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Dynamical system approach to running Λ cosmological models

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Abstract We study the dynamics of cosmological models with a time dependent cosmological term. We consider five classes of models; two with the non-covariant parametrization of the cosmological term Λ: $\Lambda(H)$CDM cosmologies, $\Lambda(a)$CDM cosmologies, and three with the covariant parametrization of Λ: $\Lambda(R)$CDM cosmologies, where $R(t)$ is the Ricci scalar, $\Lambda(\phi)$-cosmologies with diffusion, $\Lambda(X)$-cosmologies, where $X = \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi$ is a kinetic part of the density of the scalar field. We also consider the case of an emergent $\Lambda(a)$ relation obtained from the behaviour of trajectories in a neighbourhood of an invariant submanifold. In the study of the dynamics we used dynamical system methods for investigating how an evolutionary scenario can depend on the choice of special initial conditions. We show that the methods of dynamical systems allow one to investigate all admissible solutions of a running Λ cosmology for all initial conditions. We interpret Alcaniz and Lima's approach as a scaling cosmology. We formulate the idea of an emergent cosmological term derived directly from an approximation of the exact dynamics. We show that some non-covariant parametrization of the cosmological term like $\Lambda(a)$, $\Lambda(H)$ gives rise to the non-physical behaviour of trajectories in the phase space. This behaviour disappears if the term $\Lambda(a)$ is emergent from the covariant parametrization.

1 Introduction

Our understanding of the properties of the current universe is based on the assumption that gravitational interactions, which are extrapolated at the cosmological scales, are described successfully by the Einstein general relativity theory with the cosmological term Λ. If we assume that the geometry of the universe is described by the Robertson–Walker metric, i.e., the universe is spatially homogeneous and isotropic, then we obtain the model of the current universe in the form of standard cosmological model (the ΛCDM model). From the methodological point of view this model plays the role of an effective theory which describes well the current universe in the present accelerating epoch.

If we compare the ΛCDM model with the observational data, then we find that more than 70% of the energy budget is in the form of dark energy and well modelled in terms of an effective parameter of the cosmological constant term.

If we assume that the SCM (standard cosmological model) is an effective field theory which is valid up to a certain cutoff of mass M, and if we extrapolate of the SCM up to the Planck scale then we should have $\Lambda \sim 1$. On the other hand from the observations we find that both density parameters $\Omega_{\Lambda,0} = \frac{\Lambda}{3H_0^2}$ and $\Omega_{\text{m},0} = \frac{\rho_{\text{m},0}}{3H_0^2}$ are order one, which implies $\Lambda \propto H_0^2 \sim 10^{-120}$. We assume the natural units $G = c = \hbar = 1$ here.

In consequence we obtain the huge discrepancy between the expected and observed values of the term Λ. It is just what is called the cosmological constant problem requiring the explanation why the cosmological constant assumes such a small value today.

In this context an idea of a running cosmological constant term appears. It was developed in a series of papers by Shapiro et al. [1–4]. Shapiro and Solà [5] showed neither there is the rigorous proof indicating that the cosmological constant is running, nor there are strong arguments for a non-running one. Therefore one can study different theoretical possibilities of the running Λ term given in a phenomenological form and investigate cosmological implications of such an assumption. Such models are a simple generalization of the standard cosmological model in which the term Λ is constant.

The corresponding form of the $\Lambda(t)$ dependence can be motivated by quantum field theory [5–7] or by some theoretical motivations [8,9]. Padmanabhan [10] and Vishwakarma [11] also suggested that $\Lambda \propto H^2$ from the dimensional considerations.

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The relation $\Lambda(t)$ is not given directly but through a function which describes the evolution of the universe. One can consider two classes of models with the non-covariant parametrizations of the Λ term:

- the cosmological models in which dependence on time is hidden and $\Lambda(t) = \Lambda(H(t))$ or $\Lambda(t) = \Lambda(a(t))$ depends on the time through the Hubble parameter $H(t)$ or scale factor $a(t)$,

and three classes of models with covariant parametrizations of the Λ term:

- the Ricci scalar of the dark energy model, i.e., $\Lambda = \Lambda(R)$,
- the parametrization of the Λ term through the scalar field $\phi(t)$ with a self-interacting potential $V(\phi)$,
- as the special case of the previous one, the Λ term can be parametrized by a kinetic part of the energy density of the scalar field $X = \frac{1}{2} g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi$.

