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Extended scaling and residual flavor symmetry in the neutrino Majorana mass matrix

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Abstract The residual symmetry approach, along with a complex extension for some flavor invariance, is a powerful tool to uncover the flavor structure of the 3×3 neutrino Majorana mass matrix M_ν toward gaining insights into neutrino mixing. We utilize this to propose a complex extension of the real scaling ansatz for M_ν which was introduced some years ago. Unlike the latter, our proposal allows a nonzero mass for each of the three light neutrinos as well as a nonvanishing θ_{13}. The generation of light neutrino masses via the type-I seesaw mechanism is also demonstrated. A major result of this scheme is that leptonic Dirac CP-violation must be maximal while atmospheric neutrino mixing does not need to be exactly maximal. Moreover, each of the two allowed Majorana phases, to be probed by the search for nuclear $0\nu\beta\beta$ decay, has to be at one of its two CP-conserving values. There are other interesting consequences such as the allowed occurrence of a normal mass ordering which is not favored by the real scaling ansatz. Our predictions will be tested in ongoing and future neutrino oscillation experiments at T2K, NOνA and DUNE.

1 Introduction

The masses and mixing properties of the three light neutrinos are beginning to get pinned down. Though the precise mass values are still unknown, upper limits on them have been pushed down to fractions of electron volts. Furthermore, it is already known that at least one of the neutrinos must be heavier than about 50 meV. Additionally, the three angles which describe their mixing have become reasonably well known with $\theta_{12} \sim 34°$, $\theta_{23} \sim 45°$ and $\theta_{13} \sim 8°$. Understanding this mixing phenomenon (with one small and two

large angles) has emerged as a major challenge. As ongoing experiments feed in more and more information on neutrino masses and mixing, the flavor structure of the 3×3 neutrino mass matrix M_ν is being slowly uncovered. Many of its features still remain unknown nonetheless and continue to intrigue theoretical investigators. (Up-to-date overviews of these issues and their investigations along with original references may be found in the two review articles quoted in Refs. [1,2].) Especially tantalizing is the predicted phenomenon of leptonic CP-violation which likely to have implications for leptogenesis [3–5]. As yet, there is no statistically reliable definitive experimental result on leptonic CP-violation. However, hints of a near-maximal CP-violation, with the phase δ being $\simeq 3\pi/2$, have emerged from results reported by the T2K [6], NOνA [7] and Super-Kamiokande [8] experiments. Similarly, a recent global analysis [9] of all neutrino data is hinting at a nonmaximal value of $\sin^2 2\theta_{23}$. Another yet unresolved question of great interest is that of neutrino mass ordering: normal vs. inverted. In addition, one would like to know if the three neutrinos are Majorana or Dirac particles—to be presumably determined by a future observation of nuclear $0\nu\beta\beta$ decay [10].

Let us start with the minimal supposition that there are only three light and flavored left-chiral neutrinos and that they are Majorana in character. The neutrino mass term in the Lagrangian density now reads

$$-\mathcal{L}^\nu_{\text{mass}} = \frac{1}{2}\bar{\nu}^C_l (M_\nu)_{lm}\nu_m + h.c. \quad (1.1)$$

with $\nu^C_l = C\bar{\nu}_l^T$ and the subscripts l, m spanning the lepton flavor indices e, μ, τ. M_ν is a complex symmetric matrix ($M^*_\nu \neq M_\nu = M^T_\nu$) which can be put into a diagonal form by a similarity transformation with a unitary matrix U:

$$U^T M_\nu U = M^d_\nu \equiv \text{diag}(m_1, m_2, m_3). \quad (1.2)$$

Here $m_i (i = 1, 2, 3)$ are real and positive masses. We choose to work in a weak basis [11], in which the charged lepton

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mass matrix is diagonal with real and positive elements, i.e. $M_l = \text{diag.}(m_e, m_\mu, m_\tau)$ and the unphysical phases of U are absorbed into the neutrino fields. Now

$$U = U_{\text{PMNS}}$$
$$\equiv \begin{pmatrix} c_{12}c_{13} & e^{i\frac{\alpha}{2}}s_{12}c_{13} & s_{13}e^{-i(\delta-\frac{\beta}{2})} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & e^{i\frac{\alpha}{2}}(c_{12}c_{23}-s_{12}s_{13}s_{23}e^{i\delta}) & c_{13}s_{23}e^{i\frac{\beta}{2}} \\ s_{12}s_{23}-c_{12}s_{13}c_{23}e^{i\delta} & e^{i\frac{\alpha}{2}}(-c_{12}s_{23}-s_{12}s_{13}c_{23}e^{i\delta}) & c_{13}c_{23}e^{i\frac{\beta}{2}} \end{pmatrix}$$
$$(1.3)$$

with $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$ and $\theta_{ij} = [0, \pi/2]$. CP-violation enters through nontrivial values of the Dirac phase δ and of the Majorana phases α, β with $\delta, \alpha, \beta = [0, 2\pi]$. We follow the PDG convention [12] on these angles and phases except that we denote the Majorana phases by α and β. In principle there could also be a phase matrix with U_{PMNS} if we work in a weak basis where M_l is diagonal but where the unphysical phases are not absorbed in the neutrino fields. It is demonstrated later that even if we include the unphysical phase matrix, our result remains the same which is obvious, since physical results are basis independent.

Quite a few different hypotheses have been advanced over several decades on the flavor structure of M_ν, as reviewed in the first article of Refs. [1,2]. We zero in on an ansatz made some years ago [13,14] that we call Simple Real Scaling (SRS). This posits the relations

$$\frac{(M_\nu^{\text{SRS}})_{e\mu}}{(-M_\nu^{\text{SRS}})_{e\tau}} = \frac{(M_\nu^{\text{SRS}})_{\mu\mu}}{(-M_\nu^{\text{SRS}})_{\mu\tau}} = \frac{(M_\nu^{\text{SRS}})_{\tau\mu}}{(-M_\nu^{\text{SRS}})_{\tau\tau}} = k, \quad (1.4)$$

where k is a real and positive dimensionless scaling factor. It is straightforward to induce from (1.4) the form of the neutrino Majorana mass matrix:

$$M_\nu^{\text{SRS}} = \begin{pmatrix} X & -Yk & Y \\ -Yk & Zk^2 & -Zk \\ Y & -Zk & Z \end{pmatrix}. \quad (1.5)$$

Here X, Y, Z are complex mass dimensional quantities that are a priori unknown. We consistently denote complex (real) quantities by capital (small) letters throughout. We have chosen appropriate negative signs in (1.4) and (1.5) to be in conformity with the PDG convention [12] on the form of U_{PMNS} that emerges from (1.5). It was pointed out by Mohapatra and Rodejohann [13,14] that—in the basis where the charged lepton mass matrix is diagonal—(1.5) can be realized from the larger symmetry group $D_4 \times \mathbb{Z}_2$. This ansatz of SRS led to a sizable body of research [15–21]. But it predicts a vanishing s_{13} (and hence no measurable leptonic Dirac CP-violation) as well as an inverted neutrino mass hierarchy (i.e. $m_{2,1} > m_3$ with $m_3 = 0$. While the latter result is still allowed within current experimental bounds, a null value of s_{13} has been ruled out at

more than 10σ [22]. Thus SRS, as it stands, has to be abandoned.

We want to consider an extended version of (1.5) which allows a nonvanishing s_{13}. To this end, we employ the method of complex extension which in turn is based on the idea of the residual symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$ [23–25] of M_ν. This is explained in Sect. 3 below. As detailed in the subsequent Sect. 4, the complex extension (CES for Complex Extended Scaling) leads to the neutrino mass matrix

$$M_\nu^{\text{CES}} = \begin{pmatrix} x & -y_1k+iy_2k^{-1} & y_1+iy_2 \\ -y_1k+iy_2k^{-1} & z_1-wk^{-1}(k^2-1)-iz_2 & w-iz_2(2k)^{-1}(k^2-1) \\ y_1+iy_2 & w-iz_2(2k)^{-1}(k^2-1) & z_1+iz_2 \end{pmatrix}. \quad (1.6)$$

Here x, $y_{1,2}$, $z_{1,2}$, and w are real mass dimensional quantities that are a priori unknown. It will be shown that M_ν^{CES} of (1.6) can accommodate a nonzero value for each of m_1, m_2, m_3 and can fit the extant data on $\Delta m_{21}^2 \equiv m_2^2 - m_1^2$, $|\Delta m_{32}^2| \equiv |m_3^2 - m_2^2|$ as well as on θ_{12} and θ_{13}. The relation $\tan\theta_{23} = k^{-1}$ is a consequence so that the presently allowed range of $\tan\theta_{23}$ around unity would yield the permitted domain of the variation of the scaling parameter k to be close to 1. Furthermore, (1.6) leads to the result that $\alpha, \beta = 0$ or π, i.e. there is no Majorana CP-violation, and the verifiable/falsifiable prediction that $\cos\delta = 0$, i.e. leptonic Dirac CP-violation is maximal. We have no statement on the sign of $\sin\delta$ so that δ can be either $\pi/2$ or $3\pi/2$. Furthermore, we show that a normal mass ordering (with $m_{2,1} < m_3$) is allowed in addition to an inverted one ($m_{2,1} > m_3$) in the parameter space of the model.

