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Kinematics of radion field: a possible source of dark matter

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Abstract The discrepancy between observed virial and baryonic mass in galaxy clusters have lead to the missing mass problem. To resolve this, a new, non-baryonic matter field, known as dark matter, has been invoked. However, till date no possible constituents of the dark matter components are known. This has led to various models, by modifying gravity at large distances to explain the missing mass problem. The modification to gravity appears very naturally when effective field theory on a lower-dimensional manifold, embedded in a higher-dimensional spacetime is considered. It has been shown that in a scenario with two lower-dimensional manifolds separated by a finite distance is capable to address the missing mass problem, which in turn determines the kinematics of the brane separation. Consequences for galactic rotation curves are also described.

1 Introduction

Recent astrophysical observations strongly suggest the existence of non-baryonic dark matter at the galactic as well as extra-galactic scales (if the dark matter is baryonic in nature, the third peak in the Cosmic Microwave Background power spectrum would have been lower compared to the observed height of the spectrum [1]). These observations can be divided into two branches – (a) behavior of galactic rotation curves and (b) mass discrepancy in clusters of galaxies [2].

The first one, i.e., rotation curves of spiral galaxies, shows clear evidence of problems associated with Newtonian and general relativity prescriptions [2–4]. In these galaxies neutral hydrogen clouds are observed much beyond the extent of luminous baryonic matter. In a Newtonian description, the equilibrium of these clouds moving in a circular orbit of radius r is obtained through equality of centrifugal and grav-

itational force. For cloud velocity $v(r)$, the centrifugal force is given by v^2/r and the gravitational force by $GM(r)/r^2$, where $M(r)$ stands for total gravitational mass within radius r. Equating these two will lead to the mass profile of the galaxy: $M(r) = rv^2/G$. This immediately posed serious problem, for at large distances from the center of the galaxy, the velocity remains nearly constant $v \sim 200$ km/s, which suggests that mass inside radius r should increase monotonically with r, even though at large distance very little luminous matter can be detected [2–4].

The mass discrepancy of galaxy clusters also provides direct hint for existence of dark matter. The mass of galaxy clusters, which are the largest virialized structures in the universe, can be determined in two possible ways – (i) from the knowledge about motion of the member galaxies one can estimate the virial mass M_V, second, (ii) estimating mass of individual galaxies and then summing over them in order to obtain total baryonic mass M. Almost without any exception M_V turns out to be much large compared to M, typically one has $M_V/M \sim 20 - 30$ [2–4]. Recently, new methods have been developed to determine the mass of galaxy clusters; these are (i) dynamical analysis of hot X-ray emitting gas [5] and (ii) gravitational lensing of background galaxies [6] – these methods also lead to similar results. Thus dynamical mass of galaxy clusters are always found to be in excess compared to their visible or baryonic mass. This missing mass issue can be explained through postulating that every galaxy and galaxy cluster is embedded in a halo made up of dark matter. Thus the difference $M_V - M$ is originating from the mass of the dark matter halo the galaxy cluster is embedded in.

The physical properties and possible candidates for dark matter can be summarized as follows: dark matter is assumed to be non-relativistic (hence cold and pressure-less), interacting only through gravity. Among many others, the most popular choice being weakly interacting massive particles. Among different models, the one with sterile neutrinos (with masses of several keV) has attracted much attention [7,8]. Despite some success it comes with its own limitations. In

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the sterile neutrino scenario the X-ray produced from their decay can enhance production of molecular hydrogen and thereby speeding up cooling of gas and early star formation [9]. Even after a decade long experimental and observational efforts no non-gravitational signature for the dark matter has ever been found. Thus a priori the possibility of breaking down of gravitational theories at galactic scale cannot be excluded [10–17].

A possible and viable way to modify the behavior of gravity in our four-dimensional spacetime is by introducing extra spatial dimensions. The extra dimensions were first introduced to explain the hierarchy problem (i.e., observed large difference between the weak and Planck energy scales) [18–20]. However, the initial works did not incorporate gravity, but they used large extra dimensions (and hence a large volume factor) to reduce the Planck scale to TeV scale. The introduction of gravity, i.e., warped extra dimensions, drastically altered the situation. In [21] it was first shown that an anti-de Sitter solution in higher-dimensional spacetime (henceforth referred to as bulk) leads to exponential suppression of the energy scales on the visible four-dimensional embedded sub-manifold (called a brane) thereby solving the hierarchy problem. Even though this scenario of a warped geometry model solves the hierarchy problem, it also introduces additional correction terms to the gravitational field equations, leading to deviations from Einstein's theory at high energy, with interesting cosmological and black hole physics applications [22–30]. This conclusion is not bound to Einstein's gravity alone but it holds in higher curvature gravity theories[1] as well [29,30,38]. Since the gravitational field equations get modified due to the introduction of extra dimensions it is legitimate to ask whether it can solve the problem of missing mass in galaxy clusters. Several works in this direction exist and can explain the velocity profile of galaxy clusters. However, they emerge through the following setup:

- Obtaining effective gravitational field equations on a lower-dimensional hypersurface, starting from the full bulk spacetime, which involves additional contributions from the bulk Weyl tensor. The bulk Weyl tensor in spherically symmetric systems leads to a component behaving as mass and is known as "dark mass" (we should emphasize that this notion extends beyond Einstein's gravity and holds for any arbitrary dimensional reduction

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[1] In addition to the introduction of extra dimensions we could also modify the gravity theory without invoking ghosts, which uniquely fixes the gravitational Lagrangian to be Lanczos–Lovelock Lagrangian. These Lagrangians have special thermodynamic properties and also modify the behavior of four-dimensional gravity [31–37]. However, in this work we shall confine ourselves exclusively within the framework of Einstein gravity and shall try to explain the missing mass problem from kinematics of the radion field.

[29,30,38]). It has been shown in [39] that the introduction of the "dark mass" term is capable to yield an effect similar to the dark matter. Some related aspects were also explored in [40–43], keeping the conclusions unchanged.

- In the second approach, the bulk spacetime is always taken to be anti-de Sitter such that bulk Weyl tensor vanishes. Unlike the previous case, which required S^1/Z_2 orbifold symmetry, arbitrary embedding has been considered in [44] following [45]. This again introduces additional corrections to the gravitational field equations. These additional correction terms in turn lead to the observed virial mass for galaxy clusters.

However, all these approaches are valid for a single brane system. In this work we generalize previous results for a two brane system. This approach not only gives a handle on the hierarchy problem at the level of Planck scale but is also capable of explaining the missing mass problem at the scale of galaxy clusters. Moreover, in this setup the additional corrections will depend on the radion field (for a comprehensive discussion see [26]), which represents the separation between the two branes. Hence in our setup the missing mass problem for galaxy clusters can also shed some light on the kinematics of the separation between the two branes.

