THE EUROPEAN
PHYSICAL JOURNAL C

CrossMark

Holographic free energy and thermodynamic geometry

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Abstract We obtain the free energy and thermodynamic geometry of holographic superconductors in $2 + 1$ dimensions. The gravitational theory in the bulk dual to this $2 + 1$-dimensional strongly coupled theory lives in the $3+1$ dimensions and is that of a charged AdS black hole together with a massive charged scalar field. The matching method is applied to obtain the nature of the fields near the horizon using which the holographic free energy is computed through the gauge/gravity duality. The critical temperature is obtained for a set of values of the matching point of the near horizon and the boundary behaviour of the fields in the probe limit approximation which neglects the back reaction of the matter fields on the background spacetime geometry. The thermodynamic geometry is then computed from the free energy of the boundary theory. From the divergence of the thermodynamic scalar curvature, the critical temperature is obtained once again. We then compare this result for the critical temperature with that obtained from the matching method.

1 Introduction

There has been an immense amount of interest in studying strongly coupled systems using one of the most fascinating developments in modern theoretical physics, the gauge/gravity correspondence. The correspondence [1–4] gives an exact duality between a gravity theory living in a $(d + 1)$-dimensional AdS spacetime with a field theory having conformal invariance sitting on the d-dimensional boundary of this spacetime. In recent years, it was first demonstrated in [5] that the formation of a scalar field condensate near the horizon of a black hole is possible below a certain critical temperature for a charged AdS black hole coupled mini-

mally to a complex scalar field. The gauge/gravity duality then implies that the scalar operator dual to the scalar field in the bulk acquires a non-zero vacuum expectation value in the boundary field theory. This is what is known as the holographic superconductor phase transition [6–8]. Thereafter, a lot of work has been done to study the properties of the holographic Fermi liquid [9,10], holographic insulator/superconductor phase transition [11], transport properties of holographic superconductor both in the probe limit and away from the probe limit, that is, including the effects of back reaction of matter fields on the background spacetime [12–18].

Another interesting development that has taken place recently is the association of a geometrical structure with thermodynamic systems in equilibrium. This was first realized through the work in [19–21]. It was shown that one can get a Riemannian metric with an Euclidean signature from the equilibrium state of a thermodynamic system. The Riemannian scalar curvature can then be computed and captures the details of interactions of the thermodynamic system. It turns out that this framework based on a geometrical structure gives a handle to study critical phenomena [21].

In this paper, we set out to study the properties of $2 + 1$-dimensional holographic superconductors using the formalism of the thermodynamic geometry. We employ the matching method [16,22] to obtain the behaviour of the matter fields near the horizon of the black hole. This in turn is used to compute the critical temperature and the condensation operator. We obtain the critical temperature for a set of values of the matching point where the near horizon and boundary values of the fields are matched. The analysis is based on the probe limit approximation (which neglects the back reaction of the matter fields on the background spacetime geometry) and is carried out for two sets of boundary conditions for the condensation operator, namely $\psi_- \neq 0$, $\psi_+ = 0$ and $\psi_+ \neq 0$, $\psi_- = 0$. The choice $\psi_+ = 0$ essentially implies that ψ_- is dual to the expectation value of the condensation operator at

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the boundary. The requirement $\psi_+ = 0$ is necessary because we want to turn the condensate on without it being sourced. We then proceed to compute the free energy of this $2 + 1$-dimensional holographic superconductor. The trick here is to relate the free energy of the theory on the boundary to the value of the on-shell action of the Abelian Higgs sector of the full Euclidean action with proper boundary terms [23,24]. From this, we compute the thermodynamic metric using the formalism of [21]. The computation is once again carried out for both sets of boundary conditions for the condensation operator as mentioned earlier. The scalar curvature is computed next and the temperature at which the scalar curvature diverges is said to be the critical temperature in this approach. This temperature is then compared with that obtained from the matching method. The analysis gives us yet another way of comparing the results with those obtained from other analytical techniques, namely, the Sturm–Liouville eigenvalue method [25–30] and the matching method [15,16,22,31].

This paper is organized as follows. In Sect. 2, we provide the basic set up for the holographic superconductors in the background of a $3 + 1$-dimensional electrically charged black hole in anti-de Sitter spacetime. In Sect. 3, using the probe limit approximation, we compute the critical temperature using the matching method, where the matching has been carried out at several points between the boundary and the horizon. In Sect. 4, we analytically obtain free energy expression in terms of the chemical potential and the charge density. In Sect. 5, we calculate the thermodynamic metric and the scalar curvature. We conclude finally in Sect. 6.

2 Basic set up

The model for a holographic superconductor needs a complex scalar field coupled to a $U(1)$ gauge field in anti-de Sitter spacetime. In $3 + 1$ dimensions, the action for this reads

$$
S = \int d^4x \sqrt{-g}
$$
$$
\times \left[\frac{1}{2\kappa^2}(R - 2\Lambda) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - (D_\mu\psi)^*D^\mu\psi - m^2\psi^*\psi \right]
\tag{1}
$$

where $\Lambda = -\frac{3}{L^2}$ is the cosmological constant, $\kappa^2 = 8\pi G$, G being the Newton universal gravitational constant, A_μ and ψ represent the gauge field and scalar fields, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$; ($\mu, \nu = 0, 1, 2, 3, 4$) is the field strength tensor, $D_\mu\psi = \partial_\mu\psi - iqA_\mu\psi$ is the covariant derivative.