The rest of the paper is organized as follows. In Sect. 2 we elucidate the meaning of the residual $\mathbb{Z}_2 \times \mathbb{Z}_2$ discrete symmetry of M_ν in terms of its invariance under two separate similarity transformations. Simple real scaling and its real generalization are discussed in Sect. 3. Section 4 contains a presentation of the procedure of complex extension; this is first illustrated for $\mu\tau$ interchange symmetry and then applied to the scaling transformation to lead to the proposed M_ν^{CES} of (1.6) as well as its main consequences, namely $\tan\theta_{23} = k^{-1}$ and $\cos\delta = 0$ plus the allowed occurrence of a normal mass ordering. The origin of the neutrino mass matrix M_ν^{CES} in our scheme from the type-I seesaw mechanism is shown in Sect. 5. Detailed phenomenological implications of M_ν^{CES} are worked out numerically in Sect. 6 and fitted with the current data yielding various 3σ-allowed regions in the parameter space; the application of our results to forthcoming experiments on nuclear $0\nu\beta\beta$ decay and neutrino oscillations is also discussed in the same section. Section 7 summarizes our conclusions.

2 Meaning of residual flavor symmetry of M_ν

It would be useful to focus on the feature [23–25] of M_ν that it has a residual (sometimes called 'remnant' [26]) $\mathbb{Z}_2 \times \mathbb{Z}_2$ flavor symmetry and at the same time review the representation content of the latter. Such an exercise will enable us to set up the theoretical machinery needed to apply the idea to SRS. In addition, this will lead us to its real generalization as well as to its complex extension.

Let G be a generic 3×3 unitary matrix representation of some horizontal symmetry of M_ν effected through the similarity transformation

$$G^T M_\nu G = M_\nu. \tag{2.1}$$

Equations (1.2) and (2.1) lead to the conclusion that the unitary matrix $U' \equiv GU$ also puts M_ν into a diagonal form by a similarity transformation, i.e. $U'^T M_\nu U' = M_\nu^d$. It can then be shown [23–25] that, if m_1, m_2 and m_3 are nondegenerate, G has eigenvalues ± 1 and is diagonalized by U. Thus

$$GU = Ud, \tag{2.2}$$
$$d^2 = I. \tag{2.3}$$

Here d is a 3×3 diagonal matrix in flavor space with $d_{lm} = \pm \delta_{lm}$. There are eight possible distinct forms for d. Two of these are trivial – being the unit and the negative unit matrices. Of the remaining six, three are negatives of the other three. Finally, we have three G_a's (a = 1, 2, 3) but it is sufficient to consider any two of those as independent on account of the relation $G_a = \epsilon_{abc} G_b G_c$. The two independent G_a (chosen here as $G_{2,3}$) are representations of a residual $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry in the Majorana mass term of the neutrino Lagrangian. It follows from (2.2) and (2.3) that

$$G^2 = I, \tag{2.4}$$
$$\det G = \pm 1. \tag{2.5}$$

The eigenvalue equation (2.2) needs to be considered for the two independent d's, i.e. d_2 and d_3, corresponding respectively to G_2 and G_3. Suppose we choose

$$d_2 = \text{diag}\,(-1, 1, -1), \tag{2.6}$$
$$d_3 = \text{diag}\,(-1, -1, 1), \tag{2.7}$$

for $\det G = 1$. (The choice for the case $\det G = -1$ is a trivial extension with $-d_2$ and $-d_3$.) Now

$$G_{2,3} = U d_{2,3} U^\dagger \tag{2.8}$$

can be obtained by use of the explicit form of U as given in (1.3). For instance, let us consider the situation for $\mu\tau$

interchange symmetry [27–29] which implies $\theta_{23} = \pi/4$ and $\theta_{13} = 0$. Now we obtain

$$G_2 = \begin{pmatrix} -\cos 2\theta_{12} & 2^{-\frac{1}{2}} \sin 2\theta_{12} & -2^{-\frac{1}{2}} \sin 2\theta_{12} \\ 2^{-\frac{1}{2}} \sin 2\theta_{12} & -\frac{1}{2}(1 - \cos 2\theta_{12}) & -\frac{1}{2}(1 + \cos 2\theta_{12}) \\ -2^{-\frac{1}{2}} \sin 2\theta_{12} & -\frac{1}{2}(1 + \cos 2\theta_{12}) & -\frac{1}{2}(1 - \cos 2\theta_{12}) \end{pmatrix},$$
$$G_3^{\mu\tau} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.9}$$

The above G_3 explicitly implements $\mu\tau$ interchange in the neutrino flavor basis and hence has been labeled with the superscript $\mu\tau$. Thus one can now identify one of the two residual $\mathbb{Z}_2$s as $\mathbb{Z}_2^{\mu\tau}$. The full residual symmetry in this case is $\mathbb{Z}_2 \times \mathbb{Z}_2^{\mu\tau}$. Our aim would be to perform a similar task with scaling symmetry in obtaining a $\mathbb{Z}_2^{\text{scaling}}$. It may be mentioned that some authors [30] have generalized $G_3^{\mu\tau}$ to

$$G_3^{G\mu\tau} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos 2\theta_{23} & \sin 2\theta_{23} \\ 0 & \sin 2\theta_{23} & \cos 2\theta_{23} \end{pmatrix}, \tag{2.10}$$

which can accommodate an arbitrary θ_{23} but still has $\theta_{13} = 0$. A somewhat different use of the residual symmetry approach with another pair of $\mathbb{Z}_2$ was made in Refs. [31–33].

A comment on the use of the residual $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry would be in order. One could start from any arbitrary ansatz on U_{PMNS}, reconstruct the residual $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry and work out the consequences. However, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry emerging from an arbitrary ansatz may not follow from a larger symmetry group or have some deeper flavor meaning. The SRS ansatz has been shown to follow [13,14] from a larger flavor symmetry group $D_4 \times \mathbb{Z}_2$.

3 Simple real scaling and its real generalization

SRS and the corresponding M_ν^{SRS}, cf. (1.5), were already introduced in Sect. 1. It is evident from (1.5) that the latter has a vanishing determinant, i.e. one null eigenvalue. The corresponding eigenvector, given that θ_{12} and θ_{23} are known to be hugely nonzero, can be identified only with the third column of U^{SRS} and written as

$$C_3^{\text{SRS}} = \begin{pmatrix} 0 \\ (1 + k^2)^{-\frac{1}{2}} e^{i\frac{\beta}{2}} \\ k(1 + k^2)^{-\frac{1}{2}} e^{i\frac{\beta}{2}} \end{pmatrix}. \tag{3.1}$$

Two immediate consequences are that $m_3 = 0$, i.e. the neutrino mass ordering is inverted ($m_{2,1} > m_3$), and $\theta_{13} = 0$. The full U^{SRS} can be written with an undetermined angle θ_{12} and the corresponding c_{12}, s_{12} as

$$U^{\text{SRS}} = \begin{pmatrix} c_{12} & s_{12}e^{i\frac{\alpha}{2}} & 0 \\ -k(1+k^2)^{-\frac{1}{2}}s_{12} & k(1+k^2)^{-\frac{1}{2}}c_{12}e^{i\frac{\alpha}{2}} & (1+k^2)^{-\frac{1}{2}}e^{i\frac{\beta}{2}} \\ (1+k^2)^{-\frac{1}{2}}s_{12} & -(1+k^2)^{-\frac{1}{2}}c_{12}e^{i\frac{\alpha}{2}} & k(1+k^2)^{-\frac{1}{2}}e^{i\frac{\beta}{2}} \end{pmatrix}.$$
(3.2)

A comparison between (1.3) and (3.2) immediately yields

$$\tan\theta_{23} = k^{-1}.$$
(3.3)

As we shall see, (3.3) is going to survive both the real generalization and the complex CP-transformed extension of SRS.

An expression for G_3^{scaling} as a representation for $\mathbb{Z}_2^{\text{scaling}}$ can now be derived by use of (2.8). On utilizing U^{SRS} from (3.2) and d_3 from (2.7), we have

$$G_3^{\text{scaling}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & (1-k^2)(1+k^2)^{-1} & 2k(1+k^2)^{-1} \\ 0 & 2k(1+k^2)^{-1} & -(1-k^2)(1+k^2)^{-1} \end{pmatrix}$$
$$= (G_3^{\text{scaling}})^T.$$
(3.4)

The $\mathbb{Z}_2^{\text{scaling}}$ symmetry of M_ν^{SRS} ensures that

$$\left(G_3^{\text{scaling}}\right)^T M_\nu^{\text{SRS}} G_3^{\text{scaling}} = M_\nu^{\text{SRS}}.$$
(3.5)