Further the same setup is also shown to explain the observed rotation curves of galaxies as well. Hence both problems associated with dark matter, namely the missing mass problem for galaxy clusters and the rotation curves for galaxies, can be explained by the two brane system introduced in this work via the kinematics of the radion field.

The paper is organized as follows – in Sect. 2, after providing a brief review of the setup we have derived effective gravitational field equations on the visible brane which will involve additional correction terms originating from the radion field to modify the gravitational field equations. In Sect. 3 we have explored the connection between the radion field, dark matter, and the mass profile of galaxy clusters using relativistic Boltzmann equations along with Sect. 4 describing possible applications. Then in Sect. 5 we have discussed the effect of our model on the rotation curve of galaxies while Sect. 6 deals with a few applications of our result in various contexts. Finally, we conclude with a discussion of our results.

Throughout our analysis, we have set the fundamental constant c to unity. All the Greek indices $\mu, \nu, \alpha, \ldots$ run over the brane coordinates. We will also use the standard signature $(-+ + \cdots)$ for the spacetime metric.

2 Effective gravitational field equations on the brane

The most promising candidate for getting effective gravitational field equations on the brane originates from the Gauss–

Codazzi equation. However, these equations are valid on a lower-dimensional hypersurface (i.e., on the brane) embedded in a higher-dimensional bulk. Hence this works only for a single brane system. But the brane world model, addressing the hierarchy problem, requires the existence of two branes, where the above method is not applicable. To tackle the problem of a two brane system we need to invoke the radion field (i.e., the separation between two branes), which has significant role in the effective gravitational field equations. The bulk metric ansatz incorporating the above features takes the following form:

$$\mathrm{d}s^2 = e^{2\phi(y,x)}\mathrm{d}y^2 + q_{\mu\nu}(y,x)\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{1}$$

The positive and negative tension branes are located at $y = 0$ and $y = y_0$, respectively, such that the proper distance between the two branes being given by $d_0(x) = \int_0^{y_0} \mathrm{d}y \exp\phi(x,y)$ and $q_{\mu\nu}$ stands for the induced metric on $y = $ constant hypersurfaces. The effective field equations on the brane depend on the extrinsic curvature, $K_{\mu\nu} = (1/2)\pounds_n q_{\mu\nu}$, where the normal to the surface is $n = \exp(-\phi)\partial_y$ but it also inherits a non-local bulk contribution through $E_{\mu\nu} = {}^{(5)}C_{\mu\alpha\nu\beta}n^\alpha n^\beta$, ${}^{(5)}C_{\mu\alpha\nu\beta}$ being the bulk Weyl tensor. At first glance it seems that due to non-local bulk effects the effective field equations cannot be solved in closed form, but, as we will briefly describe, it can be achieved through radion dynamics and at low energy scales [46].

We will now proceed to derive low energy gravitational field equations. As we have already stressed, unless one solves for the non-local effects from the bulk the system of equations would not close. Further it will be assumed that curvature scale on the brane, L, is much larger than that of bulk, ℓ. Then we can expand all the relevant geometrical quantities in terms of the small, dimensionless parameter $\epsilon = (\ell/L)^2$. At zeroth order of this expansion, one recovers ${}^{(0)}q_{\mu\nu}(y,x) = h_{\mu\nu}(x)\exp(-2d(y,x)/\ell)$, while at the first order one has [46]

$$^{(4)}G_\nu^\mu = -\frac{2}{\ell}\left({}^{(1)}K_\nu^\mu - \delta_\nu^\mu\,{}^{(1)}K\right) - {}^{(1)}E_\nu^\mu, \tag{2}$$

$$e^{-\phi}\partial_y\,{}^{(1)}E_{\mu\nu} = \frac{2}{\ell}\,{}^{(1)}E_{\mu\nu}, \tag{3}$$

$$e^{-\phi}\partial_y\,{}^{(1)}K_\nu^\mu = -\left(D^\mu D_\nu\phi + D^\mu\phi D_\nu\phi\right) + \frac{2}{\ell}\,{}^{(1)}K_\nu^\mu - {}^{(1)}E_\nu^\mu. \tag{4}$$

The evolution equations for ${}^{(1)}E_\nu^\mu$ and ${}^{(1)}K_\nu^\mu$ can be solved,

$$^{(1)}E_\nu^\mu = \exp(4d(y,x)/\ell)\hat{e}_\nu^\mu(x), \tag{5}$$

$$^{(1)}K_\nu^\mu(y,x) = \exp(2d(y,x)/\ell)\,{}^{(1)}K_\nu^\mu(0,x)$$
$$- \frac{\ell}{2}\left[1 - \exp(-2d(y,x)/\ell)\right]{}^{(1)}E_\nu^\mu(y,x)$$
$$- \left[D^\mu D_\nu d(y,x) - \frac{1}{\ell}\left(D^\mu d D_\nu d - \frac{1}{2}\delta_\nu^\mu(Dd)^2\right)\right], \tag{6}$$

where $\hat{e}_\nu^\mu = h^{\mu\alpha}e_{\alpha\nu}(x)$, with $e_{\alpha\nu}(x)$ being the integration constant of Eq. (3), which can be fixed using the junction conditions [46],

$$\frac{\ell}{2}\left[1 - \exp(-2d_0/\ell)\right]\exp(4d_0/\ell)\hat{e}_\nu^\mu(x)$$
$$= -\frac{\kappa^2}{2}\left[\exp(2d_0/\ell)T_\nu^{(\mathrm{hid})\mu} + T_\nu^{(\mathrm{vis})\mu}\right]$$
$$- \left(D^\mu D_\nu d_0 - \delta_\nu^\mu D^2 d_0\right)$$
$$+ \frac{1}{\ell}\left(D^\mu d_0 D_\nu d_0 + \frac{1}{2}\delta_\nu^\mu(Dd_0)^2\right) \tag{7}$$

where κ^2 stands for the bulk gravitational constant, $T_\nu^{(\mathrm{hid})\mu}$ stands for energy-momentum tensor on the hidden (positive tension) brane, and $T_\nu^{(\mathrm{vis})\mu}$ for the visible (negative tension) brane, respectively. Use of the expressions for ${}^{(1)}E_\nu^\mu$ and ${}^{(1)}K_\nu^\mu$ in Eq. (2) leads to the effective field equations on the visible brane (i.e., the brane on which the Planck scale is exponentially suppressed) in this scenario as [46]

$$^{(4)}G_\nu^\mu = \frac{\kappa^2}{\ell}\frac{1}{\Phi}T_\nu^{(\mathrm{vis})\mu} + \frac{\kappa^2}{\ell}\frac{(1+\Phi)^2}{\Phi}T_\nu^{(\mathrm{hid})\mu}$$
$$+ \frac{1}{\Phi}\left(D^\mu D_\nu\Phi - \delta_\nu^\mu D^2\Phi\right)$$
$$+ \frac{\omega(\Phi)}{\Phi^2}\left[D^\mu\Phi D_\nu\Phi - \frac{1}{2}\delta_\nu^\mu(D\Phi)^2\right] \tag{8}$$

where the scalar field $\Phi(x)$ appearing in the above effective equation is directly connected to the radion field $d_0(x)$ (representing the proper distance between the branes) such that $\omega(\Phi)$ and Φ obey the following expressions [46]:

$$\Phi = \exp\left(\frac{2d_0}{\ell}\right) - 1; \quad \omega(\Phi) = -\frac{3}{2}\frac{\Phi}{1+\Phi}. \tag{9}$$

