

Parameterized post-Newtonian approximation in a teleparallel model of dark energy with a boundary term

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Abstract We study the parameterized post-Newtonian approximation in teleparallel model of gravity with a scalar field. The scalar field is non-minimally coupled to the scalar torsion as well as to the boundary term introduced in Bahamonde and Wright (Phys Rev D 92:084034 [arXiv:1508.06580v4](https://arxiv.org/abs/1508.06580v4) [gr-qc], 2015). We show that, in contrast to the case where the scalar field is only coupled to the scalar torsion, the presence of the new coupling affects the parameterized post-Newtonian parameters. These parameters for different situations are obtained and discussed.

1 Introduction

In a teleparallel model of gravity, instead of the torsionless Levi-Civita connections, curvatureless Weitzenböck connections are used [2–4]. A teleparallel equivalent of general relativity was first introduced in [5] as an attempt for unification of electromagnetism and gravity. This theory is considered as an alternative theory of usual general relativity and has been recently employed to study the late time acceleration of the Universe [6–8]. This can be accomplished by considering modified $f(T)$ models [9–24], where T is the torsion scalar, or by introducing exotic field such as quintessence. Assuming a non-minimal coupling between the scalar field and the torsion opens new windows in studying the cosmological evolution [25–31], and can be viewed as a promising scenario for late time acceleration and super-acceleration [32].

A non-minimally coupled scalar field, like the scalar-tensor model, may alter the Newtonian potential. So it is necessary to check if the model can pass local gravitational tests such as solar system observations. This can be done in the context of the parameterized post-Newtonian formalism [35–40]. In [33, 34] it was shown that when the scalar field is only coupled to the scalar torsion, there is no deviation from general relativity in the parameterized post-Newtonian

(PPN) parameters and the theory is consistent with gravitational tests and solar system observations.

Recently a new coupling between the scalar field and a boundary term \mathcal{B} , corresponding to the torsion divergence $\mathcal{B} \propto \nabla_\mu T^\mu$, was introduced in [1], where the cosmological consequences of such a coupling for some simple power law scalar field potential and the stability of the model were discussed. There it was found that the system evolves to an attractor solution, corresponding to late time acceleration, without any fine tuning of the parameters. In this framework, the phantom divide line crossing is also possible. Thermodynamics aspects of this model were studied in [41]. This model includes two important subclasses, i.e. quintessence non-minimally coupled to the Ricci scalar and quintessence non-minimally coupled to the scalar torsion. Another important feature of this model is its ability to describe the present cosmic acceleration in the framework of Z_2 symmetry breaking by alleviating the coincidence problem [42].

In this paper, we aim to investigate whether this new boundary coupling may affect the Newtonian potential and PPN parameters: $\gamma(r)$ and $\beta(r)$.

The scheme of the paper is as follows: In the second section we introduce the model and obtain the equations of motion. In the third section, we obtain the weak field expansion of the equations in the PPN formalism and obtain and discuss their solutions for spherically symmetric metric. We show that the PPN parameters may show deviation from general relativity. We consider different special cases and derive explicit solutions for the PPN parameters in terms of the model parameters and confront them with observational data.

We use units with $\hbar = c = 1$ and choose the signature $(-, +, +, +)$ for the metric.

2 The model and the field equations

In our study we use vierbeins $e_a = e_a^\mu \partial_\mu$, whose duals, $e^a{}_\mu$, are defined through $e^a{}_\mu e_a{}^\nu = \delta_\mu^\nu$. The metric ten-

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isor is given by $g^{\mu\nu} = \eta_{ab}e_a^\mu e_b^\nu$, $\eta = \text{diag}(-1, 1, 1, 1)$. $e = \det(e^a_\mu) = \det\sqrt{-g}$. Greek indices (indicating coordinate bases) like the first Latin indices (indicating orthonormal bases) a, b, c, \dots belong to $\{0, 1, 2, 3\}$, while $i, j, k, \dots \in \{1, 2, 3\}$.

Our model is specified by the action [1]

$$S = \int \left(\frac{T}{2k^2} + \frac{1}{2}(-\partial_\mu\phi\partial^\mu\phi + \epsilon T\phi^2 + \chi\mathcal{B}\phi^2) - V(\phi) + \mathcal{L}_m \right) ed^4x, \tag{1}$$

where $k^2 = 8\pi G_N$, and G_N is the Newtonian gravitational constant. The torsion scalar is defined by

$$T = S^\rho_{\mu\nu}T_\rho{}^{\mu\nu} = \frac{1}{4}T^\rho{}_{\mu\nu}T_\rho{}^{\mu\nu} + \frac{1}{2}T^\rho{}_{\mu\nu}T^{\nu\mu}{}_\rho - T^\rho{}_{\mu\rho}T^{\nu\mu}{}_\nu, \tag{2}$$

and the boundary term is [43,44]

$$\mathcal{B} = \frac{2}{e}\partial_\mu(eT^\mu), \tag{3}$$

where $T^\mu = T^\lambda{}_\lambda{}^\mu$. The Weitzenböck torsion and connection are given by

