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A perturbative approach to neutron stars in $f(T, \mathcal{T})$-gravity

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Abstract We derive a Tolman–Oppenheimer–Volkoff equation in neutron star systems within the modified $f(T, \mathcal{T})$-gravity class of models using a perturbative approach. In our approach $f(T, \mathcal{T})$-gravity is considered to be a static spherically symmetric space-time. In this instance the metric is built from a more fundamental vierbein which can be used to relate inertial and global coordinates. A linear function $f = T(r) + \mathcal{T}(r) + \chi h(T, \mathcal{T}) + \mathcal{O}(\chi^2)$ is taken as the Lagrangian density for the gravitational action. Finally we impose the polytropic equation of state of neutron star upon the derived equations in order to derive the mass profile and mass–central density relations of the neutron star in $f(T, \mathcal{T})$-gravity.

1 Introduction

Recently it has been shown that the universe is accelerating in its expansion [1,2]. The concept of the cosmological constant together with the inclusion of dark matter yield the ΛCDM model which explains a whole host of phenomena within the universe [3–5]. We may also explain this acceleration by instead modifying the gravitational theory itself with alternative theories of gravity an example of which is $f(R)$-gravity [6–9].

Our focus of this paper is on one alternative theory of gravity called $f(T)$-gravity, which makes use of a "teleparallel" equivalent of GR (TEGR) [10] approach, in which instead of the torsion-less Levi-Civita connection, the Weitzenböck connection is used, with the dynamical objects being four linearly independent vierbeins [11,12]. The Weitzenböck connection is curvature-free and describes the torsion of a manifold.

The differences between $f(T)$ class of gravity and other gravity forms such as $f(R)$ and TEGR is in the choice of

the function $f(T)$ which is taken [8]. Comparing $f(T)$-gravity with $f(R)$-gravity it is noted that $f(T)$-gravity cannot be reformulated as a teleparallel action plus a scalar field through the conformal transformation due to the appearance of additional scalar-torsion coupling terms [13,14]. The obvious difference is that $f(T)$-gravity has a class of equations which is easier to work with because the field equations are of second order rather than fourth order like in $f(R)$-gravity class scenarios [8]. In $f(T)$-gravity more degrees of freedom are obtained, which thus corresponds to one massive vector field [15,16].

We make use of a pure tetrad [17], which means that the torsion tensor is formed by a multiple of the tetrad and its first derivative only. Under the assumption of invariance under general coordinate transformations, global Lorentz transformations, and the parity operation we construct the Lagrangian density from this torsion tensor [9,10,12,17]. Also the Lagrangian density is of second order in the torsion tensor [10,12]. Thus $f(T)$-gravity generalises the above TEGR formalism, making the gravitational Lagrangian a function of T [8–10].

Our goal for this paper is to derive a working model for the TOV equations within a new modification of $f(T)$ class gravity, called $f(T, \mathcal{T})$-gravity in a perturbative manner. We make use of a perturbative approach due to the fact that a non-physical assumption had to be taken whilst deriving the TOV equations in an analytical manner.

$f(T, \mathcal{T})$-gravity couples the gravitational sector and the standard matter one [8]. Instead of having the Ricci scalar coupled with the trace of the energy-momentum tensor $\mathcal{T}$ as is done in $f(R, \mathcal{T})$-gravity, $f(T, \mathcal{T})$-gravity couples the torsion scalar T with the trace of the matter energy-momentum tensor $\mathcal{T}$ [8,9,18]. Recently a modification to this theory has been propose, that of allowing for a general functional dependence on the energy-momentum trace scalar, $\mathcal{T}^{\mu}{}_{\mu} = \mathcal{T}$ [8,9].