Note that some parametrizations of the Λ term can also arise from another theory beyond general relativity. For example Shapiro and Solà [5] suggested that a solution, which is derived from the form of $\rho_\Lambda(H) = \rho_\Lambda^0 + \alpha(H^2 - H_0^2) + \mathcal{O}(H^4)$, can be a solution of the fundamental general relativity equations.

Another problem, which is related to the standard cosmological problem, is the problem of coincidence [12]. From the cosmological data such as measurements of distant SNIa, CMB, BAO and other astronomical observations, we find that we live in the very special age of the universe when $\rho_\Lambda \sim \rho_{\text{dm}} \sim \rho_{\text{b}}$. The appearance of any epoch with this coincidence is puzzling and we should explain why we live in such a special epoch.

The appearance of scaling solutions in the phase space suggests that in this model the problem of cosmic coincidence can be solved because during the whole evolution $\rho_\Lambda \sim \rho_{\text{m}}$.

The motivation for studying cosmology with the decaying vacuum comes from the solution of the cosmological constant problem as well as the cosmic coincidence problem – the main problems which standard cosmological model struggles. In this context, different propositions of parametrization of the Λ term are postulated. As mentioned above both the covariant contributions to the general relativity action and others violate this covariance. We study cosmological implications of such choices. And the methods of dynamical systems will be used to help us to understand better the dynamical aspects of this problem.

We are looking for such parametrizations of the Λ term for which in the phase space the de Sitter stationary state is a global attractor and a generic class of initial conditions gives rise in this attractor. It is a consequence of the fact that we are going toward a solution of the standard cosmological model without an idea of the fine tuning.

The main aim of this paper is to study dynamics of the cosmological models with the running cosmological term and dust matter. We apply dynamical systems methods to investigate theoretically possible dynamics of these models. The main advantage of these methods is the possibility of studying all solutions (cosmological evolutionary scenarios) for admissible initial conditions. The phase space is a geometrization of the dynamics whose structure informs us how generic are solutions with desired properties. In this approach we are looking for attractor solutions in the phase space representing generic solutions for the problem which gives such a parametrization of $\Lambda(t)$ which explain how the value of cosmological term achieves a small value for the current universe. We search for such an evolutionary scenario for which the Λ_{bare} is an attractor in the phase space.

The dynamics of both the above mentioned subclasses of the $\Lambda(t)$CDM cosmologies is investigated by dynamical system methods. Bonanno and Carloni have recently used these methods to study the qualitative behaviour of FRW cosmologies with time-dependent vacuum energy on cosmological scales [6]. Of course, the methods of dynamical systems are not a way to solve problems of the cosmological constant. It is only a useful tool for the visualization of the dynamics in a geometrical way which can help us to better understand the term Λ during the cosmic evolution.

We also develop the idea of an emergent relation $\Lambda(a)$ obtained from the behaviour of the trajectories of the dynamical system near the invariant submanifold $\frac{\dot{H}}{H^2} = 0$. By the emerging of a running parametrization $\Lambda(a)$ we understand its derivation directly from the true dynamics. Therefore, the corresponding parametrization is obtained from the entry of trajectories in a de Sitter state.

Measurements of the cosmic microwave background anisotropy are considered in the background of the ΛCDM model and indicates that the cosmological spatial hypersurface of the FRW geometry is very close to flat [13,14]. On the other hand, under of the assumption of flatness, the data favour rather the time-independent dark energy [15].

It is well known that if a spatially curved time variable dark energy model is used to analyse the CMB anisotropy measurements then there is a degeneracy between the spatial curvature and the parameters which govern the dark energy time variability. For this reason it seems that an in-depth analysis should be performed of the influence of curvature effects on the dynamical scenarios of different cosmological models.

For this aim we consider the following issues.