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} = e_a{}^\lambda T^a{}_{\mu\nu} \tag{4}$$

and

$$\Gamma^\lambda{}_{\mu\nu} = e_a{}^\lambda\partial_\mu e^a{}_\nu, \tag{5}$$

respectively. $S^\rho{}_{\mu\nu}$ is defined according to

$$S^\rho{}_{\mu\nu} = \frac{1}{4}(T^\rho{}_{\mu\nu} - T_{\mu\nu}{}^\rho + T_{\nu\mu}{}^\rho) + \frac{1}{2}\delta_\mu^\rho T^\sigma{}_{\nu\sigma} - \frac{1}{2}\delta_\nu^\rho T^\sigma{}_{\mu\sigma}. \tag{6}$$

Note that $R = -T + \mathcal{B}$, where R is the Ricci scalar curvature. Hence for $\chi = -\epsilon$ the model reduces to a quintessence model coupled non-minimally to the scalar curvature, while for $\chi = 0$, we recover the quintessence model coupled non-minimally to the scalar torsion.

By variation of the action (1) with respect to the vierbeins we obtain

$$\begin{aligned} &\left(\frac{2}{k^2} + 2\epsilon\phi^2\right)\left(e^{-1}e^a{}_\mu\partial_\lambda(eS_a{}^{\lambda\nu}) - T^\rho{}_{\beta\mu}S_\rho{}^{\nu\beta} - \frac{1}{4}\delta_\mu^\nu T\right) \\ &- \delta_\mu^\nu\left(-\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - V(\phi)\right) \\ &- \partial^\nu\phi\partial_\mu\phi + 4(\chi + \epsilon)\phi S_\mu{}^{\beta\nu}\partial_\beta\phi \\ &+ \chi(\delta_\mu^\nu\Box\phi^2 - \nabla^\nu\nabla_\mu\phi^2) = -\tau_\mu^\nu. \end{aligned} \tag{7}$$

τ_μ^ν is the energy-momentum tensor of matter.

The trace of (7), multiplied by $-\delta_\mu^\nu/2$, is

$$\begin{aligned} &-\delta_\mu^\nu\left(\frac{1}{k^2} + \epsilon\phi^2\right) \\ &(e^{-1}e^a{}_\alpha\partial_\lambda(eS_a{}^{\lambda\alpha})) - \frac{1}{2}\delta_\mu^\nu\partial_\alpha\phi\partial^\alpha\phi - 2\delta_\mu^\nu V(\phi) \\ &- 2\delta_\mu^\nu(\chi + \epsilon)\phi S_\alpha{}^{\beta\alpha}\partial_\beta\phi - \frac{3}{2}\chi\delta_\mu^\nu\Box\phi^2 = \frac{1}{2}\delta_\mu^\nu\tau. \end{aligned} \tag{8}$$

By combining (8) and (7) we obtain

$$\begin{aligned} &\left(\frac{2}{k^2} + 2\epsilon\phi^2\right)\left(e^{-1}e^a{}_\mu\partial_\lambda(eS_a{}^{\lambda\nu}) - T^\rho{}_{\beta\mu}S_\rho{}^{\nu\beta} - \frac{1}{4}\delta_\mu^\nu T\right) \\ &- \delta_\mu^\nu V(\phi) - \partial^\nu\phi\partial_\mu\phi + 4(\chi + \epsilon)\phi S_\mu{}^{\beta\nu}\partial_\beta\phi \\ &- \chi\nabla^\nu\nabla_\mu\phi^2 - \frac{1}{2}\chi\delta_\mu^\nu\Box\phi^2 \\ &- \delta_\mu^\nu\left(\frac{1}{k^2} + \epsilon\phi^2\right)(e^{-1}e^a{}_\alpha\partial_\lambda eS_a{}^{\lambda\alpha}) \\ &- 2\delta_\mu^\nu(\chi + \epsilon)\phi S_\alpha{}^{\beta\alpha}\partial_\beta\phi \\ &= -\tau_\mu^\nu + \frac{1}{2}\delta_\mu^\nu\tau. \end{aligned} \tag{9}$$

Note that the trace of the energy-momentum tensor is $\tau = g^{\mu\nu}\tau_{\mu\nu}$.

In the same way, variation of the action with respect the scalar field gives

$$-\frac{1}{e}\partial_\mu e g^{\mu\nu}\partial_\nu\phi - \chi\mathcal{B}\phi - \epsilon T\phi + V'(\phi) = 0. \tag{10}$$

Equations (9) and (10) are the main equations that we will work with in the following.

