Eur. Phys. J. C (2016) 76:625
DOI 10.1140/epjc/s10052-016-4468-z

THE EUROPEAN PHYSICAL JOURNAL C

CrossMark

Regular Article - Theoretical Physics

Chern–Simons invariants on hyperbolic manifolds and topological quantum field theories

L. Bonora[1,2,a], **A. A. Bytsenko**[3,b], **A. E. Gonçalves**[3,c]

[1] International School for Advanced Studies (SISSA/ISAS), Via Bonomea 265, 34136 Trieste, Italy
[2] INFN, Sezione di Trieste, Italy
[3] Departamento de Física, Universidade Estadual de Londrina, Caixa Postal 6001, Londrina-Paraná, Brazil

Received: 14 October 2016 / Accepted: 3 November 2016 / Published online: 17 November 2016

Abstract We derive formulas for the classical Chern–Simons invariant of irreducible $SU(n)$-flat connections on negatively curved locally symmetric three-manifolds. We determine the condition for which the theory remains consistent (with basic physical principles). We show that a connection between holomorphic values of Selberg-type functions at point zero, associated with R-torsion of the flat bundle, and twisted Dirac operators acting on negatively curved manifolds, can be interpreted by means of the Chern–Simons invariant. On the basis of the Labastida–Mariño–Ooguri–Vafa conjecture we analyze a representation of the Chern–Simons quantum partition function (as a generating series of quantum group invariants) in the form of an infinite product weighted by S-functions and Selberg-type functions. We consider the case of links and a knot and use the Rogers approach to discover certain symmetry and modular form identities.

Contents

[a] e-mail: bonora@sissa.it
[b] e-mail: aabyts@gmail.com
[c] e-mail: aedsongoncalves@gmail.com

1 Introduction

The Chern–Simons theory is one of the two archetypal field theories in physics, together with Yang–Mills theory, that describe the interaction of gauge fields. In 3D, the number of dimensions we wish to consider in this paper, the dynamics is so constrained as to leave room only for non-dynamical (topological) correlators. The corresponding topological quantum field theory was defined and developed by Witten [1] and Reshetikhin and Turaev [2], and it was applied to the mathematical theory of knots and links in three-dimensional manifolds. From a field theory point of view, the observables of the Chern–Simons theory are correlators of Wilson lines (beside the partition function). In this paper we will analyze some properties of the latter.

A well-known characteristic of the Chern–Simons path integral is that its well-definiteness is connected to the properties of a topological invariant in four-dimensional manifolds. Such invariant is related to the Chern–Simons form via the transgression formula. The invariant of a four-manifold in the topological field theory involves its signature and Euler characteristic (see for example [3]). The Chern character allows one to map the analytical Dirac index in terms of K-theory classes into a topological index, which can be expressed in terms of cohomological characteristic classes. This results in a connection between the Chern–Simons action and the Atiyah–Singer index theorem. Such a connection will be used in this paper in order to determine the Chern–Simons invariant of irreducible $SU(n)$-flat connections on negatively curved locally symmetric three-manifolds. Indeed a critical point of the Chern–Simons functional is just a flat connection; it corresponds to a representation of the fundamental group $\pi_1(\mathsf{X})$ associated to a three-manifold X. The value of the Chern–Simons functional at a critical point can be regarded as a topological invariant of a pair (X, ρ), where ρ is a representation of $\pi_1(\mathsf{X})$. Due to a well-known adiabatic argument, knowing these invariants allows us to compute the partition function.

On the other hand, the Chern–Simons partition function is a generating series of quantum group invariants weighted by S-functions. Recall that the Chern–Simons theory has been conjectured to be equivalent to a topological string theory $1/N$ expansion in physics. The Chern–Simons/topological string duality conjecture identifies the generating function of Gromov–Witten invariants as Chern–Simons knot invariants [4]. The existence of a sequence of integer invariants is conjectured [4,5] in a similar spirit to the Gopakumar–Vafa setting [6]. This provides essential evidence of the duality between Chern–Simons theory and topological string theory. Such an integrality conjecture is called the LMOV conjecture. In the context of this conjecture we derive a new representation of the Chern–Simons quantum partition function in the form of an infinite product in terms of Selberg-type functions.

Deeply related with the content of this paper is the problem of anomalies. In field theory anomalies may prevent the path integral from being well defined. In the case of CS in three-dimensional manifolds there are no local anomalies, but there may be global anomalies. To guarantee their absence one must restrict to integer values the (suitably normalized) coupling appearing in front of the action. For this reason it is of utmost importance to know the value of the Chern–Simons invariant in any given space–time. Strictly connected with this is the issue of existence of fermionic path integrals (fermion determinants) in three-dimensional manifolds. There are also other indeterminacies in this theory when links and knots are involved, related to the evaluation of overlapping Wilson loops. The problem of such framing anomalies was pointed out and solved by Witten [7].

Our key results More specifically the content and main results of our paper are as follows.

- In Sect. 2.1 we derive the formula for the Chern–Simons invariant of irreducible $SU(n)$-flat connections on a locally symmetric manifold of non-positive sectional curvature. For this Chern–Simons invariant our result, Eq. (2.19), determines the condition for which the quantum field theory is consistent. The results of Sect. 2.1 are preparatory for the generalization of the Chern–Simons invariant to the case of other manifolds ($X = S^3/\Gamma$, for example) and of nontrivial $U(n)$-bundle over X (Sect. 2.4).
- In Sect. 2.4 $\mathsf{X} = \Gamma\backslash\overline{\mathsf{X}}$ with $\overline{\mathsf{X}}$ is a globally symmetric space of non-compact type and Γ a discrete, torsion-free, co-compact subgroup of orientation-preserving isometries. X inherits a locally symmetric Riemannian metric g of non-positive sectional curvature. If $\mathfrak{D} : C^\infty(\mathsf{X}, V) \rightarrow C^\infty(\mathsf{X}, V)$ is a differential operator acting on the sections of the vector bundle V, then $\mathfrak{D}$ can be extended canonically to a differential operator $\mathfrak{D}_\varphi : C^\infty(\mathsf{X}, V \otimes F) \rightarrow C^\infty(\mathsf{X}, V \otimes F)$, uniquely characterized by the property that $\mathfrak{D}_\varphi$ is locally isomorphic to $\mathfrak{D} \otimes \cdots \otimes \mathfrak{D}$ (dim F times) [8]. We show that a connection between holomorphic values of Selberg-type functions at point zero, associated with R-torsion of the flat bundle, and twisted Dirac operators $\mathfrak{D}_\varphi$ on negatively curved locally symmetric spaces, can be interpreted by means of the Chern–Simons invariant. This leads to our main result, Eq. (2.18). We also briefly describe the possibility to derive the Chern–Simons invariant for locally symmetric spaces of higher rank in terms of the spectral function $\mathcal{R}(s; \varphi)$.
- The quantum $\mathfrak{sl}_N$ invariant in the case of links and a knot is analyzed in Sect. 3. On the basis of LMOV conjecture we derive a new representation of the Chern–Simons quantum partition function in the form of an infinite product in terms of Selberg-type functions. In addition, we discuss the symmetry and modular form properties of infinite-product formulas.

2 Chern–Simons invariants for negatively curved manifolds

2.1 Flat connections and gauge bundles

Flat connections on fibered hyperbolic manifolds The Chern character allows one to map the analytical Dirac index in terms of K-theory classes into a topological index which can be expressed in terms of cohomological characteristic classes. This results in a connection between the Chern–Simons action and the celebrated Atiyah–Singer index theorem. The goal of this section is to use this fact in order to present explicit formulas for the Chern classes and gauge Chern–Simons invariant of an irreducible $SU(n)$-flat connection on real compact hyperbolic three-manifolds.

Let $P = \mathsf{X} \times G$ be a trivial principal bundle over X with the gauge group $G = SU(n)$ and let $\Omega^1(\mathsf{X}; \mathfrak{g})$ be the space of all connections on P; this space is an affine space of one-forms on X with values in the Lie algebra $\mathfrak{g}$ of G. Let $\mathcal{A}_\mathsf{X} = \Omega^1(\mathsf{X}; \mathfrak{g})$ be the space of connections and $\mathcal{A}_{\mathsf{X},F} = \{A \in \mathcal{A}_\mathsf{X} | F_A = \mathrm{d}A + A \wedge A = 0\}$ be the space of flat connections on P. Then the gauge transformation group $\mathcal{G}_\mathsf{X} \cong C^\infty(\mathsf{X}, G)$ acts on $\mathcal{A}_\mathsf{X}$ via pull-back: $\mathfrak{g}^* A = \mathfrak{g}^{-1} A \mathfrak{g} + \mathfrak{g}^{-1} d\mathfrak{g}$, $\mathfrak{g} \in \mathcal{G}_\mathsf{X}$, $A \in \mathcal{A}_\mathsf{X}$. This action preserves $\mathcal{A}_{\mathsf{X},F}$.

