THE EUROPEAN
PHYSICAL JOURNAL C

CrossMark

Nonsingular black hole

Ali H. Chamseddine[1,2], **Viatcheslav Mukhanov**[3,4,5,a]

[1] Physics Department, American University of Beirut, Beirut, Lebanon
[2] I.H.E.S., 91440 Bures-sur-Yvette, France
[3] Niels Bohr International Academy, Niels Bohr Institute, Blegdamsvej 17, 2100 Copenhagen, Denmark
[4] Theoretical Physics, Ludwig-Maximilians University, Theresienstr. 37, 80333 Munich, Germany
[5] MPI for Physics, Foehringer Ring, 6, 80850 Munich, Germany

Abstract We consider the Schwarzschild black hole and show how, in a theory with limiting curvature, the physical singularity "inside it" is removed. The resulting spacetime is geodesically complete. The internal structure of this nonsingular black hole is analogous to Russian nesting dolls. Namely, after falling into the black hole of radius r_g, an observer, instead of being destroyed at the singularity, gets for a short time into the region with limiting curvature. After that he re-emerges in the near horizon region of a spacetime described by the Schwarzschild metric of a gravitational radius proportional to $r_g^{1/3}$. In the next cycle, after passing the limiting curvature, the observer finds himself within a black hole of even smaller radius proportional to $r_g^{1/9}$, and so on. Finally after a few cycles he will end up in the spacetime where he remains forever at limiting curvature.

1 Introduction

The problem of singularity within black holes has remained, since a long time, one of the most intriguing problems in theoretical physics. Although such a singularity is hidden by the event horizon, one can imagine that an observer can decide (at least in a gedanken experiment) to travel inside the black hole and the legitimate physical question which arises is: what will this observer see being inside the black hole and in particular as he approaches the singularity? In the case that the black hole has a huge mass he will have more than enough time to make the needed experiments to measure how the tidal forces are changing. If General Relativity is valid up to arbitrary high curvatures then the theory predicts that, irrespective of what any observer will do, he will finally be destroyed by the infinite curvatures. In fact, assuming universal applicability of Einstein's theory, and imposing energy

dominance conditions on the state of matter, Hawking and Penrose have proved that space-times with black holes cannot be geodesically complete [1]. It is well known that these conditions are not always valid and for instance the condensate of a scalar field or cosmological constant violates some of them. In this case the singularity can, in principle, be avoided and the spacetime can become geodesically complete. For example Ref. [2] considered the possibility of removing the singularity by forcing the contracting space inside the black hole to get to the de Sitter bouncing state. This opens the fascinating possibility of "gedanken travelling" to another universe via a black hole (of course only for those who could survive the extremely high curvatures at which the bounce is supposed to take place). However, although this idea by itself does not contradict any basic physical principles the authors of [2] were not able to provide any concrete example where such an idea could be realized constructively.

Normally the majority of research redirects the question of singularities to the yet unknown nonperturbative quantum gravity (which in turn could well be part of some yet unknown fundamental unified theory). In fact it is clear that quantum corrections to General Relativity become extremely important at Planckian curvatures and could easily modify or resolve the singularities. Therefore, one cannot say that such hopes are completely unjustified. However, until now, the perturbative treatment of these corrections has led to an extremely messy picture and did not give even the slightest constructive hints of how the problem could be treated and solved in a fully nonperturbative quantum gravity. Numerous attempts to address this question did not lead to any reliable progress. Therefore in this paper we will use a completely different approach. Instead of exploiting quantum effects we will try to resolve the problem of singularities fully at the classical level by incorporating the idea of a limiting curvature [3–8], assuming that Einstein's equations are modified at curvatures well below the Planckian curvature. There is

ᵃ e-mail: Viatcheslav.Mukhanov@physik.lmu.de

Springer

nothing that forbids this idea because Einstein's equations have been checked experimentally only for curvatures well below the Planckian ones. If the limiting curvature is below the Planck value the inevitable quantum effects, due to for instance particle production and vacuum polarization, can be ignored and the theory will be under control and would remain completely reliable up to the highest possible curvatures. In particular, in [9] we have suggested a concrete theory with limiting curvature and have shown that cosmological singularities in this theory are fully removed. In this paper we consider how a black hole is modified in our theory and what happens close to the singularity inside a black hole. We would like to point out that removing singularities can have severe consequences for questions broadly discussed in the literature, such as the so-called "information paradox" and the fate of remnants of the minimal mass which can, in principle, survive after the Hawking evaporation is over. We will discuss these questions in more detail after obtaining the solution for a nonsingular black hole.

2 Theory with limiting curvature

Consider the theory described by the action [10,11]

$$S = \int \left(-\frac{1}{2} R + \lambda \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 1 \right) + f(\chi) \right) \sqrt{-g} \, d^4 x, \tag{1}$$

where $\chi = \Box\phi$, λ is a Lagrange multiplier and we have set $8\pi G = 1$. As we have shown in Ref. [9] the usual matter does not play any significant role in resolving anisotropic singularities. Therefore to simplify the formulas we will omit here its contribution to (1). It immediately follows from variation of the Lagrange multiplier λ that the scalar ϕ always satisfies the constraint

$$g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = 1. \tag{2}$$

Therefore the term $f(\chi)$, irrespective of any power of χ, does not lead to the propagation of extra degrees of freedom which, otherwise, could be ghosts. The constraint (2) imposes a very strong restriction on the variable ϕ and in the synchronous coordinate system with metric

$$ds^2 = dt^2 - \gamma_{ik}(t, x^l) dx^i dx^k, \tag{3}$$

it has the most general solution [12]

$$\phi = \pm t + A, \tag{4}$$

unless this particular coordinate system does suffer from coordinate singularities. Thus the field ϕ plays the role of time and the constant of integration A reflects the time shift symmetry. In this coordinate system

$$\chi = \Box\phi = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial \phi}{\partial x^\nu} \right) = \frac{\dot{\gamma}}{2\gamma}, \tag{5}$$

with $\gamma = \det \gamma_{ik}$ and where by a dot we denote the time derivative. Thus, the function $f(\chi)$ allows one to introduce, in a completely covariant way, the metric and its first derivative in the "game" when we try to find a simple modification of General Relativity where singularities can be avoided. In this sense the action (1) must be treated as a modification of Einstein's gravity. The only extra new degree of freedom which appears here is mimetic Dark Matter [10] because the constraint (2) forces the longitudinal gravitational field to become dynamical even in the absence of the usual matter.