Assuming that the plane-symmetric black hole metric can be put in the form

$$
ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(dx^2 + dy^2)
\tag{2}
$$

where $f(r) = r^2(1 - \frac{r_+^3}{r^3})$ and r_+ is the horizon radius and making the ansatz for the gauge field and the scalar field to be [6]

$$
A_\mu = (\phi(r), 0, 0, 0), \quad \psi = \psi(r),
\tag{3}
$$

leads to the following equations of motion for the matter fields [30]:

$$
\phi''(r) + \frac{2}{r}\phi'(r) - \frac{2q^2\psi^2(r)\phi(r)}{f(r)} = 0
\tag{4}
$$
$$
\psi''(r) + \left(\frac{2}{r} + \frac{f'(r)}{f(r)}\right)\psi'(r) + \left(\frac{q^2\phi^2(r)}{f(r)^2} - \frac{m^2}{f(r)}\right)\psi(r)
$$
$$
= 0
\tag{5}
$$

where prime denotes derivative with respect to r. Since we shall carry out our analysis in the probe limit, we are allowed to set $q = 1$ [13].

The Hawking temperature of this black hole is

$$
T_h = \frac{f'(r_+)}{4\pi} = \frac{3}{4\pi}r_+.
\tag{6}
$$

This is interpreted as the temperature of the conformal field theory on the boundary.

For the matter fields to be regular, one requires $\phi(r_+) = 0$ and $\psi(r_+)$ to be finite at the horizon.

Near the boundary of the bulk, the matter fields obey [16]

$$
\phi_b(r) = \mu - \frac{\rho}{r},
\tag{7}
$$
$$
\psi_b(r) = \frac{\psi_-}{r^{\Delta_-}} + \frac{\psi_+}{r^{\Delta_+}}
\tag{8}
$$

where the conformal dimension $\Delta_\pm$ is given by

$$
\Delta_\pm = \frac{3 \pm \sqrt{9 + 4m^2}}{2}.
\tag{9}
$$

The parameters μ and ρ are interpreted to be dual to the chemical potential and charge density of the conformal field theory on the boundary.

At this point, we make the change of coordinates $z = \frac{1}{r}$. Under this transformation, the metric (2) takes the form

$$
ds^2 = \frac{1}{z^2}\left(-F(z)dt^2 + \frac{dz^2}{F(z)} + dx^2 + dy^2\right),
$$
$$
f(z) = \frac{1}{z^2}\left(1 - \frac{z^3}{z_h^3}\right) = \frac{1}{z^2}F(z)
\tag{10}
$$

where $F(z) = (1 - \frac{z^3}{z_h^3})$ and $z_h = \frac{1}{r_+}$.

The Hawking temperature becomes $T_h = \frac{3}{4\pi z_h}$ and the field equations (4) and (5) become[1]

$$\phi''(z) - \frac{2\psi^2(z)}{z^2 F(z)} \phi(z) = 0 \qquad (11)$$

$$\psi''(z) + \left(\frac{F'(z)}{F(z)} - \frac{2}{z} \right) \psi'(z) + \left(\frac{\phi^2(z)}{F^2(z)} - \frac{m^2}{z^2 F(z)} \right) \psi(z)$$
$$= 0 \qquad (12)$$

where prime now denotes derivative with respect to z. In the next section we shall employ the matching method in the interval $(0, z_h)$ to obtain the critical temperature below which the scalar field condensation takes place. The boundary condition $\phi(r_+) = 0$ in z-coordinate translates to $\phi(z = z_h) = 0$. The asymptotic behaviour of the fields read

$$\phi_b(z) = \mu - \rho z, \qquad (13)$$

$$\psi_b(z) = \psi_- z^{\Delta_-} + \psi_+ z^{\Delta_+}. \qquad (14)$$

3 Critical temperature from matching method

To apply the matching method, we require the fields to be finite at the horizon. The Taylor series expansions of these fields near the horizon read

$$\phi_h(z) = \phi(z_h) + \phi'(z_h)(z - z_h) + \frac{\phi''(z_h)}{2}(z - z_h)^2 + \cdots, \qquad (15)$$

$$\psi_h(z) = \psi(z_h) + \psi'(z_h)(z - z_h) + \frac{\psi''(z_h)}{2}(z - z_h)^2 + \cdots \qquad (16)$$

To compute the undetermined coefficients, we use the boundary condition $\phi(z_h) = 0$ along with $f(z_h) = 0$ and Eqs. (11) and (12). This yields[2]

$$\phi''(z_h) = -\frac{2}{3z_h} \phi'(z_h) \psi^2(z_h), \qquad (17)$$

$$\psi'(z_h) = -\frac{m^2}{3z_h} \psi(z_h); \qquad$$

$$\psi''(z_h) = \frac{\psi(z_h)}{18z_h^2} [m^4 + 6m^2 - z^4 \phi'^2(z_h)]. \qquad (18)$$

