficients $w_i(x^k, t)$ as follows from (38). For such vacuum LC-configurations, the functions h_4 and w_i are general and should be constrained only by the conditions (20). This constrains substantially the class of admissible w_i if h_3 depends only on x^k (we can perform a similar analysis as in subsection 3). The corresponding quadratic line element is

$$\begin{aligned}\mathrm{d}s^2 &= g_{\alpha\beta}(x^k, t)\mathrm{d}u^\alpha \mathrm{d}u^\beta = e^{\psi(x^k)}[(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2] \\ &\quad + \omega^2(x^k, y^3, t)\Big[\underline{h}_3(x^k, y^3) h_3(x^k)(\mathrm{d}y^3)^2 \\ &\quad + h_4(x^k, t)\ \left(\mathrm{d}t + \partial_i \check{A}(x^k, t)\ \mathrm{d}x^i\right)^2\Big],\end{aligned} \tag{77}$$

where we introduce a function $\check{A} = \check{A}(x^k, t)$ for which $w_i = \partial_i \check{A}$ satisfies $\partial_i w_j = \partial_j w_i$ in (37) and ω is a solution of

$$\partial_i \omega - (\partial_i \check{A})\ \partial_t \omega = 0.$$

In the second case a very different class of (off-) diagonal solutions result if we choose, after corresponding coordinate transformations, $\varpi = \ln\left|\partial_t h_3 / \sqrt{|h_3 h_4|}\right| = {}^0\varpi = const$ and $\partial_t \varpi = 0$. For such configurations, we can consider $\partial_t h_3 \neq 0$ and solve (75) for

$$\sqrt{|h_4|} = {}^0h\ \partial_t(\sqrt{|h_3|}), \tag{78}$$

with ${}^0h$ equals a non-vanishing constant. Such v-metrics are generated by any $f(x^i, t)$ satisfying $\partial_t f \neq 0$, when

$$h_3 = f^2\left(x^i, t\right) \quad \text{and} \quad h_4 = -({}^0h)^2\ \left[\partial_t f\left(x^i, t\right)\right]^2, \tag{79}$$

where the signs are fixed in such a way that for $N_i^a \to 0$ we obtain diagonal metrics with signature $(+,+,+,-)$. The coefficients (38) for (76) became trivial if $\alpha_i = \beta = 0$, and $w_i(x^k, t)$ is any functions solving (20). The equations in the last system for the LC-conditions are equivalent to

$$\begin{aligned}\partial_t w_i &= 2\partial_i \ln|f| - 2w_i \partial_t(\ln|f|), \\ \partial_k w_i - \partial_i w_k &= 2(w_k \partial_i - w_i \partial_k)\ln|f|,\end{aligned} \tag{80}$$

for any $n_i(x^k)$ when $\partial_i n_k = \partial_k n_i$. Constraints of type $n_k \partial_3 \underline{h}_3 = \partial_k \underline{h}_3$ have to be imposed for a non-trivial multiple $\underline{h}_3$ depending on y^3.

The corresponding quadratic line element is

$$\begin{aligned}\mathrm{d}s^2 &= g_{\alpha\beta}(x^k, t)\mathrm{d}u^\alpha \mathrm{d}u^\beta = e^{\psi(x^k)}[(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2] + \\ &\quad + \omega^2(x^k, y^3, t)\Big[\underline{h}_3(x^k, y^3) f^2\left(x^i, t\right)(\mathrm{d}y^3)^2 - ({}^0h)^2 \\ &\quad \times \left[\partial_t f\left(x^i, t\right)\right]^2\ \left(\mathrm{d}t + w_i(x^k, t)\ \mathrm{d}x^i\right)^2\Big],\end{aligned} \tag{81}$$

where w_i are taken to solve the conditions (80)with $\partial_i w_j = \partial_j w_i$ and ω is a solution of

$$\partial_i \omega - w_i\ \partial_t \omega = 0.$$

We conclude that off-diagonal interactions in effective EYMH systems result in vanishing cosmological constant as is demonstrated in the general solutions (77) and (81) presented above for LC-configurations. Such constructions can be generalized to include inhomogeneous effective vacuum configurations with non-trivial nonholonomically induced torsion (16). Effects of this nature exist in TMTs when the analogous EYMH systems are described by an action with two measures (27) related to an action (25) via a N-adapted conformal transform (26). Additionally, a subclass of cosmological solutions satisfying the conditions (5) and (6) can be generated if, for instance, we restrict the generating functions in (81) to satisfy via frame/coordinate transforms $f^2\left(x^i, t\right) \to f^2(t), w_i(x^k, t) \to w_i(t), \omega \to 1$ and the integration functions are changed into integration constants.

4.3 Examples of (off-)diagonal nonholonomic deformations of cosmological metrics

In this section, we present details of how AFDM are employed to construct a new class of inhomogeneous and anisotropic cosmological solutions with target d-metrics $\widehat{\mathbf{g}}$ (21), with certain well-defined limits for $\eta_\alpha \to 1$, to a primed metric $^\circ\mathbf{g}$. These can be interpreted as conformal, frame or coordinate transformations of the well-known metrics like

FLRW, Bianchi, Kasner, or another metric corresponding to a particular cosmological solution [29–32].

4.3.1 Off-diagonal deformations of FLRW configurations in TMTs

We show how N-anholonomic FLRW deformations can be constructed to define three classes of generic off-diagonal cosmological solutions for modified EYMH systems in TMTs. Similar models for one measure theories are presented in [22].

FLRW metrics: For convenience, we introduce the necessary notations to describe the primed standard FLRW metric, when written in the diagonal form,

$$ {}^F\mathring{\mathbf{g}} = a^2(t)\left(\frac{\mathrm{d}r\otimes\mathrm{d}r}{1-Kr^2}+r^2\mathrm{d}\theta\otimes\mathrm{d}\theta\right) + a^2(t)r^2\sin^2\theta\mathrm{d}\varphi\otimes\mathrm{d}\varphi - \mathrm{d}t\otimes\mathrm{d}t, \quad (82)$$

with $K=\pm1,0$ and spherical coordinates $x^1=r, x^2=\theta, y^3=\varphi, y^4=t$ and

$$ {}^F\mathring{g}_1 = a^2/(1-\kappa r^2),\quad {}^F\mathring{g}_2 = a^2r^2/(1-\kappa r^2),$$
$$ {}^F\mathring{h}_3 = a^2(t)r^2\sin^2\theta,\quad {}^F\mathring{h}_4=-1,\quad {}^F\mathring{N}_i^a=0.$$

For simplicity, we take $K=0$ and choose Cartesian coordinates $(x^1=x, x^2=z, y^3=y, y^4=t)$, when the coefficients of ${}^F\mathring{\mathbf{g}}$ are taken, respectively, in the form ${}^F\mathring{g}_1 = {}^F\mathring{g}_2 = a^2$, ${}^F\mathring{h}_3 = a^2$ ${}_Fh_4=-1$ and ${}^F\mathring{N}_i^a=0$. In this case, the non-trivial coefficients of the primed diagonal metric depend only on the time-like coordinate t and takes the form

$$ {}^F\mathring{\mathbf{g}} = a^2(t)\,(\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}z\otimes\mathrm{d}z+\mathrm{d}y\otimes\mathrm{d}y)-\mathrm{d}t\otimes\mathrm{d}t. \quad (83)$$

Here we also note that instead of FLRW we can consider any other 'primed' metric $^\circ\mathbf{g}$, which can be a Bianchi, Kasner or a metric of a particular cosmological solution [30,33–36].

The metrics (82) and/or (83) define exact homogeneous cosmological solutions of Eqs. (19) and (20) with source $\boldsymbol{\Upsilon}_{\alpha\beta}=\frac{\kappa}{2}\,T_{\alpha\beta}$ for a perfect fluid energy-momentum stress tensor,

$$ T^\alpha{}_\beta = diag[-p,-p,\rho,-p]. \quad (84)$$

Here ρ and p are the proper energy density and pressure in the fluid rest frame. The Einstein equations corresponding to ansatz (82) take the form of two coupled nonlinear ODEs (the Friedmann equations),

$$ H^2 \equiv \left(\frac{\partial_t a}{a}\right)^2 = \frac{1}{3}\rho-\frac{\kappa}{a^2} \quad (85)$$

and

$$ \partial_t H + H^2 = \frac{\partial^2_{tt}a}{a} = -\frac{1}{6}(\rho+3p). \quad (86)$$

The Hubble constant $H\equiv\partial_t a/a$ has the units of inverse time and is positive (negative) for an expanding (collapsing) universe. Equations (85) and (86) are related via the condition $\nabla_\alpha T^\alpha{}_\beta=0$, for which the considered diagonal homogeneous ansatz is written as

$$ \partial_t\rho+3H(\rho+p)=0.$$

Here we note that the strong energy conditions for matter, $\rho+3p\geq0$, or equivalently, the equation of state, $w=p/\rho\geq-1/3$, must be satisfied for an expanding universe.