We can choose the function $f(\chi)$ in such a way as to bound the derivative of the metric determinant in the synchronous coordinate system. Because these derivatives enter in an essential way in the coordinate independent curvature invariants (see below) this opens the possibility to have a nonsingular modification of gravity. After many trials, the simplest way we were able to find to construct such a theory is to use a Born–Infeld type function, where

$$f(\chi) = 1 - \sqrt{1 - \chi^2} + g(\chi), \tag{6}$$

and χ is restricted by $\chi^2 \leq 1$ for obvious reasons. The function $g(\chi)$ is less restrictive but it has at least to satisfy two necessary conditions. First, it must be chosen in such a way as to remove the χ^2 terms in the Taylor expansion of $f(\chi)$ because these would lead to an unwanted modification to Einstein's theory at low curvatures. Second, the function $g(\chi)$ must remove the singularity in $df/d\chi$ at $\chi = 1$, otherwise the curvature invariants would blow up at this point. In the theory with $f(\chi)$ given in (6) the limiting curvature would be of the order of the Planck curvature, where the quantum effects are extremely important. To avoid this problem we will assume that the limiting curvature is at least a few orders of magnitude below the Planckian value and this would allow for justifying why vacuum polarization effects and particle production effects could be ignored. Taking for $g(\chi)$ a function which leads to particularly simple equations,

$$g(\chi) = \frac{1}{2}\chi^2 - \chi \arcsin \chi, \tag{7}$$

and introducing a limiting curvature, characterized by χ_m^2, as an extra free scale in the theory we will take, after rescaling $\chi \to \sqrt{\frac{2}{3}} \frac{\chi}{\chi_m}$ and $f \to \chi_m^2 f$,

$$f(\chi) = \chi_m^2 \left(1 + \frac{1}{3} \frac{\chi^2}{\chi_m^2} - \sqrt{\frac{2}{3}} \frac{\chi}{\chi_m} \arcsin \left(\sqrt{\frac{2}{3}} \frac{\chi}{\chi_m} \right) \right.$$
$$\left. - \sqrt{1 - \frac{2}{3} \frac{\chi^2}{\chi_m^2}} \right). \tag{8}$$

As we have already seen in [9] this choice of f removes singularities in the Friedmann and Kasner universes. In this paper we will consider what happens with singularities for black holes.

Variation of the action (1) with respect to the metric $g^{\mu\nu}$ gives the modified Einstein's equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \tilde{T}_{\mu\nu}, \tag{9}$$

where

$$\tilde{T}_{\mu\nu} = 2\lambda\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}(\chi f' - f + g^{\rho\sigma}\partial_\rho f'\partial_\sigma\phi) \\ - (\partial_\mu f'\partial_\nu\phi + \partial_\nu f'\partial_\mu\phi) \tag{10}$$

characterizes the modification to General Relativity and we have denoted $f' = \mathrm{d}f/\mathrm{d}\chi$. For metric (3) the time-time and space-space components of the curvature are [12]

$$R_0^0 = -\frac{1}{2}\dot{\varkappa} - \frac{1}{4}\varkappa_i^k\varkappa_k^i, \quad R_k^i = -\frac{1}{2\sqrt{\gamma}}\frac{\mathrm{d}(\sqrt{\gamma}\varkappa_k^i)}{\mathrm{d}t} - P_k^i, \tag{11}$$

where $\varkappa_k^i = \gamma^{im}\dot{\gamma}_{mk}$, $\varkappa = \varkappa_i^i = \dot{\gamma}/\gamma$ and P_k^i is the three dimensional Ricci tensor for the metric γ_{ik}. The corresponding components of $\tilde{T}_\nu^\mu$ for solution (4) are

$$\tilde{T}_0^0 = 2\lambda + \chi f' - f - \dot{\chi}f'', \\ \tilde{T}_k^i = (\chi f' - f + \dot{\chi}f'')\delta_k^i. \tag{12}$$

The 0–0 equation

$$R_0^0 - \frac{1}{2}R = \tilde{T}_0^0 \tag{13}$$

then takes the form

$$\frac{1}{8}(\varkappa^2 - \varkappa_i^k\varkappa_k^i + 4P) = 2\lambda + \chi f' - f - \dot{\chi}f'', \tag{14}$$

and the space-space equation

$$R_k^i = \tilde{T}_k^i - \frac{1}{2}\tilde{T}_\alpha^\alpha\delta_k^i \tag{15}$$

becomes

$$\frac{1}{2\sqrt{\gamma}}\frac{\partial(\sqrt{\gamma}\varkappa_k^i)}{\partial t} + P_k^i = (\lambda + \chi f' - f)\delta_k^i \tag{16}$$

Variation of the action with respect to ϕ gives

$$\frac{1}{\sqrt{\gamma}}\partial_0(2\sqrt{\gamma}\lambda) = \Box f' = \frac{1}{\sqrt{\gamma}}\partial_0(\sqrt{\gamma}f''\dot{\chi}) - \Delta f', \tag{17}$$

where $\Delta f'$ is the covariant Laplacian of f' for the metric γ_{ik} and this equation can be used to determine the Lagrange multiplier λ. Up to this point we did not make any assumptions as regards to the metric γ_{ik}. However, for our purposes it will be enough to consider only the case when the determinant of the metric is factorizable, that is, $\gamma(t, x^i) = \gamma_1(t)\gamma_2(x^i)$.