In the rest of our analysis, we shall set $m^2 = -2$, which in turn implies $\Delta_- = 1$ and $\Delta_+ = 2$. This is consistent with

the Breitenlohner–Freedman bound [32,33]. Hence the near horizon expansions of these fields up to $\mathcal{O}(z^2)$ read

$$\phi_h(z) = \phi'(z_h) \left[(z - z_h) - \frac{\psi^2(z_h)}{3z_h}(z - z_h)^2 \right], \qquad (19)$$

$$\psi_h(z) = \psi(z_h) \left[1 + \frac{2}{3z_h}(z - z_h) - \frac{(8 + z_h^4 \phi'^2(z_h))}{36z_h^2}(z - z_h)^2 \right]. \qquad (20)$$

The matching method involves matching the near horizon expression of the fields with the asymptotic solution of these field at any arbitrary point between the horizon and the boundary, say $z = \frac{z_h}{2}$. In our analysis, we shall match the solution at $z = \frac{z_h}{\lambda}$, where λ lies between $[1, \infty]$. We shall later on set specific values of λ. The matching conditions are

$$\phi_h\left(\frac{z_h}{\lambda}\right) = \phi_b\left(\frac{z_h}{\lambda}\right), \quad \phi'_h\left(\frac{z_h}{\lambda}\right) = \phi'_b\left(\frac{z_h}{\lambda}\right), \qquad (21)$$

$$\psi_h\left(\frac{z_h}{\lambda}\right) = \psi_b\left(\frac{z_h}{\lambda}\right), \quad \psi'_h\left(\frac{z_h}{\lambda}\right) = \psi'_b\left(\frac{z_h}{\lambda}\right). \qquad (22)$$

From Eq. (21), we obtain the following relations:

$$\psi^2(z_h) = \frac{3\lambda}{1 - \lambda^2} \left(\frac{\mu}{z_h \phi'(z_h)} + 1 \right), \qquad (23)$$

$$\rho = \frac{\lambda}{z_h(\lambda + 1)} \left[2\mu + z_h \phi'(z_h) \left(1 - \frac{1}{\lambda} \right) \right]. \qquad (24)$$

Similarly, from Eq. (22), we get

$$\psi_{-/+} = \left[1 - \frac{(\lambda - 1)}{3\lambda} \frac{(3\lambda\Delta + 2)}{\Delta(\lambda - 1) + 2} \right] \left(\frac{\lambda}{z_h} \right)^\Delta \psi(z_h), \qquad (25)$$

$$\phi'^2(z_h) = \frac{1}{z_h^4} \left[\frac{12\lambda}{(\lambda - 1)} \frac{\lambda\Delta - 2(1 - \Delta)}{\Delta(\lambda - 1) + 2} - 8 \right]$$

$$\Rightarrow \phi'(z_h) = -\frac{\chi(\lambda, \Delta)}{z_h^2} \qquad (26)$$

where

$$\chi(\lambda, \Delta) = \sqrt{\frac{12\lambda}{(\lambda - 1)} \frac{\lambda\Delta - 2(1 - \Delta)}{\Delta(\lambda - 1) + 2} - 8}. \qquad (27)$$

In Eq. (25), ψ_- is for $\Delta = \Delta_- = 1$ and ψ_+ is for $\Delta = \Delta_+ = 2$. Note that we consider the negative sign before the square root of $\phi'(z_h)$ because $\phi'(z_h)$ is the electric field due to the charge of the black hole.

Substituting $\phi'(z_h)$ from Eq. (26) in Eqs. (23) and (24), we obtain

$$\psi(z_h) = \sqrt{\frac{3\lambda}{\lambda^2 - 1} \left(\frac{\mu z_h}{\chi(\lambda, \Delta)} - 1 \right)}, \qquad (28)$$

$$\rho = \frac{\lambda}{z_h(\lambda + 1)} \left[2\mu - \frac{\chi(\lambda, \Delta)}{z_h} \left(1 - \frac{1}{\lambda} \right) \right]. \qquad (29)$$

Using Eqs. (23)–(26) and $T = \frac{3}{4\pi z_h}$, we obtain the condensation operator and the critical temperature in terms of the

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[1] There is an error in the factor of $\frac{1}{2}$ written down in front of the matter part of the action in [24] as it does not lead to the equation of motion (11) for the field ϕ with the correct numerical factor.

[2] There are errors in sign in the undetermined coefficients in [24]. Also factors of z_h are missing.

chemical potential and the charge density

$$\langle \mathcal{O} \rangle = \gamma_{(\mu)} T_c^{\Delta} \left(1 - \frac{T}{T_c} \right)^{1/2} ; \quad T_c = \xi_{(\mu)} \mu, \tag{30}$$

$$\langle \mathcal{O} \rangle = \gamma_{(\rho)} T_c^{\Delta} \left(1 - \frac{T}{T_c} \right)^{1/2} ; \quad T_c = \xi_{(\rho)} \sqrt{\rho} \tag{31}$$

where

$$\gamma_{(\mu)} = \sqrt{\frac{3\lambda^2}{\lambda^2 - 1}} \left[1 - \frac{(\lambda - 1)}{3\lambda} \frac{(3\lambda\Delta + 2)}{\Delta(\lambda - 1) + 2} \right] \left(\frac{4\pi\lambda}{3} \right)^{\Delta},$$
$$\xi_{(\mu)} = \frac{3}{4\pi \chi(\lambda, \Delta)}, \tag{32}$$

$$\gamma_{(\rho)} = \sqrt{\frac{3\lambda}{\lambda - 1}} \left[1 - \frac{(\lambda - 1)}{3\lambda} \frac{(3\lambda\Delta + 2)}{\Delta(\lambda - 1) + 2} \right] \left(\frac{4\pi\lambda}{3} \right)^{\Delta},$$
$$\xi_{(\rho)} = \frac{3}{4\pi \sqrt{\chi(\lambda, \Delta)}}. \tag{33}$$