It is known that the Chern–Simons invariant is a real valued function on the space of connections $\mathcal{A}_\mathsf{X}$ on a trivial principal bundle over an oriented three-manifold, and it is given by

$$\mathrm{CS}(A) = \frac{1}{8\pi^2} \int_\mathsf{X} \mathrm{Tr} \left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right). \qquad (2.1)$$

A well-known formula related to the CS integrand is

$$d\,\mathrm{Tr}\,(A \wedge \mathrm{d}A + (2/3)A \wedge A \wedge A) = \mathrm{Tr}(F_A \wedge F_A). \quad (2.2)$$

Equation (2.2) provides another description of the Chern–Simons invariant. Indeed, let X be a closed manifold, and M be an oriented four-manifold with $\partial\mathsf{M} = \mathsf{X}$. We denote an extension of A over M by $\underline{A}$. Then the Stokes theorem gives

$$\mathrm{CS}(\underline{A}) = \frac{1}{8\pi^2}\int_{\mathsf{M}} \mathrm{Tr}(F_{\underline{A}} \wedge F_{\underline{A}}). \quad (2.3)$$

We note the difference of the Chern–Simons invariants under the action of the gauge transformation group $\mathcal{G}_{\mathsf{X}}$. In the case $A \in \mathcal{A}_{\mathsf{X}}$ and $\mathfrak{g} \in \mathcal{G}_{\mathsf{X}}$ the following formula holds [9]:

$$\begin{aligned}\mathrm{CS}(\mathfrak{g}^*A) - CS(A) &= \frac{1}{8\pi^2}\int_{\partial\mathsf{X}} \mathrm{Tr}(A \wedge d\mathfrak{g}\mathfrak{g}^{-1}) \\ &-\frac{1}{24\pi^2}\int_{\mathsf{X}} \mathrm{Tr}(\mathfrak{g}^{-1}d\mathfrak{g} \wedge \mathfrak{g}^{-1}d\mathfrak{g} \wedge \mathfrak{g}^{-1}d\mathfrak{g}). \end{aligned} \quad (2.4)$$

The last term in Eq. (2.4) is known as the Wess–Zumino–Witten term; the integrand of this term represents the generator of $H^3(G;\mathbb{Z}) \cong \mathbb{Z}$. If the manifold X is closed, i.e. $\partial\mathsf{X} = \emptyset$, the Wess–Zumino–Witten term takes its value in $\mathbb{Z}$. In this case the function $CS : \mathcal{A}_{\mathsf{X}}/\mathcal{G}_{\mathsf{X}} \to \mathbb{R}/\mathbb{Z}$ is well defined. On the other hand, when $\partial\mathsf{X} \neq \emptyset$, although the Chern–Simons invariant does not give a well-defined function on the space $\mathcal{A}_{\mathsf{X}}/\mathcal{G}_{\mathsf{X}}$ (with values in $\mathbb{R}/\mathbb{Z}$), one can regard it as the section of a certain line bundle over the moduli space of connections $\mathcal{A}_{\mathsf{X}}/\mathcal{G}_{\mathsf{X}}$.

Let us consider now the moduli space of flat connections $\mathcal{X}_{\mathsf{X}} = \mathcal{A}_{\mathsf{X},F}/\mathcal{G}_{\mathsf{X}}$ on X. This space has an alternative topological description: the holonomies of the parallel transport of flat connections on P give the identification of $\mathcal{X}_{\mathsf{X}}$ with the space of conjugacy classes of representations of $\pi_1(\mathsf{X})$ into G, since any principal G-bundle P over a compact oriented three-manifold X is trivial [9]. We use the notation $\mathrm{CS}(\rho) := \mathrm{CS}(A_\rho)$ for a representation ρ of $\pi_1(\mathsf{X})$, where A_ρ is a flat connection corresponding to a representation ρ. This gives a topological invariant for a pair (X, ρ). Since the Chern–Simons invariant is additive with respect to the sum of representations, we have

$$\mathrm{CS}(\rho_1 \oplus \rho_2) = \mathrm{CS}(\rho_1) + \mathrm{CS}(\rho_2). \quad (2.5)$$

Let X be a compact oriented hyperbolic three-manifold, and ρ be an irreducible representation of $\pi_1(\mathsf{X})$ into $SU(n)$. Denote the corresponding flat vector bundle by E_ρ, and a flat extension of A_ρ over M ($\partial\mathsf{M} = \mathsf{X}$) corresponding to ρ by $\underline{A}_\rho$. The second Chern character $\mathrm{ch}_2(\underline{E}_\rho) = \mathrm{CS}(\underline{A})$ of $\underline{E}_\rho$ can be expressed in terms of the first and second Chern classes

$$\mathrm{ch}_2(\underline{E}_\rho) = (1/2)c_1(\underline{E}_\rho)^2 - c_2(\underline{E}_\rho), \quad (2.6)$$

while the Chern character is given by

$$\begin{aligned}\mathrm{ch}(\underline{E}_\rho) &= \mathrm{rank}\,\underline{E}_\rho + c_1(\underline{E}_\rho) + \mathrm{ch}_2(\underline{E}_\rho) \\ &= \dim\rho + c_1(\underline{E}_\rho) + \mathrm{ch}_2(\underline{E}_\rho). \end{aligned} \quad (2.7)$$

The crucial point in our calculation is the Atiyah–Patodi–Singer result for a manifold with boundary [10–12]: the Dirac index is given by

$$\mathrm{Index}\,\mathfrak{D}_\rho = \int_{\mathsf{M}} \mathrm{ch}(\underline{E}_\rho)\widehat{A}(\mathsf{M}) - \frac{1}{2}(\eta(0,\mathfrak{D}_\rho) + h(0,\mathfrak{D}_\rho)). \quad (2.8)$$

Here $\widehat{A}(\mathsf{M}) \equiv \widehat{A}(\Omega(\mathsf{M}))$-genus is the usual polynomial in terms of the Riemannian curvature $\Omega(\mathsf{M})$ of a four-manifold M with boundary $\partial\mathsf{M} = \mathsf{X}$. It is given by $\widehat{A}(\mathsf{M}) = \left(\det\left(\frac{\Omega(\mathsf{M})/4\pi}{\sinh\Omega(\mathsf{M})/4\pi}\right)\right)^{1/2} = 1-(1/24)p_1(\mathsf{M})$, where $p_1(\mathsf{M}) \equiv p_1(\Omega(\mathsf{M}))$ is the first Pontrjagin class. $h(0,\mathfrak{D}_\rho)$ is the dimension of the space of harmonic spinors on X ($h(0,\mathfrak{D}_\rho) = \mathrm{dimKer}\,\mathfrak{D}_\rho$ = multiplicity of the 0-eigenvalue of $\mathfrak{D}_\rho$ acting on X with coefficients in ρ). The η-invariant was introduced by Atiyah, Patodi, and Singer [10–12] treating index theory on even dimensional manifolds with boundary and it first appears there as a boundary correction in the usual local index formula. Let as before X be a closed odd dimensional spin manifold (which in their index theorem is the boundary of an even dimensional spin manifold). $\eta_{\mathsf{X}}(s,\mathfrak{D}) := \eta(s,\mathfrak{D}_{\rho=\mathrm{trivial}})$ is analytic in s and has a meromorphic continuation to $s \in \mathbb{C}$; it is regular at $s = 0$, and its value there is the η-invariant. The result (2.8) holds for any Dirac operator on a $\mathrm{Spin}^{\mathbb{C}}$ manifold coupled to a vector bundle with connection (the metric of manifolds is supposed to be a product near the boundary). One can attach an η-invariant to any operator of Dirac type on a compact Riemannian manifold of odd dimension. (On even dimension manifolds Dirac operators have symmetric spectrum and, therefore, trivial η-invariant.)

η is a spectral invariant which measures the symmetry of the spectrum of an operator $\mathfrak{D}$, and admits a meromorphic extension to the whole s-plane, with at most simple poles at $(\dim\mathsf{X} - k)/(\mathrm{ord}\,\mathfrak{D})$ $(k = 0, 1, 2, \ldots)$ and locally computable residues. For X a compact oriented (4n-1)-dimensional Riemannian manifold of constant negative curvature, a remarkable formula relating η to the closed geodesics on X has been proved in [13]. Citing [8], the appropriate class of Riemannian manifolds for which a result of this type can be expected is that of non-positively curved locally symmetric manifolds, while the class of self-adjoint operators whose eta invariants are interesting to compute is that of Dirac-type operators, even with additional coefficients in locally flat bundles. It is one of the purpose of this paper to formulate and prove for Chern–Simons invariants (2.12) below such an extension as the one in [13] (see Sect. 2.4).