It may be noted that (3.5) does not lead uniquely to the form (1.5). Further, while the form of G_3^{scaling} follows uniquely from U^{SRS} of (3.2) via the relation been G_3 and d_3, the reverse is not the case. Indeed, though the third column of U, reconstructed from G_3^{scaling}, must be C_3^{SRS} of (3.1) since $(d_3)_{33} = 1$, its first two columns could be an arbitrary orthogonal pair. That occurs because of the degeneracy of the (1,1) and (2,2) elements in d_3. The full residual symmetry of M_ν^{SRS} is $\mathbb{Z}_2^k \times \mathbb{Z}_2^{\text{scaling}}$, where a representation for $\mathbb{Z}_2^k$ is $G_2^k = U^{\text{SRS}}d_2 U^{\text{SRS}\dagger}$. Explicitly,

$$G_2^k = \begin{pmatrix} -\cos 2\theta_{12} & k(1+k^2)^{-1}\sin 2\theta_{12} & -(1+k^2)^{-1}\sin 2\theta_{12} \\ k(1+k^2)^{-1}\sin 2\theta_{12} & -k^2(1+k^2)^{-1}(1-\cos 2\theta_{12}) & -k(1+k^2)^{-1}(1+\cos 2\theta_{12}) \\ -(1+k^2)^{-1}\sin 2\theta_{12} & -k(1+k^2)^{-1}(1+\cos 2\theta_{12}) & -(1+k^2)^{-1}(k^2-\cos 2\theta_{12}) \end{pmatrix},$$
(3.6)

which obeys

$$(G_2^k)^T M_\nu^{\text{SRS}} G_2^k = M_\nu^{\text{SRS}}.$$
(3.7)

A good check is that, for $k = 1$, the scaling procedure just reduces to $\mu\tau$ interchange with the additional constraint $M_{\mu\mu}^\nu = M_{\mu\tau}^\nu$. But the point of real interest is that M_ν^{SRS} of (1.5) is not the most general form obeying (3.5). The latter may be worked out to be

$$M_\nu^{\text{GRS}} = \begin{pmatrix} x & -Yk & Y \\ -Yk & Z - Wk^{-1}(k^2-1) & W \\ Y & W & Z \end{pmatrix},$$
(3.8)

where W is another a priori unknown mass dimensional complex quantity. We call this form of M_ν the Generalized Real Scaling ansatz and denote it by the superscript GRS. Evidently, the specific choice $W = -Zk$ reduces M_ν^{GRS} to M_ν^{SRS}. The neutrino mass matrix M_ν^{GRS} of (3.8) has interesting properties. For one thing, it has a determinant which does not appear to vanish. Therefore, we take all neutrino masses to be nonzero and can accommodate a nonzero m_3 and in principle a normal mass ordering with $m_{2,1} < m_3$. However, being invariant under a similarity transformation by G_3^{scaling} of (3.4), the third column of the corresponding U^{GRS} is constrained to be C_3^{SRS} of (3.1). Consequently, one obtains a vanishing θ_{13} which is now experimentally known to be nonzero. Thus M_ν^{GRS} of (3.8) is unacceptable. A more extended version of scaling in the neutrino mass matrix is needed to describe nature. This is what will be provided in the next section.

4 Complex extension of scaling ansatz

It would be useful to first recall how the complex extension of $\mu\tau$ interchange symmetry was originally made [27–29]. The $\mu\tau$ interchange invariant $M_\nu^{\mu\tau}$ obeys the condition

$$(G_3^{\mu\tau})^T M_\nu^{\mu\tau} G_3^{\mu\tau} = M_\nu^{\mu\tau}$$
(4.1)

with $G_3^{\mu\tau}$ given by (2.9). Equation (4.1) forces $M_\nu^{\mu\tau}$ to have the form

$$M_\nu^{\mu\tau} = \begin{pmatrix} A & B & B \\ B & C & D \\ B & D & C \end{pmatrix},$$
(4.2)

with A, B, C, D as mass dimensional complex quantities. It is well known that (4.2) leads to $\theta_{13} = 0$ and cannot be accepted as it stands.

Grimus and Lavoura made an alternative proposal, namely the complex-extended invariance relation

$$(G_3^{\mu\tau})^T M_\nu G_3^{\mu\tau} = M_\nu^*.$$
(4.3)

This was justified [27–29] by means of a non-standard CP-transformation [34–40] on the ν_e field which is generally represented as[1]

$$\nu_{L\alpha} \rightarrow i G_{\alpha\beta}\gamma^0 \nu_{L\beta}^C$$
(4.4)

---

[1] It is a theoretically interesting question whether such an extended CP-invariance can arise from an automorphism of a larger flavor symmetry like in the top-down approach of Ref. [42]. But we do not explore this possibility here.

with $G_{\alpha\beta}$ as the matrix element of the flavor symmetry. Equation (4.4) along with (1.1) leads to (4.3) if $G_{\alpha\beta}$ is considered as $G_3^{\mu\tau}$. Suffice it to say that (4.3) leads to a complex-extended $\mu\tau$ ($CE\mu\tau$) symmetric form of M_ν:

$$
\begin{pmatrix}
-(U_{e1}^{CES})^* & -(U_{e2}^{CES})^* & -(U_{e3}^{CES})^* \\
\frac{1-k^2}{1+k^2}(U_{\mu1}^{CES})^* + \frac{2k}{1+k^2}(U_{\tau1}^{CES})^* & \frac{1-k^2}{1+k^2}(U_{\mu2}^{CES})^* + \frac{2k}{1+k^2}(U_{\tau2}^{CES})^* & \frac{1-k^2}{1+k^2}(U_{\mu3}^{CES})^* + \frac{2k}{1+k^2}(U_{\tau3}^{CES})^* \\
\frac{2k}{1+k^2}(U_{\mu1}^{CES})^* - \frac{1-k^2}{1+k^2}(U_{\tau1}^{CES})^* & \frac{2k}{1+k^2}(U_{\mu2}^{CES})^* - \frac{1-k^2}{1+k^2}(U_{\tau2}^{CES})^* & \frac{2k}{1+k^2}(U_{\mu3}^{CES})^* - \frac{1-k^2}{1+k^2}(U_{\tau3}^{CES})^*
\end{pmatrix}
$$
$$
= \begin{pmatrix}
\tilde{d}_1 U_{e1}^{CES} & \tilde{d}_2 U_{e2}^{CES} & \tilde{d}_3 U_{e3}^{CES} \\
\tilde{d}_1 U_{\mu1}^{CES} & \tilde{d}_2 U_{\mu2}^{CES} & \tilde{d}_3 U_{\mu3}^{CES} \\
\tilde{d}_1 U_{\tau1}^{CES} & \tilde{d}_2 U_{\tau2}^{CES} & \tilde{d}_3 U_{\tau3}^{CES}
\end{pmatrix}. \tag{4.9}
$$

$$
M_\nu^{CE\mu\tau} = \begin{pmatrix} a & B & B^* \\ B & C & d \\ B^* & d & C^* \end{pmatrix}, \tag{4.5}
$$

where a, d are real and B, C are complex mass dimensional quantities in general. Once again, since the determinant does not vanish, we take all neutrino masses to be nonzero. The observable consequences of (4.5) are: $\theta_{23} = \pi/4$, $\cos\delta = 0$, α, $\beta = 0$ or π while θ_{13} is in general nonzero. A further extension of this approach has recently been made [26,41] allowing nonmaximal values for θ_{23} and Dirac CP-violation.

We have derived (3.3), i.e. $\tan\theta_{23} = k^{-1}$, so that atmospheric neutrino mixing is not forced to be strictly maximal. On the other hand, the observed fact that $\tan\theta_{23}$ is not far from unity implies that so is k. Our proposed relation, in place of (4.3), is

$$
\left(G_3^{scaling}\right)^T M_\nu G_3^{scaling} = M_\nu^*, \tag{4.6}
$$

with $G_3^{scaling}$ as given in (3.4) and, as stated earlier, in the basis where the charged lepton mass matrix is diagonal and positive. The general form of M_ν^{CES}, as given in (1.6), follows in consequence. It is important to note that M_ν^{CES} of (1.6) has a structure that is quite different from that of either M_ν^{SRS} of (1.5) or M_ν^{GRS} of (3.8). If all imaginary parts in M_ν^{CES} are set equal to zero, a form similar to that of M_ν^{GRS} is recovered but with all real entries, while those in M_ν^{GRS} of (3.8) are in general complex. Therefore, no special choice in M_ν^{CES} can yield M_ν^{SRS} or M_ν^{GRS} in their respective generalities.