Then both χ and $\varkappa$ depend only on time and $\Delta f'$ vanishes; hence integrating (17) we obtain

$$\lambda = \frac{C}{2\sqrt{\gamma}} + \frac{1}{2}f''\dot{\chi}. \tag{18}$$

where C is a constant of integration corresponding to mimetic cold matter. Because this matter behaves exactly like dust we can neglect it for the reasons explained above and set $C = 0$. By subtracting from Eq. (16) one third of its trace we find

$$\frac{\partial}{\partial t}\left(\sqrt{\gamma}\left(\varkappa_k^i - \frac{1}{3}\varkappa\delta_k^i\right)\right) = -2\left(P_k^i - \frac{1}{3}P\delta_k^i\right)\sqrt{\gamma}, \tag{19}$$

from which it follows that

$$\varkappa_k^i = \frac{1}{3}\varkappa\delta_k^i + \frac{\lambda_k^i}{\sqrt{\gamma}}, \tag{20}$$

where

$$\lambda_k^i = -2\int\left(P_k^i - \frac{1}{3}P\delta_k^i\right)\sqrt{\gamma}\,\mathrm{d}t \tag{21}$$

and it is traceless, $\lambda_i^i = 0$. Substituting expression (20) together with (18) into (14) we obtain

$$\frac{1}{12}\varkappa^2 + f - \chi f' = \frac{\lambda_k^i\lambda_i^k}{8\gamma} - \frac{1}{2}P. \tag{22}$$

Taking into account that $\chi = \dot{\gamma}/2\gamma = \varkappa/2$ we infer that (22) is a first order non-linear differential equation for γ, which involves the separate components of the metric only via the spatial scalar curvature P. Substituting the function f from (8) into this equation leads to the particularly simple equation

$$\chi_m^2\left(1 - \sqrt{1 - \frac{2}{3}\frac{\chi^2}{\chi_m^2}}\right) = \varepsilon, \tag{23}$$

where

$$\varepsilon = \frac{\lambda_k^i\lambda_i^k}{8\gamma} - \frac{1}{2}P \tag{24}$$

does not depend on the time derivative of the metric. By squaring (23) and recalling that $\chi = \dot{\gamma}/2\gamma$ we finally arrive at the master equation

$$\frac{1}{12}\left(\frac{\dot{\gamma}}{\gamma}\right)^2 = \varepsilon\left(1 - \frac{\varepsilon}{\varepsilon_m}\right), \tag{25}$$

which will be used to analyze the black hole solution and where we have denoted $\varepsilon_m = 2\chi_m^2$.

3 Schwarzschild solution in general relativity and the boundary conditions for ϕ

In the empty spherically symmetrical space solution of Einstein's equations is unique and is given by the Schwarzschild metric

$$ds^2 = \left(1 - \frac{r_g}{r}\right)dt_S^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2 d\Omega^2, \tag{26}$$

where r_g is the gravitational radius and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the line element on the surface of unit sphere. This metric is regular both outside the black hole $r > r_g$ and inside the black hole for $r_g > r > 0$ and only becomes singular on the horizon at $r = r_g$. Since the singularity occurs "inside the black hole" it is enough for us to consider only the internal part of this black hole, where the metric (26) is well applicable and happens to be most convenient for analyzing the internal structure of a nonsingular black hole. For $r < r_g$ the coordinates r and t exchange their roles and r becomes a time-like coordinate while t_S becomes a space-like one. Inside the black hole the decrease of the "radial coordinate" from $r = r_g$ to $r = 0$ corresponds to time increase. Inversely, if we assume that time grows with r then the same Schwarzschild solution describes the white hole, which is just a time reversed black hole. Let us rename the coordinate in (26) as $r \to r_g\tau^2$ and $t_S \to R$. Then inside the Schwarzschild black hole the metric (26) becomes

$$ds^2 = 4r_g^2\tau^2 N^{-2}(\tau)d\tau^2 - N^2(\tau)dR^2 - \tau^4 r_g^2 d\Omega^2, \tag{27}$$

where

$$N^2(\tau) = \frac{1 - \tau^2}{\tau^2}, \tag{28}$$

and where for negative τ, changing to the interval $-1 \leq \tau \leq 0$ describes the collapse "inside" the black hole until the *space-like singularity* is reached *at the moment of time $\tau = 0$*. In fact the spacetime described by metric (27) is obviously non-static and the Riemann squared tensor equals

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{12}{(r_g\tau^3)^4} = \frac{12r_g^2}{r^6} \tag{29}$$

and blows up at the moment of time $\tau = 0$ or, as sometimes incorrectly stated, in the center of the black hole at $r = 0$. The Planck curvature is reached at the moment $|\tau| \simeq r_g^{-1/3}$ or at $r \simeq r_g^{1/3}$. Introducing the proper time

$$t = \int 2r_g\tau N^{-1}(\tau)d\tau = \int \frac{2r_g\tau^2}{\sqrt{1 - \tau^2}}d\tau$$
$$= r_g\left(\arcsin\tau - \tau\sqrt{1 - \tau^2}\right), \tag{30}$$

we can rewrite the metric (27) in the form

$$ds^2 = dt^2 - a^2(t)dR^2 - b^2(t)d\Omega^2, \tag{31}$$

where for the Schwarzschild black hole

$$a^2(t) = \frac{1 - \tau^2(t)}{\tau^2(t)}, \quad b^2(t) = \tau^4(t)r_g^2. \tag{32}$$

The coordinate system (31) is obviously synchronous and happens to be the one most convenient to find a nonsingular generalization of the Schwarzschild solution in the theory

with limiting curvature. Therefore we will use metric (31) and determine the functions $a^2(t)$ and $b^2(t)$, which will be modified in the vicinity of the singularity compared to (32).