We will assume $d_0(x)$, the brane separation to be finite and everywhere non-zero. This suggests that $\Phi(x)$ should always be greater than zero and shall never diverge. Finally we also have a differential equation satisfied by Φ from the trace of Eq. (7), which can be written as [46]

$$D_\mu D^\mu\Phi = \frac{\kappa^2}{\ell}\frac{1}{2\omega+3}\left(T^{(\mathrm{vis})} + T^{(\mathrm{hid})}\right)$$
$$- \frac{1}{2\omega+3}\frac{\mathrm{d}\omega}{\mathrm{d}\Phi}D_\mu\Phi D^\mu\Phi \tag{10}$$

where $\omega(\Phi)$ has been defined in Eq. (9) and T^{vis} and T^{hid} stands for the trace of the energy-momentum tensor on the hidden and visible branes, respectively. In the above expressions D_μ stands for the four-dimensional covariant derivative, also $D^2\Phi$ stands for $D^\mu D_\mu \Phi$ and $(D\Phi)^2 = D_\mu \Phi D^\mu \Phi$.

The above effective field equations for gravity have been obtained following [46], where no stabilization mechanism for the radion field was proposed. In this work as well we would like to emphasize that we are working with the radion field in the absence of any stabilization mechanism. However, as already emphasized in [46], in order to provide a possible resolution to the gauge hierarchy problem one requires stabilization of the radion field. Even though we will not explicitly invoke a stabilization mechanism, we will outline how stabilization can be achieved and argue that it will not drastically alter the results.

In such a situation with a stabilized radion field, the field $\Phi(x)$ appearing in the above equations can be thought of as fluctuations of the radion field around its stabilized value [47]. In particular stabilization of the radion field can be achieved by first introducing a bulk scalar field following [48] and then solving for it. Substitution of the solution in the action and subsequent integration over the extra spatial dimension lead to a potential for Φ. The same will appear in the above equations through the projection of the bulk energy-momentum tensor, which would involve the bulk scalar field and shall lead to an additional potential on the right hand side of the above equations, whose minima would be the stabilized value for $\Phi = \Phi_c$. Choosing $\Phi = \Phi_c + \Phi(x)$, where $\Phi(x)$ represents small fluctuations around the stabilized value, one ends up with similar equations as above with bulk terms having contributions similar to $T^{(\text{vis})}$ and $T^{(\text{hid})}$, respectively. Thus the final results, to leading order, will remain unaffected by the introduction of a stabilization mechanism. Even though the fact that the virial mass of galaxy clusters scale with r will hold, the subleading correction terms in the case of galactic motion will change due to the presence of a stabilization mechanism due to the appearance of extra bulk inherited terms in the above equations. It would be an interesting exercise to work out the above steps explicitly and obtain the relevant corrections due to the stabilization mechanism, which we will pursue elsewhere.

As illustrated above for the two brane system the non-local terms get mapped to the radion field, the separation between the two branes. Hence ultimately one arrives at a system of closed field equations for a two brane system. The field equations as presented in Eq. (8) are closed since the radion field Φ satisfies its own field equation Eq. (10). Hence the problematic non-local terms in a single brane approach get converted to the radion field in a two brane approach and make the system of gravitational field equations at low energy closed.

We are mainly interested in spherically symmetric spacetime, in which generically the line element takes the following form:

$$\mathrm{d}s^2 = -e^\nu \mathrm{d}t^2 + e^\lambda \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2. \tag{11}$$

This particular form of the metric is used extensively in various physical contexts, for example in obtaining a black hole solution, particle orbit, perihelion precession of planetary orbits, bending of light and in various other astrophysical phenomena [49–51]. Given this metric ansatz we can compute all the derivatives of the scalar field and being in a static situation, the brane separation is assumed to depend on the radial coordinate only. Thus we will only have terms involving a derivative with respect to r (denoted by a prime). First we can rewrite the scalar field equation, which will be a differential equation for Φ. We will also assume that there is no matter on the hidden brane, but only on the visible brane, which is assumed to be a perfect fluid. Thus on the visible brane we have an energy-momentum tensor $T_\mu^{\nu(\text{vis})} = \text{diag}(-\rho, p, p_\perp, p_\perp)$, with the trace being given by $T = -\rho + p + 2p_\perp$. From now on we will remove the label 'vis' from the energy-momentum tensor, since only on the visible brane the energy-momentum tensor is non-zero. With these inputs and the above spherically symmetric metric ansatz we obtain the scalar field equation as,

$$\partial_r^2 \Phi + \frac{2}{r} \partial_r \Phi + \left(\frac{\nu' - \lambda'}{2}\right) \partial_r \Phi = \frac{\kappa^2}{\ell} \frac{1 + \Phi}{3} T^{(\text{vis})} e^\lambda$$
$$+ \frac{1}{2(1 + \Phi)} (\partial_r \Phi)^2 \tag{12}$$

Having derived the scalar field equation, next we need to obtain the field equations for gravity with the metric ansatz given by Eq. (11). These will be differential equations for $\nu(r)$ and $\mu(r)$, respectively. We can separate out the time-time component, the radial component, and the transverse components leading to

$$-e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} = \frac{\kappa^2}{\ell} \frac{\rho + \rho_0}{\Phi} + e^{-\lambda} \frac{\nu'}{2} \frac{\Phi'}{\Phi}$$
$$+ \frac{\kappa^2}{\ell} \frac{1 + \Phi}{3\Phi} T - \frac{e^{-\lambda} \Phi'^2}{4\Phi(1 + \Phi)}, \tag{13}$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = \frac{\kappa^2}{\ell} \frac{p - \rho_0}{\Phi} - \frac{2}{r} e^{-\lambda} \frac{\Phi'}{\Phi}$$
$$- e^{-\lambda} \frac{\nu'}{2} \frac{\Phi'}{\Phi} - \frac{3}{4} \frac{\Phi'^2}{\Phi(1 + \Phi)} e^{-\lambda}, \tag{14}$$

$$
e^{-\lambda}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right) = \frac{\kappa^2}{\ell\Phi} 2\left(p_\perp - \rho_0\right)
$$
$$
+ \frac{2}{r}e^{-\lambda}\frac{\Phi'}{\Phi} - \frac{2\kappa^2}{3\ell}\frac{1+\Phi}{\Phi}T + \frac{1}{2}\frac{e^{-\lambda}\Phi'^2}{\Phi(1+\Phi)}, \tag{15}
$$

where the primes denote derivatives with respect to the radial coordinate. In the above field equations along with the perfect fluid, we have contributions from the brane cosmological constant. Here we have inserted a brane energy density ρ_0, where ρ_0 and the brane cosmological constant are related via $\rho_0 = \Lambda/8\pi G$. Here G is the four-dimensional gravitational constant. Finally, we have contribution from the radion field itself, since it appears on the right hand side of the gravitational field equations. Having derived the field equations we will now proceed to determine the effect of the radion field on the kinematics of galaxy clusters and hence its implications for the missing mass problem.