Grimus and Lavoura [27–29] had proved a corollary of complex-extended invariance. This can be stated with respect to a relation such as (4.3) or (4.6) as

$$
G_3^{scaling} U^* = U\tilde{d} \tag{4.7}
$$

with $\tilde{d}$ as a diagonal matrix. Once again, $\tilde{d}_{lm} = \pm\delta_{lm}$ if the neutrino masses m_1, m_2, m_3 are all nonzero and nondegenerate. The key difference between (4.7) and (2.2) is the complex conjugation of U in the LHS. Let us take

$$
\tilde{d} = \text{diag}\,(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3), \tag{4.8}
$$

where each $\tilde{d}_i$ ($i = 1, 2, 3$) can be $+1$ or -1. With $G_3 = G_3^{scaling}$, (4.7) can be written explicitly:

It is evident from (4.9) that the choice $\tilde{d}_1 = 1$ leads to an imaginary U_{e1} in contradiction with the real (1,1) element of (1.3); this choice is hence excluded. Note that the choice of U_{PMNS} in (1.3) is simply due to the choice of the weak basis where the neutrino fields are phase rotated. However, in the appendix, we demonstrate that the physical results derived here are basis independent, i.e., the inclusion of an unphysical phase matrix does not impair our predictions. There are now four permitted cases a, b, c, d with the following four combinations allowed for $\tilde{d}$:

$$
\tilde{d}_a \equiv \text{diag}\,(-1, 1, 1), \tag{4.10}
$$
$$
\tilde{d}_b \equiv \text{diag}\,(-1, 1, -1), \tag{4.11}
$$
$$
\tilde{d}_c \equiv \text{diag}\,(-1, -1, 1), \tag{4.12}
$$
$$
\tilde{d}_d \equiv \text{diag}\,(-1, -1, -1). \tag{4.13}
$$

The above can be written compactly as

$$
\tilde{d}_{a,b,c,d} = \text{diag}(-1, \eta, \xi) \tag{4.14}
$$
$$
\eta_{a,b} = 1, \eta_{c,d} = -1, \tag{4.15}
$$
$$
\xi_{a,c} = 1, \xi_{b,d} = -1. \tag{4.16}
$$

Comparing with (1.3), we obtain

$$
e^{-i\alpha} = -\eta, \tag{4.17}
$$
$$
e^{i(2\delta-\beta)} = -\xi. \tag{4.18}
$$

Thus we are led to the result that $\alpha = \pi, 0$ for $\eta = +1, -1$ respectively; in a similar manner $2\delta - \beta = \pi, 0$ for $\xi = +1, -1$, respectively. We can derive from (4.9) altogether six independent constraint conditions as linear relations among various elements of U^{CES} and $(U^{CES})^*$. These are listed in Table 1.

More information is obtained by use of the explicit expressions of $U_{l\alpha}^{CES}$ from (1.3).

Consider the real and imaginary parts of the constraint condition on $U_{\tau3}^{CES}$ given in the bottom line in Table 1. Since c_{13} is known to be nonzero, it can be canceled from both

Table 1 Constraint equations on elements of the mixing matrix

Element of U^{CES}	Constraint condition
$\mu 1$	$2k U^{\mathrm{CES}}_{\mu 1} = (1-k^2) U^{\mathrm{CES}}_{\tau 1} - (1+k^2)(U^{\mathrm{CES}}_{\tau 1})^*$
$\tau 1$	$2k U^{\mathrm{CES}}_{\tau 1} = -(1-k^2) U^{\mathrm{CES}}_{\mu 1} - (1+k^2)(U^{\mathrm{CES}}_{\mu 1})^*$
$\mu 2$	$2k U^{\mathrm{CES}}_{\mu 2} = (1-k^2) U^{\mathrm{CES}}_{\tau 2} + \eta(1+k^2)(U^{\mathrm{CES}}_{\tau 2})^*$
$\tau 2$	$2k U^{\mathrm{CES}}_{\tau 2} = -(1-k^2) U^{\mathrm{CES}}_{\mu 2} + \eta(1+k^2)(U^{\mathrm{CES}}_{\mu 2})^*$
$\mu 3$	$2k U^{\mathrm{CES}}_{\mu 3} = (1-k^2) U^{\mathrm{CES}}_{\tau 3} + \xi(1+k^2)(U^{\mathrm{CES}}_{\tau 3})^*$
$\tau 3$	$2k U^{\mathrm{CES}}_{\tau 3} = -(1-k^2) U^{\mathrm{CES}}_{\mu 3} + \eta(1+k^2)(U^{\mathrm{CES}}_{\mu 3})^*$

sides. Now, from the respective real and imaginary parts, we have the relations

$$2k c_{23} \cos\frac{\beta}{2} = [k^2(1+\xi) - 1 + \xi] s_{23} \cos\frac{\beta}{2}, \tag{4.19}$$

$$2k c_{23} \sin\frac{\beta}{2} = [k^2(1-\xi) - 1 - \xi] s_{23} \sin\frac{\beta}{2}. \tag{4.20}$$

Since $\xi^2 = 1$, the product of the above two equations leads to the result

$$\sin\beta = 0, \tag{4.21}$$

or $\beta = 0$ or π. There are now four options:

$$\beta = 0, \xi = 1 \Rightarrow \tan\theta_{23} = k^{-1}, \tag{4.22}$$

$$\beta = 0, \xi = -1 \Rightarrow \tan\theta_{23} = -k, \tag{4.23}$$

$$\beta = \pi, \xi = 1 \Rightarrow \tan\theta_{23} = -k, \tag{4.24}$$

$$\beta = \pi, \xi = -1 \Rightarrow \tan\theta_{23} = k^{-1}. \tag{4.25}$$

The option $\beta = 0, \xi = 1$ for cases a and c, cf. (4.10) and (4.12), as well as $\beta = \pi, \xi = 1$ for cases b and d, cf. (4.11) and (4.13), yield the scaling relation (3.3) while the other two options require $\tan\theta_{23}$ to equal $-k$. As will be shown below, the latter possibility is inconsistent with other constraint conditions. Our final result on the Majorana phases is that both α and β are restricted to be 0 or π. A combination of information from $0\nu\beta\beta$ decay, the cosmological upper bound on $\Sigma_i m_i$ and the effective mass $\Sigma_i |U_{ei}|^2 m_i$ measured in single β-decay is expected to experimentally constrain [43] these phases.

To proceed, consider the constraint condition on $U^{\mathrm{CES}}_{\tau 2}$ given in the fourth line from the top of Table 1. The corresponding real and imaginary parts, respectively, yield

$$2k\left[c_{12}s_{23}\cos\frac{\alpha}{2} + s_{12}s_{13}c_{23}\cos\left(\delta+\frac{\alpha}{2}\right)\right]$$
$$= [1-k^2 - \eta(1+k^2)]\left[c_{12}c_{23}\cos\frac{\alpha}{2} - s_{12}s_{13}s_{23}\cos\left(\delta+\frac{\alpha}{2}\right)\right], \tag{4.26}$$

$$2k\left[c_{12}s_{23}\sin\frac{\alpha}{2} + s_{12}s_{13}c_{23}\sin\left(\delta+\frac{\alpha}{2}\right)\right]$$
$$= [1-k^2 - \eta(1+k^2)]\left[c_{12}c_{23}\sin\frac{\alpha}{2} - s_{12}s_{13}s_{23}\sin\left(\delta+\frac{\alpha}{2}\right)\right]. \tag{4.27}$$

Let us now take the two cases at hand.

Case 1: $\eta = 1, \alpha = \pi$
On utilizing that each of s_{12}, s_{13} and c_{23} is nonzero, one obtains from (4.26) and (4.27) the respective relations

$$(\tan\theta_{23} - k^{-1}) \sin\delta = 0, \tag{4.28}$$

$$c_{12}(\tan\theta_{23} - k^{-1}) + s_{12}s_{13}(1 + k^{-1}\tan\theta_{23})\cos\delta = 0. \tag{4.29}$$

Case 2: $\eta = -1, \alpha = 0$
It is easy to see that here one obtains the same pair of equations, namely (4.28) and (4.29), but in a reverse sequence.

Equation (4.29) has important implications. If $\tan\theta_{23}$ is put equal to $-k$, instead of k^{-1}, one is led to $c_{12} = 0$ in contradiction with experiment [9]. Therefore, the two options $\beta = 0, \xi = 1$ and $\beta = \pi, \xi = -1$ need to be retained with the other two options $\beta = 0, \xi = -1$ and $\beta = \pi, \xi = 1$ discarded. Now that $\tan\theta_{23}$ does equal k^{-1}, i.e. (3.3) holds, from (4.29) we have

$$\cos\delta = 0, \tag{4.30}$$

i.e. leptonic Dirac CP-violation is maximal with δ being either $\pi/2$ or $3\pi/2$. However, we are unable to distinguish between these two options since we have no statement on the sign of $\sin\delta$.

We have checked that (4.30) consistently follows from the remaining four constraint equations of Table 1 and that no new condition emerges. Finally, we are left with four options, as shown in Table 2. Each of these implies (4.30), i.e. the maximality of leptonic Dirac CP-violation which enters via U_{PMNS}.