First of all we notice that at $\chi^2 \ll \chi_m^2 = \varepsilon_m/2$ our theory coincides with General Relativity in the leading order and therefore the functions given in (32) satisfy Eq. (25) until we start to approach the limiting curvature. To determine where the Schwarzschild solution must be valid let us assume that

$$\phi = t + A \tag{33}$$

and calculate

$$\chi = \Box\phi = \frac{\dot{\gamma}}{2\gamma} = \frac{1}{2}\frac{d\ln(a^2b^4)}{dt}$$
$$= \frac{\sqrt{1 - \tau^2}}{4r_g\tau^2}\frac{d\ln((1 - \tau^2)\tau^6 r_g^4)}{d\tau} = \frac{3 - 4\tau^2}{2r_g\tau^3\sqrt{1 - \tau^2}}. \tag{34}$$

It then follows that for $1 > |\tau| > (\varepsilon_m r_g^2)^{-1/6}$ we have $\chi^2 \ll \varepsilon_m$ and the Schwarzschild metric is a good approximation of the exact solution in (25). However, one may immediately notice that in the near horizon region (for $\tau^2 \to 1$) χ^2 grows unbounded for the Schwarzschild solution although the horizon is nothing more than a coordinate singularity. It seems that the curvature must grow giving rise to a "firewall" in our theory, thus completely modifying the Schwarzschild solution, even for large black holes. However, this "firewall" is completely fake and its appearance is related to taking the wrong solution (33) for ϕ which corresponds to an unjustified "concentration" of this field in the near horizon region that significantly changes the Schwarzschild solution even outside the Schwarzschild radius. We have noted above that the solution (33) is a generic solution, *but only* if the synchronous coordinate system has *no coordinate singularities*. Obviously the coordinate system (31) does not satisfy this requirement because $\gamma = (1 - \tau^2)\tau^6 r_g^4$ vanishes as $\tau^2 \to 1$.

To find the synchronous coordinate system which is free of fictitious coordinate singularities we make a coordinate transformation introducing instead of t and R, the new coordinates T and $\bar{R}$ defined by

$$T = R + \int \frac{\sqrt{1 + a^2}}{a}dt, \quad \bar{R} = R + \int \frac{dt}{a\sqrt{1 + a^2}}. \tag{35}$$

Then in the new *synchronous coordinates* the metric (31) becomes

$$ds^2 = dT^2 - (1 + a^2)d\bar{R}^2 - b^2 d\Omega^2, \tag{36}$$

where a^2 and b^2 are now functions which depend on the argument $T - \bar{R}$. For the Schwarzschild solution (32) this metric takes the form

$$ds^2 = dT^2 - \tau^{-2}d\bar{R}^2 - \tau^4 r_g^2 d\Omega^2, \tag{37}$$

where the relation between τ and $T - \bar{R}$ can be found by substituting (32) in (35) and taking into account (30):

$$T - \bar{R} = \frac{2}{3} r_g \tau^3. \tag{38}$$

This metric describes the Schwarzschild solution in the Lemaitre coordinate system which is synchronous, regular on the horizon and covers both external and internal parts of the black hole. Therefore, instead of (33), the solution for ϕ with correct asymptotic behavior far away from the black hole is given by

$$\phi = T = R + \int \frac{\sqrt{1 + a^2}}{a} \mathrm{d}t. \tag{39}$$

Although the Lemaitre coordinates cover the whole manifold and have no coordinate singularities, they are not very convenient for investigating the internal structure of nonsingular black holes because the metric components depend on both space and time coordinates in non separable way. This leads to equations which have a very complicated structure due to the spatial curvature terms. Therefore we continue to work in the coordinate system (31) but taking the correct solution for ϕ. It is easy to see that (39) satisfies the constraint (2) for an arbitrary $a(t)$ as it should. Calculating χ for solution (39) in the coordinate system (31) we find that it is not equal to $\dot{\gamma}/2\gamma$ anymore and is now given by

$$\chi = \Box\phi = \frac{\dot{\gamma}}{2\gamma}\sqrt{1 + \frac{1}{a^2}} + \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{1 + \frac{1}{a^2}}. \tag{40}$$

In the case of Schwarzschild black hole we obtain

$$\chi = \frac{3}{2r_g\tau^3}, \tag{41}$$

and on the horizon we have $\chi^2 \ll \varepsilon_m$ for $r_g \gg \varepsilon_m^{-1/2}$. Thus for large black holes corrections to Einstein's equations are negligible on the horizon and the fake firewall does not arise. Only for very small black holes with a minimal mass determined by the limiting curvature, the Schwarzschild solution will be completely modified in our theory. The result is not surprising because in this case the limiting curvature is already reached on the horizon. Notice that in the case of large black holes, away from the horizon, $a^2 \propto \tau^{-2}$ and as we will see it continues to grow after the bounce in a nonsingular black hole. Therefore the terms with $1/a^2$ in (40) can be neglected once we are far enough from the original horizon and later. This can be seen by comparing, for instance, (41) with (34) which coincides to order $O(\tau^2)$ for $\tau^2 \ll 1$. Hence, with good accuracy we can set

$$\chi = \frac{\dot{\gamma}}{2\gamma}, \tag{42}$$

and use (25) to investigate the future of a nonsingular black hole. If this approximation fails, we would need to work directly with Eq. (22) with χ given in (40). Fortunately, as we

will show, the approximation holds very well and improves with time and therefore, we can avoid extremely messy calculations which would be needed otherwise.

Finally, to complete this section we would like to give the approximate explicit leading order expression for the Schwarzschild metric entirely in terms of time t, in the near horizon and close to singularity regions. As we will see this metric will be helpful to understand what happens within the black hole after reaching the limiting curvature and the bounce.