Equations (30)–(33) present the general expressions relating the condensation operator and the critical temperature in terms of the parameter λ. We shall now study the above equations for two different boundary conditions, namely, $\psi_- \neq 0$, $\psi_+ = 0$ and $\psi_+ \neq 0$, $\psi_- = 0$. Let us first consider the case $\psi_- = \langle \mathcal{O} \rangle$, $\psi_+ = 0$. As mentioned earlier setting $\psi_+ = 0$ implies that the condensate ψ_- forms in the absence of the source term ψ_+. For simplicity, we choose the matching point to be the middle point between the horizon and the boundary, that is, $z = \frac{z_h}{2}$, which implies setting $\lambda = 2$ in the above expressions. This gives $\chi = \sqrt{8}$ for the value of $\Delta = \Delta_- = 1$. Hence the relation between the condensation operator and the critical temperature reads

$$\langle \mathcal{O}_- \rangle_{(\mu)} = \frac{80\pi}{27} T_c \left(1 - \frac{T}{T_c} \right)^{1/2}, \quad T_c = 0.084\mu, \tag{34}$$

$$\langle \mathcal{O}_- \rangle_{(\rho)} = \frac{40\pi\sqrt{6}}{27} T_c \left(1 - \frac{T}{T_c} \right)^{1/2}, \quad T_c = 0.142\sqrt{\rho}. \tag{35}$$

We also carry out our calculations for other values of the matching point by choosing different values of λ. For example, $z = 0.10 z_h$ (that is, $\lambda = 10$), we obtain $T_c = 0.169\sqrt{\rho}$, which agrees fairly well with the numerical result $T_c = 0.225\sqrt{\rho}$ [15]. For the other case, that is, $\psi_+ = \langle \mathcal{O} \rangle$, $\psi_- = 0$, we once again choose the matching point to be the middle point between the horizon and the boundary, that is, $z = \frac{z_h}{2}$. This gives $\chi = \sqrt{28}$. Hence the relation between the condensation operator and the critical temperature reads

$$\langle \mathcal{O}_+ \rangle_{(\mu)} = \frac{160\pi^2}{27} T_c^2 \left(1 - \frac{T}{T_c} \right)^{1/2}, \quad T_c = 0.0451\mu, \tag{36}$$

$$\langle \mathcal{O}_+ \rangle_{(\rho)} = \frac{80\pi^2\sqrt{6}}{27} T_c^2 \left(1 - \frac{T}{T_c} \right)^{1/2}, \quad T_c = 0.104\sqrt{\rho}. \tag{37}$$

As before we carry out our analysis for other values of the matching point by choosing different values of λ. For example, $z = 0.33 z_h$ (that is, $\lambda = 3$), we obtain $T_c = 0.119\sqrt{\rho}$, which is in very good agreement with the numerical $T_c = 0.118\sqrt{\rho}$ [6]. The analytical results for different values of λ are presented in Tables 1 and 2. We therefore observe that the results from the matching method depends crucially on the matching point. We would like to mention that there is a priori no way to determine a suitable matching point. This is a lacuna of this method in contrast to the more analytically sound Sturm–Liouville approach [15,27].

4 Free energy of the holographic superconductor

We now proceed to compute the free energy at a finite temperature of the field theory living on the boundary of the $3 + 1$-bulk theory. To do this the holographic approach is used in relating the free energy (Ω) of the boundary field theory to the product of the temperature (T) and the on-shell value of the Abelian Higgs sector of the Euclidean action (S_E) [23].

To proceed further, we first write down the action for the action for the Abelian Higgs sector

$$S_M = \int d^4x \sqrt{-g} \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - (D_\mu \psi)^* D^\mu \psi - m^2 \psi^* \psi \right]. \tag{38}$$

Using the ansatz, $m^2 = -2$ and $q = 1$, we get

$$S_M = \int d^4x \left[\frac{\phi'^2(z)}{2} - \frac{F(z)\psi'^2(z)}{z^2} + \frac{\phi^2(z)\psi^2(z)}{z^2 F(z)} + \frac{2\psi^2(z)}{z^4} \right]. \tag{39}$$

Applying the boundary condition ($\phi(z_h) = 0$) and the equations of motion (11), (12), we obtain the on-shell value of the action S_E to be

$$S_o = \int d^3x \left[-\frac{1}{2} \phi(z)\phi'(z) \mid_{z=0} + \frac{F(z)\psi(z)\psi'(z)}{z^2} \mid_{z=0} - \int_0^{z_h} dz \frac{\phi^2(z)\psi^2(z)}{z^2 F(z)} \right]. \tag{40}$$

Substituting the asymptotic behaviour of $\phi(z)$ and $\psi(z)$ in the above action, we get

$$S_o = \int d^3x \left[\frac{\mu\rho}{2} + 3\psi_+\psi_- + \left(\frac{\psi_-^2}{z} \right) \mid_{z=0} - \int_0^{z_h} dz \frac{\phi^2(z)\psi^2(z)}{z^2(1 - z^3/z_h^3)} \right]. \tag{41}$$