Table 2 Predictions of the CP phases

$\tilde{d}$	α	β	$\cos\delta$
$\tilde{d}_a = \mathrm{diag}\,(-1,+1,+1)$	π	0	0
$\tilde{d}_b = \mathrm{diag}\,(-1,+1,-1)$	π	π	0
$\tilde{d}_c = \mathrm{diag}\,(-1,-1,+1)$	0	0	0
$\tilde{d}_d = \mathrm{diag}\,(-1,-1,-1)$	0	π	0

Table 3 Parameters of M_ν^{CES} in terms of the parameters of m_D and M_R

$x = -\left(\frac{a^2}{M_1} + \frac{e^2}{M_2} + \frac{f^2}{M_3}\right)$

$y_1 = \frac{1}{k}\left(\frac{ab_1}{M_1} + \frac{ec_1}{M_2} + \frac{fd_1}{M_3}\right)$

$y_2 = k\left(\frac{ab_2}{M_1} + \frac{ec_2}{M_2} + \frac{fd_2}{M_3}\right)$

$z_1 = -\frac{1}{k^2}\left(\frac{b_1^2}{M_1} + \frac{c_1^2}{M_2} + \frac{d_1^2}{M_3}\right) + k^2\left(\frac{b_2^2}{M_1} + \frac{c_2^2}{M_2} + \frac{d_2^2}{M_3}\right)$

$z_2 = \frac{2b_1b_2}{M_1} + \frac{2c_1c_2}{M_2} + \frac{2d_1d_2}{M_3}$

$w = \frac{1}{k}\left(\frac{b_1^2}{M_1} + \frac{c_1^2}{M_2} + \frac{d_1^2}{M_3}\right) + k\left(\frac{b_2^2}{M_1} + \frac{c_2^2}{M_2} + \frac{d_2^2}{M_3}\right)$

---

5 Origin of neutrino masses from type-I seesaw

In this section we discuss the realization of the complex-extended scaling neutrino mass matrix M_ν^{CES} through the type-I seesaw mechanism via three heavy right-handed neutrino fields N_{lR} ($l = 1, 2, 3$) with a Majorana mass matrix M_R. We choose a weak basis in which the charged lepton and the right-handed neutrino mass matrices are diagonal and nondegenarate. With m_D as the Dirac mass matrix and $M_R = \text{diag}\,(M_1, M_2, M_3)$, the neutrino mass terms read

$$-\mathcal{L}_{\text{mass}}^{\nu,N} = \bar{N}_{lR}(m_D)_{l\alpha}L_\alpha + \frac{1}{2}\bar{N}_{lR}(M_R)_l\delta_{lm}N_{mR}^C + \text{h.c.} \tag{5.1}$$

The effective light neutrino mass matrix is given by the standard seesaw relation,

$$M_\nu = -m_D^T M_R^{-1} m_D. \tag{5.2}$$

We represent the G, introduced earlier for left-handed fields, generically by G_L and define a corresponding G_R for N_R. The residual CP-transformations on the neutrino fields are defined by [41]

$$\nu_{L\alpha} \rightarrow i(G_L)_{\alpha\beta}\gamma^0\nu_{L\beta}^C, \quad N_{R\alpha} \rightarrow i(G_R)_{\alpha\beta}\gamma^0N_{R\beta}^C. \tag{5.3}$$

The invariance of the mass terms of (5.1) under the CP-transformations defined in (5.3) leads to the relations

$$G_R^\dagger m_D G_L = m_D^*, \quad G_R^\dagger M_R G_R^* = M_R^*. \tag{5.4}$$

Equations (5.2) and (5.4) together imply $G_L^T M_\nu G_L = M_\nu^*$. Now, specifying G_L by G_3^{scaling}, we obtain the key equation

$$\left(G_3^{\text{scaling}}\right)^T M_\nu G_3^{\text{scaling}} = M_\nu^*. \tag{5.5}$$

Since we choose the right-handed neutrino mass matrix M_R to be diagonal, the symmetry matrix G_R is diagonal with entries ± 1, i.e.

$$G_R = \text{diag}\,(\pm 1, \pm 1, \pm 1). \tag{5.6}$$

Hence there are eight different structures of G_R. Correspondingly, from the first relation of (5.4), there are eight different structures of m_D. Unlike the complex transformations of m_D

and M_R in (5.4), we now have real symmetry transformations $G_R^\dagger m_D G_L = m_D$ and $G_R^\dagger M_R G_R^* = M_R$. It can be shown by tedious algebra that, except for $G_R = \text{diag}\,(-1, -1, -1)$, all other structures of G_R are incompatible with scaling symmetry, i.e. cannot generate M_ν^{GRS}. Thus we take $G_R = \text{diag}\,(-1, -1, -1)$ as the only viable residual symmetry on the right-handed neutrino field. Now, $G_R^\dagger m_D G_L = m_D^*$ can be written as

$$m_D G_L = -m_D^*, \tag{5.7}$$

which is basically a complex extension of the Joshipura–Rodejohann result $m_D G_L = -m_D$ [15–21]. In our context, (5.7) can be rewritten as

$$m_D G_3^{\text{scaling}} = -m_D^*. \tag{5.8}$$

The most general m_D that satisfies (5.8) is

$$m_D^{\text{CES}} = \begin{pmatrix} a & b_1 + ib_2 & -b_1/k + ib_2k \\ e & c_1 + ic_2 & -c_1/k + ic_2k \\ f & d_1 + id_2 & -d_1/k + id_2k \end{pmatrix}, \tag{5.9}$$

where a, $b_{1,2}$, $c_{1,2}$, $d_{1,2}$, e, and f are arbitrary real mass dimensional quantities. Using (5.2), M_ν^{CES} of (1.6) is obtained with the parameters as given in Table 3. Some detailed interesting consequences of m_D^{CES}, specifically with respect to leptogenesis, will be studied elsewhere.

We notice that, for both types of ordering, the neutrino masses become hierarchical, i.e. $m_{2,1} \ll m_3$ for normal ordering and $m_{2,1} \gg m_3$ for inverted ordering, for low values of the lightest neutrino mass. However, they tend toward quasi-degeneracy $m_1 \sim m_2 \sim m_3$ as the latter increases to its permitted maximum value ~ 0.07 eV. This is clear from the mass bands shown in Fig. 1.

6 Phenomenological constraints and consequences

We need to numerically pin down the mass dimensional six real parameters x, y_1, y_2, z_1, z_2 and w of M_ν^{CES} by inputting the 3σ ranges of quantities measured in neutrino oscillation experiments. To that end, we use the values from a recent global analysis [9]. In addition, we use the cosmological upper limit [44] of 0.23 eV on the sum $m_1 + m_2 + m_3$ of the masses of the neutrinos.

These input numbers are shown in Table 4. In terms of output, we obtain the 3σ allowed intervals of the above mentioned six real parameters and from those the 3σ allowed ranges of the individual neutrino masses m_1, m_2, m_3. Both types of neutrino mass ordering, normal as well as inverted, are found to be allowed. All these values are listed in Tables 5 and 6 respectively for the two separate categories of mass ordering.

Fig. 1 Plots of the mass band for normal (*left*) and inverted (*right*) mass ordering. We have choosen to plot the lightest eigenvalue also in the ordinate to bring three mass bands together. Color code: *green* (m_3), *red* (m_2) and *blue* (m_1)
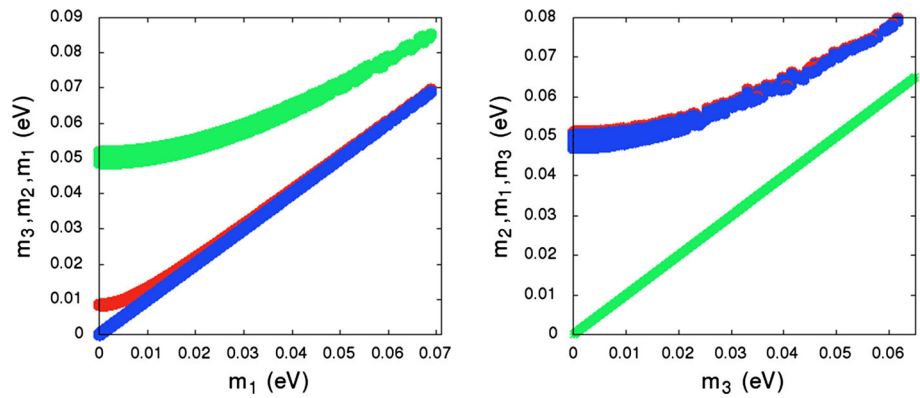

Table 4 Input values used

θ_{12} degrees	θ_{23} degrees	θ_{13} degrees	$\Delta m^2_{21} 10^{-5} \text{eV}^2$	$|\Delta m^2_{31}| 10^{-3} (\text{eV}^2)$	$\Sigma_i m_i (\text{eV})$
$31.29 - 35.91$	$38.3 - 53.3$	$7.87 - 9.11$	$7.02 - 8.09$	$2.32 - 2.59$	< 0.23

Table 5 Output values obtained for normal mass ordering with the best fit m's given within brackets

x (eV)	y_1 (eV)	y_2 (eV)	z_1 (eV)	z_2 (eV)	w (eV)
$-0.20 - +0.21$	$-0.12 - +0.11$	$-0.05 - +0.05$	$-0.17 - +0.17$	$-0.18 - +0.17$	$-0.16 - +0.15$
m_1 (eV)		m_2 (eV)		m_3 (eV)	
$9.2 \times 10^{-5} - 0.071 \, (0.052)$		$0.01 - 0.077 \, (0.054)$		$0.051 - 0.082 \, (0.072)$	

Table 6 Output values obtained for inverted mass ordering with the best fit m's given within brackets

x (eV)	y_1 (eV)	y_2 (eV)	z_1 (eV)	z_2 (eV)	w (eV)
$-0.44 - +0.46$	$-0.16 - +0.16$	$-0.14 - +0.14$	$-0.01 - +0.01$	$-0.01 - +0.01$	$-0.05 - +0.06$
m_1 (eV)		m_2 (eV)		m_3 (eV)	
$0.049 - 0.079 \, (0.068)$		$0.051 - 0.085 \, (0.069)$		$8.2 \times 10^{-5} - 0.068 \, (0.048)$	

Neutrinoless double beta decay $0\nu\beta\beta$

This is the lepton number violating process

$$(A, Z) \longrightarrow (A, Z + 2) + 2e^- \tag{6.1}$$

with no final state neutrinos. An observation of the decay will confirm the Majorana nature of neutrinos which is yet to be established. The corresponding half-life [45,46] is given by

$$\frac{1}{T^{0\nu}_1/2} = G_{0\nu}|M_{0\nu}|^2|M^\nu_{ee}|^2 m_e^{-2}. \tag{6.2}$$

where $G_{0\nu}$ is a phase space factor, $M_{0\nu}$ is the nuclear matrix element (NME), m_e is the electron mass and finally $|M^\nu_{ee}|$ is the (1,1) element of M^ν which can also be written as $\Sigma_i m_i U^2_{ei}$. Following the PDG parametrization of the mixing matrix U_{PMNS}, one can write M^ν_{ee} as

$$M^\nu_{ee} = c^2_{12}c^2_{13}m_1 + s^2_{12}c^2_{13}m_2 e^{i\alpha} + s^2_{13}m_3 e^{i(\beta-2\delta)}. \tag{6.3}$$

There are several ongoing experiments which have put significant upper limits on $|M^\nu_{ee}|$. Some recent experiments

like KamLAND-Zen [47] and EXO [48] have improved this upper bound to 0.35 eV. However, the most significant upper bound on $|M^\nu_{ee}|$ to date is put by GERDA phase-I data [49] to be 0.22 eV; this is likely to be lowered by GERDA phase -II data [50] to around 0.098 eV.