- We explore idea of the reducing dynamics to the form of the 2D dynamical system of the Newtonian type as soon as possible. In this system, the energy integral is related with the curvature index (or density parameter for the curvature fluid) and therefore energy levels will determine evolutional paths in the configuration space. All information, which is concerning these types of evolution, can be directly taken from the geometry of potential function $V(a)$ because the curvature effect of new types of evolution emerges. For example we can obtain oscillating models, models with bounce, oscillating models without the initial and final singularity etc.

– From the cosmological point of view, it is interesting to find in the phase space attractors, which position is caused by curvature effects. In the generic case these attractors lie on the invariant submanifold, which represents the surface of the flat model. However, our dynamical analysis gives us an opportunity to detect curvature attractors beyond the invariant submanifold, which represents the evolution of the flat models, which are studied in detail by the phase portraits of the lower dimension.

2 $\Lambda(H)$CDM cosmologies as dynamical systems

From the theoretical point of view if we do not know the exact form of the $\Lambda(t)$ relation we study the dynamical properties of cosmological models in which the Λ-dependence on the cosmological time t is through the Hubble parameter or scale factor, i.e. $\Lambda(t) = \Lambda(H(t))$ or $\Lambda(t) = \Lambda(a(t))$. The connection of such models with the mentioned ones in the previous section will be demonstrated, in which the choice of a $\Lambda(t)$ form was motivated by physics. Cosmological models with a quadratic Λ-dependence on the cosmological time are revealed as a special solution in the phase space.

In the investigation of the dynamics of $\Lambda(H)$ cosmologies we apply the dynamical system methods [16]. We investigate all solutions which are admissible for all physically admitted initial conditions. The global characteristics of the dynamics are given in the form of phase portraits, which reflect the phase space structure of all solutions of the problem. The phase space structure contains all information as regards dependence of solutions on initial conditions, its stability, genericity, etc. Then we can distinguish some generic (typical) cases as well as non-generic (fine-tuned) ones, which physical realizations require a tuning of the initial conditions. The methods of dynamical systems allow us to study the stability of the solutions in a simple way by investigation of te linearization of the system around the non-degenerate critical points of the system.

If the dynamical system is in the form $\dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt} = f(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$ and f is of class C^∞, then the solution of this system is a vector field $\mathbf{x}(t; \mathbf{x}_0)$ where $\mathbf{x}(t_0)$ is a vector of initial conditions. Beyond this regular solution there are singular ones. They are special and obtained from the condition of vanishing of its right-hand sides.

The $\Lambda(H)$CDM cosmological models have recently been investigated intensively in the contemporary cosmology [8, 17–19]. Among these class of models there is one with a particular form of $\Lambda(t) = \Lambda + \alpha H^2$. It was studied in detail in [17]. Its generalization to the relation of $\Lambda(H)$ given in the form of a Taylor series of the Hubble parameter can be found in [20].

It is also interesting that motivations for studying such a class of models can be taken from Urbanowski's expansion formula for decaying false vacuum energy, which can be identified with the cosmological constant term [7]. It is sufficient to interpret the time t in terms of the Hubble time scale $t = t_H \equiv \frac{1}{H}$. Therefore, $\Lambda(H)$CDM cosmologies can be understood as some kind of effective theories of the influence of vacuum decay in the universe [21]. This approach is interesting especially in the context of both the dark energy and the dark matter problem because the problem of cosmological constant cannot be investigated in isolation from the problem of dark matter.

In $\Lambda(H)$ cosmologies, in general, a scaling relation on matter is modified and differs from the canonical relation $\rho_{m=}\rho_{m,0}a^{-3}$ in the ΛCDM model. The deviation from the canonical relation here is characterized by a positive constant ϵ such that $\rho_m = \rho_{m,0}a^{-3+\epsilon}$ [22].