Note that the term $(\frac{\psi^2}{z})|_{z=0}$ diverges and therefore one needs to add a counter term at the boundary to cancel this divergence. For the boundary condition $\psi_- = 0$, $\psi_+ \neq 0$, the counter term comes from the counter action [23,24]

$$S_c = -\int d^3x(\sqrt{-h}\psi^2(z))|_{z=0} \tag{42}$$

where h is the determinant of the induced metric on the AdS boundary. Using the asymptotic behaviour of $\psi(z)$ (Eq. 14) and evaluating this term, we obtain

$$S_c = -\int d^3x \left[2\psi_+\psi_- + \left(\frac{\psi_-^2}{z}\right)|_{z=0}\right]. \tag{43}$$

The free energy of the $2+1$-boundary field theory can now be obtained by adding S_o and S_c. This yields

$$\Omega = -T(S_o + S_c)$$
$$= \beta T V_2 \left[-\frac{\mu\rho}{2} - \psi_+\psi_- + I\right]$$
$$= \beta T V_2 \left[-\frac{\mu\rho}{2} + I\right] \tag{44}$$

where in the second equality we have set $\int d^3x = \beta V_2$, V_2 being the volume of the 2-dimensional space of the boundary and in the last equality we have used the fact that $\psi_- = 0$. The integral I reads

$$I = \int_0^{z_h} dz \frac{\phi^2(z)\psi^2(z)}{z^2(1 - z^3/z_h^3)} = \int_0^{z_h/\lambda} dz \frac{\phi_b^2(z)\psi_b^2(z)}{z^2(1 - z^3/z_h^3)}$$
$$+ \int_{z_h/\lambda}^{z_h} dz \frac{\phi_h^2(z)\psi_h^2(z)}{z^2(1 - z^3/z_h^3)} \equiv \mathcal{I}_1 + \mathcal{I}_2. \tag{45}$$

For the other boundary condition, namely, $\psi_+ = 0$, $\psi_- \neq 0$, the counter action reads [23,24,34]

$$S_c = -\int d^3x \left(\sqrt{-h}z\psi(z)\psi'(z)\right)|_{z=0}$$
$$= -\int d^3x \left[3\psi_+\psi_- + \left(\frac{\psi_-^2}{z}\right)|_{z=0}\right]. \tag{46}$$

This cancels the divergence term of the on-shell action (41) and yields

$$\Omega = \beta T V_2 \left[-\frac{\mu\rho}{2} + I\right]. \tag{47}$$

To evaluate the integral, we rewrite the matter field in the following form:

$$\psi(z_h) = \chi_1 \sqrt{\frac{\mu z_h}{\chi} - 1}, \quad \psi_{-/+} = \chi_2 \frac{\psi(z_h)}{z_h^\Delta} \tag{48}$$

where $\chi_1 = \sqrt{\frac{3\lambda^2}{\lambda^2-1}}$ and $\chi_2 = \lambda^\Delta \left[1 - \frac{(\lambda-1)}{3\lambda}\frac{3\lambda\Delta+2}{\Delta(\lambda-1)+2}\right]$. Now using the substitution $z = z_h l$, we obtain

$$\mathcal{I}_1 = \int_0^{z_h/\lambda} dz \frac{\phi_b^2(z)\psi_b^2(z)}{z^2(1 - z^3/z_h^3)}$$
$$= \int_0^{z_h/\lambda} dz \frac{(\mu - \rho z)^2 \psi_{-/+}^2 z^{2\Delta}}{z^2(1 - z^3/z_h^3)}$$
$$= \psi_{-/+}^2 z_h^{2\Delta-1}[\mu^2\mathcal{A}_1 + \rho^2 z_h^2 \mathcal{A}_2 - 2\mu\rho z_h \mathcal{A}_3]$$
$$= \chi_2^2 \frac{\psi^2(z_h)}{z_h}\left[B_1\mu^2 + B_2\frac{\mu}{z_h} + B_3\frac{1}{z_h^2}\right]$$
$$= \chi_1^2\chi_2^2 \left[C_1\mu^3 + C_2\frac{\mu^2}{z_h} + C_3\frac{\mu}{z_h^2} + C_4\frac{1}{z_h^3}\right] \tag{49}$$

where the constants are given by the following relations:

$$\mathcal{A}_1 = \int_0^{1/\lambda} \frac{l^{2\Delta-2}dl}{(1-l^3)}; \quad \mathcal{A}_2 = \int_0^{1/\lambda} \frac{l^{2\Delta}dl}{(1-l^3)};$$
$$\mathcal{A}_3 = \int_0^{1/\lambda} \frac{l^{2\Delta-1}dl}{(1-l^3)},$$
$$B_1 = \mathcal{A}_1 + \frac{4\mathcal{A}_2\lambda^2}{(1+\lambda)^2} - \frac{4\mathcal{A}_3\lambda}{(1+\lambda)},$$
$$B_2 = \frac{2\chi\mathcal{A}_3(\lambda-1)}{(1+\lambda)} - \frac{4\chi\mathcal{A}_2(1-1/\lambda)}{(1+1/\lambda)^2},$$
$$B_3 = \frac{\mathcal{A}_2\chi^2(\lambda-1)^2}{(\lambda+1)^2},$$
$$C_1 = \frac{B_1}{\chi}; \quad C_2 = \frac{B_2}{\chi} - B_1; \quad C_3 = \frac{B_3}{\chi} - B_2; \quad C_4 = -B_3. \tag{50}$$