As seen above, the internal part of the singular black hole is described by metric (31), (32) for $-1 < \tau < 0$. According to (30) the proper time t runs in the interval $-\pi/2 < t < 0$. Consider the near horizon region corresponding to $1 + \tau \ll 1$. Then, as follows from (30),

$$1 + \tau \simeq \frac{1}{8}\left(\frac{\pi}{2} + \frac{t}{r_g}\right)^2 \equiv \frac{1}{8}\left(\frac{\bar{t}}{r_g}\right)^2, \tag{43}$$

and, in this approximation, the metric takes the form

$$\mathrm{d}s^2 = \mathrm{d}\bar{t}^2 - \frac{1}{4}\left(\frac{\bar{t}}{r_g}\right)^2 \mathrm{d}R^2 - r_g^2 \mathrm{d}\Omega^2, \tag{44}$$

in the near horizon region for $\bar{t} \ll r_g$. Notice that the numerical coefficient in front of $\mathrm{d}R^2$ has no physical meaning because it can be rescaled by $R \to \alpha R$. On the other hand, the coefficient in front of the angular part of the metric cannot be rescaled and determines the spatial curvature in the near horizon region which gives a contribution of order $1/r_g^4$ to the Riemann squared curvature. For large black holes the curvature on the horizon is rather small. Now we turn to the region close to the singularity $|\tau| \ll 1$, where

$$t \simeq \frac{2}{3} r_g \tau^3 \tag{45}$$

and the metric (31), (32) becomes

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \left(\frac{t}{t_0}\right)^{-2/3} \mathrm{d}R^2 - \left(\frac{t}{t_0}\right)^{4/3} r_g^2 \mathrm{d}\Omega^2, \tag{46}$$

where $t_0 = 2r_g/3$. As one can see from (41) the limiting curvature is reached when at $t \sim r_g \tau^3 \sim -\varepsilon_m^{-1/2}$ so that χ^2 becomes of order ε_m and $R_{\alpha\beta\gamma\delta}^2 \sim \varepsilon_m^2$ (see (29)). Before that, the Schwarzschild solution is a good approximation of the exact solution in the theory with limiting curvature. Considering the asymptotic expressions (44) and (46) we can view the evolution of the internal part of the black hole as a change of one Kasner solution (44) with $p_i = (1, 0, 0)$ in the near horizon region to the other Kasner solution (46) with $p_i' = 2/3 - p_i$, close to the singularity region [9]. This change happens around $t \sim O(1)r_g$ and is due to the spatial curvature term which, as we will see shortly, is only important in this region between the two asymptotics.

4 Black hole with limiting curvature

When the limiting curvature is reached, General Relativity is no longer valid, and the Schwarzschild solution is modified. To find how, and to see what happens when we approach the limiting curvature and beyond, we have to solve Eq. (25), which we quote again for convenience of the reader

$$\frac{1}{12}\left(\frac{\dot{\gamma}}{\gamma}\right)^2 = \varepsilon\left(1 - \frac{\varepsilon}{\varepsilon_m}\right), \tag{47}$$

where we now have

$$\varepsilon = \frac{\lambda_k^i \lambda_i^k}{8\gamma} - \frac{1}{2}P \tag{48}$$

and

$$\frac{\lambda_k^i}{\sqrt{\gamma}} = -\frac{2}{\sqrt{\gamma}}\int\left(P_k^i - \frac{1}{3}P\delta_k^i\right)\sqrt{\gamma}dt. \tag{49}$$

One can easily check that as $\varepsilon_m \to \infty$ the Schwarzschild solution is the exact solution of these equations. The spatial curvature components for the metric (31) are

$$P_1^1 = 0, \quad P_2^2 = P_3^3 = \frac{1}{b^2}, \quad P = \frac{2}{b^2}, \tag{50}$$

and therefore

$$\frac{\lambda_k^i}{\sqrt{\gamma}} = -\frac{2\tilde{\lambda}_{(i)}\delta_k^i}{ab^2}F(t), \quad F(t) = \int a dt, \tag{51}$$

where

$$\tilde{\lambda}_{(i)} = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right). \tag{52}$$

To determine the constant of integration in $F(t)$ consider the times t satisfying

$$|t| \gg \varepsilon_m^{-1/2},$$

for which the Schwarzschild solution is valid in the leading approximation. Then using (32) for a and (30) to express $dt/d\tau$, we find

$$\int a dt = \int a(\tau)\frac{dt}{d\tau}d\tau = r_g\tau^2 + C \tag{53}$$

and the constant of integration C can be found from Eq. (20) for $\varkappa_1^1$,

$$\varkappa_1^1 = \frac{1}{3}\varkappa + \frac{\lambda_1^1}{\sqrt{\gamma}} = \frac{1}{3}\varkappa + \frac{4(r_g\tau^2 + C)}{3ab^2}. \tag{54}$$

In fact, taking into account that

$$\varkappa_1^1 = \gamma^{11}\dot{\gamma}_{11} = \frac{d\ln a^2}{dt}, \quad \varkappa = \frac{d\ln(a^2b^4)}{dt} \tag{55}$$

and replacing d/dt by the derivative with respect to τ, Eq. (55) simplifies to

$$\frac{\sqrt{1-\tau^2}}{2r_g\tau^2}\frac{d\ln(a/b)}{d\tau} = \frac{(r_g\tau^2 + C)}{ab^2}. \tag{56}$$

Substituting for a and b from (32) and comparing, we find that

$$C = -\frac{3}{2}r_g, \tag{57}$$

and hence

$$\frac{\lambda_k^i}{\sqrt{\gamma}} = \frac{2\tilde{\lambda}_{(i)}\delta_k^i r_g}{ab^2}\left(\frac{3}{2} - \tau^2\right). \tag{58}$$