FRW cosmologies with a running cosmological term $\Lambda(t)$ such that $\rho_{vac} = \Lambda(t)$ and $p_{vac} = -\Lambda(t)$ can be formulated in the form of a non-autonomous dynamical system,

$$\frac{dH}{dt} \equiv \dot{H} = -H^2 - \frac{1}{6}(\rho_m + 3p_m) + \frac{1}{3}\Lambda(t) \tag{1}$$

$$\frac{d\rho_m}{dt} \equiv \dot{\rho}_m = -3H(\rho_m + p_m) - \dot{\Lambda}, \tag{2}$$

where ρ_m and p_m are the energy density and the pressure of matter, respectively, and a dot denotes differentiation with respect to the cosmological time t. In this paper, we assume that $8\pi G = c = 1$. In this model the energy-momentum tensor is not conserved because of the presence of an interaction in both matter and dark energy sector. System (1)–(2) has a first integral called the conservation condition in the form

$$\rho_m - 3H^2 = -\Lambda(t). \tag{3}$$

Note that the solution $\rho_m = 0$ is a solution of (2) only if $\Lambda = \text{const}$. Of course system (1)–(2) does not form a closed dynamical system, while a concrete form of the $\Lambda(t)$ relation is not postulated. Therefore, this cosmology belongs to a more general class of models in which the energy-momentum tensor of matter is not conserved.

Let us consider that both visible matter and dark matter are given in the form of dust, i.e. $p_m = 0$ and

$$\Lambda(t) = \Lambda(H(t)). \tag{4}$$

Due to the above simplifying assumption (4), system (1)–(2) with the first integral in the form (3) assumes the form of a two-dimensional closed dynamical system,

$$\dot{H} = -H^2 - \frac{1}{6}\rho_{\rm m} + \frac{1}{3}\Lambda(t), \quad (5)$$

$$\dot{\rho}_{\rm m} = -3H\rho_{\rm m} - \Lambda'(H)\left(-H^2 - \frac{1}{6}\rho_{\rm m} + \frac{\Lambda(H)}{3}\right), \quad (6)$$

where $\Lambda'(H) = \frac{{\rm d}\Lambda}{{\rm d}H}$ and $\rho_{\rm m} - 3H^2 = -\Lambda(H)$ are the first integrals of system (5)–(6).

Let us consider $\Lambda(H)$ given in the form of a Taylor series with respect to the Hubble parameter H, i.e.

$$\Lambda(H) = \sum_{n=1}^{\infty} \frac{1}{n!}\frac{{\rm d}^n}{{\rm d}H^n}\Lambda(H)|_0 H^n. \quad (7)$$

We assume additionally that the model dynamics has a reflection symmetry, $H \to -H$, i.e., $a(t)$ is a solution of the system and $a(-t)$ is also its solution. Therefore, only even terms of type H^{2n} are present in the expansion series (7). Finally, we assume the following form of the energy density parametrization through the Hubble parameter H [1]:

$$\rho_\Lambda(H) = \Lambda_{\rm bare} + \alpha_2 H^2 + \alpha_4 H^4 + \cdots . \quad (8)$$

There are also some physical motivations for such a choice of the $\Lambda(H)$ parametrization (see [19]).

It would be useful for the further dynamical analysis of the system under consideration to re-parametrize the time variable

$$\tau \longmapsto \tau = \ln a \quad (9)$$

and to rewrite the dynamical system (5)–(6) in the new variables

$$x = H^2, \quad y = \rho_{\rm m}. \quad (10)$$

Then we obtain the following dynamical system:

$$x' \equiv \frac{{\rm d}x}{{\rm d}\ln a} = 2\left[-x - \frac{1}{6}y + \frac{1}{3}(\Lambda + \alpha_2 x + \alpha_4 x^2 + \cdots)\right], \quad (11)$$

$$y' \equiv \frac{{\rm d}y}{{\rm d}\ln a} = -3y - \frac{1}{3}(\alpha_2 + 2\alpha_4 x + \cdots) \times \left[-x - \frac{1}{6}y + \frac{1}{3}(\Lambda + \alpha_2 x + \alpha_4 + \cdots)\right] \quad (12)$$

and

$$y - 3x = -(\Lambda + \alpha_2 x + \alpha_4 x^2 + \cdots) \quad (13)$$

where instead of $\Lambda_{\rm bare}$ we write simply Λ, which represents a constant contribution to the $\Lambda(H)$ given by the expansion in the Taylor series (7).