For a trivial representation ρ one can choose a trivial flat connection $\underline{A}$; then $F_{\underline{A}} = 0$. For this choice using Eqs. (2.6) and (2.7) we have

$$\int_{\mathsf{M}} \mathrm{ch}(\underline{E}_\rho)\widehat{A}(\mathsf{M}) = \int_{\mathsf{M}} (\mathrm{rank}\underline{E}_\rho + c_1(\underline{E}_\rho) + \mathrm{ch}_2(\underline{E}_\rho))(1 - \frac{1}{24}p_1(\mathsf{M})) = -\frac{1}{8\pi^2}\int_{\mathsf{M}} \mathrm{Tr}(F_{\underline{A}_\rho} \wedge F_{\underline{A}_\rho}) - \frac{\dim\rho}{24}\int_{\mathsf{M}} p_1(\mathsf{M}) \tag{2.9}$$

while $$\int_{\mathsf{M}} \mathrm{ch}(\underline{E})\widehat{A}(\mathsf{M}) = -\frac{1}{24}\int_{\mathsf{M}} p_1(\mathsf{M}) - \frac{\mathbf{1}}{\mathbf{2}}\eta(\mathbf{0}, \mathfrak{D}). \tag{2.10}$$

Note that in this formula the zero eigenvalue multiplicity of $\mathfrak{D}_\rho$ acting on X with coefficients in ρ has been excluded ($s \in i\,\mathrm{Spec}'(\mathfrak{D}), \mathrm{Spec}'(\mathfrak{D}) \equiv \mathrm{Spec}(\mathfrak{D}) - \{0\}$). From Eqs. (2.9) and (2.10) we obtain

$$\mathrm{Index}\,\mathfrak{D}_\rho - \dim\rho \cdot \mathrm{Index}\,\mathfrak{D} = \mathrm{CS}(\underline{A}_\rho) - \frac{1}{2}(\eta(0, \mathfrak{D}_\rho) - \dim\rho \cdot \eta(0, \mathfrak{D})). \tag{2.11}$$

Then the Chern–Simons invariant can be derived from Eq. (2.11),

$$\mathrm{CS}(\underline{A}_\rho) = (1/2)(\eta(0, \mathfrak{D}) - \dim\rho \cdot \eta(0, \mathfrak{D}_\rho)) + \text{modulo} \quad \mathbb{Z}. \tag{2.12}$$

2.2 The Chern–Simons- and the η-invariants

Gluing properties for η Let M' and M'' be oriented manifolds and let $-\mathsf{M}'$ be the manifold with opposite orientation with respect to M''. If M' and M'' have a common boundary we can glue them along it and form a new oriented and closed manifold $\mathsf{N} = \mathsf{M}' \bigcup (-\mathsf{M}'')$. Then the following gluing formula holds:

$$\exp(i\pi\eta(\mathsf{M}')) = \exp(i\pi\eta(\mathsf{M}'')) \cdot \exp(i\pi\eta(\mathsf{N})), \tag{2.13}$$

This property is instrumental for the main conclusion of this part of the paper contained in résumé: A critical point of the Chern–Simons functional is just a flat connection, and it corresponds to a representation of the fundamental group $\pi_1(\mathsf{X})$. Thus, the value of this functional at a critical point can be regarded as a topological invariant of a pair (X, ρ), where ρ is a representation of $\pi_1(\mathsf{X})$. This is the Chern–Simons invariant of a flat connection on X. Taking into account a pair $(\mathsf{M}, \partial\mathsf{M} = \mathsf{X})$ we have derived Eq. (2.12) for the Chern–Simons invariants of irreducible $SU(n)$-flat connections on a locally symmetric manifolds of non-positive section curvature.

By making use Eq. (2.12) one can rewrite the multiplicative structure of eta invariants, associated with twisted Dirac operators $\mathfrak{D}_\rho$ [1]

$$\exp(i2\pi\mathrm{CS}(\underline{A}_\rho)) = \exp(i\pi\eta(0, \mathfrak{D}_\rho)) \cdot \exp(-i\pi\,\dim\rho \cdot \eta(0, \mathfrak{D})). \tag{2.14}$$

Dirac operators on locally symmetric spaces of rank one One can repeat the technique and arguments discussed in Sect. 2.1 for the construction of the eta-functions and the Chern–Simons invariant. The Chern–Simons invariant admits a representation in terms of Selberg-type spectral function $Z(s, \mathfrak{D}_\rho)$, which is a meromorphic function on $\mathbb{C}$ and given for $\mathrm{Re}(s^2) \gg 0$ [8,14]. $\log Z(s, \mathfrak{D}_\rho)$ has a meromorphic continuation given by the identity

$$\log Z(s, \mathfrak{D}_\rho) = \log\det{}'\left(\frac{\mathfrak{D}_\rho - is}{\mathfrak{D}_\rho + is}\right) + i\pi\eta(s, \mathfrak{D}_\rho), \tag{2.15}$$

$Z(s, \mathfrak{D}_\rho)$ satisfies the functional equation $Z(s, \mathfrak{D}_\rho)Z(-s, \mathfrak{D}_\rho) = \exp\left(2\pi i\eta(s, \mathfrak{D}_\rho)\right)$, where the "twisted" zeta-function $Z(s, \mathfrak{D}_\rho)$ is meromorphic on $\mathbb{C}$. Zeta-functions are given by the formulas [8]

$$\log Z(s, \mathfrak{D}) = (-1)^q \sum_{[\gamma]\in\mathcal{E}_1(\Gamma)} \frac{L(\gamma, \mathfrak{D})}{|\det(I - P_h(\gamma))|^{1/2}}\frac{e^{-s\ell_\gamma}}{m_\gamma}, \tag{2.16}$$

$$\log Z(s, \mathfrak{D}_\rho) = (-1)^q \sum_{[\gamma]\in\mathcal{E}_1(\Gamma)} \mathrm{Tr}\,\rho(\gamma)\frac{L(\gamma, \mathfrak{D})}{|\det(I - P_h(\gamma))|^{1/2}}\frac{e^{-s\ell_\gamma}}{m_\gamma}, \tag{2.17}$$

where $\ell(\gamma)$ is the length of the closed geodesic c_γ in the free homotopy class corresponding to $[\gamma]$, $m(\gamma)$ is the multiplicity of c_γ, $L(\gamma, \mathfrak{D})$ are the Lefschitz numbers, and $P_h(\gamma)$ is the hyperbolic part of the linear Poincaré map $P(\gamma)$ (see for details [8]).

Taking into account that the Dirac operator is Hermitian, and the function $\mathrm{CS}(\underline{A}_\rho)$ is real, it is possible to formulate the following result:

[1] Cf. Eq. (2.13): the η-invariant of locally symmetric manifolds of non-positive curvature can be expressed as spectral values of zeta-functions constructed out of the periodic geodesics [8,14]. In this case a necessary regularization can be given by the geodesic spectrum. For the time being we assumed the other regularization for the η-invariant, which involves the "dual" data, namely the spectrum of the Laplace operator associated to the metric [14].

$$\exp(i2\pi CS(\underline{A}_\rho)) = \frac{Z(0,\mathfrak{D})^{\dim\rho}}{Z(0,\mathfrak{D}_\rho)} = (-1)^{\dim\ker(\mathfrak{D}_\rho - \dim\rho\cdot\mathfrak{D})}, \quad (2.18)$$

as follows from Eq. (2.12). In particular $Z(0,\mathfrak{D})^{\dim\rho}\cdot Z(0,\mathfrak{D}_\rho)^{-1} = \pm 1$ ([8], Corollary 7.5). There is an indeterminacy of sign unless

$$\dim\ker(\mathfrak{D}_\rho - \dim\rho\cdot\mathfrak{D}) = 2n, \quad n\in\mathbb{Z}. \quad (2.19)$$

2.3 Determinant line bundles

In the previous subsections we have studied the connection of the CS invariant with the η-invariant. The latter, on the other hand, features also in the context of anomaly formulas, which are related to the geometry of the determinant line bundles. In this subsection we would like to recall such a connection. We have in mind in particular a three-dimensional manifold X, as above, but it is possible to stick to a more general treatment.