Off-diagonal effective EYMH cosmological solutions of type 1: In this case the d-metric is of the type (51) with $\partial_t h_a\neq0$, $\partial_t\varpi\neq0$ and $\Upsilon-4s^2\neq0$, when

$$ h_3=\frac{{}^s\widetilde{\Psi}^2}{4(\Lambda_0-4s^2)}=\eta_3\,{}^F\mathring{h}_3 \quad\text{and}\quad h_4=\frac{(\partial_t\,{}^s\widetilde{\Psi})^2}{{}^s\Xi}=\eta_4\,{}^F\mathring{h}_4$$

correspond to an effective cosmological constant $\Lambda_0-4s^2\neq0$ with redefined generating functions, ${}^s\Psi\longleftrightarrow{}^s\widetilde{\Psi}$. The left label "s" emphasizes that such values encode contributions from effective gauge fields, where

$$ {}^s\Psi^2=(\Lambda_0-4s^2)^{-1}\int\mathrm{d}t(\boldsymbol{\Upsilon}-4s^2)\partial_t({}^s\widetilde{\Psi}^2) \text{ and}$$
$$ {}^s\widetilde{\Psi}^2=(\Lambda_0-4s^2)\int\mathrm{d}t(\Upsilon-4s^2)^{-1}\partial_t({}^s\Psi^2). \quad (87)$$

The functional

$$ {}^s\Xi[\Upsilon,{}^s\widetilde{\Psi}]=\int\mathrm{d}t(\Upsilon-4s^2)\partial_t({}^s\widetilde{\Psi}^2)$$

in the formula for h_4 in (48) can be considered as a redefined source, $\Upsilon-4s^2\to{}^s\Xi$, for a prescribed generating function $\widetilde{\Psi}$, when $\Upsilon-4s^2=\partial_t({}^s\Xi)/\partial_t({}^s\widetilde{\Psi}^2)$. Such effective sources contain information on effective EYMH interactions in TMTs. For convenience we work with a couple of generating data, $({}^s\Psi,\,{}^v\Lambda-4s^2)$ and $[{}^s\widetilde{\Psi},\Lambda_0-4s^2,{}^s\Xi]$ related by Eqs. (87) for a prescribed effective cosmological constant Λ_0 and the parameter s for gauge fields. Such values have to be fixed in a form which is compatible with experimental/ observational data, and result in a small vacuum density. Summarizing the results for off-diagonal nonholonomic deformations of the prime metric (83), we get a quadratic line element

$$ \mathrm{d}s^2=g_{\alpha\beta}(x^k,t)\mathrm{d}u^\alpha\mathrm{d}u^\beta=e^{\psi(x,z)}[(\mathrm{d}x)^2+(\mathrm{d}z)^2] + ({}^s\omega)^2\frac{{}^s\widetilde{\Psi}^2}{4(\Lambda_0-4s^2)}\Big[\mathrm{d}y + \left({}_1n_k(x,z)+{}_2\widetilde{n}_k(x,z)\int\mathrm{d}t\frac{(\partial_4\,{}^s\widetilde{\Psi})^2}{({}^s\widetilde{\Psi})^3\,{}^s\Xi}\right)\mathrm{d}x^k\Big]^2 + ({}^s\omega)^2\frac{(\partial_t\,{}^s\widetilde{\Psi})^2}{{}^s\Xi}\left[\mathrm{d}t+\frac{\partial_i\,{}^s\Xi}{\partial_t\,{}^s\Xi}\mathrm{d}x^i\right]^2, \quad (88)$$

where ${}^{s}\omega(x,z,y,t)$ is a solution of (44) which for our data is written in the form

$$\partial_i\omega - n_i\partial_3\omega - w_i\partial_t\omega = 0.$$

For the N-connection coefficients, we have

$$n_i = {}_1n_k(x,z) + {}_2\widetilde{n}_k(x,z)\int dt\frac{(\partial_4\,{}^{s}\widetilde{\Psi})^2}{({}^{s}\widetilde{\Psi})^3\,{}^{s}\Xi} \quad \text{and} \quad w_i = \frac{\partial_i\,{}^{s}\Xi}{\partial_t\,{}^{s}\Xi}.$$

The function $\psi(x,z)$ in (88) is a solution of (68), i.e. of $\partial^2_{xx}\psi + \partial^2_{zz}\psi = 2(\Upsilon - 4s^2)$.

To understand possible physical implications of d-metrics (88) it is more convenient to use the so-called polarization functions η_α and $\eta_i^a := N_i^a - \mathring{N}_i^a$ as in (21) and parameterize such solutions in the form

$$\begin{aligned}\widehat{\mathbf{g}} &= \eta_1\,{}^{F}\mathring{g}_1 dx\otimes dx + \eta_2\,{}^{F}\mathring{g}_2 dz\otimes dz \\ &\quad + ({}^{s}\omega)^2\left[\eta_3\,{}^{F}\mathring{h}_3\mathbf{e}^3\otimes\mathbf{e}^3 + \eta_4\,{}^{F}\mathring{h}_4\mathbf{e}^4\otimes\mathbf{e}^4\right], \\ \mathbf{e}^3 &= dy + \eta_1^3 dx + \eta_2^3 dz,\ \mathbf{e}^4 = dt + \eta_1^4 dx + \eta_2^4 dz, \end{aligned} \tag{89}$$

where

$$\eta_1 = \eta_2 = a^{-2}(t)e^{\psi(x,z)}, \quad \eta_3 = {}^{s}\widetilde{\Psi}^2/4(\Lambda_0 - 4s^2)a^2(t),$$
$$\eta_4 = (\partial_t\,{}^{s}\widetilde{\Psi})^2/\,{}^{s}\Xi, \quad \eta_i^3 = n_i, \quad \eta_i^4 = w_i,$$

are determined by the above solutions for the coefficients of the target d-metric.

Solutions (89) describe general off-diagonal deformations of the FLRW metrics in TMTs encoding modified EYMH interactions. Such interactions may result in changing the topology and symmetries, and they are characterized by inhomogeneous, locally anistropic configurations or non-perturbative effect. The problem of the physical interpretation of such cosmological off-diagonal solutions is simplified to some extent if we consider small deformations with polarizations of the type $\eta_\alpha \approx 1 + \chi_\alpha$ and $\eta_i^a \approx 0 + \chi_i^a$ for small values $|\chi_\alpha| \ll 1$ and $|\chi_i^a| \ll 1$, by which we obtain small deformations of the FLRW universes by certain generalized two measure interactions and/or modified gravity theories with effective EYMH fields. Nevertheless, even in such cases the target configuration may encode nonlinear and nonholonomic parametric effects as results of rescaling (87) of generating functions. This way we model nonlinear nonholonomic transformations of a FLRW universe into an effective and small-deformed one with small values of effective cosmological constant, nonlinear anisotropic processes and other effects of similar magnitude.

Off-diagonal cosmological solutions of type 2 and "losing" information on effective EYMH: This class of solutions are characterized by the condition $\partial_t h_3 = 0$. Equation (69) can be solved only if $\Upsilon - 4s^2 = 0$, i.e. when the contributions from effective YM fields compensate other (effective) modified gravity and/or matter field sources. We take the function $w_i(x^k,t)$ as a solution of (71), or its equivalent (76), because the coefficients β and α_i from (38) are zero. To find non-trivial values of n_i we integrate (70) for $\partial_t h_3 = 0$ for any given h_3 and find $n_i = {}^1n_k\left(x^i\right) + {}^2n_k\left(x^i\right)\int h_4 dt$. Also, we take $g_1 = g_2 = e^{\psi(x^k)}$, with $\psi(x^k)$ determined by (68) for a given source $(\Upsilon - 4s^2)$.

In summary, this class of solutions can be chosen to be defined by the ansatz

$$\begin{aligned}\widehat{\mathbf{g}} &= e^{\psi(x^k)}dx^i\otimes dx^i + {}^0h_3(x^k)\mathbf{e}^3\otimes\mathbf{e}^3 + h_4(x^k,t)\mathbf{e}^4\otimes\mathbf{e}^4, \\ \mathbf{e}^3 &= dy + \left[{}^1n_k\left(x^i\right) + {}^2n_k\left(x^i\right)\int h_4 dt\right]dx^i, \\ \mathbf{e}^4 &= dt + w_i(x^k,t)dx^i, \end{aligned} \tag{90}$$

for arbitrary generating functions $h_4(x^k,t), w_i(x^k,t)$, ${}^0h_3(x^k)$ and integration functions ${}^1n_k\left(x^i\right)$ and ${}^2n_k\left(x^i\right)$. In general, such solutions carry non-trivial nonholonomically induced torsion (16).

The conditions (20) constrain (90) to a subclass of LC-solutions resulting in the following equations:

$$\begin{aligned}&{}^2n_k\left(x^i\right) = 0 \quad \text{and} \quad \partial_i\,{}^1n_k = \partial_k\,{}^1n_i, \\ &\partial_t w_i + \partial_i({}^0h_3) = 0 \quad \text{and} \quad \partial_i\,w_k = \partial_k\,w_i, \end{aligned} \tag{91}$$

for any $w_i(x^k,t)$ and ${}^0h_3(x^k)$. This class of constraints on solutions (90) do not involve the generating function $h_4(x^k,t)$ but only the N-connection coefficients for a prescribed value ${}^0h_3(x^k)$.

Another metric to consider is the prime FLRW metric as in (82) and/or (83) and repeat the constructions for the metric (89) but with the difference that we take $\partial_t h_3 = 0$. However, we study here another possibility, i.e., to begin with a prime metric which is not a solution of gravitational field equations and finally to generate off-diagonal cosmological solution with effective non-trivial nonholonomic vacuum configuration. Let us consider ${}^\circ g_i = 1, {}^\circ h_3 = 1$, ${}^\circ h_4(t) = -a^{-2}(t)$, which, by TMT with effective EYMH anisotropic and inhomogeneous nonlinear interactions, result in target d-metrics of the type (90). Using polarization functions we write

$$\eta_i = e^{\psi(x^k)}, \quad \eta_3 = {}^0h_3(x^k), \quad \eta_4 = h_4(x^k,t)a^2(t),$$
$$\eta_i^3 = {}^1n_k\left(x^i\right) + {}^2n_k\left(x^i\right)\int h_4 dt, \quad \eta_i^4 = w_i(x^k,t),$$

with ${}^\circ w_i(t) = 0$ and ${}^\circ n_i(t) = 0$. Such cosmological solutions are constructed as nonholonomic deformations of a conformal transformation (with multiplication on factor $a^{-2}(t)$) of the FLRW metric (82). We work with polarization functions $\eta_4(x^k,t)$ when $h_4 = \eta_4\,{}^\circ h_4(t) \to -a^{-2}(t)h_4(x^k,t)$ for $\eta_4 \to 1$. The solutions are written in the form

$$\begin{aligned}\widehat{\mathbf{g}} &= \eta_1\,dx\otimes dx + \eta_2\,dz\otimes dz \\ &\quad + ({}^{s}\omega)^2\left[\eta_3\mathbf{e}^3\otimes\mathbf{e}^3 - \eta_4\,a^{-2}(t)\mathbf{e}^4\otimes\mathbf{e}^4\right], \\ \mathbf{e}^3 &= dy + \eta_1^3 dx + \eta_2^3 dz,\ \mathbf{e}^4 = dt + \eta_1^4 dx + \eta_2^4 dz, \end{aligned} \tag{92}$$

where

$$\eta_1 = \eta_2 = a^{-2}(t)e^{\psi(x,z)}, \eta_3 = a^2(t)({}^0h_3),$$
$$\eta_4 = h_4 a^2, \eta_i^3 = n_i, \eta_i^4 = w_i,$$

are the coefficients of the target d-metric (90).