This expression, which we derived in the region where Einstein theory is applicable, can also be used "deep inside the black hole" for $\tau^2 \ll 1$ if we neglect the τ^2 term inside the brackets

$$\frac{\lambda_k^i}{\sqrt{\gamma}} = \frac{3\tilde{\lambda}_{(i)}\delta_k^i r_g}{ab^2}. \tag{59}$$

Substituting this expression in (48) and using (50) for the spatial curvature term we finally obtain

$$\varepsilon = \frac{3r_g^2}{4a^2b^4} - \frac{1}{b^2}. \tag{60}$$

It is clear that for

$$a^2b^2 \ll r_g^2, \tag{61}$$

the spatial curvature term can be neglected. For instance, for the Schwarzschild black hole this condition takes the form

$$(1 - \tau^2)\tau^2 \ll 1 \tag{62}$$

and hence deep inside the black hole ($\tau^2 \ll 1$) and close to the horizon ($(1 - \tau^2) \ll 1$) the spatial curvature term in (60) is negligible. Thus ignoring this term and taking into account that $\gamma = a^2b^4\sin^2\theta = \gamma_t\sin^2\theta$, hence, $\dot{\gamma}/\gamma = \dot{\gamma}_t/\gamma_t$ and after substitution of (60) in (47) we obtain the equation

$$\left(\frac{\dot{\gamma}_t}{\gamma_t}\right)^2 = \frac{9r_g^2}{\gamma_t}\left(1 - \frac{3r_g^2}{4\varepsilon_m\gamma_t}\right), \tag{63}$$

which can easily be integrated to give the solution

$$\gamma_t = \frac{3r_g^2}{4\varepsilon_m}(1 + 3\varepsilon_m t^2). \tag{64}$$

The corresponding metric components $a^2(t)$ and $b^2(t)$ can be obtained directly from (20). For instance the equation for $\varkappa_1^1$ takes the following explicit form:

$$\frac{d\ln a^2}{dt} = \frac{1}{3}\frac{\dot{\gamma}_t}{\gamma_t} + \frac{2r_g}{\sqrt{\gamma_t}}. \tag{65}$$

Integrating this equation, using γ_t from (64), we find

$$a^2(t) = \left(\frac{3r_g^2}{4\varepsilon_m}(1+3\varepsilon_m t^2)\right)^{1/3} \exp\left(\frac{4}{3}\left(\sinh^{-1}\left(\sqrt{3\varepsilon_m}t\right)\right.\right.$$
$$\left.\left. + \ln\left(\frac{4}{3}\sqrt{3\varepsilon_m}\right)\right)\right), \tag{66}$$

where the constant of integration is fixed by requiring that before the bounce for $|t| \gg \varepsilon_m^{-1/2}$ the asymptotic form of the solution must be given by (46). Similarly we obtain

$$b^2(t) = \left(\frac{3r_g^2}{4\varepsilon_m}(1+3\varepsilon_m t^2)\right)^{1/3} \exp\left(-\frac{2}{3}\left(\sinh^{-1}\left(\sqrt{3\varepsilon_m}t\right)\right.\right.$$
$$\left.\left. + \ln\left(\frac{4}{3}\sqrt{3\varepsilon_m}\right)\right)\right). \tag{67}$$

Thus, the singularity is avoided and instead of it we have a bounce of duration $\Delta t \simeq \varepsilon_m^{-1/2}$. During this time the curvature is not very different from the limiting curvature but drastically drops after that. In fact, as follows from (66) and (67), after the bounce for $t \gg \varepsilon_m^{-1/2}$ the metric is

$$ds^2 = dt^2 - Q_0^2\left(\frac{t}{t_0}\right)^2 dR^2 - \frac{1}{Q_0}r_g^2 d\Omega^2, \tag{68}$$

where $Q_0 = (\frac{16}{3}\varepsilon_m r_g^2)^{2/3}$. If the size of the black hole r_g is much larger than $\varepsilon_m^{-1/2}$ then $Q_0 \gg 1$. The asymptotic form (68) is valid only when the spatial curvature could be neglected and the condition (61) is satisfied. It holds until the time $t \sim r_g/Q_0^{1/2}$ where it is violated. For $t \gg \varepsilon_m^{-1/2}$ we have $\chi^2 \ll \varepsilon_m$. Moreover, using the formulas from the appendix it can be readily checked that for the solution (68)

$$R_{\alpha\beta\gamma\delta}^2 \sim \frac{Q_0^2}{r_g^4} \sim \left(\frac{\varepsilon_m}{r_g}\right)^{4/3} \tag{69}$$

and it follows that $R_{\alpha\beta\gamma\delta}^2 \ll \varepsilon_m^2$ for large black holes with $r_g \gg \varepsilon_m^{-1/2}$. Hence, at these times the corrections to Einstein equations are negligible and (68) must be a solution of Einstein equations in empty space for a spherically symmetric metric. We know, however, that such a solution is unique and is described by the Schwarzschild metric. In fact, rescaling $R \rightarrow \tilde{R} = 3Q_0^{1/2}R$ and introducing

$$R_{g_1} = \frac{r_g}{Q_0^{1/2}} = \frac{r_g^{1/3}}{(16\varepsilon_m/3)^{1/3}}, \tag{70}$$

we can rewrite (68) as

$$ds^2 = dt^2 - \frac{1}{4}\left(\frac{t}{R_{g_1}}\right)^2 d\tilde{R}^2 - R_{g_1}^2 d\Omega^2. \tag{71}$$

Comparing this metric to (44) we can identify its spacetime with the inner side of the near horizon asymptotic of the Schwarzschild solution with the gravitational radius $R_{g_1} \propto r_g^{1/3}$. As pointed out above, at the moment of time