In the sequel we suppose that Y is a compact Spin$^{\mathbb{C}}$-manifold with nonempty boundary. The Dirac operator $\mathfrak{D}$ on a closed Spin$^{\mathbb{C}}$-manifold Y (coupled to a vector bundle with connection) is self-adjoint and has a discrete spectrum Spec $(\mathfrak{D})$. We suppose that the metric of Y near the boundary has explicit product structure, and in a neighborhood of the boundary there is a given isometry with $(-1,0]\times\partial\mathsf{Y}$. $\eta(s,\mathfrak{D})$ is analytic for Re $(s) > -2$ [15,16], and we set

$$\tau_{\mathsf{Y}} = \exp(i2\pi\xi_{\mathsf{Y}}) \equiv \exp\left(i\pi(\eta(0,\mathfrak{D}) + \dim\ker\mathfrak{D})\right)\in\mathbb{C}. \quad (2.20)$$

Under a smooth variation of parameters (for example, the metric on Y) the η-invariant jumps by integers (the general theory shows that $|\tau_{\mathsf{Y}}| = 1$), whereas ξ (mod 1) is smooth. Therefore the invariant (2.20) is defined and we have $\tau_{\mathsf{Y}}\in\det^{-1}_{\partial\mathsf{Y}}$, where $\det_{\partial\mathsf{Y}}$ is the determinant line of the Dirac operator $\mathfrak{D}_{\partial\mathsf{Y}}$ on the boundary.

Fiber bundles Let us discuss some aspects of Dirac operators in the case of fiber bundles. Let $\pi:\mathsf{W}\to\mathsf{Z}$ be a smooth fiber bundle with a Riemannian metric on the tangent bundle $T(\mathsf{W}/\mathsf{Z})$, which is endowed with spin structure. Here and in the following W/Z denotes the fiber of $\mathsf{W}\to\mathsf{Z}$. A spin structure on a manifold means a spin structure on its tangent bundle, in this case the tangent bundle $T(\mathsf{W}/\mathsf{Z})$ along the fibers. Every point in Z determines a Dirac operator acting on the corresponding fiber. We will eventually identify the fiber $\partial\mathsf{W}/\mathsf{Z}$ with a three-dimensional manifold X without boundary, but for the time being we keep as general as possible.

Assume that the Riemannian metric on the fibers is a product near the boundary. The determinant line carries the Quillen metric and a canonical connection ∇ [15] and the exponentiated ξ-invariant is a smooth section $\tau_{\mathsf{W}/\mathsf{Z}}:\mathsf{Z}\to\det^{-1}_{\partial\mathsf{W}/\mathsf{Z}}$.

For Dirac operators coupled to complex bundles in K-theory one can express the Chern character of the index in terms of the Chern character of a complex vector bundle E by means of the formula [17]:

$$\mathrm{ch}\,\pi_!^{\mathsf{W}/\mathsf{Z}}([E]) = \pi_*^{\mathsf{W}/\mathsf{Z}}(\widehat{A}(\mathsf{W}/\mathsf{Z})\mathrm{ch}\,(E)), \quad (2.21)$$

where π_* is the pushforward map in rational cohomology. Note that for a family of closed manifolds this is a result of Atiyah–Patodi–Singer (see Eq. (2.8)).

Let the fibers $\partial\mathsf{W}/\mathsf{Z}$ be odd-dimensional and closed. Then the determinant line bundle $\det\mathfrak{D}_{\partial\mathsf{W}/\mathsf{Z}}(E)$ is well defined as a smooth line bundle (it carries a canonical metric and connection) [15]. The complex Dirac operator for the fibers $\partial\mathsf{W}/\mathsf{Z}$ is self-adjoint and there is a geometrical invariant $\tau_{\mathsf{W}/\mathsf{Z}}$ defined by Atiyah–Patodi–Singer,

$$\tau_{\mathsf{W}/\mathsf{Z}}(E) = \exp(i\pi(\eta(0,\mathfrak{D}) + \dim\ker\mathfrak{D})_{\partial\mathsf{W}/\mathsf{Z}}). \quad (2.22)$$

The 2-form curvature of the determinant line bundle is [17]

$$\Omega(\det\mathfrak{D}_{\mathsf{W}/\mathsf{Z}}(E)) = i2\pi\int_{\mathsf{W}/\mathsf{Z}}\widehat{A}(\Omega(\mathsf{W}/\mathsf{Z}))\mathrm{ch}\,(\Omega(E))_{(2)}\in\Omega^2(\mathsf{Z}), \quad (2.23)$$

where $\Omega(\mathsf{W}/\mathsf{Z})$ and $\Omega(E)$ are the curvature forms.

Now it is possible to apply this geometric setup to compute the holonomy on Z. Let $\partial\mathsf{Y}\to S^1$ be a loop of manifolds. A metric and spin structure on $\partial\mathsf{Y}$ could be induced by a metric and bounding spin structure on S^1. The holonomy of the determinant line bundles around the loop takes the form

$$\mathrm{hol}\det\mathfrak{D}_{\partial\mathsf{Y}/S^1}(E) = a\!-\!\lim\tau^{-1}_{\partial\mathsf{Y}}(E), \quad (2.24)$$

where a-lim is the adiabatic limit, i.e. the limit as the metric on S^1 blows up ($\varepsilon\to 0: g_{S^1}\to g_{S^1}\varepsilon^{-2}$). For the flat determinant line bundles no adiabatic limit is required. A nontrivial result for (2.24) means that the determinant line bundle (fermion determinant; see below) is nontrivial, which is tantamount to saying that there is a global anomaly. Equation (2.24) is in fact known as the global anomaly formula [15,16,18].

Let us summarize. The differential geometry of determinant line bundles has been developed in [19] in a special case and in [15,16] in general. In [20,21] the results on ξ-invariants were used to re-demonstrate the holonomy formula for determinant line bundles, known as Witten's global anomaly formula [18]. For a family of Dirac operators the exponentiated ξ-invariant is a section of the inverse determinant line bundle over the parameter space. In [20] the usual formula for the variation of the ξ-invariant was been generalized to a formula for the covariant derivative. The variational formula relates the exponentiated ξ-invariant to the natural connection on the (inverse) determinant line bundle [20]. One can use such a connection to compute the holonomy, or global anomaly. The latter can expressed as the adiabatic limit of the exponentiated ξ-invariant.

Returning to Chern–Simons invariants, we have seen that the latter are strictly connected to the η invariants and to the global anomaly formula. In the original papers, Atiyah, Patodi, and Singer discuss the relationship of η-invariants (and so exponentiated ξ-invariants) to classical Chern–Simons invariants for closed manifolds. It has been shown [10–12] that certain ratios of exponentiated ξ-invariants are topological invariants which live in K^{-1}-theory with $\mathbb{R}/\mathbb{Z}$ coefficients. The exponentiated ξ-invariant is local and therefore it can serve as an action for a field theory, the same one can say for the Chern–Simons invariant. But there is also a crucial difference: the Chern–Simons invariant is multiplicative in coverings, whereas the exponentiated ξ-invariant is not (nevertheless the gluing law does exhibit some local properties of the η-invariant).

The last considerations lead us to the physical interpretation of the material collected so far.

Note on fermion theories In theories containing fermions interacting with a gauge potential A, by formally integrating out the fermionic fields, one gets an expression which is interpreted as the determinant of the corresponding Dirac operator (fermion determinant). One of the most important problems in quantum field theory is the definition of such a determinant. In some cases they are ill-defined due to anomalies. In a three-dimensional manifold X we can assume that there are no local anomalies and we have only to worry about global anomalies. Formal calculations show for the determinant, as a result of the fermion integration, the exponential of a term to be precisely proportional to the CS action for A; see [22][2] and references therein. Thus the fermion determinant is well defined if this exponential is and this is so if condition (2.19) is satisfied. For instance, this global anomaly vanishes if the number of integrated out Dirac fermions is even (or the number of integrated out Majorana fermions are a multiple of four). This, however, is not enough. Since $\exp(2i\pi \mathrm{CS}(\underline{A}_\rho))$ must have the same value whatever is the manifold M over which we perform the integral (2.3), Eq. (2.13) requires

$$\exp(i\pi\eta(\mathsf{N})) = 1 \tag{2.25}$$

for any three-manifold N without boundary. This condition, relying on the Dai–Freed theorem, has been analyzed in [23]. Whether a theory satisfies or not (2.25) depends on the number of fermions and the type of fermions in it, i.e. whether they are Majorana or Dirac.

The same anomaly (the so-called parity anomaly) also originates from the presence of massless Majorana fermions on the three-dimensional space X. They give rise to a determinant line bundle which leads precisely to the calculation outlined above. The result shows up in the form of the η invariant (or, better, the τ invariant), but the analysis is parallel to the previous one and leads to the same conclusions.

In [23] the previous results are used to analyze the 3+1 dimensional theories that describe the so-called topological insulators and topological superconductors. These theories are defined on a $3+1$ manifold with a boundary and it is usually necessary to enforce complementarity between the fermions in the bulk and those in the boundary in order to cancel the global anomalies and satisfy the condition analogous to (2.25).