The class of solutions (89) represent the off-diagonal deformations of the FLRW metrics in TMTs encoding effective gauge and scalar field interactions when the effective cosmological constant is fixed to be zero. We generate solutions with non-Killing symmetry for non-trivial v-conformal factors ${}^s\omega(x,z,y,t)$ subject to the constraints

$$\partial_i\, {}^s\omega - \eta_i^3\, (\partial_3\, {}^s\omega) - \eta_i^4\, (\partial_t\, {}^s\omega) = 0.$$

The LC-conditions (91) constraints substantially the time dependence of $\eta_i^4 = w_i(x^k,t)$. The class of solutions with non-trivial nonholonomic torsion (16) allow arbitrary dependencies on t for N-connection coefficients w_i.

Off-diagonal metrics (90) result only with time-like dependence in the coefficients i.e., when $h_4 = h_4(t)$, $w_i = w_i(t)$ and $n_i(t)$ are determined with some constant values of ${}^0h_4$, ${}^1n_k$, ${}^2n_k$. Such conditions are relevant for the Levi-Civita configurations if $w_i = const$. This defines solutions of the Einstein equations with nonholonomic vacuum encoding TMTs contributions and effective EYMH interactions. They transform nonholonomically a FLRW universe into certain effective vacuum Einstein configurations which in this particular case are diagonalizable by coordinate transformations.

Off-diagonal cosmological solutions of type 3 and effective matter fields interactions: Non-vacuum metrics with $\partial_t h_3 \neq 0$ and $\partial_t h_4 = 0$ are generated by taking the ansatz

$$\widehat{\mathbf{g}} = e^{\psi(x^k)}\mathrm{d}x^i \otimes \mathrm{d}x^i + h_3(x^k,t)\, \mathbf{e}^3\otimes\mathbf{e}^3 - {}^0h_4(x^k)\, \mathbf{e}^4\otimes\mathbf{e}^4,$$
$$\mathbf{e}^3 = \mathrm{d}y^3 + n_i(x^k,t)\mathrm{d}x^i,\ \mathbf{e}^4 = \mathrm{d}t + w_i(x^k,t)\mathrm{d}x^i, \qquad (93)$$

where $g_1 = g_2 = e^{\psi(x^k)}$, where $\psi(x^k)$ is a solution of (33) for any given ${}^v\Upsilon(x^k) - 4s^2$. The function $h_3(x^k,t)$ is constrained to satisfy Eq. (34), which for $\partial_t h_4 = 0$ leads to

$$\partial_{tt}^2 h_3 - \frac{(\partial_t h_3)^2}{2h_3} - 2\, {}^0h_4\, h_3[\Upsilon(x^k,t) - 4s^2] = 0, \qquad (94)$$

where the constant s^2 is introduced as an additional source in order to take into account possible contributions resulting from (anti) self-dual fields. The N-connection coefficients are

$$w_i = \partial_i \widetilde{\Psi}/\partial_t \widetilde{\Psi},$$
$$n_i = {}^1n_k\left(x^i\right) + {}^2n_k\left(x^i\right)\int [1/(\sqrt{|h_3|})^3]\mathrm{d}t,$$

where $\widetilde{\Psi} = \ln|\partial_t h_3/\sqrt{|{}^0h_4 h_3|}|$.

The Levi-Civita configurations for solutions (93) are selected by the conditions (37) which, for this case, are satisfied if

$${}^2n_k\left(x^i\right) = 0,\ \ \partial_i\, {}^1n_k = \partial_k\, {}^1n_i,$$

and

$$\partial_t\left(w_i[\widetilde{\Psi}]\right) + w_i[\widetilde{\Psi}]\partial_t\left(h_3[\widetilde{\Psi}]\right) + \partial_i h_3[\widetilde{\Psi}] = 0,$$
$$\partial_i\, w_k[\widetilde{\Psi}] = \partial_k\, w_i[\widetilde{\Psi}].$$

Such conditions are similar to (91) but for a different relation of v-coefficients of d-metrics to another type of generating function $\widetilde{\Psi}$. They are always satisfied for cosmological solutions with $\widetilde{\Psi} = \widetilde{\Psi}(t)$ or if $\widetilde{\Psi} = const$ (in the last case $w_i(x^k,t)$ can be any functions as follows from (35) with zero β and α_i; see (38)). We have

$$\widehat{\mathbf{g}} = \eta_1\, {}^F\mathring{g}_1\mathrm{d}x\otimes\mathrm{d}x + \eta_2\, {}^F\mathring{g}_2\mathrm{d}z\otimes\mathrm{d}z$$
$$+ ({}^s\omega)^2\left[\eta_3\, {}^F\mathring{h}_3\mathbf{e}^3\otimes\mathbf{e}^3 + \eta_4\, {}^F\mathring{h}_4\mathbf{e}^4\otimes\mathbf{e}^4\right],$$
$$\mathbf{e}^3 = \mathrm{d}y + \eta_1^3\mathrm{d}x + \eta_2^3\mathrm{d}z,\quad \mathbf{e}^4 = \mathrm{d}t + \eta_1^4\mathrm{d}x + \eta_2^4\mathrm{d}z, \qquad (95)$$

where

$$\eta_1 = \eta_2 = a^{-2}(t)e^{\psi(x,z)},\quad \eta_3 = {}^s\widetilde{\Psi}^2/4(\Lambda_0 - 4s^2)a^2(t),$$
$$\eta_4 = (\partial_t\, {}^s\widetilde{\Psi})^2/\, {}^s\Xi,\quad \eta_i^3 = n_i,\quad \eta_i^4 = w_i.$$

Any solution $h_3(x^k,t)$ of Eq. (94) generates a d-metric or (95) which depends on the parameter $(\Lambda_0 - 4s^2) \neq 0$. The singular case with $\Lambda_0 = 4s^2$ can be described by a d-metric (93) when h_3 is a solution of $\partial_{tt}^2 h_3 - (\partial_t h_3)^2/2h_3 = 0$. For such configurations, we lose information as regards Λ_0 and s^2 but certain encodings of matter field interactions are possible in the function $\psi(x^k)$ if the right side source $\partial_{xx}^2\psi + \partial_{zz}^2\psi = 2(\Upsilon - 4s^2)$ is changed to the non-trivial case $2({}^v\Upsilon - 4s^2)$ for an N-adapted and anisotropic source ${}^v\Upsilon(x^k)$.

Finally, we emphasize that off-diagonal deformations of FLRW metrics in TMTs with effective EYMH interactions sources of the type $\Upsilon - 4s^2$ can be used for driving to zero an effective cosmological constant or for modelling parametric transforms to configurations with small effective vacuum energy.

4.3.2 Nonhomogeneous EYMH effects in Bianchi cosmology in TMTs

Spatially homogeneous but anisotropic relativistic cosmological models were constructed following the Bianchi classification corresponding to symmetry properties of their spatial hypersurfaces [30,37,38]. Such cosmological metrics are parameterized by orthonormal tetrad (vierbein) bases $e_{\alpha''} = e^{\underline{\alpha}}_{\ \alpha''}\partial/\partial u^{\underline{\alpha}}$, if

$${}^B\mathbf{g}_{\alpha''\beta''} = {}^Be^{\underline{\alpha}}_{\ \alpha''}\ {}^Be^{\underline{\beta}}_{\ \beta''}\ {}^Bg_{\underline{\alpha}\underline{\beta}} = diag[1,1,1,-1] \qquad (96)$$

and

$$\left[{}^Be_{\alpha''},\ {}^Be_{\beta''}\right] = {}^Bw^{\gamma''}_{\ \alpha''\beta''}(t)\ \ {}^Be_{\gamma''}$$

are satisfied and the 'structure constants' depend on time-like variables,

$$ {}^{B}w^{\gamma''}_{\alpha''\beta''}(t) = \epsilon_{\alpha''\beta''\tau''}n^{\tau''\gamma''}(t) + \delta^{\gamma''}_{\beta''}b_{\alpha''}(t) - \delta^{\gamma''}_{\alpha''}b_{\beta''}(t). \quad (97) $$

The values ${}^{B}w^{\gamma''}_{\alpha''\beta''}(t)$ are determined by some diagonal tensor, $n^{\tau''\gamma''}$, and vector, $b_{\alpha''}$, fields used for the classification mentioned. Depending on the parametrization of such tensor and vector objects, one constructs the so-called Bianchi universes which are either open or closed similar to the homogeneous and isotropic FLRW case. With non-trivial limits from observational cosmology, the so-called Bianchi I, V, VII_0, VII_h and IX universes and their corresponding cosmologies exist.