$t \sim R_{g_1}$ the spatial curvature term in (60) becomes dominant and changes the asymptotic solution (71) to another one which can be written by analogy with (46). We simply take into account that in the corresponding Schwarzschild black hole with radius R_{g_1} the singularity would be reached at $t = \frac{\pi}{2}R_{g_1}$ and we can write

$$ds^2 = dt^2 - \left(\frac{t - \frac{\pi}{2}R_{g_1}}{t_1}\right)^{-2/3} d\tilde{R}^2$$
$$- \left(\frac{t - \frac{\pi}{2}R_{g_1}}{t_1}\right)^{4/3} R_{g_1}^2 d\Omega^2, \tag{72}$$

where $t_1 = \frac{2}{3}R_{g_1}$. This solution is valid until the limiting curvature is reached, that is, for $\frac{\pi}{2}R_{g_1} - t \gg \varepsilon_m^{-1/2}$. After we start to approach the limiting curvature the solution changes and it is described by the formulas (66) and (67) with the obvious replacements $r_g \rightarrow R_{g_1}$, $t - \frac{\pi}{2}R_{g_1}$. To return to the original scale factor $a^2(t)$ we rescale $\tilde{R}$ back to R. As a result, after *the second bounce*, we again re-emerge inside the near horizon region described by the metric

$$ds^2 = dt^2 - 9Q_0Q_1^2\left(\frac{t - \frac{\pi}{2}R_{g_1}}{t_1}\right)^2 dR^2 - \frac{1}{Q_1}R_{g_1}^2 d\Omega^2, \tag{73}$$

for $t - \frac{\pi}{2}R_{g_1} \gg \varepsilon_m^{-1/2}$, where

$$Q_1 = \left(\frac{16}{3}\varepsilon_m R_{g_1}^2\right)^{2/3} = \left(\frac{16}{3}\varepsilon_m \frac{r_g^2}{Q_0}\right)^{2/3} = Q_0^{1/3}. \tag{74}$$

Obviously, the metric (73) describes the near horizon Schwarzschild geometry with the gravitational radius

$$R_{g_2} = \frac{R_{g_1}}{Q_1^{1/2}} = \frac{r_g}{Q_0^{\frac{1}{2}\left(1+\frac{1}{3}\right)}} = \frac{r_g^{1/9}}{(16\varepsilon_m/3)^{4/9}}. \tag{75}$$

Repeating the steps above we find that the spacetime structure inside the nonsingular black hole is similar to "a Russian nesting doll". Namely, its geometry is a time sequence of the internal Schwarzschild geometries separated by "layers with limiting curvature" of width $\Delta t \simeq \varepsilon_m^{-1/2}$. The Schwarzschild radii characterizing these subsequent geometries decrease and are proportional to r_g, $r_g^{1/3}$, $r_g^{1/9}$, $r_g^{1/27}$, ... After the $n+1$th bounce (the first bounce takes place at $t = 0$), which happens at the moment

$$T_n = \frac{\pi}{2}(R_{g_1} + R_{g_2} + \cdots + R_{g_n}), \tag{76}$$

the gravitational radius is equal to

$$R_{g(n+1)} = \frac{R_{g(n)}}{Q_n^{1/2}} = \frac{r_g}{Q_0^{\frac{1}{2}(1+\frac{1}{3}+\cdots+\frac{1}{3^n})}}$$
$$= \frac{1}{(16\varepsilon_m/3)^{1/2}}\exp\left(\frac{1}{4\cdot 3^n}\ln Q_0\right), \tag{77}$$

and when the gravitational radius becomes comparable with the minimal possible one

$$R_{g_{\min}} = \frac{1}{(16\varepsilon_m/3)^{1/2}}, \qquad (78)$$

the approximations we used to obtain the picture described above breaks down. In fact, after

$$n_{\max} \sim \ln \ln Q_0 \qquad (79)$$

bounces the width of the layers with limiting curvature is of the order of the size of the black hole and we cannot use anymore the Schwarzschild solution in between the layers. After that the limiting curvature is reached and never drops to small values. The corresponding geometry is similar to the one which describes the minimal black hole in our theory [14].

5 Summary and speculations

We have shown that in the theory with limiting curvature the internal structure of a black hole is significantly modified compared to a singular Schwarzschild black hole. Namely, the curious observer who decides to travel inside the Schwarzschild eternal black hole after first crossing the horizon will find himself in a non-static space of infinite volume (for eternal black hole), but exists for finite time $t \sim r_g$. At the beginning the curvature of large black holes is very low but grows and finally, after time $t \sim r_g$, becomes infinite and one ends up in a singularity, which happens not "at the point in the center of black hole" but at the moment of time $t = 0$. In this sense the evolution and singularity within a black hole is similar to a Kasner universe. The spacetime in this case is not geodesically complete. In our theory with limiting curvature, the Einstein equations are only significantly modified when the curvature starts to approach its limiting value. The singularity is removed and the curvature does not grow indefinitely. In fact, the singularity is replaced by a "time layer" of duration $\Delta t \sim \varepsilon_m^{-1/2}$, which would be of the order of Planck time if the limiting curvature would be the Planckian one. After that the curvature drops down to the value which an observer would find immediately after crossing the horizon of the smaller black hole of radius $r_g^{1/3}$. The subsequent evolution repeats the previous cycle but this time inside a black hole of this smaller radius. Once again, instead of ending at the singularity we pass through a layer of limiting curvature and find ourselves inside a black hole of even smaller radius $\sim r_g^{1/9}$ and so on. Finally when the size of the black hole becomes of the order of the width of a time layer $\sim \varepsilon_m^{-1/2}$, we end inside the black hole of minimal possible mass and stay there forever at limiting curvature. Notice that the number of the "layers" which we have to pass to reach inside this minimal black hole is not big even for large black holes. For instance, for a galactic mass black hole of radius $r_g \sim 10^{49}$ (in Planck units) after the crossing of limiting curvature we find ourselves in black holes of radii $r_g^{1/3} \sim 10^{16}$, $r_g^{1/9} \sim 10^5$, $r_g^{1/27} \sim 10^2$ correspondingly. Finally at the fourth layer $r_g^{1/81} \sim O(1)$, and we cannot trust anymore the approximations used to arrive at the above picture and we end up within a minimal black hole at limiting curvature, which after that never drops significantly. The spacetime of a nonsingular black hole is geodesically complete and the singularity problem is resolved.