Our interest is in studying the behaviour of spherically symmetric compact objects in this theory. We propose the use

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of a linear function, namely $f(T, \mathcal{T}) = \alpha T(r) + \beta \mathcal{T}(r) + \varphi$, where α and β are arbitrary constants which may be varied to align our star's behaviour with current observations. φ is then considered to be the cosmological constant. We consider the linear modification since it is the natural first functional form to consider, and the right place to start to understand how the trace of the stress-energy tensor might effect $f(T, \mathcal{T})$ gravity. In particular, our focus is on neutron stars in $f(T, \mathcal{T})$ gravity.

Besides the possibility of the existence of these exotic stars, this is also a good place to study the behaviour of modified theories of gravity in terms of constraints. Moreover, this also opens the door to considerations of stiff matter in early phase transitions [19].

The plan of this paper is as follows: in Sect. 2 we go over the mathematical tools and give an overview of $f(T, \mathcal{T})$-gravity. In Sect. 3 we discuss the rotated tetrad taken and discuss how the equations of motion in $f(T, \mathcal{T})$-gravity are derived perturbatively. In Sect. 4 the two TOV equations are derived and discussed along with the schwarzschild solution, while the results are then used in Sect. 5 where we output the numerical results given by the yielded TOV equations. Finally we discuss the results in Sect. 6.

2 $f(T, \mathcal{T})$-gravity overview

$f(T, \mathcal{T})$-gravity generalises $f(T)$-gravity and thus is based on the Weitzenbock's geometry. We will follow a similar notation style as that given in Refs. [7,8,10,20–22]. Using: Greek indices $\mu, \nu, \ldots$ and capital Latin indices $i, \kappa, \ldots$ over all general coordinate and inertial coordinate labels, respectively [7,8,20,21].

Torsion tensor [20,21,23] is given by

$$T^\lambda_{\ \mu\nu}(e^\lambda_{\ \mu}, \omega^\lambda_{\ i\mu}) = \partial_\mu e^\lambda_{\ \nu} - \partial_\nu e^\lambda_{\ \mu} + \omega^\lambda_{\ i\mu} e^i_{\ \nu} - \omega^i_{\ \lambda\nu} e^i_\mu, \qquad (1)$$

where $\omega^\lambda_{\ i\mu}$ is the spin connection [23]. The torsion tensor has vanishing curvature. Therefore by doing so all the information of the gravitational field is embedded in the torsion tensor [22], while the gravitational Lagrangian is the torsion scalar [23]. The contorsion tensor is then defined as

$$K^{\mu\nu}_{\ \ \rho} = -\frac{1}{2}(T^{\mu\nu}_{\ \ \rho} - T^{\nu\mu}_{\ \ \rho} - T_\rho^{\ \mu\nu}), \qquad (2)$$

while the superpotential of teleparallel gravity is defined by [20,21]

$$S_\rho^{\ \mu\nu} = \frac{1}{2}(K^{\mu\nu}_{\ \ \rho} + \delta^\mu_\rho T^{\alpha\nu}_{\ \ \alpha} - \delta^\nu_\rho T^{\alpha\mu}_{\ \ \alpha}). \qquad (3)$$

Unlike the contorsion tensor, the superpotential tensor does not have any apparent physical meaning, instead is it purely introduced to reduce the size of the Lagrangian.

The torsion scalar [20–22] is then given as

$$T = S_\rho^{\ \mu\nu} T^\rho_{\ \mu\nu}. \qquad (4)$$

As in the analogous $f(R, T)$ theories [24], we further generalised the gravitational Lagrangian by taking an arbitrary function f and thus giving [25,26]

$$S = -\frac{1}{16\pi G} \int \mathrm{d}^4 x e \left[f(T, \mathcal{T}) + \mathcal{L}_\mathrm{m} \right]. \qquad (5)$$