The AFDM allows us to generalize any Bianchi metric ${}^{B}\mathbf{g}_{\alpha''\beta''}$ (96) into locally anisotropic solutions. As the first step, we transform a set of coefficients ${}^{B}g_{\underline{\alpha}\underline{\beta}}(t)$ into the prime metric using frame transformations, ${}^{B}\mathring{\mathbf{g}}_{\alpha\beta} = {}^{B}e^{\underline{\alpha}}_{\ \alpha''}\ {}^{B}e^{\underline{\beta}}_{\ \beta''}\ {}^{B}g_{\underline{\alpha}\underline{\beta}}$. One also needs to solve certain quadratic algebraic equations for ${}^{B}e^{\underline{\alpha}}_{\ \alpha''}$ in order to define frame coefficients depending on the coordinate t, and ${}^{B}\mathring{\mathbf{g}}_{\alpha\beta}$ is parameterized as a prime metric,

$$ {}^{B}\mathring{\mathbf{g}}_{\alpha\beta} = {}^{B}\mathring{g}_{i}dx^{i}\otimes dx^{i} + {}^{B}\mathring{h}_{a}(t)\ {}^{B}\mathring{\mathbf{e}}^{a}\otimes {}^{B}\mathring{\mathbf{e}}^{a}, \quad (98) $$
$$ \mathring{\mathbf{e}}^{3} = dy^{3} + {}^{B}\mathring{n}_{i}(t)dx^{i},\ \mathring{\mathbf{e}}^{4} = dt + {}^{B}\mathring{w}_{i}(t)dx^{i}. $$

We generalize these anisotropic homogeneous cosmological metrics to generic off-diagonal locally anisotropic and inhomogeneous configurations defining cosmological solutions in TMTs with effective EYMH interactions.

The target ansatz is considered to be of the type (21),

$$ \widehat{\mathbf{g}} = \left[\eta_i\ {}^{B}\mathring{g}_i, ({}^{s}\omega)^2\ \eta_a\ {}^{B}\mathring{h}_a;\ {}^{B}\mathring{n}_i + \eta^3_1,\ {}^{B}\mathring{w}_i + \eta^4_i\right], $$

with prime data determined by te coefficients of (98). We construct metrics $\widehat{\mathbf{g}}$ defining generic off-diagonal solutions of the nonholonomic EYMH system in TMTs, (68)–(72) with source (67), following the same procedure as in Sect. 3. In terms of polarization functions, such solutions take the following form:

$$ \widehat{\mathbf{g}} = \eta_1\ {}^{B}\mathring{g}_1 dx^1\otimes dx^1 + \eta_2\ {}^{B}\mathring{g}_2 dx^2\otimes dx^2 + ({}^{B}\omega)^2\left[\eta_3\ {}^{B}\mathring{h}_3\mathbf{e}^3\otimes\mathbf{e}^3 + \eta_4\ {}^{B}\mathring{h}_4\mathbf{e}^4\otimes\mathbf{e}^4\right], $$
$$ \mathbf{e}^3 = dy^3 + ({}^{B}\mathring{n}_1 + \eta^3_1)dx^1 + ({}^{B}\mathring{n}_2 + \eta^3_2)dx^2, \quad (99) $$
$$ \mathbf{e}^4 = dt + ({}^{B}\mathring{w}_1 + \eta^4_1)dx^1 + ({}^{B}\mathring{w}_2 + \eta^4_2)dx^2. $$

The off-diagonal deformations of Bianchi metrics determined by the sources $\Upsilon - 4s^2 \neq 0$, and $\Lambda_0 - 4s^2 \neq 0$, with $\partial_t h_a \neq 0, \partial_t \varpi \neq 0$ are computed as

$$ {}^{B}g_1 = \eta_1\ {}^{B}\mathring{g}_1 = e^{\psi(x^k)}, $$
$$ {}^{B}g_2 = \eta_2\ {}^{B}\mathring{g}_2 = e^{\psi(x^k)}, $$

for $\psi(x^k)$ being a solution of the Poisson equation $\partial^2_{11}\psi + \partial^2_{22}\psi = 2(\Upsilon - 4s^2)$;

$$ {}^{B}h_3 = \eta_3\ {}^{B}\mathring{h}_3 = \frac{{}^{B}\widetilde{\Psi}^2}{4(\Lambda_0 - 4s^2)} \text{ and} $$
$$ {}^{B}h_4 = \eta_4\ {}^{B}\mathring{h}_4 = \frac{(\partial_t\ {}^{B}\widetilde{\Psi})^2}{{}^{B}\Xi}, $$

are computed for an effective cosmological constant $\Lambda_0 - 4s^2 \neq 0$ with generating function

$$ {}^{B}\Psi^2 = (\Lambda_0 - 4s^2)^{-1}\int dt(\Upsilon - 4s^2)\partial_t({}^{B}\widetilde{\Psi}^2) \text{ or} $$
$$ {}^{B}\widetilde{\Psi}^2 = (\Lambda_0 - 4s^2)\int dt(\Upsilon - 4s^2)^{-1}\partial_t({}^{B}\Psi^2). $$

We put the left label "B" in our formulae in order to emphasize that certain values contain information on prime metrics. For simplicity, we omit "s" even when gauge and Higgs fields contributions are present.

The functional

$$ {}^{B}\Xi[\Upsilon,\ {}^{B}\widetilde{\Psi}] = \int dt(\Upsilon - 4s^2)\partial_t({}^{B}\widetilde{\Psi}^2) $$

can be considered as a redefined source, $\Upsilon - 4s^2 \rightarrow {}^{B}\Xi$, for a prescribed generating function ${}^{B}\widetilde{\Psi}$ for locally anisotropic and inhomogeneous Bianchi configurations, when $\Upsilon - 4s^2 = \partial_t({}^{B}\Xi)/\partial_t({}^{B}\widetilde{\Psi}^2)$. This allows to compute the N-connection coefficients

$$ {}^{B}n_k(x^k, t) = {}^{B}\mathring{n}_k + \eta^3_k = {}_1n_k(x^i) + {}_2\widetilde{n}_k(x^i) \int dt \frac{(\partial_t\ {}^{B}\widetilde{\Psi})^2}{({}^{B}\widetilde{\Psi})^3\ {}^{B}\Xi} \text{ and} \quad (100) $$
$$ {}^{B}w_i(x^k, t) = {}^{B}\mathring{w}_i + \eta^4_i = \frac{\partial_i\ {}^{B}\Xi}{\partial_t\ {}^{B}\Xi}, $$

which is constrained additionally to define LC-configurations following the procedure described in Sect. 3.

The v-conformal factor ${}^{B}\omega(x^k, y^3, t)$ is a solution of (44) with coefficients (100) when

$$ \partial_i\ {}^{B}\omega - {}^{B}n_i\ \partial_3\ {}^{B}\omega - {}^{B}w_i\ \partial_t\ {}^{B}\omega = 0. $$

Having constructed an inhomogeneous locally anisotropic cosmological metric $\widehat{\mathbf{g}}(x^k, t)$ (99), we consider additional assumptions on generating and integration functions when the coefficients are homogeneous but with nonholonomically deformed Bianchi symmetries. This is possible if we choose at the end "pure" time dependencies ${}^{B}\widetilde{\Psi}(t), \Upsilon(t), {}^{B}h_a(t), w_i(t)$ and constant values ${}^{B}g_k$ and ${}^{B}n_i$.

4.3.3 Kasner type metrics

Another class of anisotropic cosmological metrics is determined by the Kasner solution and various generalizations

[39–41]. Such 4-d metrics are written in the form

$$ {}^{K}\mathbf{g}=t^{2p_1}dx\otimes dx+t^{2p_3}dz\otimes dz+t^{2p_2}dy\otimes dy-dt\otimes dt, \quad (101) $$

with ${}^{K}g_1=t^{2p_1}$, ${}^{K}g_2=t^{2p_3}$, ${}^{K}h_3=t^{2p_2}$, ${}^{K}h_4=-1$ and ${}^{K}N_i^a=0$. The constants p_1, p_2, p_3 define solutions of the vacuum Einstein equations if the following conditions are satisfied:

$$ 2\,{}^{3}P={}^{2}P-{}^{1}P, \quad (102) $$

for $\left({}^{1}P\right)^2=(p_1)^2+(p_2)^2+(p_3)^2$, ${}^{2}P=p_1+p_2+p_3$, ${}^{3}P=p_1p_2+p_2p_3+p_1p_3$. Following the anholonomic deformation method, we generalize such solutions to generic off-diagonal cosmological configurations as in Sect. 4.2 when $\Upsilon=4s^2$.

The data for a primary metric are taken as $\mathring{g}_1=1$, $\mathring{g}_2=t^{2(p_3-p_1)}$, $\mathring{h}_3=t^{2(p_2-p_1)}$, $\mathring{h}_4=-t^{-2p_1}$ and $\mathring{N}_i^a=0$ with constants p_1, p_2 and p_3 considered for (101). For simplicity, let us analyze solutions with $p_3=p_1$ and consider an example when a Kasner universe is generalized to locally anisotropic configurations characterized with gravitational polarizations

$$ \eta_i=1,\quad \eta_3=f\left(x^i,t\right),\quad \eta_4={}^{0}h^2\left[\partial_t f\left(x^i,t\right)\right]^2, $$
$$ \eta_i^3=n_i(x^k,t),\quad \eta_i^4=w_i(x^i,t). $$

For $h_a=\eta_a\,{}^{\circ}h_a$ and $N_i^a=\eta_i^a+{}^{\circ}N_i^a$, the target metric is of type (81) generated for $\Upsilon=4s^2$,

$$ \widehat{\mathbf{g}}=dx^1\otimes dx^1+dx^2\otimes dx^2+f^2\left(x^i,t\right)t^{-2p_1}\mathbf{e}^3\otimes\mathbf{e}^3 $$
$$ -{}^{0}h^2\left[\partial_t f\left(x^i,t\right)\right]^2 t^{-2p_1}\mathbf{e}^4\otimes\mathbf{e}^4, $$
$$ \mathbf{e}^3=dy^3+n_k\left(x^i,t\right)dx^i,\quad \mathbf{e}^4=dt+w_i(x^k,t)dx^i, \quad (103) $$

where $w_i=w_i(x^i,t)$ are arbitrary functions and

$$ n_k={}^{1}n_k(x^i)+{}^{2}n_k(x^i)\int dt\left[\partial_t \ln|f\left(x^i,t\right)|\right]^2. $$

The coefficient h_4 is determined by h_3 following the formula $\sqrt{|h_4|}={}^{0}h\,\partial_t\sqrt{|h_3|}$, which holds true for η_a for arbitrary generating function $f\left(x^i,t\right)$ if $p_2=p_1$. Additional constraints on $f\left(x^i,t\right)$ are needed if the last condition is not satisfied. In the limit of trivial polarizations, this d-metric results in a conformally transformed metric (with factor t^{2p_1}) of the Kasner solution (101). In general, such primed metrics are not a solution of the Einstein equations for the Levi-Civita connection but it is possible to choose gravitational polarizations that generate vacuum off-diagonal Einstein fields even when the conditions of type (102) are not satisfied.