For an evaporating black hole the derivation of Hawking radiation remains unchanged for a large black hole [13]. However, when it reaches the minimal size of order $\varepsilon_m^{-1/2}$ the near horizon geometry changes and we expect that the minimal remnants of it must be stable. This question obviously requires further investigation [14]. If we take the limiting curvature, which is a free parameter in our theory, to be at least a few orders of magnitude below the Planck scale, the answer to it can be obtained using standard methods of quantum field theory in external gravitational field. In fact, in this case the unknown nonperturbative quantum gravity does not play an essential role and its need in such a case becomes unclear because the uncontrollable Planckian curvatures are never reached. This opens up the possibility of resolving the information paradox without involving the "mysteriously imprinted" correlations in Hawking radiation which is supposed to take care of returning all information back to the Minkowski space after the disappearance of the black hole. In our case the smallest black hole remnant has enough space "inside it" to hide all the information as regards to the original matter from which the black hole was formed together with the information as regards to the negative energy quanta (with respect to an outside observer) which never escapes from the black hole and reduce its mass in the process of Hawking evaporation. The evolution in this case remains unitary on complete Cauchy hypersurfaces which inevitably goes inside the black hole remnant. The picture here is very similar to the one described as a possible option in [2]. The content of the minimal mass black hole can be significantly different depending on the way how the remnant was formed. However, an infinite degeneracy of the black hole remnants is completely irrelevant for an outside observer who calculates, for instance, the scattering processes with participation of these minimal black holes, because this degeneracy is entirely related to events which happen in the absolute future of this observer.

6 Appendix

For convenience of the reader we quote below the explicit expressions for curvature invariants which can be used to

verify statements about the behavior of the curvature in a nonsingular black hole with the metric (31). The scalar curvature is given by the expression

$$R = -\dot{\varkappa} - \frac{1}{3}\varkappa^2 - \frac{2}{3}\frac{F^2}{\gamma} - \frac{2}{b^2},$$

where $F = \int a \mathrm{d}t$. The square of the Ricci tensor is given by

$$R_{\alpha\beta}R^{\alpha\beta} = \frac{1}{3}\dot{\varkappa}^2 + \frac{1}{6}\varkappa^2\dot{\varkappa} + \frac{1}{36}\varkappa^4 + \frac{2}{3}\left(\dot{\varkappa} + \frac{1}{6}\varkappa^2\right)\frac{F^2}{\gamma}$$
$$+ \frac{4}{9}\frac{F^4}{\gamma^2} + \frac{1}{b^2}\left(\frac{2}{3}\dot{\varkappa} + \frac{1}{3}\varkappa^2\right) + \frac{4a^2}{3\gamma}$$

and the square of the Riemann tensor is

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \left(\frac{\varkappa^4}{54} + \frac{\dot{\varkappa}^2}{3} + \frac{\varkappa^2\dot{\varkappa}}{9}\right) + \frac{2}{9}\left(4\dot{\varkappa} + \varkappa^2\right)\frac{F^2}{\gamma}$$
$$- \frac{16\varkappa}{27}\frac{F^3}{\gamma^{\frac{3}{2}}} + \frac{4F^4}{3\gamma^2} + \frac{8a^2}{3\gamma} + \frac{16a}{9}\frac{F^2}{\gamma^{\frac{3}{2}}}$$
$$- \frac{8\varkappa a F}{9\gamma} + \frac{1}{b^2}\left(\frac{4}{b^2} + \frac{2\varkappa^2}{9} + \frac{8F^2}{9\gamma} - \frac{8\varkappa F}{9\sqrt{\gamma}}\right).$$

References

1. S.W. Hawking, R. Penrose, The singularities of gravitational collapse and cosmology. Proc. R. Soc. Lond. A **314**, 529 (1970)
2. V. Frolov, M. Markov, V. Mukhanov, Through a black hole into a new universe? Phys. Lett. B **216**, 272 (1989)
3. M. Markov, Pis'ma Zh. Eskp. Teor. Fiz. **36**, 214 (1982)
4. M. Markov, Pis'ma Zh. Eskp. Teor. Fiz. **46**, 341 (1987)
5. M. Markov, JETP Lett. **36**, 265 (1982)
6. M. Markov, JETP Lett. **46**, 431 (1987)
7. V. Mukhanov, R. Brandenberger, A nonsingular universe. Phys. Rev. Lett. **68**, 1969 (1992)
8. V. Mukhanov, R. Brandenberger, A. Sornborger, A cosmological theory without singularities. Phys. Rev. D **48**, 1629 (1993)
9. A. Chamseddine, V. Mukhanov, Resolving cosmological singularities. JCAP **1703**(03), 009 (2017)
10. A.H. Chamseddine, V. Mukhanov, Mimetic dark matter. JHEP **1311**, 135 (2013)
11. A.H. Chamseddine, V. Mukhanov, A. Vikman, Cosmology with mimetic matter. JCAP **1406**, 017 (2014)
12. L. Landau, E. Lifshitz, *The Classical Theory of Fields*, 4th edn. (Butterworth, Heinemann, 1980)
13. G.W. Gibbons, S.W. Hawking, Cosmological event horizons, thermodynamics and particle creation. Phys. Rev. D **15**, 2738 (1977)
14. A.H. Chamseddine, V. Mukhanov, In preparation