Now, with the help of the first integral (13) we rewrite system (11)–(12) to the new form

$$x' = 2\left(-x - \frac{1}{6}y + \frac{3x - y}{3}\right) = -y, \quad (14)$$

$$y' = -3y - (\alpha_2 + 2\alpha_4 x + \cdots)\frac{3x - y}{9}. \quad (15)$$

Therefore, all trajectories of the system on the plane (x, y) are determined by the first integral (13).

The dynamical system (11)–(12) in a finite domain has a critical point of the one type: a stationary solution $x = x_0$, $y = y_0 = 0$ representing a de Sitter universe. In the original variables $(H, \rho_{\rm m})$ we have two solutions: the stable expanding de Sitter universe and the unstable contracting de Sitter universe, both lying on the H axis. Note that if stationary solutions exist then they always lie on the intersection of the x axis ($y = 0$) with the trajectory of the flat model represented by the first integral (13), i.e., they are solutions of the following polynomial equation:

$$x - \frac{1}{3}(\Lambda + \alpha_2 x + \alpha_4 x^2 + \cdots) = 0 \quad (16)$$

and $y = 0$ (empty universe).

Note that the static critical point which represents the static Einstein universe does not satisfy the first integral (13) because both y and Λ are positive. Let us notice that if we substitute y into (11) then the dynamics is reduced to a form of a one-dimensional dynamical system,

$$\frac{{\rm d}x}{{\rm d}\tau} = -(3x - \Lambda - \alpha_2 x - \alpha_4 x^2 - \cdots). \quad (17)$$

$$y = 3x - (\Lambda + \alpha_2 x + \alpha_4 x^2 + \cdots). \quad (18)$$

Following the Hartmann–Grobman theorem [16] a system in the neighbourhood of critical points is well approximated by its linear part obtained by its linearization around this critical point.

On the other hand, a linear part dominates for small x in a right-hand side. Let us consider the dynamical system (17) truncated on this linear contribution, then the Hartman–Grobman theorem [16] guarantees us that the dynamical system in a neighbourhood of the critical point is a good approximation of the behaviour near the critical points. This system has the simple form

$$\frac{{\rm d}x}{{\rm d}\tau} = x(\alpha_2 - 3) + \Lambda, \quad (19)$$

$$y = (3 - \alpha_2)x - \Lambda. \quad (20)$$

System (19)–(20) has the single critical point of the form

$$x_0 = \frac{\Lambda}{3 - \alpha_2}, \quad y = 0. \quad (21)$$

It represents an empty de Sitter universe.

Let us now shift the position of this critical point to the origin by introducing the new variable $x \to X = x - x_0$. Then we obtain

$$\frac{dX}{d\tau} = (\alpha_2 - 3)X, \tag{22}$$

which possesses the exact solution of the form

$$X = X_0 e^{\tau(\alpha_2 - 3)} = X_0 a^{-3+\alpha_2}, \tag{23}$$

where α_2 is constant. Of course this critical point is asymptotically stable if $\alpha_2 < 3$. The trajectories approaching this critical point at $\tau = \ln a \to \infty$ has the attractor solution $X = X_0 a^{\alpha_2 - 3}$ or $x = X + x_0$, where $x_0 = \frac{\Lambda}{3-\alpha_2}$ or $X = 0$ (see Fig. 1). This attractor solution is crucial for the construction of a new model of a decaying Lambda effect strictly connected with the dark matter problem [9,21].

The solution (23) has a natural interpretation: in a neighbourhood of a global attractor of system (17), trajectories behave as the universal solution, which motivates the Alcaniz–Lima approach in which

$$x = H^2 = \frac{\tilde{\rho}_{m,0}}{3} a^{-3+\alpha_2} + \frac{\rho_{\Lambda,0}}{3}, \tag{24}$$

where $\tilde{\rho}_{m,0} = \frac{3}{3-\alpha_2}\rho_{m,0}$.