To generate homogeneous but anisotropic solutions we eliminate dependencies on space coordinates and consider arbitrary $w_i=w_i(t)$ and constant ${}^{1}n_k$ and ${}^{2}n_k$, when

$$ n_k={}^{1}n_k+{}^{2}n_k\int dt\,[\partial_t \ln|f(t)|]^2. $$

For LC-configurations, we take ${}^{2}n_k=0$ and impose constraints of type (80) on $w_i(t)$.

In a similar manner, we construct various nonholonomic deformations of the Kasner universes of types 1–3 and/or and generalize them to solutions of type (103).

5 Effective TMT large field inflation with ${}^{c}\alpha$-attractors

We consider a broad class of (off-) diagonal attractor solutions that arise naturally in (modified) gravity theories and TMTs and define what we imply by natural inflationary models. In this work, we study cosmological attractors as they are considered for cosmological models in Refs. [10,12,13]. The use of the word attractor needs to be clarified as a similar term is widely used in the theory of dynamical systems, for certain equilibrium configurations with critical points in the phase space, i.e., critical points which are stable. Our use of the word attractor solutions is in the same spirit as Refs. [10,12,13]. What the authors of that work mean by cosmological attractors (see, for instance, Ref. [10]) can be stated in their own words: "Several large classes of theories have been found, all of which have the same observational predictions in the leading order in 1/N. We called these theories "cosmological attractors." In our approach, the use of the word "attractor" is similar but in a more general context for generic off-diagonal solutions. Certain configurations in our work are determined by solutions, in general, with nonholonomically induced torsion and can be restricted to LC-configurations. Under such assumptions, these configurations appear again in other models under consideration by us. We group all such models as having "cosmological attractor configurations" since the configurations are common to these class of models. It is implicit that such solutions satisfy the conditions for "standard" cosmological attractor configurations (in the sense of Linde et al.) only for certain subclasses of nonholonomic constraints when the models are determined by imposing constraints on the corresponding generating and integration functions and integration constants. For general nonholonomic constraints, such configurations do not define cosmological attractor configurations in the sense of the above mentioned original work [10,12,13] but positively can be considered to possess similar properties for small off-diagonal deformations (perturbations) of the metrics. The important point is that such models have the same observational predictions in the leading order of $1/N$. In this section, we shall define and study cosmological attractor configurations for modified gravity theories in terms of a parameter ${}^{c}\alpha$ that determines the curvature and cut-off. Henceforth, in order to make our manuscript more transparent, wherever we use the word "attractor", it will simply imply that certain classes of theories and respective off-diagonal cosmological solutions are generated which, under specific conditions on the parameter space, lead to the same observational predictions.

5.1 Nonholonomic conformal transforms and cosmological attractors

Attractor type configurations are possible to construct for a certain classes of nonlinear scalar potentials in (28). We use the term "configuration" because in that formula and in Eq. (29) there are considered N-elongated derivatives. The equations are written with respect to nonholonomic bases and for generalized Ricci scalar curvature. As such additional assumptions are necessary in order to extract a "standard " cosmological attractor considered, for instance, in [10]. To begin with, we take the effective potential (24),

$$ {}^{e}V = {}^{q}V = q^{2}(\tanh\phi), \qquad (104)$$

for an arbitrary function q and study the model with the lagrangian

$$ {}^{1}_{q}L = -\frac{1}{\kappa}\widehat{R}(\mathbf{g}) + \frac{1}{2}\mathbf{g}^{\mu\nu}\mathbf{e}_{\mu}\phi\,\mathbf{e}_{\nu}\phi - q^{2}(\tanh\phi). \qquad (105)$$

Equations (29) impose the condition ${}^{1}_{q}L = M = const.$[9] Attractor models are usually constructed in terms of two fields. In addition to $\phi(u^{\mu})$ we consider a second field $\chi(u^{\mu})$. The fields (ϕ, χ) are subject to additional nonholonomic constraints involving the generating function $\Psi = e^{\varpi}$ (39), some possible re-definitions (49) of effective matter field sources Υ and the effective cosmological constant Λ_{0}.

The theory (105) is related to a class of models

$$ {}^{1}_{\chi}L = \frac{1}{2}\Big[\mathbf{g}^{\mu\nu}(\mathbf{e}_{\mu}\phi\,\mathbf{e}_{\nu}\phi - \mathbf{e}_{\mu}\chi\,\mathbf{e}_{\nu}\chi) + (\phi^{2}-\chi^{2})\widehat{R}(\mathbf{g}) + q^{2}(\phi/\chi)(\phi^{2}-\chi^{2})^{2}\Big] \qquad (106)$$

by the gauge condition

$$ \phi^{2} - \chi^{2} = 1. \qquad (107)$$

The Lagrange density ${}^{1}_{\chi}L$ possesses a $SO(1,1)$ symmetry which is deformed by the term $q^{2}(\phi/\chi)$. In turn, the Lagrange density ${}^{1}_{q}L$ may restore the $SO(1,1)$ symmetry at a critical point because for large ϕ there exist asymptotic limits, $\tanh\phi \to \pm 1$ and $q^{2}(\tanh\phi) \to const$. The terms proportional to q^{2} can be transformed into effective sources and cosmological constant via eventual rescaling of generating functions. Employing self-duality for gauge field configurations with source $\Upsilon - 4s^{2}$, (67), and using the gauge (107) with $\breve{\phi} = \sinh\phi$ and $\breve{\chi} = \cosh\chi$, we can approximate ${}^{1}_{\chi}L$ by

$$ {}^{1}_{\breve{\phi}}L = \frac{1}{2}\Big[-\widehat{R}(\mathbf{g}) + \mathbf{g}^{\mu\nu}\mathbf{e}_{\mu}\breve{\phi}\,\mathbf{e}_{\nu}\breve{\phi} + (\Lambda_{0} - 4s^{2})\Big].$$

Another important property of the Lagrange density ${}^{1}_{\chi}L$ is that for a fixed value $q = q_{0}$ there is local conformal invariance under N-adapted transforms,

$$ \tilde{\mathbf{g}}_{\mu\nu} = e^{-2\sigma(u)}\mathbf{g}_{\mu\nu},\ \tilde{\chi} = e^{\sigma(u)}\chi,\ \tilde{\phi} = e^{\sigma(u)}\phi. \qquad (108)$$

Such a theory describes antigravity if $\phi^{2} - \chi^{2} > 0$, i.e., χ represents the cut-off for possible values of the scalar field ϕ.

By identifying σ from (108) with $\widehat{\sigma}$ in (26) when $e^{-2\widehat{\sigma}(u)} = 2U/(V+M) = \Phi/\sqrt{|\mathbf{g}_{\alpha\beta}|}$ we present a model of a TMT theory of the type (27) derived for the action

$$ S = \int d^{4}u\Big[{}^{1}_{q}L\Phi + {}^{2}L\sqrt{|\mathbf{g}_{\alpha\beta}|} + N\phi\varepsilon^{\mu\nu\alpha\beta}F^{a}_{\mu\nu}F^{a}_{\alpha\beta}\Big] \simeq \int d^{4}u\Big[{}^{1}_{\chi}L\Phi + {}^{2}L\sqrt{|\mathbf{g}_{\alpha\beta}|} + N\phi\varepsilon^{\mu\nu\alpha\beta}F^{a}_{\mu\nu}F^{a}_{\alpha\beta}\Big] \simeq \int d^{4}u\Big[{}^{1}_{\breve{\phi}}L\Phi + {}^{2}L\sqrt{|\mathbf{g}_{\alpha\beta}|} + N\phi\varepsilon^{\mu\nu\alpha\beta}F^{a}_{\mu\nu}F^{a}_{\alpha\beta}\Big]. \qquad (109)$$

The explicit construction depends on the type of generating functions, conformal transforms, effective sources, asymptotic limits and gauge conditions employed in our theory. This way we construct different toy TMT models with EYMHs which for data (ϕ, χ) possess attractor properties and the parameters defining such attractors encode off-diagonal gravitational and (effective) matter field interactions. It is a very difficult technical task to construct cosmological solutions in such theories. Nevertheless, transforming any variant (109) into an effective gravitational theory (25) with source ${}^{e}\mathbf{T}_{\alpha\beta} \to {}^{q}\mathbf{T}_{\alpha\beta}$ (18) corresponding to ${}^{q}V$ contributions, (104), the effective EYMH equations (56)–(58) can be integrated in very general forms following the AFDM. Their solutions depend on integration functions and integration constants.

It is very surprising that Eqs. (106)–(109) and their physical consequences are similar to those for holonomic models considered in Refs. [10,12,13]. Our solutions encoding cosmological attractor configurations were derived for a class of modified theories with generalized off-diagonal metrics and nonlinear and distinguished linear connections and contributions from EYMHs for different TMT modes. It is not obvious that such nonlinear systems may have a similar cosmological attractor behaviour like in the original work with diagonal solutions. Generic off-diagonal models can be elaborated following our geometric techniques with N-adapted

[9] In this section, we use natural units $1/\kappa = 1/2$.

nonholonomic variables and splitting of corresponding systems of nonlinear PDEs. In such variables, it is possible to generate new classes of inhomogeneous and anisotropic solutions. Our goal was to find such classes of nonholonomic constraints and subclasses of generating and integration functions, and constants, when solutions with "hat" values and TMT–EYMH contributions really preserve the main physical properties of cosmological attractors. This emphasizes the general importance of the results on cosmological attractors in the cited work due to Linde et al. Our main conclusion is that, for a corresponding class of nonholonomic constraints, a cosmological attractor configuration may "survive" for very general off-diagonal and matter source deformations, in various classes of TMT theories and effective Einstein like ones encoding modified gravity theories.