3 Post-Newtonian formalism

To investigate the post-Newtonian approximation [35–40] of the model, the perturbation is specified by the velocity of the source matter $|\vec{v}|$ such that e.g. $\mathcal{O}(n) \sim |\vec{v}|^n$. The matter source is assumed to be a perfect fluid obeying the post-Newtonian hydrodynamics:

$$\tau_{\mu\nu} = (\rho + \rho\Pi + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{11}$$

where ρ is energy density, p is the pressure and Π is the specific internal energy. u^μ is the four-vector velocity of the fluid. The velocity of the source matter is $v^i = \frac{u^i}{u^0}$. The orders of smallness of the energy-momentum tensor ingredients are [35–40]

$$\rho \sim \Pi \sim \frac{p}{\rho} \sim U \sim \mathcal{O}(2) \tag{12}$$

where U is the Newtonian gravitational potential. The components of the energy-momentum tensor are given by

$$\tau_0^0 = -\rho - \rho v^2 - \rho\Pi + \mathcal{O}(6)$$

$$\begin{aligned} \tau_0^i &= -\rho v^i + \mathcal{O}(5) \\ \tau_i^j &= \rho v^j v_i + p \delta_i^j + \mathcal{O}(6). \end{aligned} \tag{13}$$

We expand the metric around the Minkowski flat background as [33,34]

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(2)}_{\mu\nu} + h^{(3)}_{\mu\nu} + h^{(4)}_{\mu\nu} + \mathcal{O}(5). \tag{14}$$

Note $h^{(1)}_{\mu\nu} = 0$ [35–40]. Accordingly, the vierbeins may be expanded as [34]

$$e^a_{\mu} = \delta^a_{\mu} + B^{(2)a}_{\mu} + B^{(3)a}_{\mu} + B^{(4)a}_{\mu} + \mathcal{O}(5). \tag{15}$$

Note $B^{(1)a}_{\mu} = 0$. In our analysis we need non-zero components of the metric up to order 4, i.e. $h^{(2)}_{ij}, h^{(2)}_{00}, h^{(3)}_{0i}, h^{(4)}_{00}$. We also use the notation $B_{\mu\nu} = \eta_{\mu\sigma} B^{\sigma}_{\nu}$ and $\delta_a^{\sigma} B^a_{\nu} = B^{\sigma}_{\nu}$. By comparing (14) and (15) we derive (like in [33,34], $B^{(2)}_{ij}$ is assumed to be diagonal)

$$\begin{aligned} h^{(2)}_{ij} &= 2B^{(2)}_{ij} \\ h^{(2)}_{00} &= 2B^{(2)}_{00} \\ h^{(3)}_{0i} &= 2B^{(3)}_{0i} \\ h^{(4)}_{00} &= 2B^{(4)}_{00} - (B^{(2)}_{00})^2. \end{aligned} \tag{16}$$

We introduce two functions A , and γ (which is one of the PPN parameters) through [33]

$$\begin{aligned} B^{(2)}_{00} &= A \\ B^{(2)}_{ij} &= \gamma A \delta_{ij}. \end{aligned} \tag{17}$$

The scalar field is expanded as

$$\phi = \phi_0 + \psi, \tag{18}$$

where

$$\psi = \psi^{(2)} + \psi^{(4)} + \mathcal{O}(6), \tag{19}$$

and ϕ_0 is a constant cosmological background. ϕ_0 is of order $\mathcal{O}(0)$ and may evolve in times of order of the Hubble time, so in solar system tests we assume that it is static. The time derivatives, $\partial_0 = \frac{\partial}{\partial t}$, of the other fields are weighted with order $\mathcal{O}(1)$ [35–40].

The potential around the background is

$$V(\phi) = V(\phi_0) + V'(\phi_0)\psi + \frac{V''(\phi_0)}{2}\psi^2 + \mathcal{O}(6). \tag{20}$$

Defining $V(\phi_0) = V_0, \frac{V^{(n)}(\phi_0)}{n!} = V_n$ we find

$$V' = V_1 + 2V_2\psi + 3V_3\psi^2 + \mathcal{O}(6). \tag{21}$$

After these preliminaries, let us solve Eqs. (9) and (10) order by order in the PPN formalism. At zeroth order (9) and (10) imply

$$V_0 = V_1 = 0. \tag{22}$$

The 0-0 component of (9) gives

$$\begin{aligned} &\left(\frac{2}{k^2} + 2\epsilon\phi^2\right) \left(e^{-1} e^a_{00} \partial_{\lambda} (e S_a^{\lambda 0}) - T^{\rho}_{\beta 0} S_{\rho}^{0\beta} - \frac{1}{4} T\right) \\ &- V(\phi) - \partial^0 \phi \partial_0 \phi \\ &+ 4(\chi + \epsilon)\phi S_0^{\beta 0} \partial_{\beta} \phi - \chi \nabla^0 \nabla_0 \phi^2 - \frac{1}{2} \chi \square \phi^2 \\ &- \left(\frac{1}{k^2} + \epsilon\phi^2\right) (e^{-1} e^a_{\alpha} \partial_{\lambda} e S_a^{\lambda\alpha}) \\ &- 2(\chi + \epsilon)\phi S_{\alpha}^{j\alpha} \partial_j \phi = -\tau_0^0 + \frac{1}{2} \tau, \end{aligned} \tag{23}$$

which at order 2 reduces to

$$\begin{aligned} &\left(\frac{1}{k^2} + \epsilon\phi_0^2\right) \partial_j S_0^{j0} - V(\phi) - \frac{1}{2} \chi \square \phi^2 - \left(\frac{1}{k^2} + \epsilon\phi^2\right) \partial_j S_i^{ji} \\ &= \frac{\rho}{2}, \end{aligned} \tag{24}$$