2.4 Adiabatic limit and twisted spectral functions

Suppose that $\mathsf{X} = \Gamma\backslash\overline{\mathsf{X}}$ with $\overline{\mathsf{X}}$ a globally symmetric space of non-compact type and Γ a discrete, torsion-free, co-compact subgroup of orientation-preserving isometries. Thus X inherits a locally symmetric Riemannian metric g of non-positive sectional curvature. In addition the connected components of the periodic set of the geodesic flow Φ, acting on the unit tangent bundle $T\mathsf{X}$, are parametrized by the nontrivial conjugacy classes $[\gamma]$ in $\Gamma = \pi_1(\mathsf{X})$. Therefore each connected component X_γ is itself a closed locally symmetric manifold of non-positive sectional curvature.

Suppose that $\varphi : \Gamma \to U(F)$ be a unitary representation of Γ on F. The Hermitian vector bundle $E = \mathsf{X} \times_\Gamma F$ over X inherits a flat connection from the trivial connection on $\overline{\mathsf{X}} \times F$. For any vector bundle E over X let $\overline{E}$ denote the pull-back to $\overline{\mathsf{X}}$. If $\mathfrak{D} : C^\infty(\mathsf{X}, V) \to C^\infty(\mathsf{X}, V)$ is a differential operator acting on the sections of the vector bundle V, then $\mathfrak{D}$ extends canonically to a differential operator $\mathfrak{D}_\varphi : C^\infty(\mathsf{X}, V \otimes F) \to C^\infty(\mathsf{X}, V \otimes F)$, uniquely characterized by the property that $\mathfrak{D}_\varphi$ is locally isomorphic to $\mathfrak{D} \otimes \cdots \otimes \mathfrak{D}$ (dim F times).

Example 2.1 Equation (2.12) suggests the generalization of the Chern–Simons invariant (2.1) to the case of nontrivial $U(n)$-bundle over X. As an example, for any representation $\rho : \Gamma \to U(n)$, a vector bundle $\underline{E}_\rho$ over a certain four-manifold M with $\partial\mathsf{M} = \mathsf{X} = S^3/\Gamma$ (which is an extension of a flat vector bundle E_ρ over S^3/Γ) has been constructed in [9]. In the case $\underline{A}_\rho$ is any extension of a flat connection A_ρ corresponding to ρ the index theorem for the twisted Dirac operator $\mathfrak{D}_\rho$ is given by (cf. Eq. (2.8))

$$\textbf{Index}\, \mathfrak{D}_\rho = \int_{\mathsf{M}} \mathrm{ch}(\underline{E}_\rho)\widehat{A}(\mathsf{M}) - \sum_{\gamma\neq 1} \frac{\chi_\rho(\gamma)}{|\Gamma|(2-\chi_r(\gamma))}. \tag{2.26}$$

[2] The result is obtained by considering a theory of a massive fermion and taking the mass to 0.

Here r denotes the two-dimensional representation induced by the inclusion $\Gamma \to SU(2)$, χ denotes the character of a representation, while $|\Gamma|$ is the order of Γ (see [9,24] for more details).

In connection with a real compact hyperbolic manifold X consider a locally homogeneous Dirac bundle E over X and the corresponding Dirac operator $\mathfrak{D} : C^\infty(\mathsf{X}, E) \to C^\infty(\mathsf{X}, E)$. As before, assume that $\mathsf{X} = \partial\mathsf{M}$, that E extends to a Clifford bundle on M.[3] and that $\varphi : \pi_1(\mathsf{X}) \to U(F)$ extends to a representation of $\pi_1(\mathsf{M})$. Let $\underline{A}_\varphi$ be an extension of a flat connection A_φ corresponding to φ.

The Cayley transform determinant and adiabatic limit Let us consider a determinant construction for a self-adjoint operator on a finite dimensional Hilbert space. The classical Cayley transform [25] for such an operator $\mathfrak{D}$ is the unitary operator $C = (\mathfrak{D} - i)/(\mathfrak{D} + i)$. For $s \in \mathbb{C}$ we have a family of operators

$$C(s) = \frac{\mathfrak{D} - is}{\mathfrak{D} + is}. \tag{2.27}$$

This family is meromorphic, has poles at $s \in i\mathrm{Spec}'(\mathfrak{D})$ ($\mathrm{Spec}'(\mathfrak{D}) \equiv \mathrm{Spec}(\mathfrak{D}) - \{0\}$), these poles being simple and having residue $\mathrm{Res}_{-i\lambda} C(s) = 2i\lambda P_\lambda$, where P_λ is projection onto the $i\lambda$ eigenspace. One has (see for details [8])

$$\log \det{}' C(s) = \sum_{\lambda \in \mathrm{Spec}'(\mathfrak{D})} m(\lambda) \log\left(\frac{\lambda - is}{\lambda + is}\right), \tag{2.28}$$

where $m(\lambda)$ denote the multiplicity.

Let $\mathfrak{D}$ be a Dirac operator, as defined above; the family of operators $C(s) = (\mathfrak{D}) - is)/(\mathfrak{D} + is)$ is meromorphic with simple poles at $s \in i\mathrm{Spec}'(\mathfrak{D})$. The determinant satisfies the functional identity $\det'(\mathfrak{D}+is)/(\mathfrak{D}-is) \cdot \det'(\mathfrak{D}-is)/(\mathfrak{D}+is) = 1$. The following result holds [8] (Proposition 2.2):

$$\lim_{x \to +\infty} \det{}' C(x) = e^{-i\pi\eta(0,\mathfrak{D})}. \tag{2.29}$$

Set $\varepsilon = x^{-1}$. Then if one replaces the metric g on X by $g_\varepsilon = g\varepsilon^{-1}$ then Eq. (2.29) says that the adiabatic limit (**a**–lim) of the Cayley transform of $\mathfrak{D}_\varepsilon$ is $\exp(-i\pi\eta(0, \mathfrak{D}))$ (cf. Eq. (2.24))

$$\mathbf{a}\!-\!\lim_{\varepsilon \to 0} \det{}' \frac{\mathfrak{D} - i\varepsilon^{-1}}{\mathfrak{D} + i\varepsilon^{-1}} = e^{-i\pi\eta(0,\mathfrak{D})}. \tag{2.30}$$

[3] A Clifford module bundle is called a Dirac bundle if it has a connection ∇ satisfying the compatibility condition $\nabla_z(v \cdot s) = (\nabla_z^R v) \cdot s + v \cdot (\nabla_z s)$. Here s is a local section of E, v is a local section of $C\ell(\mathsf{M})$, z a vector field and $(\cdot)$ denote the module multiplication. On a Dirac bundle one then has a Dirac operator defined by $\mathfrak{D}s = \sum_j e_j \cdot (\nabla_{e_j} s)$, where $\{e_j\}$ is any local orthonormal frame for M.

Locally symmetric spaces of higher rank It has been shown [26,27] that for variety flows the zeta-function associated to any cyclic flat bundle is actually meromorphic on a neighborhood of $[0, \infty)$, regular at $s = 0$, and its value at $s = 0$ coincides with R-torsion with coefficients in the given flat bundle, and thus is a topological invariant. Recall that Ray and Singer defined an analytic torsion $\tau_\rho^{\mathrm{an}}(\mathsf{X}) \in (0, \infty)$ for every closed Riemannian manifold X and orthogonal representation $\rho : \pi_1(\mathsf{X}) \to O(n)$ [28]. Because of the analogy with the Lefschetz fixed point formula, Fried proved that the geodesic flow of a closed manifold of constant negative curvature has the Lefschetz property [29]. He also conjectured that this remains true for any closed locally homogeneous Riemannian manifold.

Fried's conjecture has been proved and an adequate theory of Selberg-type zeta-functions for locally symmetric spaces of higher rank was constructed in [14]. Difficulties have been avoided by constructing certain *super* Selberg zeta-functions, $Z^\ell(s, \mathfrak{D}_\varphi)$, $0 \le \ell \le 2m < \dim \mathsf{X}$, as alternating products of formal Selberg-like functions, which reduce to Selberg zeta-functions only in the three-dimensional rank one case. Each function $Z^\ell(s, \mathfrak{D}_\varphi)$ is meromorphic on $\mathbb{C}$ and, moreover, satisfies a functional equation (see [14] for details). Not surprisingly, the functional equations play a crucial role in identifying the special value of the Selberg-type spectral function $\mathcal{R}(s; \varphi)$ with the R-torsion. Finally, $\mathcal{R}(s; \varphi)$ can be expressed as an alternating product of Z^ℓ,

$$\mathcal{R}(s; \varphi) = \prod_{\ell=0}^{\dim \mathsf{X}-1} Z^\ell(s - \dim \mathsf{X} + 1 + \ell,\ \mathfrak{D}_\varphi)^{(-1)^\ell}. \tag{2.31}$$

On this basis we conjecture that the Chern–Simons invariant for locally symmetric spaces of higher rank admits a representation in terms of the spectral function $\mathcal{R}(s; \varphi)$. This is an interesting and important conclusion and we hope to come back to this analysis in the future.