5.2 Effective interactions and cosmological attractors

We can fix different gauge conditions but obtain the same results. For instance, we can work with $\chi(x) = 1$ instead of (107), and the scalar field $\check{\phi}$. With respect to N-adapted Jordan frames, the total Lagrangian is

$$ {}^{1}_{J}L = -\frac{1}{2}\widehat{R}({}^{J}\mathbf{g})\,(1-\check{\phi}^2) + \frac{1}{2}\mathbf{g}^{\mu\nu}\mathbf{e}_\mu\check{\phi}\,\mathbf{e}_\nu\check{\phi} + q^2(\check{\phi})\,(\check{\phi}^2-1)^2. $$

We change the d-metric into a conformally equivalent metric with equivalent Einstein frame formulation in terms of ${}^{E}\mathbf{g}_{\mu\nu}$, when ${}^{E}\mathbf{g}_{\mu\nu} = (1-\check{\phi}^2)\,{}^{J}\mathbf{g}_{\mu\nu}$. The Lagrangian ${}^{1}_{J}L$ transforms into ${}^{1}_{E}L$ where

$$ {}^{1}_{E}L = -\frac{1}{2}\widehat{R}(\mathbf{g}) + \frac{1}{(1-\check{\phi}^2)^2}\mathbf{g}^{\mu\nu}\mathbf{e}_\mu\check{\phi}\,\mathbf{e}_\nu\check{\phi} + q^2(\check{\phi}). \quad (110) $$

Equivalently, ${}^{1}_{E}L$ transforms into ${}^{1}_{q}L$ (105) if the scalar fields are redefined as follows:

$$ \frac{d\phi}{d\check{\phi}} = (1-\check{\phi}^2)^{-1}, \text{ i.e. } \check{\phi} = \tanh\phi. $$

In the theory ${}^{1}_{E}L$ (110) there is an ultra violet (UV) cut-off $\Lambda = 1$, i.e. $\Lambda = M_p$ in terms of the Planck mass and if ϕ become greater than 1 we get a TMT theory with antigravity. Such models were studied in the literature [36,42] and other papers before the concept of cosmological attractors was introduced. Our main goal is to study how "nonholonomically deformed" cosmological attractors can be modelled by nonholonomic constraints, generating functions and effective sources in such a way that the criteria are satisfied for "standard" cosmological attractors to emerge in the sense of Refs. [10,12,13]. We do not consider in this work similar constructions with nonholonomic variables but only emphasize that certain antigravity effects can be modelled by off-diagonal gravitational interactions and effective polarization of physical constants [24]. The previous formulae show that ϕ becomes infinitely large if $\check{\phi} \to 1$ but the effects of the cut-off can be ignored if $|\check{\phi}| \ll 1$, when $\check{\phi} \approx \phi$. Excluding some very singular behaviour near the boundary of the moduli space, the asymptotic behaviour of ${}^{q}V(\phi)$ (104) at large ϕ is universal. This universality exists for TMT models with effective EYMH interactions as follows from above equivalence (under well-defined conditions) of theories (109) and (25).

The goal of this section is to study cosmological effects of (in general, locally anisotropic and inhomogeneous) attractors parameterized by a constant ${}^{c}\alpha \lesssim O(1)$. Attractor configurations can be introduced in several inequivalent ways. We will generalize the constructions following [10] and analyze possible off-diagonal solutions determined by the sources ${}^{q}V(\phi)$ and ${}^{c}\alpha$ parameters.

We consider the Lagrangian

$$ {}^{\alpha}_{E}L = -\frac{1}{2}\widehat{R}(\mathbf{g}) + \frac{{}^{c}\alpha}{(1-\tilde{\phi}^2)^2}\mathbf{g}^{\mu\nu}\mathbf{e}_\mu\tilde{\phi}\,\mathbf{e}_\nu\tilde{\phi} + q^2(\tilde{\phi}), $$

which is given also in the Einstein frame as (110) but contains a cut-off ${}^{c}\alpha$. We label the scalar field as $\tilde{\phi}$ (instead of $\check{\phi}$) in order to emphasize that we shall analyze a special class of solutions with ${}^{c}\alpha$-dependence. We obtain a ${}^{c}\alpha$-attractor configuration by rescaling the scalar field,

$$ \frac{d\tilde{\phi}}{d\check{\phi}} = (1-\check{\phi}^2)^{-1}, \text{ i.e. } \check{\phi} = \tanh\tilde{\phi}, \quad \text{and/ or redefining} $$

$$ \frac{\tilde{\phi}}{\sqrt{{}^{c}\alpha}} = \tanh\frac{\phi}{\sqrt{{}^{c}\alpha}}, $$

which leads to effective theories of the type

$$ \begin{aligned} {}^{\alpha}_{E}L &= -\frac{1}{2}\widehat{R}(\mathbf{g}) + \frac{{}^{c}\alpha}{(1-\tilde{\phi}^2/{}^{c}\alpha)^2}\mathbf{g}^{\mu\nu}\mathbf{e}_\mu\check{\phi}\,\mathbf{e}_\nu\check{\phi} + q^2(\frac{\check{\phi}}{\sqrt{{}^{c}\alpha}}) \\ &= -\frac{1}{2}\widehat{R}(\mathbf{g}) + \frac{1}{2}\mathbf{g}^{\mu\nu}\mathbf{e}_\mu\phi\,\mathbf{e}_\nu\phi + q^2(\tanh\frac{\phi}{\sqrt{{}^{c}\alpha}}), \end{aligned} $$

with a shifted cut-off position at $\Lambda = \sqrt{{}^{c}\alpha}$.

5.3 Off-diagonal attractor type cosmological solutions

As alluded to in the previous subsection, the asymptotic behaviour of ${}^{q}V(\phi)$ (104) at large ϕ is universal. This universality allows one to construct various classes of generic off-diagonal cosmological metrics in modified models of gravity with effective EYMH interactions using the conformal factor transformation (26). This is possible even when the generating functions and sources are very different for different classes of effective matter field interactions with nonlinear scalar potentials. The goal of this section is to prove how q-terms of the type ${}^{q}V(\phi, {}^{c}\alpha)$ for attractors are encoded in various classes of solutions studied in previous section. This holds for any

$$e^{-2\,{}^{c}\widehat{\sigma}(u)} = 2U/\left[\,{}^{q}V(\phi,\ {}^{c}\alpha) + M\right] = {}^{c}\Phi/\sqrt{|\,{}^{c}\mathbf{g}_{\alpha\beta}|},$$

where the left label "c" indicates that certain values refer to attractor configurations with ${}^{c}\alpha$-scale. The physical cosmological d-metric ${}^{c}\mathbf{g}_{\alpha\beta}$ is computed to be

$$ {}^{c}\mathbf{g}_{\mu\nu} = e^{2\,{}^{c}\widehat{\sigma}(u)}\,{}^{c}\widehat{\mathbf{g}}_{\mu\nu} = \frac{{}^{q}V(\phi,\ {}^{c}\alpha) + M}{2U}\,{}^{c}\widehat{\mathbf{g}}_{\mu\nu}. \quad (111)$$

Having computed ${}^{c}\mathbf{g}_{\mu\nu}$ for the data $\left[\,{}^{c}\widehat{\mathbf{g}}_{\mu\nu},\ {}^{q}V, M, U\right]$, we construct a corresponding TMT model when the second measure is taken to be

$$ {}^{c}\Phi = \frac{2U}{{}^{q}V(\phi,\ {}^{c}\alpha) + M}\sqrt{|\,{}^{c}\mathbf{g}_{\alpha\beta}|}. \quad (112)$$

Equations (111) and (112) can be applied to generate solutions for the TMT system (29)–(31) if ${}^{c}\widehat{\mathbf{g}}_{\mu\nu}$ is known as an attractor cosmological metric (in general, nonhomogeneous and locally anisotropic) for effective EYMH interactions.