resulting in

$$-\left(\frac{1}{k^2} + \epsilon\phi_0^2\right) \nabla^2 A - \chi \phi_0 \nabla^2 \psi^{(2)} = -\frac{1}{k^2} \nabla^2 U, \tag{25}$$

where the potential is given by

$$\nabla^2 U = -\frac{k^2}{2} \rho. \tag{26}$$

To obtain (25), we have used

$$\begin{aligned} S^{(2)0}_{j0} &= -\partial_j(\gamma A), \quad S^{(2)j}_{ij} = \partial_i((1 - \gamma)A), \\ S^{(2)i}_{0j} &= 0 \\ \partial_{\mu} e^{(2)} &= \partial_{\mu}((3\gamma - 1)A), \quad T^{(2)0}_{i0} = -\partial_i A, \\ S^{(2)0}_{0i} &= \partial_i(\gamma A). \end{aligned} \tag{27}$$

By taking the trace of the i - j component of (9), at order 2, we obtain

$$\begin{aligned} &-3\left(\frac{1}{k^2} + \epsilon\phi_0^2\right) \partial_j S_0^{j0} - \left(\frac{1}{k^2} + \epsilon\phi_0^2\right) \partial_j S_i^{ji} - 5\chi \phi_0 \nabla^2 \psi^{(2)} \\ &= -\frac{3}{2} \rho, \end{aligned} \tag{28}$$

which reduces to

$$\left(\frac{1}{k^2} + \epsilon\phi_0^2\right) \nabla^2((4\gamma - 1)A) - 5\chi \phi_0 \nabla^2 \psi^{(2)} = \frac{3}{k^2} \nabla^2 U. \tag{29}$$

At the second order perturbation, the boundary term \mathcal{B} , defined in (3), is derived:

$$\mathcal{B}^{(2)} = 2\nabla^2((1 - 2\gamma)A). \tag{30}$$

Hence from (10) the equation of motion of the scalar field becomes

$$-\nabla^2\psi^{(2)} + 2V_2\psi^{(2)} = 2\chi(1 - 2\gamma)A\phi_0. \tag{31}$$

Equations (25), (29), and (31) are our three main equations for determining A , γ , and $\psi^{(2)}$. Using these three equations, for a given U , A is derived:

$$A = \frac{2}{(1 + \epsilon\phi_0^2k^2)(1 + \gamma)}U, \tag{32}$$

and $\psi^{(2)}$ is obtained:

$$\psi^{(2)} = \frac{\gamma - 1}{k^2\chi\phi_0(\gamma + 1)}U. \tag{33}$$

γ is determined by the equation

$$\left(1 - \frac{6k^2\chi^2\phi_0^2}{1 + \epsilon k^2\phi_0^2}\right)\nabla^2(\Gamma U) - 2V_2(\Gamma U) = -\frac{k^4\chi^2\phi_0^2}{1 + \epsilon k^2\phi_0^2}\rho, \tag{34}$$

where $\Gamma := \frac{\gamma-1}{\gamma+1}$. (34) is a nonhomogeneous screened Poisson equation whose solution is

$$\Gamma U = \frac{k^4\chi^2\phi_0^2}{1 + \epsilon k^2\phi_0^2 - 6k^2\chi^2\phi_0^2} \int \frac{\exp(-\lambda|\vec{r} - \vec{r}'|)}{4\pi|\vec{r} - \vec{r}'|} \rho(x', t) d^3x', \tag{35}$$

where

$$\lambda = \sqrt{\frac{2V_2(1 + k^2\epsilon\phi_0^2)}{1 + k^2\epsilon\phi_0^2 - 6k^2\chi^2\phi_0^2}}. \tag{36}$$

Equation (32) allows us to take

$$G = \frac{2}{(1 + k^2\epsilon\phi_0^2)(\gamma + 1)}, \tag{37}$$

where G is defined through

$$h_{00}^{(2)} = 2A = 2GU. \tag{38}$$

So one can define an effective G_{eff} through

$$G_{\text{eff}} = GG_N. \tag{39}$$

The 0- i component of (7) at the third order gives

$$\left(\frac{2}{k^2} + 2\epsilon\phi_0^2\right)\partial_\mu S_0^{\mu i} = -\tau^{(3)}_0{}^i = \rho v^i, \tag{40}$$

which, by using

$$\begin{aligned} T^{(3)0}{}_{ij} &= \partial_i B^{(3)0}{}_j - \partial_j B^{(3)0}{}_i \\ T^{(3)i}{}_{j0} &= \partial_j B^{(3)i}{}_0 - \delta_j^i \partial_0(\gamma A) \\ T^{(3)i}{}_{i0} &= -3\partial_0(\gamma A) + 3\partial_i B^{(3)i}{}_0, \end{aligned} \tag{41}$$

reduces to

$$\begin{aligned} \left(\frac{2}{k^2} + 2\epsilon\phi_0^2\right)\left(\partial_0\partial_i(\gamma A) - \frac{1}{2}\nabla^2 B^{(3)0}{}_i + \frac{1}{2}\partial^j\partial_i B^{(3)0}{}_j\right) \\ = \rho v_i. \end{aligned} \tag{42}$$