The function $f(T, \mathcal{T})$ is taken to be equal to $T(r) + \mathcal{T}(r) + \chi h(T, \mathcal{T}) + \mathcal{O}(\chi^2)$ where χ is a very small parameter which will aid in differentiating between the zeroth and first order terms [27], and $h(T, \mathcal{T})$ is an arbitrary function of the torsion scalar T and the trace $\mathcal{T}$ of the energy-momentum tensor $\overset{e-m}{\mathcal{T}}$ given by $\mathcal{T} = \delta^\nu_\mu \mathcal{T}_\nu^{\ \mu}$. $\mathcal{L}_\mathrm{m}$ is the matter Lagrangian density [22,25]. In this instance f is an arbitrary function of the torsion scalar T and the trace of the energy-momentum tensor $\mathcal{T}$ [25]. The variation of the action defined in Eq. (5) with respect to the tetrad leads to the field equations [27]

$$\begin{aligned}
&e^\rho_i S_\rho^{\ \mu\nu} \partial_\mu T \chi h_{TT} \\
&\quad + e^\rho_i S_\rho^{\ \mu\nu} \chi h_{T\mathcal{T}} \mathcal{T} + e^{-1} \partial_\mu \left(e e^\rho_i S_\rho^{\ \mu\nu} \right)(1 + \chi h_T) \\
&\quad + e^\mu_i T^\lambda_{\ \mu\kappa} S_\lambda^{\ \nu\kappa}(1 + \chi h_T) \\
&\quad - \frac{e^\nu_i T(r) + \mathcal{T}(r) + \chi h(T, \mathcal{T})}{4} + (1 + \chi h_T) \omega^i_{\ \lambda\nu} S_i^{\ \nu\mu} \\
&\quad - \frac{(1 + \chi h_\mathcal{T})}{2} \left(e^\lambda_i \mathcal{T}_\lambda^{\ \nu} + p(r) e^\nu_i \right) = -4\pi e^\lambda_i \overset{e-m}{\mathcal{T}}{}^{\ \nu}_\lambda, \qquad (6)
\end{aligned}$$

where $h_T = \dfrac{\partial h}{\partial T}, h_\mathcal{T} = \dfrac{\partial h}{\partial \mathcal{T}}$, and $h_{T\mathcal{T}} = \dfrac{\partial^2 h}{\partial T \partial \mathcal{T}}$.

3 Perturbative equations of motion in $f(T, \mathcal{T})$-gravity

In perturbative theory the field equations may be expanded perturbatively in χ [27] and therefore the metric components take on the form of the expansions $A(r)_\chi = A(r) + \chi A(r)_1 + \cdots$ and $B(r)_\chi = B(r) + \chi B(r)_1 + \cdots$ [27]. The energy-momentum tensor in the field equations is still the energy-momentum tensor of the perfect fluid. The hydrodynamic quantities are also defined perturbatively by $\rho(r)_\chi = \rho(r) + \chi \rho(r)_1 + \cdots$ and $p(r)_\chi = p(r) + \chi p(r)_1 + \cdots$ [27].

A spherically symmetric metric, which has a diagonal structure, is considered for our system [28],

$$\mathrm{d}s^2 = -e^{A(r)_\chi} \mathrm{d}t^2 + e^{B(r)_\chi} \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2 \mathrm{d}\phi^2, \qquad (7)$$

and we consider the fluid inside the star to be that of a perfect fluid which yields a diagonal energy-momentum tensor

$$\overset{e-m}{\mathcal{T}}{}^{\ \nu}_\lambda = \mathrm{diag}(-\rho(r)_\chi, p(r)_\chi, p(r)_\chi, p(r)_\chi), \qquad (8)$$

where $\rho(r)_\chi$ and $p(r)_\chi$ are the energy density and pressure of the fluid, respectively [28]. These also make up the matter functions which, along with the metric functions, $A(r)$ and $B(r)$, are also taken to be independent of time [22]. Thus the system is taken to be in equilibrium [5,28].