3 Infinite products for the quantum $\mathfrak{sl}_N$ invariant

In this section we consider more general correlators in a CS theory, and view them as generating series of quantum group invariants weighted by S-functions. The quantum group invariants can be defined over any semi-simple Lie algebra $\mathfrak{g}$. In the $SU(N)$ Chern–Simons gauge theory we study the quantum $\mathfrak{sl}_N$ invariants, which can be identified as the so-called colored HOMFLY polynomials[4]

[4] The framed HOMFLY polynomial of links (an invariant of framed oriented links), is denoted by $\mathcal{H}(\mathcal{L})$, and can be normalized as follows: $\mathcal{H}(\bigcirc) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})/(q^{-\frac{1}{2}} - q^{\frac{1}{2}})$. (These invariants can be recursively computed through the HOMFLY skein.) The colored HOMFLY polynomials are defined through the *satellite knot*. A satellite of knot $\mathcal{K}$ is

One important corollary of the LMOV conjecture is the possibility to express a Chern–Simons partition function as an infinite product. In this article we derive such a product. During the calculations we use the characters of the symplectic groups. The latter were found by Weyl [30] using a transcendental method (based on integration over the group manifold). However, the appropriate characters may also be obtained by algebraic methods [31]. Following [32] we have used algebraic methods. This allows us to exploit the Hopf algebra methods to determine (sub)group branching rules and the decomposition of tensor products.

The motivation for studying an infinite-product formula, associated to topological string partition functions, based on a guess on the modular property of partition function, stimulated by properties of S-functions.

Preliminaries To derive the infinite-product formula, we need some preliminary material. First of all we denote by $\mathcal{Y}$ the set of all Young diagrams. Let χ_A be the character of the irreducible representation of the symmetric group labeled by a partition A. Given a partition μ, define $m_j = \mathrm{card}(\mu_k = j; k \geq 1)$. (The order of the conjugate class of type μ is given by $\mathfrak{z}_\mu = \prod_{j\geq 1} j^{m_j} m_j!$.) The symmetric power functions of a given set of variables $X = \{x_j\}_{j\geq 1}$ are defined as the direct limit of the Newton polynomials: $p_n(X) = \sum_{j\geq 1} x_j^n$, $p_\mu(X) = \prod_{i\geq 1} p_{\mu_i}(X)$, and we have the following formula which determines the Schur function and the orthogonality property of the character:

$$s_A(X) = \sum_\mu \frac{\chi_A(C_\mu)}{\mathfrak{z}_\mu} p_\mu(X),$$
$$\sum_\mu \frac{\chi_A(C_\mu)\chi_B(C_\mu)}{\mathfrak{z}_\mu} = \delta_{A,B}, \tag{3.1}$$

where C_μ denotes the conjugate class of the symmetric group $S_{|\mu|}$ corresponding to partition μ (for details see Sect. 3 of [33]).

Given $X = \{x_i\}_{i\geq 1}$, $Y = \{y_j\}_{j\geq 1}$, define $X * Y = \{x_i \cdot y_j\}_{i\geq 1, j\geq 1}$. We also define $X^d = \{x_i^d\}_{i\geq 1}$. The dth Adams operation of a Schur function is given by $s_A(X^d)$. (An Adams operation is type of algebraic construction; the basic idea of this operation is to implement some fundamental identities in S-functions. In particular, $s_A(X^d)$ means operation of a power sum on a polynomial.)

We use the following conventions for the notation:

- $\mathcal{L}$ will denote a link and L the number of components in $\mathcal{L}$.
- The irreducible $U_q(\mathfrak{sl}_N)$ module associated to $\mathcal{L}$ will be labeled by their highest weights, thus by Young diagrams. We usually denote it by a vector form $\overrightarrow{A} = (A^1, \ldots, A^L)$.
- Let $\overrightarrow{X} = (x_1, \ldots, x_L)$ be a set of L variables, each of which is associated to a component of $\mathcal{L}$ and $\overrightarrow{\mu} = (\mu^1, \ldots, \mu^L) \in \mathcal{Y}^L$ be a tuple of L partitions. We write

$$[\overrightarrow{\mu}] = \prod_{\alpha=1}^L [\mu^\alpha], \quad \mathfrak{z}_{\overrightarrow{\mu}} = \prod_{\alpha=1}^L \mathfrak{z}_{\mu^\alpha},$$
$$\chi_{\overrightarrow{A}}(C_{\overrightarrow{\mu}}) = \prod_{\alpha=1}^L \chi_{A^\alpha}(C_{\mu^\alpha}),$$

$$s_{\overrightarrow{A}}(\overrightarrow{X}) = \prod_{\alpha=1}^L s_{A^\alpha}(x_\alpha), \quad p_\mu(X) = \prod_{i=1}^{\ell(\mu)} p_{\mu_i}(X),$$
$$p_{\overrightarrow{\mu}}(\overrightarrow{X}) = \prod_{\alpha=1}^L p_{\mu^\alpha}(x_\alpha).$$

The case of links and a knot The quantum $\mathfrak{sl}_N$ invariant for the irreducible module $V_{A^1}, \ldots, V_{A^L}$, labeled by the corresponding partitions $A^1, \ldots, A^L$, can be identified as the HOMFLY invariants for the link decorated by $Q_{A^1}, \ldots, Q_{A^L}$. The quantum $\mathfrak{sl}_N$ invariants of the link is given by $P_{\overrightarrow{A}}(\mathcal{L}; q, t) = \mathcal{H}(\mathcal{L} \star \otimes_{\alpha=1}^L Q_{A^\alpha})$. The colored HOMFLY polynomial of the link $\mathcal{L}$ can be defined by [33]

$$P_{\overrightarrow{A}} = q^{-\sum_{\alpha=1}^L k_{A^\alpha}\omega(\mathcal{K}_\alpha)} t^{-\sum_{\alpha=1}^L |A^\alpha| \omega(\mathcal{K}_\alpha)} \langle \mathcal{L} \star \otimes_{\alpha=1}^L Q_{A^\alpha} \rangle, \tag{3.2}$$

where $\omega(\mathcal{K}_\alpha)$ is the number of the α-component $\mathcal{K}_\alpha$ of $\mathcal{L}$ and the bracket $\langle \mathcal{L} \star \otimes_{\alpha=1}^L Q_{A^\alpha} \rangle$ denotes the framed HOMFLY polynomial of the satellite link $\mathcal{L} \star \otimes_{\alpha=1}^L Q_{A^\alpha}$. We can define the following invariants:

$$\mathsf{W}_{\overrightarrow{\mu}}(\mathcal{L}; q, t) = \sum_{\overrightarrow{A}=(A^1,\ldots,A^L)} \left(\prod_{\alpha=1}^L \chi_{A^\alpha}(C_{\mu^\alpha}) \right) P_{\overrightarrow{A}}(\mathcal{L}; q, t). \tag{3.3}$$

The Chern–Simons partition function $\mathsf{W}_{\mathrm{CS}}^{\mathrm{SL}}(\mathcal{L}; q, t)$ and the free energy $F(\mathcal{L}; q, t)$ of the link $\mathcal{L}$ are the following generating series of quantum group invariants weighted by Schur functions $s_{\overrightarrow{A}}$ and by the invariants $\mathsf{W}_{\overrightarrow{\mu}}$:

$$\mathsf{W}_{\mathrm{CS}}^{\mathrm{SL}}(\mathcal{L}; q, t) = 1 + \sum_{\overrightarrow{A}} P_{\overrightarrow{A}}(\mathcal{L}; q, t) s_{\overrightarrow{A}}(\overrightarrow{X})$$

Footnote 4 continued
determined by choosing a diagram Q in the annulus. Draw Q on the annular neighborhood of $\mathcal{K}$ determined by the framing, to give a satellite knot $\mathcal{K} \star Q$. One can refer to this construction as *decorating $\mathcal{K}$ with the pattern Q*. The HOMFLY polynomial $\mathcal{H}(\mathcal{K} \star Q)$ of the satellite depends on Q only as an element of the skein $\mathcal{C}$ of the annulus. $\{Q_\lambda\}_{\lambda\in\mathcal{Y}}$ form a basis of $\mathcal{C}$. $\mathcal{C}$ can be regarded as the parameter space for these invariants of $\mathcal{K}$, and can be called the HOMFLY *satellite invariants of $\mathcal{K}$*.

$$= 1 + \sum_{\vec{\mu}} \frac{\mathsf{W}_{\vec{\mu}}(\mathcal{L}; q, t)}{\mathfrak{z}_{\vec{\mu}}} p_{\vec{\mu}}(\vec{X}), \tag{3.4}$$

$$F(\mathcal{L}; q, t) = \log \mathsf{W}_{\mathrm{CS}}(\mathcal{L}; q, t) = \sum_{\vec{\mu}} \frac{F_{\vec{\mu}}(\mathcal{L}; q, t)}{\mathfrak{z}_{\vec{\mu}}} p_{\vec{\mu}}(\vec{X}). \tag{3.5}$$