To simplify computations one may employ the gauge condition

$$\begin{aligned} -\partial^j B^{(2)i}{}_j + \frac{1}{2}\partial^i B^{(2)\mu}{}_\mu &= \frac{\chi k^2\phi_0}{k^2 + \epsilon\phi_0^2}\partial^i\psi^{(2)} \\ -\partial_j B^{(3)j}{}_0 + \frac{1}{2}\partial_0 B^{(2)j}{}_j &= \frac{\chi k^2\phi_0}{k^2 + \epsilon\phi_0^2}\partial_0\psi^{(2)}, \end{aligned} \tag{43}$$

which determines $B^{(3)0}{}_j$ in terms of second order parameters. This gauge is compatible with Eqs. (25) and (29).

Using

$$\begin{aligned} S^{(4)i}{}_{ji} &= \gamma A\partial_j(\gamma A) + A\partial_j A - \partial_j B^{(4)0}{}_0 + \partial_0 B^{(3)0}{}_j \\ S^{(4)0}{}_{j0} &= \gamma A\partial_j(\gamma A) \\ S^{(3)i}{}_{i0} &= -\frac{3}{2}\partial_0(\gamma A), \end{aligned} \tag{44}$$

one can find that (23) at the fourth order gives

$$\begin{aligned} \left(\frac{1}{k^2} + \epsilon\phi_0^2\right)\left(\nabla^2 B^{(4)0}{}_0 + \nabla^2(\gamma A)^2\right. \\ \left.- 3\nabla(\gamma A) \cdot \nabla A - A\nabla^2 A\right) \\ - 4\epsilon\phi_0\psi^{(2)}\nabla^2 A - 2(\chi + \epsilon)\phi_0\nabla\psi^{(2)} \\ \cdot \nabla A - \frac{\chi}{2}\nabla^2(\psi^{(2)})^2 - V_2(\psi^{(2)})^2 \\ - \chi\phi_0\nabla^2\psi^{(4)} + 3\chi\phi_0\nabla((\gamma - 1)A) \cdot \nabla\psi^{(2)} + \left(\partial_0\psi^{(2)}\right)^2 \\ + 3\chi\phi_0\partial_0^2\psi^{(2)} + \left(\frac{1}{k^2} + \epsilon\phi_0^2\right)\partial_0\left(3\partial_0(\gamma A) - \partial_i B^{(3)i}{}_0\right) \\ - \left(\frac{1}{k^2} + \epsilon\phi_0^2\right)\partial^j\partial_0 B^{(3)0}{}_j = \frac{1}{2}\tau^{(4)} - \tau^{(4)0}{}_0. \end{aligned} \tag{45}$$

Also, the scalar field equation at the fourth order is

$$\begin{aligned} -\nabla^2\psi^{(4)} + 2V_2\psi^{(4)} + \partial_0^2\psi^{(2)} &= \chi\phi_0 B^{(4)} \\ + \psi^{(2)}B^{(2)} + \epsilon\phi_0 T^{(4)} - 3V_3(\psi^{(2)})^2. \end{aligned} \tag{46}$$

By using

$$B^{(4)} = -8\nabla^2(\gamma^2 A^2) + 14\nabla \cdot (\gamma A \nabla A) + 2(1 - 5\gamma)A\nabla^2 A + 12\gamma A \nabla^2(\gamma A) - \nabla^2 B^{(4)0} + 6\partial_0^2(\gamma A) - 2\partial_i \partial_0 B^{(3)i}_0 \tag{47}$$

and

$$T^{(4)} = 2\nabla(\gamma A) \cdot \nabla((2 - \gamma)A), \tag{48}$$

(46) becomes

$$\begin{aligned} & -\nabla^2 \psi^{(4)} + 2V_2 \psi^{(4)} + \partial_0^2 \psi^{(2)} \\ & = 6\chi \phi_0 \partial_0^2(\gamma A) - 2\chi \phi_0 \partial_i \partial_0 B^{(3)i}_0 \\ & 2\chi \phi_0 \left(-4\gamma^2 A^2 + 7\nabla \cdot (\gamma A \nabla A) \right. \\ & \quad \left. + (1 - 5\gamma)A\nabla^2 A + 6\gamma A \nabla^2(\gamma A) \right) \\ & \quad + 2\psi^{(2)} \nabla^2((1 - 2\gamma)A) \\ & \quad + 2\epsilon \phi_0 \nabla(\gamma A) \cdot \nabla((2 - \gamma)A) - 3V_3(\psi^{(2)})^2 \\ & \quad - 2\chi \phi_0 \nabla^2 B^{(4)0} + 3\nabla(1 - \gamma)A \cdot \nabla \psi^{(2)}. \end{aligned} \tag{49}$$

Equations (45) and (49) are our main results in the fourth order. These equations together with (42) and (43) in the third order, and (32)–(34), in the second order must be solved to give the post-Newtonian parameters.