In our model there are four sets of values of the CP-violating phases α and β for each neutrino mass ordering. Since $|M^\nu_{ee}|$ is sensitive to the CP phases, we get four different plots for each mass ordering as shown in Fig. 2. The same plots are valid for both types of mass ordering provided the horizontal axis is taken to represent the lightest neutrino mass m_1 or m_3—depending on the ordering.

As mentioned earlier, we have used the upper bound of 0.23 eV on $\Sigma_i m_i$. These plots lead to upper bounds on the lightest neutrino mass for both cases of mass ordering. For hierarchical neutrinos, $|M^\nu_{ee}|$ is found to lead to an upper limit which is below the reach of the GERDA phase-II data. The latter appears close to being obtainable only for a quaside-generate neutrino mass spectrum ($m_{\text{lightest}} > 0.07$ eV). How-ever, the value predicted in our model could be probed by a
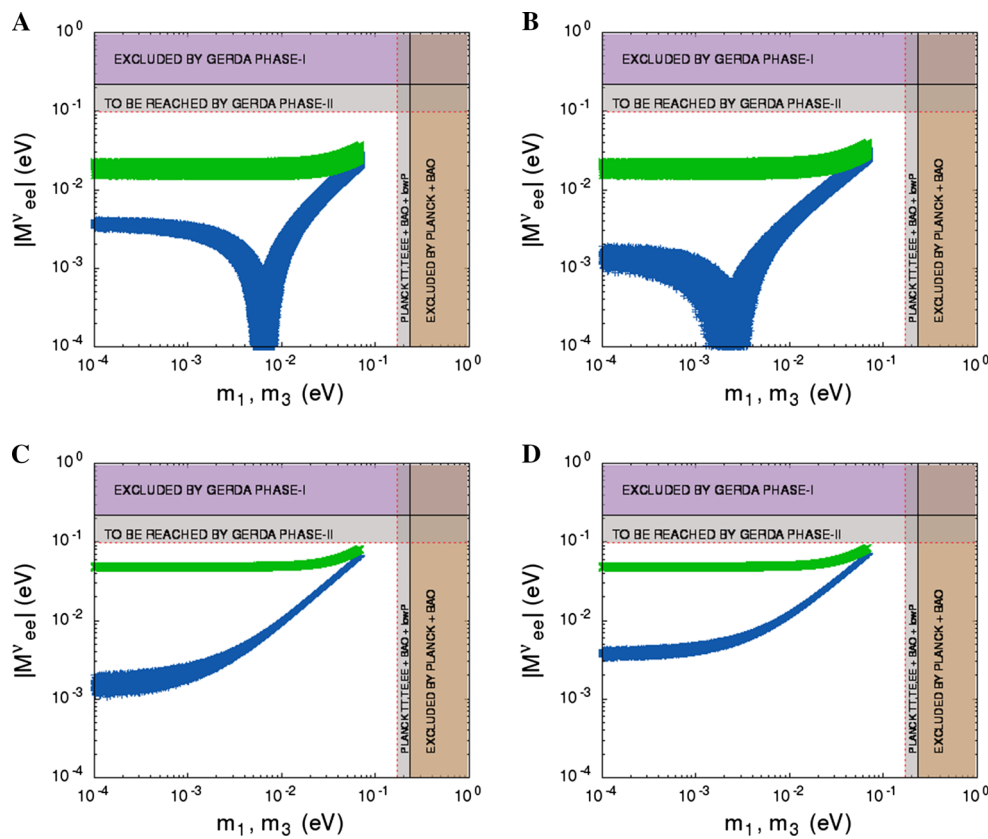
Fig. 2 Plot of $|M_{ee}^v|$ vs. the lightest neutrino mass: the *top* two figures represent Case A (*left*) and Case B (*right*) while the figures in the *lower panel* represent Case C (*left*) and Case D (*right*)

combination of GERDA and MAJORANA experiments [51]. In order to explain the nature of the plots analytically, let us first consider the inverted mass ordering: In this case, with the approximations $m_3 \simeq 0$ and $m_1 \simeq m_2$, the probed effective mass $|M_{ee}^v|$ simplifies to

$$|M_{ee}^v| = \sqrt{|\Delta m_{32}|^2} c_{13}^2 \left[\left\{ 1 - s_{12}^2 (1 - \cos \alpha) \right\}^2 + s_{12}^4 \sin^2 \alpha \right]^{1/2}. \quad (6.4)$$

Clearly, $|M_{ee}^v|$ is insensitive to the phases β and δ. On the other hand, for $\alpha = 0$ and π (6.4) simplifies to

$$|M_{ee}^v| = \sqrt{|\Delta m_{32}|^2} c_{13}^2 \quad (6.5)$$

and

$$|M_{ee}^v| = \sqrt{|\Delta m_{32}|^2} c_{13}^2 \left[\left\{ 1 - 2s_{12}^2 \right\}^2 \right], \quad (6.6)$$

respectively. Hence, for $\alpha = \pi$ (cases A, B), $|M_{ee}^v|$ is suppressed as compared to the case $\alpha = 0$ (C, D). For a normal mass ordering, in addition to the s_{13} suppression, there is a significant interference between the first two terms, thus lowering the value of $|M_{ee}^v|$. However, if $\alpha = 0$, the first two terms interfere constructively and then we obtain a lower

bound ($\sim 10^{-3}$ eV for Case C and $\sim 5 \times 10^{-3}$ eV for Case D) despite this being a case of normal mass ordering. This is one of the remarkable results of the present analysis. On the other hand, for $\alpha = \pi$, the first two terms interfere destructively, for the case of a normal mass ordering; consequently, a sizable cancelation between them brings down the value of $|M_{ee}^v|$ and results in the kinks shown by the lower curves in the top two figures.

CP asymmetry in neutrino oscillations

Here we discuss the determination of our predicted maximal Dirac CP-violating phase δ by means of neutrino oscillation studies. This δ will show up in the asymmetry parameter $A_{\alpha\beta}$, defined as

$$A_{\alpha\beta} = P(\nu_\alpha \rightarrow \nu_\beta) - P(\bar{\nu_\alpha} \rightarrow \bar{\nu_\beta}), \quad (6.7)$$

where $\alpha, \beta = (e, \mu, \tau)$ are flavor indices and the P are transition probabilities. Let us consider first $\nu_\mu \rightarrow \nu_e$ oscillation in vacuum. The transition probability can now be written (with the superscript zero indicating oscillations in vacuum) as