The equation of conservation of energy is given by

$$\frac{\mathrm{d}p(r)}{\mathrm{d}r} = -(\rho(r) + p(r))\frac{\mathrm{d}A(r)}{\mathrm{d}r}. \tag{9}$$

Following Ref. [17] the rotated tetrad

$$e^a_\mu = \begin{pmatrix} e^{\frac{A(r)_\chi}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{B(r)_\chi}{2}}\sin\theta\cos\phi & e^{\frac{B(r)_\chi}{2}}\sin\theta\sin\phi & e^{\frac{B(r)_\chi}{2}}\cos\theta \\ 0 & -r\cos\theta\cos\phi & -r\cos\theta\sin\phi & r\sin\theta \\ 0 & r\sin\theta\sin\phi & -r\sin\theta\cos\phi & 0 \end{pmatrix}$$

is used.

This form of vierbein is considered because it allows us more degrees of freedom [29] and it allows us to acquire a static and spherically symmetric wormhole solution in our standard formulation of $f(T, \mathcal{T})$-gravity [29,30].

Also, because this is a pure form of tetrad [23], the spin connection elements of the tetrad vanish and thus ensure that the spin connection terms need not be included [23].

Inserting this vierbein into the field equations, from Eq. (4) we get the resulting torsion scalar,

$$T(r) = \frac{2e^{-B(r)}}{r^2}\left(1 - e^{\frac{B(r)}{2}}\right)\left(1 - e^{\frac{B(r)}{2}} + rA'(r)\right), \tag{10}$$

where the prime denotes the derivative with respect to r. The resulting field equation components turn out to be as follows. The tt component, given by $i = \nu = 0$, results in

$$4\pi\rho(r)_\chi = \frac{T(r)_\chi + \mathcal{T}(r)_\chi + \chi h}{4} + \frac{e^{-B(r)}}{2r^2}(1 + \chi h_\mathcal{T})$$
$$\times\left[-2 + 2e^{\frac{B(r)}{2}} + rA'(r)\left(e^{\frac{B(r)}{2}} - 1\right) + rB'(r)\right]$$
$$+ \frac{(1 + \chi h_\mathcal{T})}{2}(\rho(r) - p(r))$$
$$+ \frac{e^{-B(r)}\chi}{r}\left(e^{\frac{B(r)}{2}} - 1\right)(h_{TT}T'(r) + h_{T\mathcal{T}}\mathcal{T}'(r)). \tag{11}$$

The r-r component, given by $i = \nu = 1$ results in

$$4\pi p(r)_\chi = \frac{T(r)_\chi + \mathcal{T}(r)_\chi + \chi h}{4} + \frac{e^{-B(r)}}{2r^2}(1 + \chi h_\mathcal{T})$$
$$\times\left[2\left(e^{\frac{B(r)}{2}} - 1\right) + rA'(r)\left(e^{\frac{B(r)}{2}} - 2\right)\right]$$
$$- p(r)(1 + \chi h_\mathcal{T}). \tag{12}$$

Note that the zeroth order quantities are given without a subscript.

4 Perturbative derivation in $f(T, \mathcal{T})$-gravity

We will now make use of the equations of motion given by Eqs. (11) and (12) by first considering a solution for $\rho(r)_\chi$ and $p(r)_\chi$ up to order χ. The zeroth order quantities are considered from these two equations and they are given by

$$4\pi\rho(r) = \frac{T(r) + \mathcal{T}(r)}{4} + \frac{e^{-B(r)}}{2r^2}\left[-2 + 2e^{\frac{B(r)}{2}} + rA'(r)\right.$$
$$\left.\times\left(e^{\frac{B(r)}{2}} - 1\right) + rB'(r)\right] + \frac{(\rho(r) - p(r))}{2} \tag{13}$$

and

$$4\pi p(r) = \frac{T(r) + \mathcal{T}(r)}{4} + \frac{e^{-B(r)}}{2r^2}\left[2\left(e^{\frac{B(r)}{2}} - 1\right)\right.$$
$$\left. + rA'(r)\left(e^{\frac{B(r)}{2}} - 2\right)\right] - p(r). \tag{14}$$

At this point the torsion scalar given by Eq. (10) and $\mathcal{T}(r) = \rho(r) - 3p(r)$ are inserted into the two equations which after manipulation results in

$$4\pi\rho(r) = \frac{e^{-B(r)}}{2r^2}(-1 + e^{B(r)} + rB'(r))$$
$$+ \frac{1}{4}(3\rho(r) - 5p(r)) \tag{15}$$

and

$$4\pi p(r) = \frac{e^{-B(r)}}{2r^2}(-1 + e^{B(r)} - rA'(r))$$
$$+ \frac{1}{4}(\rho(r) - 7p(r)), \tag{16}$$

respectively.