The evaluation of the integral $\mathcal{I}_2$ can be done in a similar way and yields

$$\mathcal{I}_2 = \int_{z_h/\lambda}^{z_h} dz \frac{\phi_h^2(z)\psi_h^2(z)}{z^2(1 - z^3/z_h^3)} = \phi'^2(z_h)\psi^2(z_h)z_h \left[\mathcal{A}_4 + \left(\frac{4}{3} - \frac{2\psi^2(z_h)}{3}\right)\mathcal{A}_5\right.$$
$$+ \left(\frac{\psi^4(z_h)}{9} + \frac{4}{9} - \frac{8+\chi^2}{18} - \frac{8\psi^2(z_h)}{9}\right)\mathcal{A}_6 + \left(\frac{4\psi^4(z_h)}{27} - \frac{2\psi^2(z_h)}{3}\left(\frac{4}{9} - \frac{8+\chi^2}{18}\right) - \frac{8+\chi^2}{27}\right)\mathcal{A}_7$$
$$+ \left(\frac{(8+\chi^2)^2}{36^2} + \left(\frac{4}{9} - \frac{8+\chi^2}{18}\right)\frac{\psi^4(z_h)}{9} + \frac{2(8+\chi^2)\psi^2(z_h)}{81}\right)\mathcal{A}_8 + \frac{(8+\chi^2)^2\psi^4(z_h)}{9(36)^2}\mathcal{A}_{10}$$
$$+ \left(\frac{-(8+\chi^2)\psi^4(z_h)}{243} - \frac{2(8+\chi^2)^2\psi^2(z_h)}{3(36)^2}\right)\mathcal{A}_9 \right] \tag{51}$$

where the constants $\mathcal{A}_n$ ($n = 4, 5, 6, 7, 8, 9, 10$) are given by the following relations:

$$\mathcal{A}_n = \int_{1/\lambda}^1 \frac{(l-1)^{n-2}\mathrm{d}l}{l^2(1-l^3)}. \tag{52}$$

After simplification of the above expression, we get

$$
\begin{aligned}
\mathcal{I}_2 &= \phi'^2(z_h)\psi^2(z_h)z_h[B_4 + B_5\psi^2(z_h) + B_6\psi^4(z_h)]\\
&= \phi'^2(z_h)\psi^2(z_h)z_h[C_5 + C_6\mu z_h + C_7\mu^2 z_h^2]\\
&= \chi_1^2\chi_3^2\left[D_1\mu^3 + D_2\frac{\mu^2}{z_h} + D_3\frac{\mu}{z_h^2} + D_4\frac{1}{z_h^3}\right]
\end{aligned}
\tag{53}
$$

where the constants in the above expression are given by the following relations:

$$
\begin{aligned}
B_4 &= \mathcal{A}_4 + \frac{4}{3}\mathcal{A}_5 + \left(\frac{4}{9} - \frac{8+\chi^2}{18}\right)\\
&\quad \mathcal{A}_6 - \frac{8+\chi^2}{27}\mathcal{A}_7 + \frac{(8+\chi^2)^2}{36^2}\mathcal{A}_8,\\
B_5 &= -\frac{2}{3}\mathcal{A}_5 - \frac{8}{9}\mathcal{A}_6 - \frac{2}{3}\left(\frac{4}{9} - \frac{8+\chi^2}{18}\right)\\
&\quad \mathcal{A}_7 + \frac{16+2\chi^2}{81}\mathcal{A}_8 - \frac{2}{3}\frac{(8+\chi^2)^2}{36^2}\mathcal{A}_9,\\
B_6 &= \frac{1}{9}\mathcal{A}_6 + \frac{4}{27}\mathcal{A}_7 + \left(\frac{4}{9} - \frac{8+\chi^2}{18}\right)\\
&\quad \frac{\mathcal{A}_8}{9} - \frac{8+\chi^2}{243}\mathcal{A}_9 + \frac{(8+\chi^2)^2}{36^2}\frac{\mathcal{A}_{10}}{9},\\
C_5 &= B_4 - \chi_1^2 B_5 + \chi_1^4 B_6 \; ;\\
C_6 &= \frac{\chi_1^2}{\chi}B_5 - \frac{2\chi_1^4}{\chi}B_6, \quad C_7 = \frac{\chi_1^4}{\chi^2}B_6,\\
D_1 &= \frac{C_7}{\chi}, \quad D_2 = \frac{C_6}{\chi} - C_7;\\
D_3 &= \frac{C_5}{\chi} - C_6, \quad D_4 = -C_5.
\end{aligned}
\tag{54}
$$