3 Virial theorem in galaxy clusters, kinematics of the radion field and dark matter

It is well known that the galaxy clusters are the largest virialized systems in the universe [2]. We will further assume them to be isolated, spherically symmetric systems such that the spacetime metric near them can be presented by the ansatz in Eq. (11). Galaxies within the galaxy cluster are treated as identical, point particles satisfying general relativistic collision-less Boltzmann equation.

The Boltzmann equation requires setting up appropriate phase space for a multi-particle system along with the corresponding distribution function $f(x, p)$, where x is the position of the particles in the spacetime manifold with its four-momentum $p \in T_x$, where T_x is the tangent space at x. Further the distribution function is assumed to be continuous, non-negative and describing a state of the system. The distribution function is defined on the phase space, yielding the number dN of the particles of the system, within a volume dV located at x and have four-momentum p within a three surface element $d\vec{p}$ in momentum space. All the observables can be constructed out of various moments of the distribution function. Further details can be found in [39].

For the static and spherically symmetric line element as in Eq. (11) the distribution function can depend on the radial coordinate only and hence the relativistic Boltzmann equation reduces to the following form [39]:

$$
u_r\frac{\partial f}{\partial r} - \left(\frac{1}{2}u_t^2\frac{\partial\nu}{\partial r} - \frac{u_\theta^2 + u_\phi^2}{r}\right)\frac{\partial f}{\partial u_r} - \frac{1}{r}u_r\left(u_\theta\frac{\partial f}{\partial u_\theta} + u_\phi\frac{\partial f}{\partial u_\phi}\right)
$$
$$
- \frac{1}{r}e^{\lambda/2}u_\phi\cot\theta\left(u_\theta\frac{\partial f}{\partial u_\phi} - u_\phi\frac{\partial f}{\partial u_\theta}\right) = 0. \tag{16}
$$

The spherical symmetry of the problem requires the coefficient of $\cot\theta$ to identically vanish. Hence the distribution function can be a function of r, u_r, and $u_\theta^2 + u_\phi^2$ only. Multiplying the above equation by $m u_r du$, where m stands for the galaxy mass and du is the velocity space element, we find after integrating over the cluster [39]

$$
- \int_0^R 4\pi\rho\left[\langle u_r^2\rangle + \langle u_\theta^2\rangle + \langle u_\phi^2\rangle\right]r^2 dr
$$
$$
+ \frac{1}{2}\int_0^R 4\pi r^3\rho\left[\langle u_t^2\rangle + \langle u_r^2\rangle\right]\frac{\partial\nu}{\partial r}dr = 0 \tag{17}
$$

where R stands for the radius of the galaxy cluster. Using the distribution function, the energy-momentum tensor of the matter becomes

$$
T_{ab} = \int f m u_a u_b du, \tag{18}
$$

which leads to the following expressions for the energy density and pressure:

$$
\rho_{\text{eff}} = \rho\langle u_t^2\rangle; \quad p_{\text{eff}}^{(r)} = \rho\langle u_r^2\rangle; \quad p_{\text{eff}}^{(\perp)} = \rho\langle u_\theta^2\rangle = \rho\langle u_\phi^2\rangle. \tag{19}
$$

Using these expressions for the energy density and pressure in the gravitational field equations presented in Eqs. (13), (14) and (15) and finally adding all of them together we arrive at

$$
e^{-\lambda}\left(\nu'' + 2\frac{\nu'}{r} + \frac{\nu'^2}{2} - \frac{\nu'\lambda'}{2}\right) = \frac{\kappa^2}{\ell\Phi}\left(\rho_{\text{eff}} + p_{\text{eff}}^{(r)} + 2p_{\text{eff}}^{(\perp)}\right)
$$
$$
- \frac{1}{2}\frac{e^{-\lambda}\Phi'^2}{\Phi(1+\Phi)} - \frac{\kappa^2}{3\ell}\frac{1+\Phi}{\Phi}
$$
$$
\times\left(-\rho_{\text{eff}} + p_{\text{eff}}^{(r)} + p_{\text{eff}}^{(\perp)} - 4\rho_0\right) - \frac{\kappa^2}{\ell\Phi}\rho_0. \tag{20}
$$

To obtain Eq. (20), we have used the expression for the trace of the energy-momentum tensor. We also recall that ρ_0 stands for the vacuum energy density. At this stage it is useful to introduce certain assumptions, since actually we are interested in a post-Newtonian formulation of the effective gravitational field equations. The two assumptions are: (a) ν and λ are small so that any quadratic expressions constructed out of them can be neglected in comparison to the linear one. Second, (b) the velocity of the galaxies is assumed to be much smaller compared to the velocity of light, which suggests $\langle u_r^2\rangle, \langle u_\theta^2\rangle, \langle u_\phi^2\rangle \ll \langle u_t^2\rangle$. This in turn implies $\rho_{\text{eff}} \gg p_{\text{eff}}^{(r)}, p_{\text{eff}}^{(\perp)}$ such that all the pressure terms can be neglected in comparison to the energy density. Applying all these approximation schemes, Eq. (20) can be rewritten as

$$\frac{1}{2r^2}\frac{\partial}{\partial r}\left(r^2\nu'\right) = \frac{\kappa^2}{6\ell}\rho + \frac{2\kappa^2}{3\ell}\rho_0 - \frac{1}{4}\frac{\Phi'^2}{\Phi(1+\Phi)}$$
$$+ \frac{2\kappa^2\rho}{3\Phi\ell} - \frac{\kappa^2\rho_0}{3\ell\Phi}. \tag{21}$$

We can also perform the same schemes of approximation to Eq. (17), which leads to

$$-2K + \frac{1}{2}\int_0^R 4\pi r^3 \rho\nu' \mathrm{d}r = 0 \tag{22}$$

where K stands for the total kinetic energy of the galaxies within the galaxy cluster and obeys the following expression:

$$K = \int_0^R \mathrm{d}r\, 4\pi r^2 \rho\left[\frac{1}{2}\left\{\langle u_r^2\rangle + \langle u_\theta^2\rangle + \langle u_\phi^2\rangle\right\}\right]. \tag{23}$$