To solve these complicated equations, we consider solutions specified by $U = U(r)$, which results in

$$A = A(r), \quad \gamma = \gamma(r), \quad \psi^{(2)} = \psi^{(2)}(r). \tag{50}$$

The gauge (43) implies $\partial^j B^{(3)0}_j = 0$. Therefore (42) reduces to

$$-\left(\frac{1}{k^2} + \epsilon \phi_0^2\right) \left(\nabla^2 B^{(3)0}_i\right) = \rho v_i. \tag{51}$$

For $v^i = 0$, (51) gives $B^{(3)0}_i = 0$ (by the assumption that perturbation terms vanish at large distance). In this situation Eqs. (45) and (49) become

$$\begin{aligned} & \left(\frac{1}{k^2} + \epsilon \phi_0^2\right) \left(\nabla^2 B^{(4)0}_0 + \nabla^2(\gamma A)^2\right. \\ & \quad - 3\nabla(\gamma A) \cdot \nabla A - A\nabla^2 A) \\ & \quad - 4\epsilon \phi_0 \psi^{(2)} \nabla^2 A - 2(\chi + \epsilon) \\ & \quad \times \phi_0 \nabla \psi^{(2)} \cdot \nabla A - \frac{\chi}{2} \nabla^2(\psi^{(2)})^2 - V_2(\psi^{(2)})^2 \\ & \quad - \chi \phi_0 \nabla^2 \psi^{(4)} + 3\chi \phi_0 \nabla((\gamma - 1)A) \cdot \nabla \psi^{(2)} \\ & = \frac{1}{2} \tau^{(4)} - \tau^{(4)0}_0 \end{aligned} \tag{52}$$

and

$$-\nabla^2 \psi^{(4)} + 2V_2 \psi^{(4)}$$

$$\begin{aligned} & = 2\chi \phi_0 \left(-4\gamma^2 A^2 + 7\nabla \cdot (\gamma A \nabla A) \right. \\ & \quad \left. + (1 - 5\gamma)A\nabla^2 A + 6\gamma A \nabla^2(\gamma A) \right) \\ & \quad + 2\psi^{(2)} \nabla^2((1 - 2\gamma)A) + 2\epsilon \phi_0 \nabla(\gamma A) \\ & \quad \cdot \nabla((2 - \gamma)A) - 3V_3(\psi^{(2)})^2 \\ & \quad - 2\chi \phi_0 \nabla^2 B^{(4)0} + 3\nabla(1 - \gamma)A \cdot \nabla \psi^{(2)}, \end{aligned} \tag{53}$$

respectively. To obtain the post-Newtonian parameters we must obtain A , $\psi^{(2)}$, and $\gamma(r)$. By inserting them in (52) and (53), we obtain solutions for $B^{(4)0}_0$. To do so we consider a spherically symmetric metric with a point source.

3.1 Spherically symmetric metric

The source is assumed to be

$$\rho = M\delta(\vec{r}), \quad \Pi = 0, \quad p = 0, \quad v_i = 0, \tag{54}$$

and the metric is given by

$$\begin{aligned} g_{00} & = -1 + 2G_{\text{eff}}U - 2G_{\text{eff}}^2\beta U^2 + \text{Self} + \mathcal{O}(6) \\ g_{ij} & = \mathcal{O}(5) \\ g_{ij} & = (1 + 2G_{\text{eff}}\gamma U) \delta_{ij} + \mathcal{O}(4), \end{aligned} \tag{55}$$

where ‘‘Self’’ denotes self-energy terms of order 4, and β is the PPN parameter. The Newtonian potential is

$$U = \frac{k^2 M}{8\pi r}. \tag{56}$$

To determine γ , from (32), (33), and (35), we obtain

$$\psi^{(2)} = \frac{2\chi \phi_0}{1 + \epsilon k^2 \phi_0^2 - 6\chi^2 k^2 \phi_0^2} \exp(-\lambda r) \tag{57}$$

and

$$A = \frac{k^2 M}{4\pi(1 + \epsilon k^2 \phi_0^2)(1 + \gamma)r}, \tag{58}$$

where

$$\gamma = \frac{1 + \alpha \exp(-\lambda r)}{1 - \alpha \exp(-\lambda r)}, \tag{59}$$

in which

$$\alpha = \frac{2k^2 \chi^2 \phi_0^2}{1 + k^2 \epsilon \phi_0^2 - 6k^2 \chi^2 \phi_0^2}, \tag{60}$$

and λ is given by (36). From $h^{(2)}_{00} = 2A = 2GU$, we obtain G as (37).

To obtain $B^{(4)0}_0$, one must insert (57)–(59) in (52) and (53), and solve them together. From $B^{(4)0}_0$ we determine the other PPN parameter, β , as

$$2B^{(4)0}_0 + A^2 = 2G^2\beta(r)U^2(r). \tag{61}$$

To determine the PPN parameters γ and β , we will consider different situations.