Adding $\mathcal{I}_1$ and $\mathcal{I}_2$, we finally get

$$I = E_1\mu^3 + E_2\frac{\mu^2}{z_h} + E_3\frac{\mu}{z_h^2} + E_4\frac{1}{z_h^3} \tag{55}$$

where

$$
\begin{aligned}
E_1 &= \chi_1^2\chi_2^2 C_1 + \chi_1^2\chi_3^2 D_1, \quad E_2 = \chi_1^2\chi_2^2 C_2 + \chi_1^2\chi_3^2 D_2,\\
E_3 &= \chi_1^2\chi_2^2 C_3 + \chi_1^2\chi_3^2 D_3, \quad E_4 = \chi_1^2\chi_2^2 C_4 + \chi_1^2\chi_3^2 D_4.
\end{aligned}
\tag{56}
$$

Hence the analytical expression for the free energy in terms of the chemical potential reads

$$
\begin{aligned}
\frac{\Omega}{V_2} &= -\frac{\lambda}{\lambda+1}\frac{\mu^2}{z_h} + \frac{(\lambda-1)\chi}{2(\lambda+1)}\frac{\mu}{z_h^2} + I\\
&\equiv G_1\mu^3 + G_2\mu^2 T + G_3\mu T^2 + G_4 T^3
\end{aligned}
\tag{57}
$$

where

$$G_1 = E_1, \quad G_2 = \left(E_2 - \frac{\lambda}{\lambda+1}\right)\frac{4\pi}{3},$$

$$G_3 = \left(E_3 - \frac{(\lambda-1)\chi}{2(\lambda+1)}\right)\frac{16\pi^2}{9}, \quad G_4 = \frac{64\pi^3}{27}E_4. \tag{58}$$

This expression for the free energy can also be written in terms of the charge density as

$$\frac{\Omega}{V_2} = H_1\frac{\rho^3}{T^3} + H_2\frac{\rho^2}{T} + H_3\rho T + H_4 T^3 \tag{59}$$

where

$$
\begin{aligned}
H_1 &= \frac{G_1}{8}\left(1+\frac{1}{\lambda}\right)^3\left(\frac{3}{4\pi}\right)^3,\\
H_2 &= \frac{G_2}{4}\left(1+\frac{1}{\lambda}\right)^2\left(\frac{3}{4\pi}\right)^2\\
&\quad + \frac{3\chi G_1}{8}\left(1+\frac{1}{\lambda}\right)^2\left(1-\frac{1}{\lambda}\right)\left(\frac{3}{4\pi}\right),\\
H_3 &= \frac{G_3}{2}\left(1+\frac{1}{\lambda}\right)\left(\frac{3}{4\pi}\right) + \frac{\chi G_2}{2}\left(1-\frac{1}{\lambda^2}\right)\\
&\quad + \frac{3\chi^2 G_1}{8}\left(1+\frac{1}{\lambda}\right)\left(1-\frac{1}{\lambda}\right)^2\left(\frac{4\pi}{3}\right),\\
H_4 &= \frac{\chi G_3}{2}\left(1-\frac{1}{\lambda}\right)\left(\frac{4\pi}{3}\right) + \frac{\chi^2 G_2}{4}\left(1-\frac{1}{\lambda}\right)^2\left(\frac{4\pi}{3}\right)^2\\
&\quad + \frac{\chi^3 G_1}{8}\left(1-\frac{1}{\lambda}\right)^3\left(\frac{4\pi}{3}\right)^3 + G_4.
\end{aligned}
\tag{60}
$$

In the next section, we shall make use of these results to investigate the thermodynamic geometry of this model in the grand canonical ensemble.

5 Thermodynamic geometry

With the above results in hand, we now proceed to investigate the thermodynamic geometry of this holographic superconductor. The thermodynamic metric is defined as [19,20]

$$g_{ij} = -\frac{1}{T}\frac{\partial^2\omega(T,\rho)}{\partial x^i \partial x^j} = -\frac{1}{T}\frac{\partial^2\omega(T,\mu)}{\partial x^i \partial x^j} \tag{61}$$

where $\omega = \frac{\Omega}{V_2}$, $x^1 = T$ and $x^2 = \rho$ or μ. Hence the components of the metric in terms of μ read

$$
\begin{aligned}
g_{TT} &= -\left[2G_3\frac{\mu}{T} + 6G_4\right],\\
g_{T\mu} &= g_{\mu T} = -\left[2G_2\frac{\mu}{T} + 2G_3\right],\\
g_{\mu\mu} &= -\left[6G_1\frac{\mu}{T} + 2G_2\right],
\end{aligned}
\tag{62}
$$

Table 1 For $\Delta = \Delta_+ = 2$, the critical temperature $T_c = \xi_{(\rho)} \sqrt{\rho}$ with numerical value $\xi_{(\rho)} = 0.118$ [6]

Value of λ	Matching point	From matching method		From divergence of R	
		$\bar{\xi}_{(\mu)}$	$\bar{\xi}_{(\rho)}$	$\bar{\xi}_{(\mu)}$	$\bar{\xi}_{(\rho)}$
5	$0.20\,z_h$	0.076	0.134	0.126	0.152
3	$0.33\,z_h$	0.060	0.119	0.099	0.118
2	$0.50\,z_h$	0.045	0.104	0.084	0.104
$\frac{3}{2}$	$0.66\,z_h$	0.033	0.089	0.080	0.101
$\frac{5}{4}$	$0.80\,z_h$	0.024	0.076	0.084	0.102

Table 2 For $\Delta = \Delta_- = 1$, the critical temperature $T_c = \xi_{(\rho)} \sqrt{\rho}$ with numerical value $\xi_{(\rho)} = 0.225$ [15]