5.3.1 *Off-diagonal effective EYMH cosmological attractor solutions of type 1*

Using (89) and (111), we construct families of generic off-diagonal cosmological attractor configurations with metrics

$$ {}^{c}\mathbf{g} = e^{2\,{}^{c}\widehat{\sigma}(u)}\Big\{\eta_1\ {}^{F}\mathring{g}_1 dx\otimes dx + \eta_2\ {}^{F}\mathring{g}_2 dz\otimes dz + ({}^{c}\omega)^2 \left[\eta_3\ {}^{F}\mathring{h}_3 \mathbf{e}^3\otimes\mathbf{e}^3 + \eta_4\ {}^{F}\mathring{h}_4 \mathbf{e}^4\otimes\mathbf{e}^4\right]\Big\},$$
$$\mathbf{e}^3 = dy + \eta_1^3 dx + \eta_2^3 dz,\ \mathbf{e}^4 = dt + \eta_1^4 dx + \eta_2^4 dz, \quad (113)$$

where the gravitational polarizations and N-connection coefficients are computed to be

$$ {}^{c}\eta_1 = {}^{c}\eta_2 = a^{-2}(t) e^{\,{}^{c}\psi(x,z)},\quad \eta_3 = {}^{c}\widetilde{\Psi}^2/4(\Lambda_0^c - 4s^2)a^2(t),$$
$$ {}^{c}\eta_4 = (\partial_t\,{}^{c}\widetilde{\Psi})^2/\,{}^{c}\Xi,\quad {}^{c}\eta_i^3 = n_i,\quad {}^{c}\eta_i^4 = w_i.$$

The parameter ${}^{c}\alpha$ contributes to all data defining such nonholonomic deformations of FLRW primary metric because it is included in the effective source when $\boldsymbol{\Upsilon} \to {}^{c}\boldsymbol{\Upsilon}$ with ${}^{c}\boldsymbol{\Upsilon} - 4s^2 \neq 0$. The corresponding effective cosmological constant is labelled Λ_0^c and satisfies the condition $\Lambda_0^c - 4s^2 \neq 0$ (for the class of solutions of type 1). As a result, the generating functions is redefined to simplify the formulae, ${}^{c}\Psi \longleftrightarrow {}^{c}\widetilde{\Psi}$, with

$$ {}^{c}\Psi^2 = (\Lambda_0^c - 4s^2)^{-1}\int dt({}^{c}\boldsymbol{\Upsilon} - 4s^2)\partial_t({}^{c}\widetilde{\Psi}^2) \text{ and}$$
$$ {}^{c}\widetilde{\Psi}^2 = (\Lambda_0^c - 4s^2)\int dt({}^{c}\boldsymbol{\Upsilon} - 4s^2)^{-1}\partial_t({}^{c}\Psi^2).$$

The information on ${}^{q}V(\phi,\ {}^{c}\alpha)$ is also contained in the functional

$$ {}^{c}\Xi[\,{}^{c}\Upsilon,\ {}^{c}\widetilde{\Psi}] = \int dt({}^{c}\Upsilon - 4s^2)\partial_t({}^{c}\widetilde{\Psi}^2).$$

It is considered as a redefined effective source, ${}^{c}\boldsymbol{\Upsilon} - 4s^2 \to {}^{c}\Xi$, for a prescribed generating function ${}^{c}\widetilde{\Psi}$, for which ${}^{c}\boldsymbol{\Upsilon} - 4s^2 = \partial_t({}^{c}\Xi)/\partial_t({}^{c}\widetilde{\Psi}^2)$.

We express (113) as a d-metric (21) with coefficients relevant to the υ-metric:

$$h_3 = \frac{{}^{c}\widetilde{\Psi}^2}{4(\Lambda_0^c - 4s^2)} = {}^{c}\eta_3\ {}^{F}\mathring{h}_3 \text{ and}$$
$$h_4 = \frac{(\partial_t\,{}^{c}\widetilde{\Psi})^2}{{}^{c}\Xi} = {}^{c}\eta_4\ {}^{F}\mathring{h}_4.$$

For the off-diagonal attractor N-connection coefficients, we compute

$$n_i = {}_1n_k(x,z) + {}_2\widetilde{n}_k(x,z)\int dt\frac{(\partial_t\,{}^{c}\widetilde{\Psi})^2}{({}^{c}\widetilde{\Psi})^3\,{}^{c}\Xi} \text{ and}$$
$$w_i = \frac{\partial_i\,{}^{c}\Xi}{\partial_t\,{}^{c}\Xi}.$$

The "vertical" conformal factor ${}^{c}\omega(x,z,y,t)$ is a solution of (44) for which attractor data is written in the form

$$\partial_i\,{}^{c}\omega - n_i\,\partial_3\,{}^{c}\omega - w_i\,\partial_t\,{}^{c}\omega = 0.$$

The function ${}^{c}\psi(x,z)$ presented in the attractor's polarization functions is a solution of (68) when $\partial^2_{xx}\,{}^{c}\psi + \partial^2_{zz}\,{}^{c}\psi = 2({}^{c}\Upsilon - 4s^2)$.

Finally, we conclude that the formulae for the coefficients of the d-metric (113) depend on the type of N-adapted frame and coordinate transforms necessary to fix observational data. The conformal factor $e^{2\,{}^{c}\widehat{\sigma}(u)}$ encodes the attractor parameters in a more direct form.

5.3.2 *Generalized locally anisotropic Bianchi attractors*

Sources with attractor potential ${}^{q}V(\phi)$ (104) induce generic off-diagonal cosmological solutions, in general, with inhomogeneity and local anisotropy. For a target ansatz of type (21), we parameterize

$$ {}^{c}\widehat{\mathbf{g}} = e^{2\,{}^{c}\widehat{\sigma}}\,{}^{c}\mathbf{g} = [\,{}^{c}\eta_i\ \ {}^{B}\mathring{g}_i, ({}^{c}\omega)^2\ \ {}^{c}\eta_a\ {}^{B}\mathring{h}_a;\ {}^{B}\mathring{n}_i + {}^{c}\eta_1^3,\ {}^{B}\mathring{w}_i + {}^{c}\eta_i^4],$$

when the prime solution ${}^{B}\mathring{\mathbf{g}}$ is determined by coefficients of (98). Our purpose is to state the conditions when ${}^{c}\mathbf{g}$ from the above formula defines generic off-diagonal solutions with attractor properties in TMTs with effective EYMH interactions, i.e. of (68)–(72) with source (67) encoding an attractor potential.[10] We follow the same procedure as in Sects. 3 and 4.3.2 and write in terms of the polarization functions

[10] It is supposed that the parameter ${}^{c}\alpha$ contributes to all data defining nonholonomic deformations of a primary Bianchi metric. This parameter is included into effective source when $\Upsilon \to {}^{c}\Upsilon$ with ${}^{c}\Upsilon - 4s^2 \neq 0$ and the effective cosmological constant Λ_0^c is chosen to satisfy the condition $\Lambda_0^c - 4s^2 \neq 0$.

$$
\begin{aligned}
{}^c\widehat{\mathbf{g}} =& \ {}^c\eta_1 \, {}^B\mathring{g}_1 dx^1 \otimes dx^1 + {}^c\eta_2 \, {}^B\mathring{g}_2 dx^2 \otimes dx^2 \\
&+ ({}^B_c\omega)^2 \left[{}^c\eta_3 \, {}^B\mathring{h}_3 \, {}^c\mathbf{e}^3 \otimes {}^c\mathbf{e}^3 + {}^c\eta_4 \, {}^B\mathring{h}_4 \, {}^c\mathbf{e}^4 \otimes {}^c\mathbf{e}^4 \right], \\
{}^c\mathbf{e}^3 =& \ dy^3 + ({}^B\mathring{n}_1 + {}^c\eta^3_1) dx^1 + ({}^B\mathring{n}_2 + {}^c\eta^3_2) dx^2, \quad (114) \\
{}^c\mathbf{e}^4 =& \ dt + ({}^B\mathring{w}_1 + {}^c\eta^4_1) dx^1 + ({}^B\mathring{w}_2 + {}^c\eta^4_2) dx^2.
\end{aligned}
$$

We use double left labelling with "B" and "c" in order to emphasize possible Bianchi anisotropic and attractor-like behaviour of certain geometric/ physical objects. The off-diagonal deformations with $\partial_t \, {}^c h_a \neq 0, \partial_t \, {}^c\varpi \neq 0$ are determined by

$$
{}^B_c g_1 = {}^c\eta_1 \, {}^B\mathring{g}_1 = e^{\,{}^c\psi(x^k)}, \quad {}^B_c g_2 = {}^c\eta_2 \, {}^B\mathring{g}_2 = e^{\,{}^c\psi(x^k)},
$$

for ${}^c\psi(x^k)$ being a solution of the Poisson equation $\partial^2_{11} \, {}^c\psi + \partial^2_{22} \, {}^c\psi = 2({}^c\Upsilon - 4s^2)$, and

$$
{}^B_c h_3 = {}^c\eta_3 \, {}^B\mathring{h}_3 = \frac{{}^B_c\widetilde{\Psi}^2}{4(\Lambda^c_0 - 4s^2)} \text{ and}
$$

$$
{}^B_c h_4 = {}^c\eta_4 \, {}^B\mathring{h}_4 = \frac{(\partial_t \, {}^B_c\widetilde{\Psi})^2}{{}^B_c\Xi}.
$$

The generating functions encode data on inhomogeneous locally anisotropic interactions, attractor configurations and EYMH sources,

$$
{}^B_c\Psi^2 = (\Lambda^c_0 - 4s^2)^{-1} \int dt ({}^c\Upsilon - 4s^2) \partial_t ({}^B_c\widetilde{\Psi}^2) \text{ or}
$$

$$
{}^B_c\widetilde{\Psi}^2 = (\Lambda^c_0 - 4s^2) \int dt ({}^c\Upsilon - 4s^2)^{-1} \partial_t ({}^B_c\Psi^2),
$$

which results in a redefined source, ${}^c\Upsilon - 4s^2 \to {}^B\Xi$, with ${}^c\Upsilon - 4s^2 = \partial_t({}^B_c\Xi)/\partial_t({}^B_c\widetilde{\Psi}^2)$, when

$$
{}^B_c\Xi[\Upsilon, {}^B_c\widetilde{\Psi}] = \int dt ({}^c\Upsilon - 4s^2) \partial_t ({}^B_c\widetilde{\Psi}^2)
$$

for a prescribed generating function ${}^B_c\widetilde{\Psi}$. The N-connection coefficients in (114) are computed thus:

$$
\begin{aligned}
{}^B_c n_k(x^k, t) =& \ {}^B\mathring{n}_k + {}^c\eta^3_k = {}_1n_k(x^i) \\
&+ {}_2\widetilde{n}_k(x^i) \int dt \frac{(\partial_t \, {}^B_c\widetilde{\Psi})^2}{({}^B_c\widetilde{\Psi})^3 \, {}^B_c\Xi} \text{ and} \\
{}^B_c w_i(x^k, t) =& \ {}^B\mathring{w}_i + {}^c\eta^4_i = \frac{\partial_i({}^B_c\Xi)}{\partial_t({}^B_c\Xi)}.
\end{aligned}
$$

Following the procedure explained in Sect. 3, we impose additional constraints and extract LC-configurations.