Based on LMOV conjecture the infinite-product formula for the case of links, $\mathsf{W}^{\mathrm{SL}}_{\mathrm{CS}}(\mathcal{L}; q, t; \vec{X})$ and a knot $\mathsf{W}^{\mathrm{SL}}_{\mathrm{CS}}(\mathcal{K}; q, t; X)$ are given by [34,35]

$$\mathsf{W}^{\mathrm{SL}}_{\mathrm{CS}}(\mathcal{L}; q, t; \vec{X}) = \prod_{\vec{\mu}} \prod_{Q\in\mathbb{Z}/2} \prod_{m=1}^{\infty} \prod_{k=-\infty}^{\infty} \times \left\langle 1 - q^{k+m} t^Q \vec{X} \right\rangle^{-n_{\vec{\mu};g,Q}}, \tag{3.6}$$

$$\mathsf{W}^{\mathrm{SL}}_{\mathrm{CS}}(\mathcal{K}; q, t; X) = \prod_{\mu} \prod_{Q\in\mathbb{Z}/2} \prod_{m=1}^{\infty} \prod_{k=-\infty}^{\infty} \times \langle 1 - q^{k+m} t^Q X^{\mu} \rangle^{-m\, n_{\mu;g,Q}}. \tag{3.7}$$

Here $\vec{\mu} = (\mu^1, \ldots, \mu^L)$, the length of μ^i is ℓ_i, $\vec{X} = (x_1, \ldots, x_L)$, and $n_{\mu;g,Q}$ are invariants related to the integer invariants in the LMOV conjecture. For a given μ, $n_{\mu;g,Q}$ vanish for sufficiently large $|Q|$ due to the vanishing property of $n_{\mu;g,Q}$. The products involving Q and k are finite products for a fixed partition μ.

The symmetric product $\langle 1 - q^{k+m} t^Q X^{\mu}\rangle$ and the generalized symmetric product $\langle 1 - q^{k+m} t^Q \vec{X}\rangle$ in Eqs. (3.7) and (3.6), respectively, are defined by the formula [35]

$$\langle 1 - \psi X^{\mu}\rangle = \prod_{x_{i_1},\ldots,x_{i_{\ell(\mu)}}} (1 - \psi\, x_{i_1}^{\mu_1} \cdots x_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}}), \tag{3.8}$$

$$\langle 1 - \psi \vec{X}^{\mu}\rangle = \prod_{\alpha=1}^{L} \prod_{i_{\alpha,1},\ldots,i_{\alpha,\ell_\alpha}} \times \left(1 - \psi \prod_{\alpha=1}^{L} ((x_\alpha)_{i_{\alpha,1}}^{\mu_1^\alpha} \cdots (x_\alpha)_{i_{\alpha,\ell_\alpha}}^{\mu_{\ell_\alpha}^\alpha})\right). \tag{3.9}$$

where ψ is a generic variable.

Double series and certain modular forms In Eqs. (3.6), (3.7) and (3.8), (3.9) the blocks for affine-like denominators admit representations in the form of spectral functions of hyperbolic geometry (see for example [36]). One can successfully construct quantum homological invariants and express the formal character of the irreducible tensor representations of the classical groups in terms of the symmetric and spectral functions of hyperbolic geometry [37]. Indeed, he products in Eqs. (3.6) and (3.7) can be represented in the form of Selberg-type spectral function $\mathcal{R}(s)$ of hyperbolic three-geometry. $\mathcal{R}(s)$ is an alternating product of more complicated factors, each of which is a so-called Patterson–Selberg zeta-functions Z_{Γ^γ} [38,39] ($\mathcal{R}(s)$ can be continued meromorphically to the entire complex plane $\mathbb{C}$),

$$\prod_{n=\ell}^{\infty}(1 - q^{an+\varepsilon}) = \prod_{p=0,1} Z_{\Gamma^\gamma}(\underbrace{(a\ell+\varepsilon)(1 - i\varrho(\vartheta)) + 1 - a}_{s} + a(1 + i\varrho(\vartheta)p)^{(-1)^p} = \mathcal{R}(s = (a\ell+\varepsilon)(1 - i\varrho(\vartheta)) + 1 - a), \tag{3.10}$$

$$\prod_{n=\ell}^{\infty}(1 + q^{an+\varepsilon}) = \prod_{p=0,1} Z_{\Gamma^\gamma}(\underbrace{(a\ell+\varepsilon)(1 - i\varrho(\vartheta)) + 1 - a + i\sigma(\vartheta)}_{s} + a(1 + i\varrho(\vartheta)p)^{(-1)^p} = \mathcal{R}(s = (a\ell+\varepsilon)(1 - i\varrho(\vartheta)) + 1 - a + i\sigma(\vartheta)), \tag{3.11}$$

where $q = \exp(2\pi i\vartheta)$, $\varrho(\vartheta) = \mathrm{Re}\,\vartheta/\mathrm{Im}\,\vartheta$, $\sigma(\vartheta) = (2\,\mathrm{In}\,\vartheta)^{-1}$, a is a real number, $\varepsilon, b \in \mathbb{C}$, $\ell \in \mathbb{Z}_+$. In terms of $\mathcal{R}(s)$ functions, Eq. (3.10) (see also [37], Eq (3.41)) $\mathsf{W}^{\mathrm{SL}}_{\mathrm{CS}}(\mathcal{L}; q, t; \vec{X})$ and $\mathsf{W}^{\mathrm{SL}}_{\mathrm{CS}}(\mathcal{K}; q, t; X)$ take the form

$$\mathsf{W}^{\mathrm{SL}}_{\mathrm{CS}}(\mathcal{L}; q, t; \vec{X}) = \prod_{\vec{\mu}} \prod_{Q\in\mathbb{Z}/2} \prod_{k=-\infty}^{\infty} \prod_{\alpha=1}^{L} \prod_{i_{\alpha,1},\ldots,i_{\alpha,\ell_\alpha}} \prod_{m=1}^{\infty} \times \mathcal{R}\Big(s = (m + \Omega(q^{g-2k} t^Q \vec{X}^{\mu}; \vartheta))(1 - i\varrho(\vartheta))\Big)^{-n_{\vec{\mu};g,Q}}, \tag{3.12}$$

$$\mathsf{W}^{\mathrm{SL}}_{\mathrm{CS}}(\mathcal{K}; q, t; X) = \prod_{\mu} \prod_{Q\in\mathbb{Z}/2} \prod_{k=-\infty}^{\infty} \prod_{x_{i_1},\ldots,x_{i_{\ell(\mu)}}} \prod_{m=1}^{\infty} \times (\mathcal{R}(s = (m + \Omega(t^Q X^{\mu} q^k; \vartheta))(1 - i\varrho(\vartheta))))^{-n_{\mu;g,Q}}, \tag{3.13}$$

where $\Omega(q^k t^Q X^{\mu}; \vartheta) \equiv \log(q^k t^Q x_{i_1}^{\mu_1} \cdots x_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}})/2\pi i\vartheta$.

Calculations in the case of Kauffman polynomials, relative to the orthogonal group, can be found in Ref. [37].

To finish we consider one more topic of interest, the symmetries and the modular form identities. Both Hecke [40] and Rogers [41] recognized that certain modular forms could be represented by combinations of the following double series:

$$\sum_{(m,n)\in\Omega} (-1)^{\mathcal{H}(m,n)} q^{\mathcal{L}(m,n)+\mathcal{Q}(m,n)}. \tag{3.14}$$

Here $\mathcal{H}$, $\mathcal{L}$ are linear forms, $\mathcal{Q}$ is an indefinite quadratic form and Ω is some subset of $\mathbb{Z}\times\mathbb{Z}$. [5]

[5] We also mention on this topic the deep results of [42,43], where a number of identities in the representation theory of Kač–Moody

Infinite double series of type (3.14) and their connection with the $\mathcal{R}(s)$ function have been investigated in [36]. For the functions (see [36] for details)

$$Q_{n,k}(a_1, a_2, a_3, a_4, a_5; q) := (-1)^{n+k} q^{a_1 n + a_2 k + a_3 nk + a_4 n^2 + a_5 k^2}, \tag{3.15}$$

$$\mathcal{J}(a_1, a_2, a_3, a_4, a_5; q) := \left(\sum_{n,k \geq 0} - \sum_{n,k<0} \right) Q_{n,k}(a_1, a_2, a_3, a_4, a_5; q) \tag{3.16}$$

one can find an infinite family of identities,

$$\sum_{n,k=-\infty;\, k \geq |2n|}^{\infty} (-1)^{n+k} q^{(k^2-3n^2)/2+(n+k)/2} = \prod_{n=1}^{\infty} (1-q^n)^2 \tag{3.17}$$

$$\sum_{n,k=-\infty;\, k \geq |2n|}^{\infty} (-1)^{n+k} q^{(k^2-3n^2)/2+(n+k)/2} \stackrel{(3.16),\,(3.10)}{=\!=} \sum_{n,k=-\infty}^{\infty} Q_{n,k \geq |2n|}\left(\frac{1}{2}, \frac{1}{2}, 0, -\frac{3}{2}, \frac{1}{2}; q\right) = \mathcal{R}(s = 1 - i\varrho(\vartheta))^2. \tag{3.18}$$

The identity (3.17) was conjectured by Rogers and has been proved in [40,43]. We finally remark that Rogers' approach can be used to discover possible modular identities and symmetry properties of the infinite products considered in this paper (the simplest symmetry is $q \to q^{-1}$), by using connections between Hecke–Rogers modular form identities and functional equations for $\mathcal{R}(s)$.