3.1.1 $\chi = 0$

For $\chi = 0$, from (59) and (60), we find $\gamma = 1$, hence

$$A = \frac{k^2 M}{8\pi(1 + \epsilon k^2 \phi_0^2)r}, \quad G = \frac{1}{1 + k^2 \epsilon \phi_0^2}. \tag{62}$$

Equation (57) gives $\psi^{(2)} = 0$. So we write (52) as

$$\nabla^2 B^{(4)0}_0 - \frac{1}{2}\nabla^2(A)^2 + 2A\nabla^2 A = 0, \tag{63}$$

where $\nabla A \cdot \nabla A = \frac{1}{2}\nabla^2 A^2 - A\nabla^2 A$ has been used. Putting (62) in (63), and ignoring the gravitational self-energy, we obtain

$$B^{(4)0}_0 = -\frac{A^2}{2} + \frac{k^4 M^2}{64\pi^2(1 + \epsilon k^2 \phi_0^2)^2 r^2}. \tag{64}$$

Therefore (61) yields $\beta(r) = 1$. So for $\chi = 0$ we find

$$\beta(\chi = 0) = \gamma(\chi = 0) = 1. \tag{65}$$

Therefore there is no deviation from general relativity for the PPN parameters. This is in complete agreement with [33, 34].

3.2 $\phi_0 = 0$

For $\chi \neq 0$, we may have also a situation with no deviation in the PPN parameters from general relativity, this occurs for $\phi_0 = 0$. For example for the potentials

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\Lambda}{4}\phi^4, \quad \Lambda > 0, \tag{66}$$

and

$$V(\phi) = \Lambda\phi^n, \quad \Lambda > 0, \quad n > 1, \tag{67}$$

$V_0 = V_1 = 0$ (see (22)) leads to $\phi_0 = 0$, which by using (59–61) results in $\gamma = 1$, $G = 1$, and $\beta = 1$. Therefore in this case too, there is no deviation from general relativity for the PPN parameters.

3.2.1 $V(\phi) = 0$

If we ignore the scalar field potential, we obtain $\lambda = 0$ (see (36)), and γ becomes a constant

$$\gamma = \frac{1 - (4\chi^2 - \epsilon)k^2\phi_0^2}{1 - (8\chi^2 - \epsilon)k^2\phi_0^2}. \tag{68}$$

By solving the system of Eqs. (52) and (53) for $B^{(4)0}_0$ and by considering Eqs. (57)–(61), after some computations we find

$$\beta = \frac{P}{(1 + (2\chi^2 + \epsilon)k^2\phi_0^2)(1 - (8\chi^2 - \epsilon)k^2\phi_0^2)^2}, \tag{69}$$

where

$$P = 1 + 160\left(\chi^6 + \frac{3}{10}\epsilon\chi^5 + \frac{3}{40}\epsilon\chi^4 - \frac{3}{16}\epsilon^2\chi^3 - \frac{1}{10}\epsilon^2\chi^2 + \frac{3}{160}\epsilon^3\chi + \frac{1}{160}\epsilon^3\right)k^6\phi_0^6 + 2\left(\chi^3 - 8\chi^2 + \frac{3}{2}\chi\epsilon + \frac{3}{2}\epsilon\right)k^2\phi_0^2 + 12\left(\chi^4 - \frac{7}{3}\epsilon\chi^3 - \frac{8}{3}\epsilon\chi^2 + \frac{1}{2}\chi\epsilon^2 + \frac{1}{4}\epsilon^2\right)k^4\phi_0^4. \tag{70}$$

Let us consider some limiting values: for small χ , $\chi \ll 1$ we have

$$\beta = 1 + \frac{3\epsilon k^2 \phi_0^2}{1 + \epsilon k^2 \phi_0^2} \chi - \frac{2k^2 \phi_0^2}{1 + \epsilon k^2 \phi_0^2} \chi^2 + \mathcal{O}(\chi^3) \\ \gamma = 1 + \frac{4k^2 \phi_0^2}{1 + \epsilon k^2 \phi_0^2} \chi^2 + \mathcal{O}(\chi^4), \tag{71}$$

and for small $k\phi_0$, $k\phi_0 \ll 1$ we have

$$\beta = 1 + \chi(2\chi^2 - 2\chi + 3\epsilon)k^2\phi_0^2 + \mathcal{O}(k^4\phi_0^4) \\ \gamma = 1 + 4\chi^2 k^2 \phi_0^2 + \mathcal{O}(k^4\phi_0^4). \tag{72}$$

3.2.2 $\lambda r \gg 1$

In this limit from (57) and (59) we have $\psi^{(2)} = 0$ and $\gamma = 1$, respectively. The solution of (52) is obtained, thus:

$$B^{(4)0}_0 = \frac{1}{2}A^2 + \frac{\Omega + 1}{2\chi\phi_0}\psi^{(4)}, \tag{73}$$

where $\Omega = -1 + \frac{2\chi^2 k^2 \phi_0^2}{1 + \epsilon k^2 \phi_0^2}$. The equation of motion of the scalar field (53) becomes

$$\Omega\nabla^2\psi^{(4)} + 2V_2\psi^{(4)} = (\epsilon - 2\chi)\phi_0\nabla^2 A^2 + (8\chi - \epsilon)\phi_0 A\nabla^2 A, \tag{74}$$

whose solution, in the limit $\left| \frac{V_2 r}{\Omega} \right| \gg 1$, is

$$\psi^{(4)} = \left(\frac{k^4 M^2 \phi_0 (\epsilon - 2\chi)}{64\pi^2 (1 + \epsilon k^2 \phi_0^2) (2\chi^2 k^2 \phi_0^2 - \epsilon k^2 \phi_0^2 - 1)} \right) \frac{1}{r^2}. \tag{75}$$