$$P_{\mu e}^0 \equiv P^0(\nu_\mu \rightarrow \nu_e) = P_{atm}^0 + P_{sol}^0 + 2\sqrt{P_{atm}^0}\sqrt{P_{sol}^0} \cos(\Delta_{32} + \delta), \quad (6.8)$$
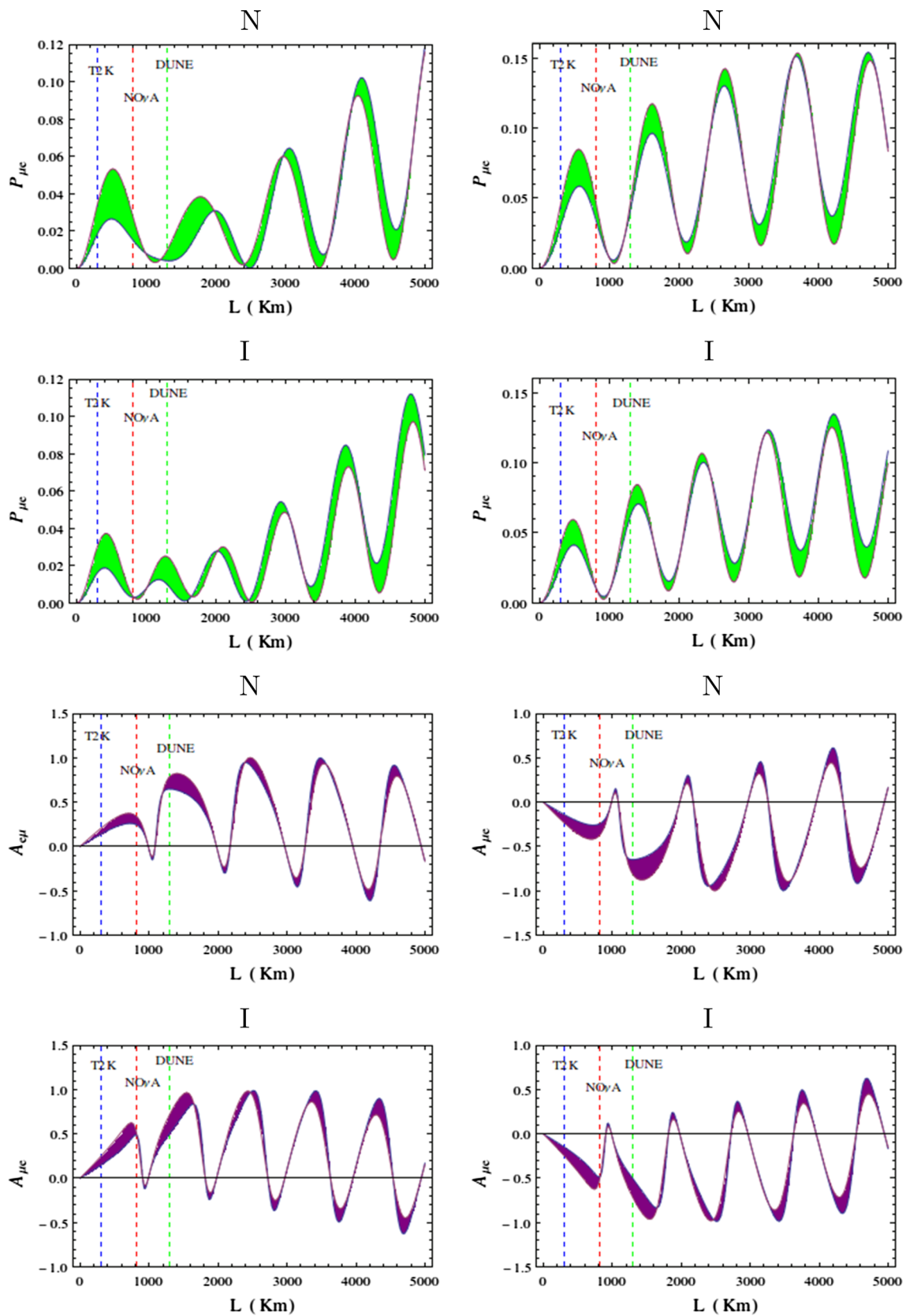
Fig. 3 Plots of the transition probability ($P_{\mu e}$) and CP-asymmetry parameter ($A_{\mu e}$) with baseline length L for $\delta = \pi/2$ (*left panel*) and $\delta = 3\pi/2$ (*right panel*) with $E = 1$ GeV. Cases for normal (*inverted*) mass ordering have been labeled on *top* by $N(I)$. The bands are caused by the atmospheric mixing angle θ_{23} being allowed to vary within the 3σ region while the other parameters are kept at their best fit values
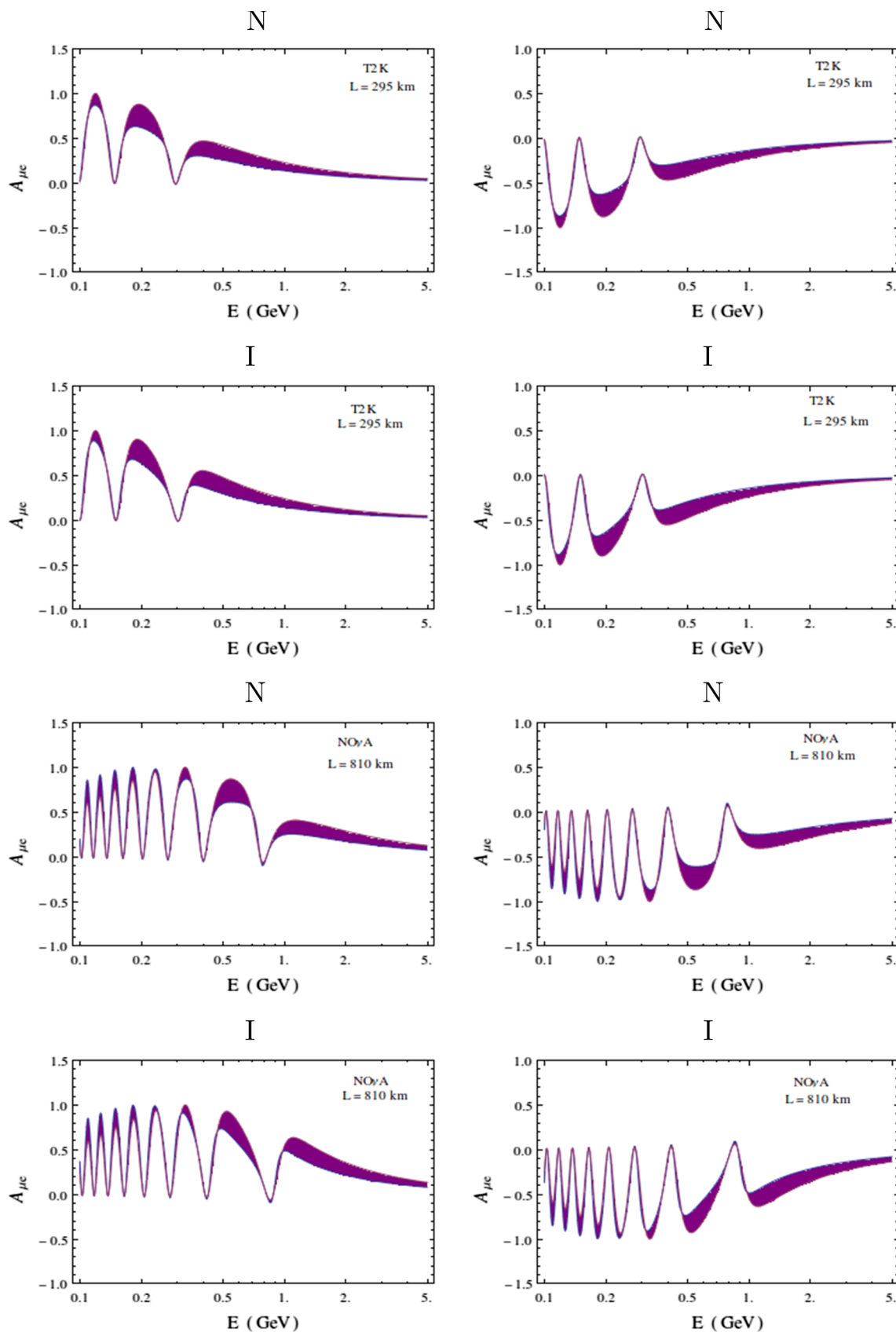
Fig. 4 Plots of the CP-asymmetry parameter $A_{\mu e}$ against beam energy E for $\delta = \pi/2$ (*left panel*) and $\delta = 3\pi/2$ (*right panel*) for various experiments as shown. Cases for normal (*inverted*) mass ordering have been labeled on *top* by $N(I)$. The atmospheric mixing angle θ_{23} is allowed to vary within the 3σ region, leading to the bands, while the other parameters are kept at their best fit values
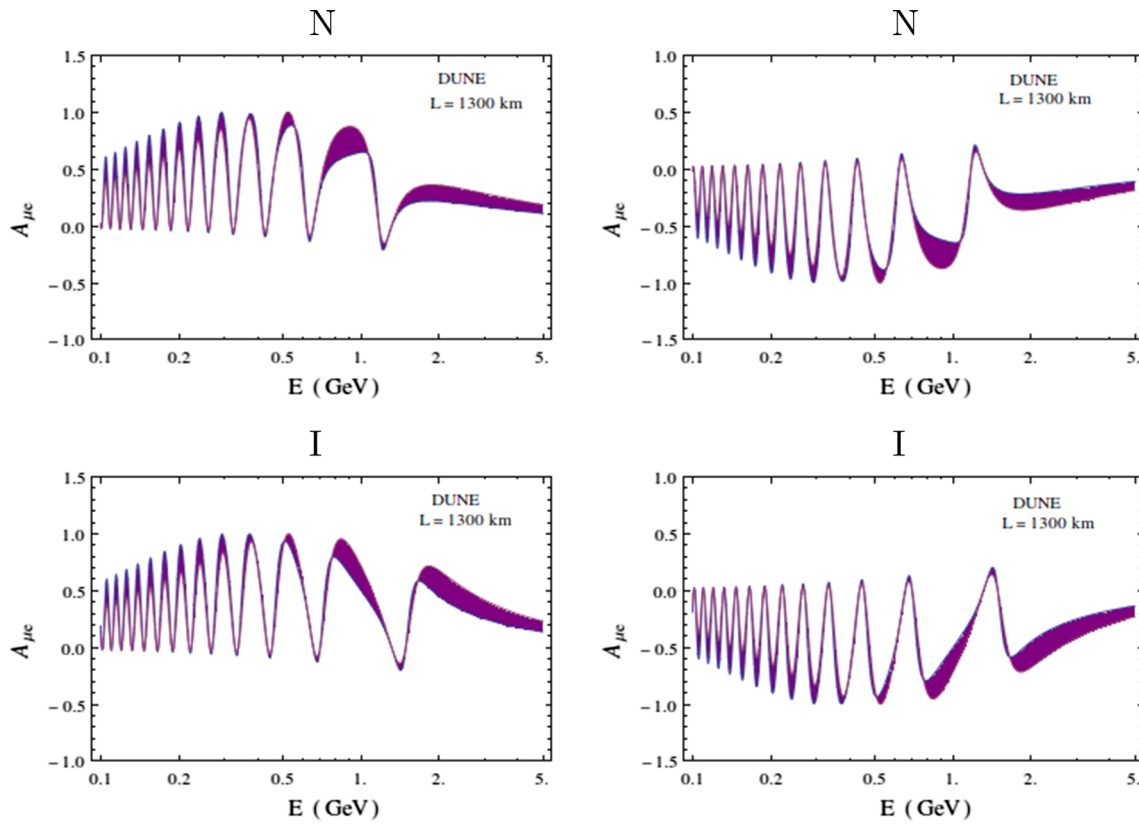
Fig. 4 continued