Acknowledgements A. A. Bonora would like to acknowledge the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq, Brazil) and Coordenação de Aperfeiçamento de Pessoal de Nível Superior (CAPES, Brazil) for financial support. L.B. would like to thank the Departamento de Fisica da Universitade Estadual de Londrina for its kind hospitality.

Footnote 5 conitinued
Lie algebras has been obtained. A family of modular functions satisfying Rogers–Ramanujan-type identities for arbitrary affine root systems has been obtained in [44]. Extensive work in the theory of partition identities shows that basic hypergeometric series provide the generating functions for numerous families of partition identities.

References

1. E. Witten, Topological quantum field theory. Commun. Math. Phys **117**, 353–386 (1988)
2. N. Yu, Reshetikhin, V.G. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups. Invent. Math. **103**, 547–597 (1991)
3. L. Crane, D. Yetter, A categorical construction of 4D topological quantum field theories, Quantum topology. In: Ser. Knots Everything, vol. 3, pp. 120–130 (World Sci. Publ, River Edge, 1993)
4. H. Ooguri, C. Vafa, Knot invariants and topological strings. Nucl. Phys. B **577**, 419–438 (2000)
5. J. M. F. Labastida, M. Mariño, C. Vafa, Knots, links and branes at large N. JHEP **11**, 007 (2000)
6. R. Gopakumar, C. Vafa, On the gauge theory/geometry correspondence. Adv. Theor. Math. Phys. **3**, 1415–1443 (1999)
7. E. Witten, Quantum field theory and the Jones polynomial. Commun. Math. Phys. **121**, 351–399 (1989)
8. H. Moscovici, R. Stanton, Eta invariants of Dirac operators on locally symmetric manifolds. Invent. Math. **95**, 629–666 (1989)
9. H. Nishi, $SU(n)$-Chern–Simons invariants of Seifert fibered 3-manifolds. Int. J. Math. **9**, 295–330 (1998)
10. M.F. Atiyah, V.K. Patodi, I.M. Singer, Spectral asymmetry and riemannian geometry. I. Math. Proc. Camb. Phil. Soc. **77**, 43–69 (1975)
11. M.F. Atiyah, V.K. Patodi, I.M. Singer, Spectral asymmetry and riemannian geometry. II. Math. Proc. Camb. Phil. Soc. **78**, 405–432 (1975)
12. M.F. Atiyah, V.K. Patodi, I.M. Singer, Spectral asymmetry and riemannian geometry. III. Math. Proc. Camb. Phil. Soc. **79**, 71–99 (1976)
13. J.J. Milson, Closed geodesics and the eta-invariant. Ann. Math. **108**, 1–39 (1978)
14. H. Moscovici, R. Stanton, R-torsion and zeta functions for locally symmetric manifolds. Invent. Math. **105**, 185–216 (1991)
15. J.-M. Bismut, D. S. Freed, The analysis of elliptic families I. Metrics and connections on determinant bundles. Commun. Math. Phys. **106**, 159–176 (1986)
16. J.-M. Bismut, D. S. Freed, The analysis of elliptic families II. Dirac operators, eta invariants, and the holonomy theorem of Witten. Commun. Math. Phys. **107**, 103–163 (1986)
17. D.S. Freed, E. Witten, Anomalies in string theory with D-branes. Asian J. Math. **3**, 819–851 (1999)
18. E. Witten, Global gravitational anomalies. Commun. Math. Phys. **100**, 197–229 (1985)
19. D. Quillen, Determinants of Cauchy–Riemann operators over a Riemann surface. Funk. Anal. i Prilozen. **19**, 37–41 (1985)
20. X. Dai, D. S. Freed, Eta invariants and determinant lines. J. Math. Phys. **35**, 5155–5194 (1994). **(Erratum: ibid 42, 2343–2344 (2001))**
21. D. S. Freed, Determinant lines bundles revisted. In: Lecture Notes in Pure and Applied Mathematics, vol. 184, pp. 187–195 (Marcel Dekker, New York, 1997)
22. L. Bonora, M. Cvitan, P. Dominis Prester, B. Lima de Souza, I. Smoli, Massive fermion model in 3d and higher spin currents. JHEP **1605**, 072 (2016)
23. E. Witten, Anomalies revisited. Lecture at strings 2015, ICTS-TITR, Bangalore, June 22, 2015. In: Witten, E. (ed.) Fermion Path Integrals And Topological Phases. arXiv:1508.04715 [cond-mat.mes-hall]
24. P.B. Kronheimer, H. Nakajima, Young–Mills instantons on ALE gravitational instantons. Math. Ann. **288**, 263–307 (1990)
25. A. Cayley, About the algebraic structure of the orthogonal group and the other classical groups in a field of characteristic zero or a prime characteristic. J. Reine Angew. Math. **32**, 119–123 (1846)

26. D. Fried, Lefschetz Formulas for Flows, in the Lefschetz Centennial Conference, part III. Contempor. Math. **58**, 19–69 (1987)
27. D. Fried, Counting circles. Dynamical systems. In: Lecture Notes in Mathematics, vol. 1348, 196–215 (Springer, Berlin, 2006)
28. D.B. Ray, I. Singer, R-torsion and the Laplacian on Riemannian manifolds. Adv. Math. **7**, 145–210 (1971)
29. D. Fried, Analytic torsion and closed geodesic on hyperbolic manifolds. Invent. Math. **84**, 523–540 (1986)
30. H. Weyl, *The Classical Groups Their Invariants and Representations*, 2nd edn. (Princeton University Press, Princeton, 1946)
31. D.E. Littlewood, Invariant theory, tensors and group characters. Philos. Trans. R. Soc. A **239**, 305–365 (1944)
32. B. Fauser, P. D. Jarvis, R. C. King, Plethysms, replicated Schur functions and series, with applications to vertex operators. J. Phys. A Math. Theor. **43**, 405202 (2010)
33. S. Zhu, Colored HOMFLY polynomial via Skein theory. JHEP **229**, 1310 (2013)
34. K. Liu, P. Peng, Proof of the Labastida–Mariño–Ooguri–Vafa conjecture. J. Differ. Geom. **85**, 479–525 (2010)
35. K. Liu, P. Peng, New structure of knot invariants. Commun. Number Theory Phys. **5**, 601–615 (2011)
36. L. Bonora, A.A. Bytsenko, M. Chaichian, Elliptic genera and characteristic q-series of superconformal field theory. Nucl. Phys. B **895**, 192–205 (2015)
37. A.A. Bytsenko, M. Chaichian, S-functions, spectral functions of hyperbolic geometry, and vertex operators with applications to structure for weyl and orthogonal group invariants. Nucl. Phys. B **907**, 258–285 (2016)
38. L. Bonora, A.A. Bytsenko, Partition functions for quantum gravity, black holes, elliptic genera and Lie algebra homologies. Nucl. Phys. B **852**, 508–537 (2011)
39. A.A. Bytsenko, M. Chaichian, R.J. Szabo, A. Tureanu, Quantum black holes. Elliptic genera and spectral partition functions. IJGMMP **11**, 1450048 (2014)
40. E. Hecke, Über den Zusammenhang zwischen elliptischen Modulfunktionen und indefiniten quadratischen Formen. Math. Werke, Vandenhoeck and Ruprecht, pp. 418–427, Göttingen (1959)
41. L.J. Rogers, On the expansion of some infinite products. Proc. Lond. Math. Soc. **24**, 337–352 (1893)
42. V.G. Kač, D.H. Peterson, Affine Lie algebras and Hecke modular forms. Bull. Am. Math. Soc. **3**, 1057–1061 (1980)
43. V.G. Kač, D.H. Peterson, Infinite-dimensional Lie algebras. Theta functions and modular forms. Adv. Math. **53**, 125–264 (1984)
44. I. Cherednik, B. Feigin, Rogers–Ramanujan type identities and Nil-DAHA. Adv. Math. **248**, 1059–1088 (2013)