At this point it is convenient to take a mass parameter ansatz. The solution is assumed to have the same form of the exterior solution for the metric function B_χ. In order to render a metric ansatz in line with the Schwarzschild metric we take the following [27]:

$$e^{-B(r)_\chi} = 1 - \frac{\Omega M(r)_\chi}{r} + \epsilon(r)_\chi, \qquad (17)$$

where Ω is an arbitrary constant and $\epsilon(r)$ is taken to be a function of r. Similar to ρ_χ, $M(r)_\chi$ is expanded in χ as $M_\chi = M + \chi M_1 + \cdots$, [27] where M is the zeroth order solution.

Taking a derivative of M_χ with respect to r the following is obtained:

$$\frac{dM_\chi}{dr} = \frac{1}{\Omega}(1 - e^{-B(r)_\chi} + \epsilon(r)_\chi + e^{-B(r)_\chi} r B'(r)_\chi + r\epsilon'(r)_\chi). \qquad (18)$$

Now we focus on the first equation of motion given by Eq. (11) where we insert the torsion scalar equation and the energy-momentum components and thus obtain the following equation:

$$4\pi\rho(r)_\chi = \frac{\Omega}{2r^2}\frac{dM_\chi}{dr} - \frac{(\epsilon_\chi(r))'}{2r^2} + \frac{1}{4}(3\rho(r)_\chi - 5p(r)_\chi)$$
$$+ \chi\left\{\frac{h}{4} + \frac{e^{-B(r)}h_T}{2r^2}\right.$$
$$\times\left[-2 + 2e^{\frac{B(r)}{2}} + rA'(r)\left(e^{\frac{B(r)}{2}} - 1\right) + rB'(r)\right]$$
$$+ \frac{h_\mathcal{T}}{2}(\rho(r) - p(r)) + \frac{e^{-B(r)}}{r}\left(e^{\frac{B(r)}{2}} - 1\right)$$
$$\times (h_{TT}T'(r) + h_{T\mathcal{T}}\mathcal{T}'(r))\left.\right\}. \qquad (19)$$

Here we invoke a linear parameter for h, given by $\alpha T(r) + \beta\mathcal{T}(r) + \varphi$, which after being inserted into this equation and after further reduction yields

$$\frac{dM_\chi}{dr} = \frac{8\pi r^2\rho(r)_\chi}{\Omega} + \frac{(\epsilon_\chi(r))'}{2r^2} - \frac{r^2}{2\Omega}(3\rho(r)_\chi - 5p(r)_\chi)$$
$$- \frac{r^2\chi}{2\Omega}\left\{\frac{\alpha e^{-B(r)}}{r^2}\right.$$
$$\times\left[2e^{\frac{B(r)}{2}} - 3 + e^{B(r)} + rA'(r)\left(e^{\frac{B(r)}{2}} - 1\right) + 2rB'(r)\right]$$
$$+ \varphi + \beta(3\rho(r) - 5p(r))\left.\right\}. \qquad (20)$$