From (75), (73), and (61), we find

$$\beta = \frac{\epsilon(\chi - 1)k^2\phi_0^2 - 1}{(2\chi^2 - \epsilon)k^2\phi_0^2 - 1}. \tag{76}$$

For small $k^2\phi_0^2, k^2\phi_0^2 \ll 1$ this gives

$$\beta = 1 + (2\chi^2 - \chi\epsilon)k^2\phi_0^2 + \mathcal{O}(k^4\phi_0^4), \tag{77}$$

and for small $\chi, \chi \ll 1$ gives

$$\beta = 1 - \frac{\epsilon k^2 \phi_0^2}{1 + \epsilon k^2 \phi_0^2} \chi + 2 \frac{k^2 \phi_0^2}{1 + \epsilon k^2 \phi_0^2} \chi^2 + \mathcal{O}(\chi^3). \tag{78}$$

Finally, let us note that, for small $\lambda r, \lambda r \ll 1$, we take $\exp(-\lambda r) \simeq 1$. In this case γ and β take the same form as (68) and (69), respectively.

3.3 Range of parameters

The most precise value for γ experimentally has been obtained from Cassini [45]. The bound on this parameter is [46]

$$|\gamma - 1| \lesssim 2.3 \times 10^{-5}. \tag{79}$$

In this experiment the gravitational interaction, in terms of Astronomical Units takes place at $r \simeq 7.44 \times 10^{-3} AU$ [47].

The parameter β is determined by lunar laser ranging experiments via the Nordtvedt effect [48]. This test indicates the bound [46]

$$|\beta - 1| \lesssim 2.3 \times 10^{-4}, \tag{80}$$

at a gravitational interaction distance $r = 1AU$ [47]. Equations (79) and (80) restrict the parameters of our model.

For $V(\phi) = 0$, (79) and (68) give

$$\left| \frac{4k^2\chi^2\phi_0^2}{1 - (8\chi^2 - \epsilon)k^2\phi_0^2} \right| \lesssim 2.3 \times 10^{-5}. \tag{81}$$

In the limiting cases (71) and (72) we find

$$\left| \frac{4k^2\phi_0^2}{1 + \epsilon k^2\phi_0^2} \chi^2 \right| \lesssim 2.3 \times 10^{-5}$$

$$\left| \frac{3\epsilon k^2\phi_0^2}{1 + \epsilon k^2\phi_0^2} \chi^2 \right| \lesssim 2.3 \times 10^{-4} \tag{82}$$

and

$$4\chi^2 k^2 \phi_0^2 \lesssim 2.3 \times 10^{-5}$$

$$\left| \chi(2\chi^2 - 2\chi + 3\epsilon)k^2\phi_0^2 \right| \lesssim 2.3 \times 10^{-4}, \tag{83}$$

respectively.

For $\lambda r \gg 1$, we have

$$\sqrt{\frac{2V_2(1 + k^2\epsilon\phi_0^2)}{1 + k^2\epsilon\phi_0^2 - 6k^2\chi^2\phi_0^2}} \gg (1AU)^{-1}, \tag{84}$$

and (80) restricts our parameters:

$$\left| \frac{\chi(2\chi - \epsilon)k^2\phi_0^2}{(2\chi^2 - \epsilon)k^2\phi_0^2 - 1} \right| \lesssim 2.3 \times 10^{-4}. \tag{85}$$

4 Conclusion

The teleparallel model of gravity with quintessence (non-minimally) coupled to the torsion and also to a boundary term (proportional to the torsion divergence) was considered (see (1)). Although the model shows some interesting aspects in cosmology and in describing the late time acceleration of the Universe, it must also pass local gravitational and solar system tests. So we studied the parameterized post-Newtonian (PPN) approximation of the model. We obtained the equations of motion (see the Sect. 2), and solve them order by order to obtain PPN parameters (see Sect. 3). Explicit expressions for the PPN parameters in a spherically symmetric metric were obtained and different possible situations were discussed. Our results show that the PPN parameters, except for some special cases, i.e. in the absence of boundary terms and also with zero scalar field background, differ from general relativity. So we conclude that coupling of the scalar field to the boundary term generally makes the model deviate from general relativity in the PPN limit.

Since T and \mathcal{B} are not invariant under local Lorentz transformations, the teleparallel model with boundary term is not invariant under Lorentz transformations unless one takes $\chi = -\epsilon$. Despite this, in spacetimes with spherical symmetry like Schwarzschild spacetime and so on, it is possible to choose good or preferred tetrads to solve this issue [49]. In scalar-tetrad theories of gravity the effect of the preferred tetrads cannot be detected via measuring the metric components [50,51]. Similarly, in our model, the PPN parameters in the standard post-Newtonian formalism do not identify the effect of the preferred tetrads. To include these effects

one must generalize the post-Newtonian approach, as was pointed out in [33].

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