We can rewrite Eq. (1) as the Newtonian equation of motion for a particle of unit mass moving in the potential $V(a)$

$$\ddot{a} = -\frac{\partial V(a)}{\partial a}. \tag{25}$$

In our case the potential $V(a)$ is given in the following form:

$$V(a) = \frac{1}{2}\left(\frac{\Lambda}{3-\alpha_2} - H_0^2\right)a^{-1+\alpha_2} - \frac{1}{2}\frac{\Lambda}{3-\alpha_2}a^2. \tag{26}$$

The first integral of (25) can be expressed by

$$\frac{\dot{a}^2}{2} + V(a) = E = \text{const}, \tag{27}$$

where E is the value of the energy level (for the positive curvature $E = -1/2$, for the negative curvature $E = 1/2$ and for the flat universe $E = 0$). Figure 2 presents the evolution of $V(a)$ for $\alpha_2 = 0.1$ and for $\alpha_2 = 1$.

In our case if we consider the curvature in the dynamical analysis then we get new solutions for the positive curvature such as the oscillating universe with the initial singularity and the universe with the bounce. But if we perturb solutions for the flat universe by a small spatial curvature then these solutions do not change qualitatively (see Fig. 2).

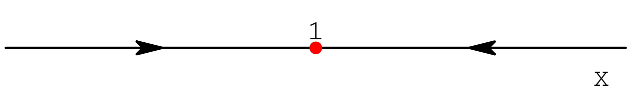


Fig. 1 A one-dimensional phase portrait of the FRW model with $\Lambda = \Lambda(H)$. Note the existence of universal behaviour of the $H^2(a)$ relation near the stable critical point (1) of the type of stable node. In a neighbourhood of this attractor we have the solution $X = H^2 - \frac{\Lambda}{3-\alpha_2} = X_0 a^{\alpha_2 - 3}$ and $\rho_m = (3-\alpha_2)H^2 - \Lambda = X_0 a^{\alpha_2 - 3}$. Therefore both ρ_m and $\rho_\Lambda - \Lambda$ are proportional (scaling solution)

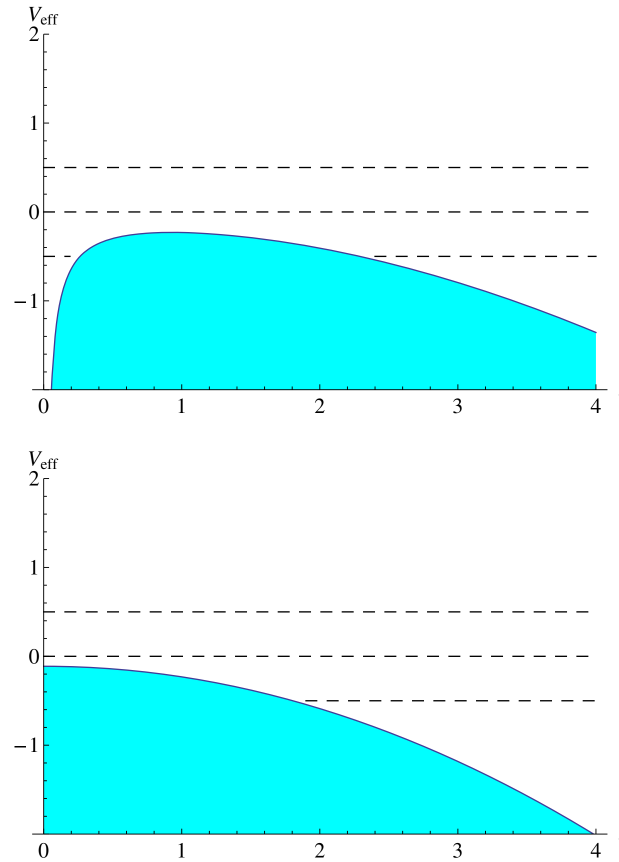


Fig. 2 The potential $V_{eff}(a)$ for $\alpha_2 = 0.1$ (*top diagram*) and $\alpha_2 = 1$ (*bottom diagram*). The *top dashed lines* ($V_{eff} = 1/2$) represent the energy level, which corresponds with the negative curvature. The *bottom dashed lines* ($V_{eff} = -1/2$) represent the energy level, which corresponds with the positive curvature. The *middle dashed lines* ($V_{eff} = 0$) represent the energy level, which corresponds with the flat universe. The forbidden domain for the motion is colored. The maximum of the potential is corresponding to a static Einstein universe in the phase space. Note that, for the case of positive curvature, the universe can oscillate with the initial singularity (the *left bottom* part of the *top diagram*) or be a universe with a bounce (the *right bottom* part of *both diagrams*)

3 $\Lambda(a(t))$CDM cosmologies as a dynamical system

In their construction many cosmological models of a decaying Λ make the ansatz $\Lambda(t) = \Lambda(a(t))$. For a review of different approaches in which ansatzes of this type appear, see Table 1.