The mass within a small volume of radial extent $\mathrm{d}r$ has the expression $\mathrm{d}M(r) = 4\pi r^2\rho \mathrm{d}r$, where in this and subsequent expressions ρ will indicate $\rho(r)$. Thus the total mass of the system can be given by the integral of $\mathrm{d}M(r)$ over the full size of the galaxy. The main contribution comes from the mass of intra-cluster gas and stars along with other particles, e.g., massive neutrinos. We can also define the gravitational potential energy Ω of the cluster as

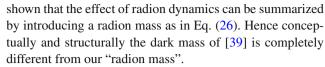$$\Omega = -\int_0^R \frac{GM(r)}{r}\mathrm{d}M(r). \tag{24}$$

Finally multiplying Eq. (21) by r^2 and integrating from 0 to r, we arrive at

$$\frac{1}{2}r^2\frac{\partial\nu}{\partial r} = \frac{\kappa^2}{6\ell}\int_0^r r^2\rho(r)\mathrm{d}r + \frac{2\kappa^2\rho_0}{3\ell}\int_0^r r^2\mathrm{d}r + \frac{\kappa^2}{4\pi\ell}M_\Phi(r), \tag{25}$$

where we have defined

$$M_\Phi(r) = \int_0^r \mathrm{d}r 4\pi r^2\left(-\frac{\ell}{4\kappa^2}\frac{\Phi'^2}{\Phi(1+\Phi)} + \frac{2\rho}{3\Phi} - \frac{\rho_0}{3\Phi}\right). \tag{26}$$

This object captures all the effect of the radion field on the gravitational mass distribution of galaxy clusters and thus may be called the "radion mass". Note that the "radion mass" defined in this work is a completely different construct compared to the "dark mass" used in the literature. The dark mass appears from non-local effects of the bulk, specifically through the bulk Weyl tensor in the effective field equation formalism. However, in this work, we have used the effective equation formalism for a two brane system as developed in [46], where the correction to the gravitational field equations originates from the radion dynamics. Pursuing these effective equations further, through the virial theorem we have

shown that the effect of radion dynamics can be summarized by introducing a radion mass as in Eq. (26). Hence conceptually and structurally the dark mass of [39] is completely different from our "radion mass".

Further, the total baryonic mass of the galaxy cluster within a radius r can be obtained by integrating the energy density over the size of the galaxy cluster, which leads to $M(r) = 4\pi\int_0^r r^2\rho(r)\mathrm{d}r$, using which we finally arrive at the following form for Eq. (25):

$$\frac{1}{2}r^2\frac{\partial\nu}{\partial r} = \frac{\kappa^2}{6\ell}\frac{M(r)}{4\pi} + \frac{2\kappa^2\Lambda}{3\ell}\frac{r^3}{3} + \frac{\kappa^2}{4\pi\ell}M_\Phi(r). \tag{27}$$

Earlier we have defined the gravitational potential associated with M, the baryonic mass. We can define an identical object using the radion mass as well, leading to a potential term Ω_Φ. Given the potentials we can introduce three radii: (a) R_V, the virial radius, obtained using the total baryonic potential and baryonic mass, (b) R_I, the inertial radius, obtained from the moment of inertia of the galaxy cluster, and finally (c) R_Φ, the radion radius obtained from the radion mass. Using these expressions and the definition for the virial mass, $M_V = \sqrt{2KR_V/G}$, yields the following expression:

$$\frac{M_V}{M} = \sqrt{\frac{\kappa^2}{24\pi G\ell} + \frac{2\kappa^2\rho_0}{9\ell G}\frac{R_V R_I^2}{M} + \frac{\kappa^2}{4\pi G\ell}\frac{R_V}{R_\Phi}\frac{M_\Phi^2}{M^2}}. \tag{28}$$

For most of the clusters, the virial mass M_V is three times compared to the baryonic mass M and thus for all practical purposes the first term inside the square root, which is of order unity can be neglected with respect to the other two. The second term yields the contribution from the brane cosmological constant, which is several orders of magnitude smaller compared to the observed mass and thus can also be neglected. Finally, the virial mass turns out to be

$$\frac{M_V}{M} \approx \frac{M_\Phi}{M}\sqrt{\frac{\kappa^2}{4\pi G\ell}\frac{R_V}{R_\Phi}}. \tag{29}$$

Among the various terms in the above expression, the virial mass M_V is determined from the study of the velocity dispersion of galaxies within the cluster and is much larger than the visible mass. The above expression shows that if the radion field kinematics is such that M_{tot} is equal to M_Φ, then that in turn will lead to the correct virial mass of the galaxy clusters. The effect of the radion field and hence of the extra dimension can also be probed through gravitational lensing.

To see that, let us explore the differential equation for Φ, which has not yet been considered. Solving that will lead to some leading order behavior of the radion field Φ, which in turn would affect M_Φ. Thus, the crucial thing is whether M_Φ behaves as r at large distance from the core of the cluster. In this case, from the above equation, we readily observe

that the galaxy virial mass would also scale as $M_V \sim r$, explaining the issue of dark matter and the galaxy rotation curve. To answer all these questions let us start by using the differential equation for Φ. There we will work under the same approximation schemes, i.e., we will neglect all the quadratic terms, e.g., $\nu' \Phi'$, Φ'^2, will set $e^\lambda \sim 1$, and shall neglect the vacuum energy contribution ρ_0 to obtain

$$\Phi'' + \frac{2}{r}\Phi' = -\frac{\kappa^2}{3\ell}(1+\Phi)\rho. \tag{30}$$

Multiplying both sides by r^2 and integrating twice we obtain (noting that Φ'^2 should not contribute)

$$\Phi = -\frac{\kappa^2}{12\pi\ell}\int^r dr \, \frac{M(r)}{r^2}. \tag{31}$$

Here $M(r)$ stands for the mass of the baryonic matter within radius r and we know from the observations that the density of the baryonic matter falls as $\rho_c(r_c/r)^\beta$, where $3 > \beta > 2$ and r_c stand for the core radius of the cluster. Thus it is straightforward to compute the mass profile, which goes as $\sim r^{3-\beta}$, except for some constant contribution. Hence finally after integration we find the radion field to vary with the radial distance as $r^{2-\beta}$. However, note that the mass of the radion field, i.e., M_Φ, under these approximations (matter is non-relativistic and the field is weak) can be obtained:

$$M_\Phi(r) = \int_0^r dr \, 4\pi r^2 \frac{2\rho}{3\Phi} = 8\pi(\beta-2)(3-\beta)\frac{\ell}{\kappa^2}r. \tag{32}$$

Thus the radion mass indeed scales linearly with radial distance, which would correctly reproduce the observed virial mass of the galaxy cluster. Due to the linear nature of the virial mass, the velocity profile does not die out at large r as expected. Hence the radion field kinematics can explain the kinematics of the galaxy cluster very well and thus the missing mass problem can be described without invoking any additional matter component.