The main task at this point is to reduce the values of $A'(r)$ and $B'(r)$ where the definitions given by Eqs. (15), (16), and (17) will be substituted, thus resulting in our first TOV equation,

$$\frac{dM_\chi}{dr} = \frac{8\pi r^2\rho(r)_\chi}{\Omega} + \frac{(\epsilon_\chi(r))'}{2r^2} - \frac{r^2}{2\Omega}(3\rho(r)_\chi - 5p(r)_\chi)$$
$$- \frac{r^2\chi\varphi}{2\Omega} + \frac{\chi\alpha}{4\Omega}\left(1 - \frac{\Omega M(r)}{r} + \epsilon(r)\right)^{\frac{1}{2}}$$

$$\times\left\{-2 + r[r + r\epsilon(r) - M(r)\Omega]^{-1}\right.$$
$$\times [r^2(p(r)(7 + 16\pi) - \rho(r)) - 2]$$
$$+ \left(1 + \epsilon(r) - \frac{\Omega M(r)}{r}\right)^{-\frac{1}{2}}$$
$$[4 - r^2(p(r)(17 + 16\pi) + \rho(r)(32\pi - 7))]\left.\right\}. \qquad (21)$$

Now we shift our focus into deriving the pressure–radius relation of the TOV equations. For this purpose, Eq. (12) is considered where a similar treatment will be given i.e. we substitute the torsion scalar equation and the energy-momentum definition to give

$$A'(r)_\chi = e^{B(r)_\chi}\left\{\frac{1}{r}(1 - e^{-B(r)_\chi})\right.$$
$$+ \frac{r}{2}(\rho(r)_\chi - 7p(r)_\chi) - 8\pi p(r)_\chi r$$
$$+ 2r\chi\left\{\frac{h}{4} + \frac{h_T e^{-B(r)}}{2r^2}\left[2\left(e^{\frac{B(r)}{2}} - 1\right)\right.\right.$$
$$+ rA'(r)\left(e^{\frac{B(r)}{2}} - 2\right) - h_\mathcal{T}p(r)\left.\right\}\left.\right\}. \qquad (22)$$

Inserting the definition of h and Eq. (17) and reducing further yields

$$A'(r)_\chi = \left\{8p(r)\pi r(\alpha\chi - 1) + \frac{\Omega M(r)}{r^2} - \frac{\epsilon(r)}{r} + \frac{r\chi\varphi}{2}\right.$$
$$- \frac{r}{2}(7p(r) - \rho(r))(1 + \beta - \alpha\chi)\left.\right\}$$
$$\times\left[1 - \frac{\Omega M(r)_\chi}{r} + \epsilon(r)_\chi\right]^{-1}. \qquad (23)$$

This result is then inserted into the continuity equation given by Eq. (9) and thus results in the second TOV equation required,

$$\frac{p(r)_\chi}{pr} = -(\rho(r)_\chi + p(r)_\chi)$$
$$\times\left\{8p(r)\pi r(\alpha\chi - 1) + \frac{\Omega M(r)}{r^2} - \frac{\epsilon(r)}{r} + \frac{r\chi\varphi}{2}\right.$$
$$- \frac{r}{2}(7p(r) - \rho(r))(1 + \beta - \alpha\chi)\left.\right\}$$
$$\times\left[1 - \frac{\Omega M(r)_\chi}{r} + \epsilon(r)_\chi\right]^{-1}. \qquad (24)$$

5 Numerical modelling of neutron stars

Through Eqs. (21) and (24) any spherically symmetric mass in $f(T, \mathcal{T})$-gravity can be investigated in terms of its physical properties. In order to obtain a mass profile relation for the TOV equations, we numerically integrate our TOV equations of stellar structure to build models of neutron stars in $f(T, \mathcal{T})$-gravity. Here we take the relativistic energy density $\rho(r)$ to be equal to

$$\rho(r) = \rho_0(r) + \frac{p(r)}{\Gamma - 1}, \tag{25}$$

where $\rho_0(r)$ is the rest matter density [31]. We take the initial conditions equal to $m(0) = 0$ and $p(0) = K\rho_{0,c}^{\Gamma}$ where $\rho_{0,c}^{\Gamma}(r) = 10^{15} \text{gm/cm}^3$ is the central density [31]. We take $\Gamma = 4/3$ and $G = 1$ and $c = 1$ [31].