where $\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E}$ is the kinematic phase factor (L being the baseline length and E being the beam energy) and P_{atm}^0, P_{sol}^0 are, respectively, defined as

$$\sqrt{P_{\text{atm}}^0} = \sin\theta_{23}\sin 2\theta_{13}\sin\Delta_{31}, \tag{6.9}$$

$$\sqrt{P_{\text{sol}}^0} = \cos\theta_{23}\cos\theta_{13}\sin 2\theta_{12}\sin\Delta_{21}. \tag{6.10}$$

For an antineutrino beam, δ is replaced by $-\delta$ and thus we have

$$\bar{P}_{\mu e}^0 \equiv P^0(\bar{\nu}_\mu \to \bar{\nu}_e) = P_{\text{atm}}^0 + P_{\text{sol}}^0$$
$$+ 2\sqrt{P_{\text{atm}}^0}\sqrt{P_{\text{sol}}^0}\cos(\Delta_{32} - \delta). \tag{6.11}$$

Now the CP-asymmetry parameter $A_{\mu e}^0$ in vacuum [52] can be calculated as

$$A_{\mu e}^0 = \frac{P_{\mu e}^0 - \bar{P}_{\mu e}^0}{P_{\mu e}^0 + \bar{P}_{\mu e}^0}$$
$$= \frac{2\sqrt{P_{\text{atm}}^0}\sqrt{P_{\text{sol}}^0}\sin\Delta_{32}\sin\delta}{P_{\text{atm}}^0 + P_{\text{sol}}^0 + 2\sqrt{P_{\text{atm}}^0}\sqrt{P_{\text{sol}}^0}\cos\Delta_{32}\cos\delta}. \tag{6.12}$$

Table 7 Predictions of the CP phases for $\tilde{d}_1 = 1$

$\tilde{d}$	α	β	$\cos\delta$
$\tilde{d}_e = \text{diag}(-1, +1, +1)$	0	0	0
$\tilde{d}_f = \text{diag}(-1, +1, -1)$	0	π	0
$\tilde{d}_g = \text{diag}(-1, -1, +1)$	π	0	0
$\tilde{d}_h = \text{diag}(-1, -1, -1)$	π	π	0

With our prediction $\cos\delta = 0$, (6.12) can be rewritten as

$$A_{\mu e}^0 = \pm\frac{2\sqrt{P_{\text{atm}}^0}\sqrt{P_{\text{sol}}^0}\sin\Delta_{32}}{P_{\text{atm}}^0 + P_{\text{sol}}^0}, \tag{6.13}$$

with a $+$ $(-)$ sign for $\delta = \pi/2$ $(3\pi/2)$.

In order to realistically describe neutrino oscillations in long baseline experiments, matter effects in neutrino propagation through the earth need to be taken into account. In that case P_{atm}^0 and P_{sol}^0 will be modified to

$$\sqrt{P_{\text{atm}}} = \sin\theta_{23}\sin 2\theta_{13}\frac{\sin(\Delta_{31} - aL)}{\Delta_{31} - aL}\Delta_{31}, \tag{6.14}$$

$$\sqrt{P_{\text{sol}}} = \cos\theta_{23}\cos\theta_{13}\sin 2\theta_{12}\frac{\sin aL}{aL}\sin\Delta_{21}, \tag{6.15}$$

respectively. Here $a = G_F N_e / \sqrt{2}$ with G_F as the Fermi constant and N_e as the number density of electrons in the medium of propagation. An approximate value of a for the earth is 3500 km^{-1} [52]. Now the same formulas for $P_{\mu e}$, $\bar{P}_{\mu e}$ and $A_{\mu e}$ will hold as in (6.8), (6.11) and (6.12) but with P_{atm}^0 and P_{sol}^0 replaced by P_{atm} and P_{sol}, respectively.

In Fig. 3 we plot $P_{\mu e}$ and $A_{\mu e}$ against the baseline length L in the two cases $\delta = \pi/2$ and $\delta = 3\pi/2$ for both normal and inverted mass ordering. The lengths corresponding to T2K, NOνA and DUNE are indicated in these figures. In Fig. 4 the CP asymmetry $A_{\mu e}$ is plotted against the beam energy E again for the cases $\delta = \pi/2$ and $\delta = 3\pi/2$ separately for the three above cited experiments; both normal and inverted mass ordering cases are included. As expected, $A_{\mu e}$ has opposite signs for $\delta = \pi/2$ and $\delta = 3\pi/2$. It is further interesting that the extrema of the CP-asymmetry parameter exhibit opposite behavior as a function of E for $\delta = \pi/2$ and $\delta = 3\pi/2$.

7 Summary

In this paper we have proposed a complex extension of the scaling ansatz for the neutrino Majorana mass matrix M^ν. To that end, we have made use of the residual $\mathbb{Z}_2 \times \mathbb{Z}_2^{\text{scaling}}$ symmetry of M^ν by obtaining the representation G_3^{scaling} from the original simple scaling ansatz on M^ν. The resultant form of the neutrino Majorana matrix is given by M_ν^{CES} of (1.6). We have shown that it admits nonzero values of all the physical neutrino masses as well as both normal and inverted types of mass ordering. We have shown how a nonvanishing θ_{13} emerges from M_ν^{CES}. The additional result $k^{-1} = \tan\theta_{23}$, k being the real positive scaling factor, has also been derived. Dirac CP-violation has been shown to be maximal with $\cos\delta = 0$, while Majorana CP-violation has been demonstrated to be absent with $\alpha, \beta = 0$ or π. The type-I seesaw mechanism which yields nonzero neutrino masses within our scheme has also been constructed. Phenomenological implications for both $0\nu\beta\beta$ decay and neutrino/antineutrino oscillation studies at long baselines have been worked out and projections made that will be testable in forthcoming experiments.

Appendix: Derivation of the results on CP-violation even with the inclusion of the unphysical matrix

As mentioned in Sect. 1, our calculations have been done in a weak basis where the unphysical phases are absorbed in the neutrino fields. However, one can also reproduce our results including this unphysical phase matrix in the calculation. In that case the U_{PMNS} of (1.3) can be written as

$$U_{\text{PMNS}} = P_\phi U, \tag{A.1}$$

with $P_\phi = \text{diag.}(e^{i\phi}, 1, 1)$. Note that there is only a single unphysical phase in the phase matrix P_ϕ, since the symmetry under consideration dictates M_ν^{CES} in (1.6) to contain seven real parameters which correspond to three nonzero masses, three mixing angles and an unphysical phase. Now for $\tilde{d}_1 = -1$, Eq. (4.9) and the (1,1) element of U_{PMNS} in (A.1) give

$$e^{-2i\phi} = 1, \tag{A.2}$$

therefore, $\phi = 0$ or π. From the (1,2) element we get

$$e^{-i(\alpha+2\phi)} = -\eta. \tag{A.3}$$

Thus for both values of ϕ, (A.3) leads to (4.17); therefore, for each $\tilde{d}$ matrix with $\tilde{d}_1 = -1$, the prediction for α, i.e., $\alpha = 0$ or π remain the same. Now, following the same way as in Sect. 4, the results presented in Table 2 can be reproduced.

Unlike the previous case, now $\tilde{d}_1 = 1$ cannot be ruled out. In this case, from the (1,1) element of U_{PMNS} in (A.1), we get $\phi = \pi/2$ or $3\pi/2$. Now, for both values of ϕ, Eq. (A.3) with $\eta = 1$ leads to $\alpha = 0$, and with $\eta = -1$ leads to $\alpha = \pi$. Since the predictions for α remain the same, so do the other parameters which are solved exactly in the same way as in Sect. 4, by use of *both* real and imaginary parts of the relevant complex equations. We put the more general statements regarding the CP phases for each $\tilde{d}$ with $\tilde{d}_1 = 1$ in Table 7. In comparison with Table 2, the values of α have changed relative to those of β, but the final result that both α and β are either 0 or π remains the same, though the value of $\tilde{d}_1$ has changed.

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