Before concluding the section, let us briefly mention the connection of the above formalism with the gauge hierarchy problem. The separation between the two branes is denoted by d, which varies with the radion field Φ, logarithmically [see Eq. (9)]. The radion field except for a constant contribution varies weakly with radial distance and hence leads to very small corrections to the distance d between the branes. Thus the graviton mass scale for the visible brane will be suppressed by a similar exponential factor as in the original scenario of Randall and Sundrum [21,52], leading to a possible resolution of the gauge hierarchy problem. Thus, as advertised earlier, the existence of an extra spatial dimension leads to a radion field, producing a possible explanation for the dark matter in galaxy clusters along with solving the gauge hierarchy problem.

4 Application: cluster mass profiles

In the previous section we have discussed galaxy clusters by assuming them to be bound gravitational systems, with approximate spherical symmetry and being virialized, i.e., in hydrostatic equilibrium. With these reasonable set of assumptions we have shown that the mass of clusters receives an additional contribution from the kinematics of the radion field and provides an alternative to the missing mass problem. In this section we will discuss one application of the above formalism, namely the mass profile of galaxy clusters and possible experimental consequences. We again start from a collision-less Boltzmann equation with spherical symmetry and in hydrostatic equilibrium to read

$$\frac{d}{dr}\left[\rho_{gas}(r)\sigma_r^2\right] + \frac{2\rho_{gas}(r)}{r}\left(\sigma_r^2 - \sigma_{\theta,\phi}^2\right) = -\rho_{gas}(r)\frac{dV(r)}{dr}. \tag{33}$$

Here $V(r)$ stands for the gravitational potential of the cluster, σ_r and $\sigma_{\theta,\phi}$ are the mass weighted velocity dispersions in the radial and tangential directions, respectively, with ρ_{gas} being the gas density. For spherically symmetric systems, $\sigma_r = \sigma_{\theta,\phi}$ and the pressure profile becomes $P(r) = \sigma_r^2 \rho_{gas}(r)$. Further if the velocity dispersion is assumed to have originated from thermal fluctuations, for a gas sphere with temperature profile $T(r)$, the velocity dispersion becomes $\sigma_r^2 = k_B T(r)/\mu m_p$, where k_B is the Boltzmann constant, $\mu \simeq 0.609$ is the mean mass, and m_p the proton mass. Thus Eq. (33) can be rewritten as

$$\frac{d}{dr}\left[\frac{k_B T(r)}{\mu m_p}\rho_{gas}(r)\right] = -\rho_{gas}(r)\frac{dV(r)}{dr}. \tag{34}$$

The potential can be divided into two parts: the Newtonian potential and the potential due to the radion field. As multiplied by $(4/3)r^2/G$, the Newtonian potential leads to the Newtonian mass M_N, which includes the mass of gas and galaxies, and in particular of the CD galaxies. Thus finally we obtain the mass profile of a virialized galaxy cluster to be

$$M_N(r) + \frac{4}{3G}r^2\frac{dV_\Phi}{dr} = -\frac{4}{3}\left(\frac{k_B T(r)}{\mu m_p G}\right)r$$
$$\times \left(\frac{d\ln\rho_{gas}(r)}{d\ln r} + \frac{d\ln T(r)}{d\ln r}\right). \tag{35}$$

Thus one needs two experimental inputs, the observed gas density profile, ρ_{gas} and the observed temperature profile $T(r)$. The gas density can be obtained from the characteristic properties of the observed X-ray surface brightness profiles, similarly from an X-ray spectral analysis one obtains the radial profile of the temperature. Thus from the X-ray

analysis one can model the galaxy distribution and obtain the baryonic contribution to the mass of the galaxy cluster. From the difference between virial mass and the above estimate one can obtain the contribution due to the radion field. At the leading order the radion mass scales linearly with the radial distance with its coefficients being $\mathcal{O}(\ell/\kappa^2)$. Thus an estimate of the radion mass will lead to a possible value for ℓ/κ^2. Assuming the bulk gravitational constant to be at the Planck scale one can possibly constrain the bulk curvature scale.

5 Effect on galaxy rotation curves

Having described a possible resolution of the missing mass problem in connection with galaxy clusters, let us now concentrate on the rotation curves of galaxies. To perform the same we would invoke some general Lie groups of transformation on a vacuum brane spacetime. In particular we will assume the metric to be static and spherically symmetric [i.e., expressed as in Eq. (11)], such that $\pounds_\xi g_{\mu\nu} = \psi(r)g_{\mu\nu}$, where the vector field ξ^μ can be time dependent. These are known as conformally symmetric vacuum brane model and we consider angular velocity of a test particle in a visible (i.e., negative tension) brane, which can be determined in terms of the conformal factor $\psi(r)$. The above essentially amounts to the assumption that each brane is conformally mapped onto itself along the vector field ξ^μ [53–55]. It turns out that the metric and the vector field ξ^μ obey the following expressions upon solving the relation $\pounds_\xi g_{\mu\nu} = \psi(r)g_{\mu\nu}$:

$$\xi^\mu = \left(\frac{1}{2}\frac{k}{B}t, \frac{r\psi(r)}{2}, 0, 0\right),\qquad(36)$$

$$e^{-\lambda} = \psi^2/B^2; \quad e^\nu = C^2 r^2 \exp\left(-2\frac{k}{B}\int\frac{dr}{r\psi}\right),\qquad(37)$$
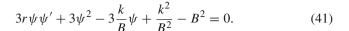
where k is a separation constant and B and C are integration constants. Substitution of these metric functions in the gravitational field equations presented in Eq. (8) leads to

$$-\frac{\psi^2}{B^2}\left(\frac{1}{r^2} + \frac{2}{r}\frac{\psi'}{\psi}\right) + \frac{1}{r^2} = -\frac{e^{-\lambda}\Phi'^2}{4\Phi(1+\Phi)},\qquad(38)$$

$$\frac{\psi^2}{B^2}\left(\frac{3}{r^2} - 2\frac{k}{B}\frac{1}{r^2\psi}\right) - \frac{1}{r^2} = -\frac{3}{4}\frac{\Phi'^2}{\Phi(1+\Phi)}e^{-\lambda},\qquad(39)$$

$$\frac{\psi^2}{B^2}\left(2\frac{\psi'}{r\psi} - 2\frac{k}{B}\frac{1}{r^2\psi} + \frac{k^2}{B^2}\frac{1}{r^2\psi^2} + \frac{1}{r^2}\right) = \frac{1}{4}\frac{e^{-\lambda}\Phi'^2}{\Phi(1+\Phi)},\qquad(40)$$

where a prime denotes differentiation by the radial coordinate r. Multiplying Eq. (40) by 2 and adding it to Eq. (39) one can readily equate it to Eq. (38), resulting in the following differential equation satisfied by $\psi(r)$:

$$3r\psi\psi' + 3\psi^2 - 3\frac{k}{B}\psi + \frac{k^2}{B^2} - B^2 = 0.\qquad(41)$$