The value of Ω is taken to be 2, as one in Ref. [27] suggests, and the value of φ is taken to be the cosmological constant, 2.036×10^{-35} [32]. Here we also consider our value of χ as being a very small but non-zero value $\sim 10^{-12}\,\text{cm}^{-2}$ [27]. The value of α in Eqs. (21) and (24) is taken to be -1 and the value of β is varied. We vary the value of β so as to manipulate the dominance of the function $\mathcal{T}(r)$.

5.1 Mass profile curve

In Fig. 1 we show the mass profile curve of a neutron star. We take $\beta = -1$ to include the GR case at this order of the perturbation and contrast with decreasing values of β.

As the function $\mathcal{T}(r) = \rho(r) - 3p(r)$ [26] is included i.e. when $\beta = -5$ is taken, a similar mass profile is generated from the TOV equations. Figure 1 shows that the neutron star at first appears to be smaller in nature, however, around the 17 km mark it surpasses the neutron star generated by the GR case to yield a larger stellar structure.

Current observations shown in the literature by [33] state that such massive neutron stars may exist. In fact the figures show that there is $\approx 4.67\%$ increase in the maximum mass value of the neutron star gained.

We further magnify this effect by taking $\beta = -10$ and as may be seen from Fig. 1, the neutron star at first behaves exactly like the previous case and again surpasses the other cases at the 17 km mark to yield a more massive neutron star. In fact by considering $\beta = -10$ the allowable maximum mass of the neutron star is increased by 5.43% over the $\beta = -5$ case and 10.35% over the GR case.

In order to better understand the physical behaviour of neutron stars we investigate how mass varies over radius for different settings of β in Fig. 1. Whichever values of β are taken, the graphical output is similar in structure to that of GR. Another aspect to note is that the general behaviour of the stellar system remains the same, however, a new degree
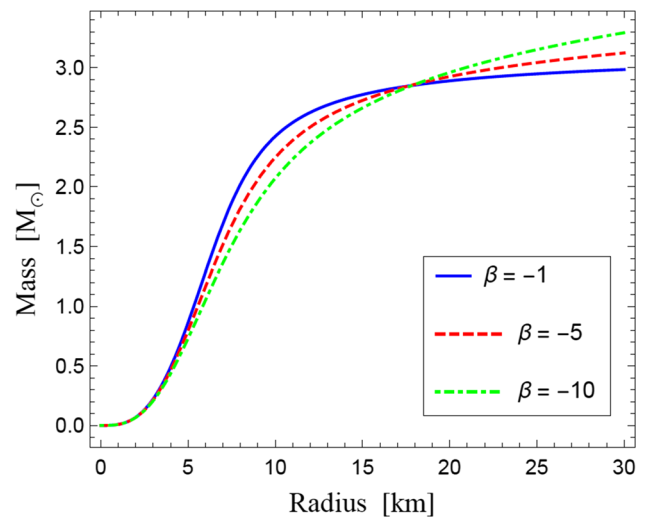

Fig. 1 Mass profile graph of a neutron star where $\alpha = -1$ and three different variations of β. The value of $\Gamma = 4/3$, $\Omega = 2$ [27] and $\varphi = 2.036 \times 10^{-35}$ [32]

of freedom is allowed depending on the maximum mass and radius of these stars.

In contrast the results gained in this paper are similar to those gained by various other authors such as found in Refs. [34–37]. The resulting mass profile curves behave in a similar manner where mass steadily increases with radius to plateau at an instance.

5.2 Radius–central density curve

Figure 1 is heavily dependent on the central density, not in behaviour but in terms of the particular values being produced. To contrast this we plot the radius–central density curve which was generated from the TOV equations. This plot is given by Fig. 2.

Again we contrast with the GR case when taking $\beta = -1$. When we decrease the value of β by taking $\beta = -5$ we may note that the central density figure is significantly lower than that of the GR case. It is noted that in both cases the curves decrease at the same rate with radius and curve at the same instance to intersect at a point close to 1 km.