Dependencies on all spacetime coordinates are modelled via a v-conformal factor ${}^B_c\omega(x^k, y^3, t)$ (in indirect form, it also contain attractor properties) as a solution of (44) with the attractor coefficients stated above when

$$
\partial_i({}^B_c\omega) - {}^B_c n_i \, (\partial_3 \, {}^B_c\omega) - {}^B_c w_i \, (\partial_t \, {}^B_c\omega) = 0.
$$

Restricting the class of generating functions, we extract homogeneous configurations but with anisotropies when parameterizations are of the type ${}^B_c\widetilde{\Psi}(t)$, ${}^c\Upsilon(t)$, ${}^B_c h_a(t)$, ${}^B_c w_i(t)$ and constant values for ${}^B_c g_k$ and ${}^B_c n_i$.

Applying the AFDM, we generate off-diagonal cosmological attractor solutions of types 2 and 3 for the conventional and other families of inflation potentials, for instance, when we use $\widetilde{q}(\frac{\phi/\sqrt{{}^c\alpha}}{1+\phi/\sqrt{{}^c\alpha}})$ instead of $q(\phi/\sqrt{{}^c\alpha})$ [10]. We note that we have used a different system of notations and our approach is based on geometric methods which allows us to construct exact solutions of modified gravitational and matter field equations. For certain well-defined conditions, we reproduce the results and "diagonal" models studied in (supersymmetric) models with dark matter and dark energy effects. Nevertheless, nonlinear parametric systems of PDEs corresponding to effective EYMH interactions in (modified) TMTs contain solutions at a richer level that were not analyzed and applied to modern cosmology. Even though the off-diagonal effects at large observational scales seem to be very small, the generic nonlinear character of cosmological solutions depending on space-like coordinates result in new nonlinear physics described by rescaling via generating functions and effective sources. Attractor type configurations offer alternative solutions of crucial importance for explaining the inflation scenarios in modern cosmology.

5.4 Cosmological implications of TMT nonholonomic attractor type configurations

Here we concentrate on observational consequences of generic off-diagonal solutions for the effective EYMH systems with attractor properties in TMTs. We have demonstrated that Lagrangians of type ${}^1_q L$ (105) and ${}^1_\chi L$ (106) and their effective energy-momentum tensors are naturally included as sources (22) in action (27) with two measures, which result in a nonholonomic modification of Einstein gravity (25). Geometrically, we reproduce such effects via re-definition of generating functions (26) and fixing a cut-off constant ${}^c\alpha$ for attractor configurations, when the effective matter field interactions are modelled for a nonholonomic off-diagonal vacuum configuration with small effective cosmological constant and gravitational η-polarizations.

In general, proposing and observing physical realizations for solutions with arbitrary η-deformations of well-known prime cosmological metrics (for instance, of FLRW, Bianchi or Kasner type ones) are difficult. Nevertheless, we have elaborated upon the large distance inflationary scenarios if $\eta \approx 1 + \varepsilon\widetilde{\eta}$ when $|\varepsilon\widetilde{\eta}| \ll 1$. We note that such configurations encode nonlinear parametric effects even when the off-diagonal and inhomogeneous terms are not taken into consideration in order to explain certain observational data. Using the results of analysis for ${}^1_q L$ (105) and LC-configurations [10], we conclude and speculate on such observational consequences:

1. TMTs and nonholonomic modifications of the EYMH theory contain inflationary model of the plateau-type and features of universal attractor property when $n_s = 1 - 2/N$ and $r = 12\,{}^c\alpha/N^2$.
2. For ${}^c\alpha = 1$ such models are related to cosmological scenarios with the Starobinsky type model and Higgs inflation [43–46]; we obtain an asymptotic theory for quadratic inflation with $n_s = 1 - 2/N, r = 8/N$, for large cut-off ${}^c\alpha$.
3. Decreasing ${}^c\alpha$, we get a universal attractor property both for diagonal and off-diagonal configurations; there are many models which have the same values n_s and r. This property is preserved for EYMH contributions, solitonic and/or gravitational waves for corresponding nonholonomic configurations.
4. In the limit of large ${}^c\alpha$, we have generated models of simplest chaotic inflation. We have shown that effective nonlinear potentials with a second attractor are other viable possibilities.
5. For intermediate values of ${}^c\alpha$, the predictions interpolate between these two critical points, thus oscillating between the sweet spots of both Planck and BICEP2 [6–8].

With respect to the old and new cosmological problem, the issues 1–5 is analyzed in the context of TMTs when the constructions are naturally extended to include effective gauge field contributions which, in turn, modify nonlinearly the sources, effective cosmological constant and generating functions. Via conformal transforms, the attractor configurations are related to inhomogeneous and locally anisotropic solutions in modified gravity theories. It is not surprising that the cosmological attractor configurations with TMT and nonhlonomic modifications of the EYMH theory are described in diagonal limits by the same parametric data as for the holonomic attractor solutions [10,12,13]. We imposed such nonholonomic constraints and selected respective generating functions which reproduce this class of cosmological solutions. Nevertheless, the constants ${}^c\alpha, n_s, r$ encode contributions from modified gravity theories and off-diagonal gravitational and matter field interactions and result in different observational consequences.

6 Concluding remarks

To mention a few, the most important physical solutions in modern gravity and cosmology theories pertaining to black holes, wormhole configurations, FLRW metrics, are constructed for diagonal metrics transforming the (modified) Einstein equations into certain nonlinear systems of second (or higher) order ODEs. The solutions generally depend on integration constants. Such constants are fixed following a certain symmetry and other physical assumptions in order to explain and describe the experimental and observational data. There are also constructed more sophisticated classes of solutions, for instance, with off-diagonal rotating metrics with Killing, Lie type symmetries and solitonic hierarchies which provide important examples of nonlinear models of gravitational and matter interactions. Nevertheless, the bulk of such analytic and numerical methods of constructing exact solutions are based on certain assumptions where the corresponding nonlinear system of PDEs are transformed into ODEs. The solutions are parameterized via integration parameters, symmetry and physical constants. The main idea is to formulate an approach to simplify the equations and find solutions depending, for instance, on a radial or a time-like variable. The drawback of this approach is that a number of nonlinear parametrical solutions are lost and thus unavailable for possible applications in cosmology and astrophysics.

The AFDM is presented as a geometric method for constructing general classes of off-diagonal metrics, auxiliary connections and adapted frames of reference when gravitational and matter field equations in various modified/ generalized gravity theories, including general relativity, are decoupled. This decoupling implies that the corresponding nonlinear system of PDEs splits into certain subclasses of equations which contain partial derivative depending only on one coordinate and relates only two unknown variables and/or generating functions. As a result, we can integrate such systems in very general off-diagonal forms when various classes of solutions are determined not only by integration constants but also by generating and integration functions, symmetry parameters and anholonomy relations. The solutions depend, in general, on all spacetime coordinates and can be with Killing or non-Killing symmetries, of different smooth classes, with singularities and non-trivial topology. We can make, for instance, certain approximations on the type of generating functions and effective source at the end, after a general form of solution has been constructed. This way we generate new classes of cosmological metrics which are homogeneous or inhomogeneous, and in general, with local anisotropies, which cannot be found if one works from the very beginning with a simplified ansatz and higher symmetries. Furthermore, the possibility to re-define the generating functions and sources via nonlinear frame transformations and parametric deformations allows one to entertain new classes of solutions and study various nonlinear physical effects.

In this paper, we studied in explicit form certain classes of modified gravity theories which can be modelled as TMTs with effective EYMH interactions. Possible scalar fields and corresponding nonlinear interaction potentials were chosen to select and reproduce attractor type solutions with cut-off constants which seem to have fundamental implication in elaborating isotropic and anisotropic inflation scenarios in modern cosmology. In general, one can work with off-diagonal configurations and consider diagonal limits for minimal and/or non-minimal coupling constants. We proved that the decoupling property holds also in TMTs, which results in the possibility of constructing various classes of off-diagonal cosmological solutions with small vacuum density. Such solutions describe spacetimes with nonholonomically induced torsion. Nevertheless we formulated well-defined criteria when additional nonholonomic constraints are introduced that allow one to extract LC-configurations. We studied nonholonomic deformations of FLRW, Bianchi and Kasner type metrics encoding TMT effects and possible contributions of effective EYMH interactions.

We have shown that attractor type cosmological solutions with cut-off parameters can be derived by nonlinear re-definitions of generating functions and effective sources in TMT if a corresponding type of nonlinear scalar potential is chosen. In general, such attractor solutions are model independent and are constructed in explicit form to accommodate effective EYMH interactions. In this way various large scale inflationary models, with anisotropic expansion and parametric nonlinear processes can be realized.

For certain conditions, the gravitational and matter field equations of TMTs are expressed as effective Einstein equations with non-minimal coupling [19]. In this presentation, we proved that in nonholonomic N-adapted variables and for additional assumptions the constructions are generalized in such form that two measure configurations serve to encode massive gravity effects and nonlinear parametric off-diagonal interactions (see Eqs. (25)–(27)). In general, such a theory also has four extra degrees of freedom with the Boulware–Deser (BD) ghosts. This problem can be circumvented if one imposes additional constraints. We imposed nonholonomic constraints for constructing cosmological attractor configurations. This procedure constrains the extra dimension degrees of freedom and encodes the TMT and massive term contributions into certain subclasses of solutions for off-diagonal effective Einstein spaces (see similar constructions for ghost-free massive $f(R)$ theories in Refs. [47–49]). We conclude that in our models the BD ghosts are absent for such special classes of nonholonomic configurations if generic off-diagonal cosmological solutions are constructed for effective Einstein equations of type (33)–(37).

There remain many open questions on how to provide viable explanations for the recent observational data from Planck and BICEP. In this work, we have shown that attractor configurations can be constructed in TMTs with effective gravitational and matter field equations. Such solutions provide a new background for investigating cosmological theories with anisotropies, inhomogeneities, dark energy and dark matter physics.

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