The above differential equation can be readily solved, yielding $r = r(\psi)$ [53–55]. However, the solution depends on the mutual dependence of k on B. We will use galaxy rotation curves as the benchmark to determine the region of interest in the (k, B) plane. In connection to rotation curves, the motion of a particle on a circular orbit and its tangential velocity is of importance. For the static and spherically symmetric spacetime the tangential velocity of a particle in circular orbit corresponds to

$$v_{\text{tg}}^2 = \frac{r\nu'}{2} = 1 - \frac{k}{B}\frac{1}{\psi},\qquad(42)$$

where the last equality follows from Eq. (37). The above relation further shows the fact that the rotational velocity is determined by the g_{rr} component alone. Since v_{tg} is determined by ψ, it is possible to write all the expressions derived earlier in terms of the tangential velocity, e.g., $\exp(\lambda) = (B^4/k^2)(1 - v_{\text{tg}}^2)^2$. From Eq. (42) it is clear that asymptotic limits exist only if $k \in (-2B^2, 2B^2)$ [55]. In this case the solution to Eq. (41) corresponds to

$$r^2 = R_0^2 \frac{\left(\frac{|\psi - \psi_2|}{|\psi - \psi_1|}\right)^m}{|3\psi^2 - 3\frac{k}{B}\psi + \frac{k^2}{B^2} - B^2|};$$

$$\psi_{1,2} = \frac{3\frac{k}{B} \pm \sqrt{12B^2 - 3\frac{k^2}{B^2}}}{6};\qquad(43)$$

$$m = \frac{3k}{B\sqrt{12B^2 - 3\frac{k^2}{B^2}}}.$$

Use of this solution leads to the following asymptotic expression for the tangential velocity:

$$v_{\text{tg},\infty} = \sqrt{1 - \frac{6k}{3k + \sqrt{12B^4 - 3k^2}}}.\qquad(44)$$

Note that, for the choices $B = 1.00000034$ and $k = 0.9$, the limiting tangential velocity is given by $v_{\text{tg},\infty} \sim 216.3$ km/s, which is of the same order as the observed galactic rotational velocities. Thus the behavior of all the metric coefficients in the solutions depend on two arbitrary constants of integration, namely, k and B. In order to obtain a numerical estimate for these parameters we assume that there exists some radius r_0 beyond which the baryonic matter density ρ_B is negligible. Requiring $\exp(\lambda) = 1 - (2GM_\text{B}/r_0)$, with $M_\text{B} = 4\pi\int_0^{r_0} dr\, r^2\rho_\text{B}$, we readily obtain

$$\frac{k^2}{B^4} = \left(1 - \frac{2GM_\text{B}}{r_0}\right)\left[1 - v_{\text{tg}}^2(r_0)\right]^2.\qquad(45)$$

Hence the ratio k^2/B^4 can be determined observationally through the tangential velocity. It follows that around and outside r_0 the radion field will dominate and hence one can introduce a "radion mass" in an identical manner. This, using the conformal symmetry and the ratio k^2/B^4 from the above equation, immediately reads

$$
\begin{aligned}
M_\Phi(r) &= \int \mathrm{d}r\, 4\pi r^2 \left(\frac{\ell}{\kappa^2}\right)\left[\frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right)\right] \\
&= 4\pi \frac{\ell}{\kappa^2}\left[r - \int \mathrm{d}r\, r^2 \frac{\psi^2(r)}{B^2}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right)\right] \\
&= \frac{4\pi\ell r}{\kappa^2}\left[1 - \left(1 - \frac{2GM_\mathrm{B}}{r_0}\right)\frac{1 - v_\mathrm{tg}^2(r_0)}{1 - v_\mathrm{tg}^2(r)}\right]. \quad (46)
\end{aligned}
$$

However, the tangential velocity v_tg is non-relativistic, i.e., much smaller than unity (in $c = 1$ units) and hence the radion mass turns out to obey the scaling relation

$$
M_\Phi(r) = \frac{8\pi G\ell}{\kappa^2}M_\mathrm{B}\frac{r}{r_0}. \quad (47)
$$

The above result explicitly shows that the "radion mass" will scale linearly with the radial distance, which stops the velocity profile from dying out at large r. However, note that the linear behavior of the radion mass is only the leading order behavior. If we had kept higher order terms, we would have corrections over and above the linear term, leading to

$$
M_\Phi(r) = \frac{8\pi G\ell}{\kappa^2}M_\mathrm{B}\frac{r}{r_0} + C_1 r^{\ell_1} + C_2 r^{\ell_2} \quad (48)
$$

where C_1 and C_2 are constants depending on κ^2/ℓ and ℓ_1, ℓ_2, and both are strictly less than unity. Given the mass profile, the corresponding velocity profile can be obtained by dividing the mass profile by r and some suitable numerical factor. The coefficients and powers of the velocity profile (and hence the mass profile) can be determined by fitting the velocity profile with the observed one. We should emphasize that the linear term alone cannot lead to a good fit; its effect is to make the velocity profile flat at large distances. Thus at smaller distances the additional correction terms in Eq. (48) are absolutely essential. Hence the effect of the radion field can only be felt at large distances, preventing the velocity profile from decaying and the sub-leading factors in Eq. (48) are important for matching with the experimental data. In particular, from Fig. 1 it turns out that all the four curves are consistent with the following choices of the power law behavior: $\ell_1 \simeq 0.1$ and $\ell_2 \simeq 0.4$, respectively. The coefficients C_1 and C_2 turn out to have the following numerical estimates: $C_1 = -25.56 \pm 4.3$ and $C_2 = 1.75 \pm 0.08$, respectively. Thus at small enough values of r the dominant contribution comes from the term $C_1 r^{\ell_1}$, while for somewhat larger values of r, $C_2 r^{\ell_2}$ dominates. Finally at large values of r the linear term,

i.e., the contribution from the radion field, becomes dominating, leading to a flat velocity profile for the galaxies. Hence the correction terms are quite significant as regards obtaining a good fit with the observational data.
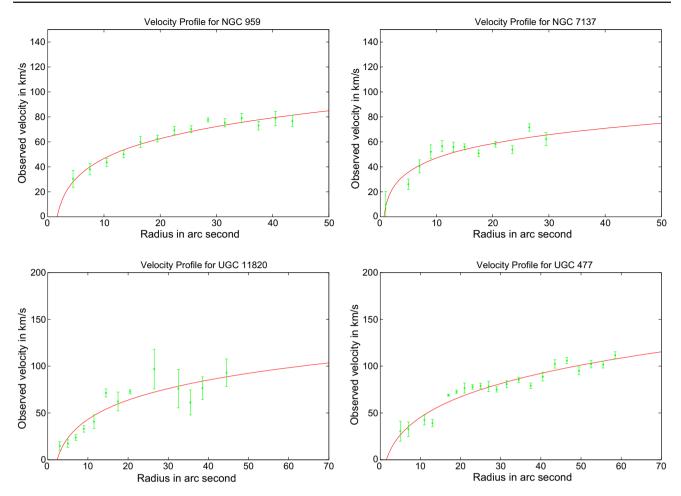
6 Application to other scenarios

In this work we have used a two brane model with the brane separation being represented by the radion field Φ. We have also assumed that our universe corresponds to the visible brane. In such a setup the effective gravitational field equations on the brane, written in a spherically symmetric context, depends on the radion field and its derivatives. The use of a collision-less Boltzmann equation leads to the result that the virial mass of the galaxy clusters scales linearly with radial distance. Thus without any dark matter we can reproduce the virial mass of galaxy clusters by invoking extra dimensions.