Value of λ	Matching point	From matching method		From divergence of R	
		$\bar{\xi}_{(\mu)}$	$\bar{\xi}_{(\rho)}$	$\bar{\xi}_{(\mu)}$	$\bar{\xi}_{(\rho)}$
5	$0.20\,z_h$	0.113	0.164	0.334	0.709
3	$0.33\,z_h$	0.102	0.156	0.475	0.340
2	$0.50\,z_h$	0.084	0.142	0.359	0.345
$\frac{3}{2}$	$0.66\,z_h$	0.065	0.124	0.355	0.284
$\frac{5}{4}$	$0.80\,z_h$	0.047	0.106	0.375	0.233

and in terms of ρ they read

$$
\begin{aligned}
g_{TT} &= -\left[12H_1\frac{\rho^3}{T^6} + 2H_2\frac{\rho^2}{T^4} + 6H_4\right], \\
g_{T\rho} &= g_{\rho T} = \left[9H_1\frac{\rho^2}{T^5} + 2H_2\frac{\rho}{T^3} - H_3\frac{1}{T}\right], \\
g_{\rho\rho} &= -\left[6H_1\frac{\rho}{T^4} + 2H_2\frac{1}{T^2}\right].
\end{aligned}
\tag{63}
$$

The scalar curvature of a general metric

$$
ds_{th}^2 = g_{11}(dx^1)^2 + 2g_{12}dx^1 dx^2 + g_{22}(dx^2)^2
\tag{64}
$$

is given by [21]

$$
\begin{aligned}
R = \frac{-1}{\sqrt{g}}\Bigg[&\frac{\partial}{\partial x^1}\left(\frac{g_{12}}{g_{11}\sqrt{g}}\frac{\partial g_{11}}{\partial x^2} - \frac{1}{\sqrt{g}}\frac{\partial g_{22}}{\partial x^1}\right) \\
&+ \frac{\partial}{\partial x^2}\left(\frac{2}{\sqrt{g}}\frac{\partial g_{22}}{\partial x^2} - \frac{1}{\sqrt{g}}\frac{\partial g_{11}}{\partial x^2} - \frac{g_{12}}{g_{11}\sqrt{g}}\frac{\partial g_{11}}{\partial x^1}\right)\Bigg].
\end{aligned}
\tag{65}
$$

To look for any singularity in R, one has to see whether the denominator of the right hand side of Eq. (65) vanishes. The condition of the divergence of R is $\det g_{ij} = 0$. For the metric (62), this gives

$$
\begin{aligned}
4\Bigg[&(3G_2G_4 - G_3^2) + (9G_1G_4 - G_2G_3)\frac{\mu}{T} \\
&+ (3G_1G_3 - G_2^2)\frac{\mu^2}{T^2}\Bigg] = 0,
\end{aligned}
\tag{66}
$$

and for the metric (63), this gives

$$
\begin{aligned}
\Bigg[&-9H_1^2\frac{\rho^4}{T^{10}} + 18H_1H_3\frac{\rho^2}{T^6} + (36H_1H_4 + 4H_2H_3)\frac{\rho}{T^4} \\
&+ (12H_2H_4 - H_3^2)\frac{1}{T^2}\Bigg] = 0.
\end{aligned}
\tag{67}
$$

The temperature for which the scalar curvature vanishes can be obtained by solving these equations. We obtain this critical temperature for the two different boundary conditions $\psi_- \neq 0$, $\psi_+ = 0$ and $\psi_+ \neq 0$, $\psi_- = 0$ for a set of values of λ and compare them with the results which have been obtained from the matching method. The Tables 1 and 2 give the results for $\Delta = \Delta_+$ and $\Delta = \Delta_-$ obtained from the matching method and the thermodynamic geometry for a set of values of the matching point for these two different boundary conditions, respectively.

Considering the boundary condition $\psi_+ \neq 0$, $\psi_- = 0$ (which implies $\Delta = \Delta_+ = 2$) and setting $\lambda = 2$, the above equations yield $T_c = 0.084\mu$. This does not agree with the result in [24]. For $\lambda = 3$ (that is, $z = 0.33z_h$), we obtain $T_c = 0.118\sqrt{\rho}$ from the divergence of the scalar curvature. This turns out to agree very well with the numerical value $T_c = 0.118\sqrt{\rho}$ [6].

6 Conclusions

We now summarize our findings. We obtain the value of the critical temperature and the condensation operator of a holographic superconductor living in a $2 + 1$ dimensions for two different sets of boundary conditions of the condensation operator using the formalism of thermodynamic geometry. The results are compared with those obtained from the

matching method. The matching point is taken to be anywhere between the horizon and the boundary and the results are obtained for a set of values of the matching point. The near horizon expressions obtained by the matching method plays a crucial role in obtaining the free energy of the holographic superconductor. This in turn is used to compute the thermodynamic geometry. It is observed that the results for the critical temperature (in terms of the charge density) from the two approaches, namely, the thermodynamic geometry approach and the matching method are exactly equal with the numerical value $T_c = 0.118\sqrt{\rho}$ when the matching point for the near horizon and the boundary behaviour of the fields is taken to be the $z = z_h/3$ between the horizon and the boundary for the $\psi_+ \neq 0$, $\psi_- = 0$ case.

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