To magnify this we again take a lower value of β by taking $\beta = -10$, this shows that the central density declines somewhat rapidly at first initially with radius and then slowly declines steadily similar to the previous cases, however, it reaches the lowest value the fastest out of the three cases considered. Thus this shows that a slightly larger neutron star is allowed in such a gravitational framework.

Much like the mass profile curves the results gained from Refs. [34–37] exhibit similar results as gained in this study. The central density value increases significantly with radius at first to gain a maximum value as we get closer to the centre of the star.
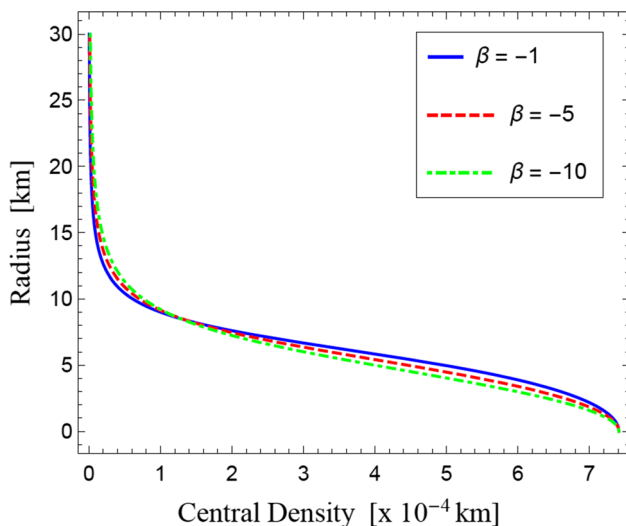
Fig. 2 Radius–central density graph of a neutron star where $\alpha = -1$ and three different variations of β. The value of $\Gamma = 4/3$, $\Omega = 2$ [27] and $\varphi = 2.036 \times 10^{-35}$ [32]

6 Conclusion

In this study the TOV equations are derived in a perturbative way for $f(T, \mathcal{T})$-gravity. Later the two equations are applied to a polytropic equation of state which yielded the characteristics of the neutron star in such a gravitational framework.

Our main goal throughout this research is to derive a working model which involved little to no assumptions in the derivation. We also wanted to retain and include as many general terms as possible. We did this also because we would like to further fine tune our results to current observations.

A reasonable boundary condition was taken in order to solve the TOV equations by numerical techniques. We apply the polytropic equation of state in order to reduce our TOV equations from a four variable equation to a three variable equation by making one of the variables dependent on the others.

Our approach considered a value of χ which is non-zero but, however, very small, $\sim 10^{-12}$ cm^{-2}. The literature shown in Ref. [27] shows that the typical value of the Ricci curvature is calculated to be roughly of a similar order. Thus assuming our value of χ to be so small is reasonable.

Our graphical representations are inspired by the work carried out by the authors of Refs. [8,38]. The graphs show that a larger neutron star is allowed in such a gravitational framework. We vary the values of β accordingly to output the variations occurring when we include the $\mathcal{T}(r)$ term. By taking a lower value of this term we note that it allows for a larger neutron star. This value will require future fine tuning in order to align with current observations.

More values of β were considered in testing. The results showed that when positive values of β were considered no

tangible neutron star would be yielded in such a gravitational framework. When lower values of β were considered the yielded stellar structures did not behave in accordance with the theory as explained in Ref. [39]. Thus the range of values for β considered to yield a tangible and proper neutron star would be $-10 \ll \beta \ll -1$.

There have not yet been many extensive studies conducted where the linear Lagrangian approach is considered as is done in this manuscript. However, there have been cosmological studies that reconstruct the Lagrangian for various state parameter conditions [26].

For future work we also hope to derive the TOV equations using a non-linear Lagrangian, however, till now we have not been able to yield working TOV equations. We also hope to derive the TOV equations in $f(T, \mathcal{T})$-gravity in an analytical manner.

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