However, in order to become a realistic model we should apply our results to other situations and look for consistency. There are mainly three issues which we want to address: (i) the advantage over other modified gravity models, (ii) reproducing the correct cosmology, and (iii) the connection with local gravity tests, in particular the fifth force proposal. We address all these issues below.

- In present day particle physics an important and long standing problem is the gauge hierarchy problem, which originates due to the large energy separation between the weak scale and the Planck scale. In our model the branes are separated by a distance d, such that the energy scale on our universe gets suppressed by $M_\mathrm{vis} \sim M_\mathrm{Pl}e^{-2kd}$, with k being related to brane tension. Thus a proper choice of k (such that $kd \sim 10$) leads to $M_\mathrm{vis} \sim M_\mathrm{weak}$ and hence solves the hierarchy problem. Along with the missing mass problem, i.e., producing a linear virial mass our model has the potential of resolving the gauge hierarchy problem as well. This is a major advantage over modified gravity models, where the modifications in gravitational field equations are due to modifying the action for gravity. These models, though able to explain the missing mass problem, usually ds not address the gauge hierarchy problem.

- The next hurdle comes from local gravity tests. This should place some constraints on the behavior of the radion field. The analysis using a spherically symmetric metric ansatz has been performed in [58] assuming dark matter to be a perfect fluid which is a perturbation over the Schwarzschild solution. We can repeat the same analysis with our radion field mass function, which is a perturbation over the vacuum Schwarzschild solution. We then can compute the correction to the perihelion precession of mercury due to dark matter which leads

Fig. 1 Best fit curves for four chosen low surface brightness galaxies, NGC 959, NGC 7137, UGC 11820, UGC 477, respectively [56,57]. On the *vertical axis* we have plotted the observed velocity in km/s and the *horizontal axis* illustrates the radius measured in arc second. The good fit shows that the assumption of spherical symmetry is a good one, also the fact that baryonic matter plus radion field explains the galactic rotation curves fairly well. It also depicts the need for the sub-leading terms in Eq. (48)

to the following constraint on the bulk curvature radius [58–60]:

$$\frac{2\ell(3-\beta)(\beta-2)}{\kappa^2 M_\odot} a(1-e^2) \leq \frac{10^{-5}}{36^2\pi} \frac{T_M}{T_E} \Delta\delta\phi \qquad (49)$$

where $\Delta\delta\phi = 0.004 \pm 0.0006$ arc second per century corresponds to an excess in the perihelion precession of Mercury [61]. a is the semi-major axis, e stands for eccentricity, and T_M and T_E are the periods of revolution of Mercury and Earth, respectively.

- Let us now briefly comment on the relation between the existence of a fifth force and dark matter. In all these models the generic feature corresponds to the existence of a scalar field which couples to dark matter and in turn couples weakly (or strongly) to standard model particles [62–64]. In our model this feature comes quite naturally, since the radion field Φ, which plays the role of dark matter, can also be thought of as a scalar field,

coupled to standard model particles through the matter energy-momentum tensor with coupling parameter $\sim \kappa^2/\ell(3+2\omega)^{-1}$. Thus effectively we require a fifth force to accommodate modifications of gravity at small scales. There exist stringent constraints on the fifth force from various experimental and observational results (see for example [65–67]). We can apply these constraints on the fifth force for scalar tensor theories of gravity and that leads to the following bound on the composite object: $(\kappa^2/12\pi G\ell)(1+\Phi) < 2.5 \times 10^{-5}$. Hence for compatibility of the radion field presented in this work with fifth force constraints, the bulk curvature ℓ, the bulk gravitational constant κ^2, and the radion field must satisfy the above mentioned inequality.

- Finally we address some cosmological implications of our work. In cosmology one averages over all the matter contributions at the scale of galaxy clusters and assumes all the matter components to be perfect fluids. The same applies to our model as well, in which the effect of a

radion field Φ at the galactic scale is to generate an effective dark matter density profile, with a given mass function. Since the mass function obeys the observed dark matter profile, therefore on average in the cosmological scale it reproduces the standard dark matter content and hence the standard cosmological models.

Thus the radion field model proposed in this work not only matches with the virial mass profile of galaxy clusters but also fits well into other scenarios. The model has the advantage over other alternative gravity models, since it can also address the hierarchy problem by exponentially suppressing Planck scale on our universe. Second, local gravity tests and fifth force phenomenology provides constraints on the bulk curvature radius consistent with the virial mass profile. Still, the results can change depending on the stabilization mechanism for the radion field, which would be an interesting future avenue to explore.

7 Discussion

Brane world models can address some of the long standing puzzles in theoretical physics, namely: (a) the hierarchy problem and (b) the cosmological constant problem. To solve the hierarchy problem we need two branes, with warped five-dimensional geometry such that the energy scale on the visible brane gets suppressed exponentially leading to TeV scale physics. For the cosmological constant the brane tension plays a crucial role. Two brane models naturally inherit an additional field, the separation between the branes (known as the radion field). The radion field is also very important in both macroscopic and microscopic physics, for it can have possible signatures in inflationary scenarios [24–26], black hole physics [68,69], collider searches [70], etc. Along with the gauge hierarchy and the cosmological constant problem, another very important problem in physics is the missing mass problem. This appears since the baryonic and the virial mass of a galaxy cluster do not coincide. In this work using a two brane setup we have shown that, along with the gauge hierarchy and cosmological constant problem, this model is also capable of addressing the missing mass problem through the kinematics of the brane separation, i.e., the radion field. Due to the presence of this additional field, the gravitational field equations on the brane get modified and yield additional correction terms on top of Einstein's field equations. By considering the relativistic Boltzmann equation we have derived the virial mass of galaxy clusters, which depends on an effective additional mass constructed out of the radion field. Moreover, these correction terms modify the structure of gravity and hence the motion under its influence at large distance, thereby producing a linear increase in the virial mass of the galaxy clusters. This in turn leads to the appropriate velocity law for galaxies within a galaxy cluster, solving the missing mass problem.

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