In this section, we would like to discuss some general properties of the corresponding dynamical systems which model a decaying Λ term. It would be convenient to introduce the dynamical system in the state variables (H, ρ),

Table 1 Different choices of the $\Lambda(a)$ parametrization for different cosmological models appearing in the literature

$\Lambda(a)$ Parametrization	References
$\Lambda \sim a^{-m}$	[23,24]
$\Lambda = M_{\text{pl}}^4 \left(\frac{r_{\text{pl}}}{R}\right)^n$	[25]
$\Lambda = \frac{c^5}{\hbar G^2}\left(\frac{l_{pl}}{a}\right)^n$	[26]
$\rho_\Lambda = \tilde{\rho}_{v,0} + \frac{\epsilon\rho_{\text{m},0}}{3-\epsilon}a^{-3+\epsilon}$	[9]
$\rho_{de} = a^{-(4+\frac{2}{\alpha})}, \quad \alpha = 2(1-\Omega_{\text{m},0}-\Omega_{\Lambda,0})$	[27]
$\rho_\Lambda = 3\alpha^2 M_p^2 a^{-2\left(1+\frac{1}{c}\right)}$	[28]
$\Lambda = \frac{\Lambda_{\text{pl}}}{(a/l_{\text{pl}})^2} \propto a^{-2}$	[29]
$\Lambda = \Lambda_1 + \Lambda_2 a^{-m}$	[30]

$$\dot{H} = -H^2 - \frac{1}{6}\rho_{\text{m}} + \frac{\Lambda_{\text{bare}}}{3} + \frac{\Lambda(a)}{3}, \tag{28}$$

$$\dot{\rho}_{\text{m}} = -3H\rho_{\text{m}} - \frac{\text{d}\Lambda}{\text{d}a}(Ha) \tag{29}$$

or

$$\frac{\text{d}H^2}{\text{d}\ln a} = 2\left(-H^2 - \frac{1}{6}\rho_{\text{m}} + \frac{1}{3}\Lambda_{\text{bare}} + \frac{1}{3}\Lambda(a)\right), \tag{30}$$

$$\frac{\text{d}\rho_{\text{m}}}{\text{d}\tau} = \frac{\text{d}\rho_{\text{m}}}{\text{d}\ln a} = -3\rho_{\text{m}} - a\frac{\text{d}\Lambda}{\text{d}a} \tag{31}$$

with the first integral of the form

$$\rho_{\text{m}} = 3H^2 - \Lambda_{\text{bare}} - \Lambda(a). \tag{32}$$

As we have prescribed the form of the $\Lambda(a)$ relation, we can start the dynamical analysis with Eq. (28). It would be convenient to rewrite it to the form of the acceleration equation, i.e.,

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\rho_{\text{m}}(a) + \frac{\Lambda_{\text{bare}}}{3} + \frac{\Lambda(a)}{3}, \tag{33}$$

where $\rho_{\text{m}}(a)$ is determined by Eq. (31) which is a linear non-homogeneous differential equation which can be solved analytically

$$\frac{\text{d}\rho_{\text{m}}}{\text{d}\tau} = -3\rho_{\text{m}} - \frac{\text{d}\Lambda}{\text{d}\tau}(a) \tag{34}$$

and

$$\rho_{\text{m}} = -\left(\int^a a^3 \text{d}\Lambda(a) + C\right)a^{-3} \tag{35}$$

Equation (33) can be rewritten in an analogous form to the Newtonian equation of motion for a particle of unit mass moving in the potential $V(a)$ (Eq. (25)), where

$$V(a) = \frac{1}{6}a^{-1}\left(\int^a a^3\frac{\text{d}\Lambda}{\text{d}a}\text{d}a + C\right) - \frac{\Lambda_{\text{bare}}}{6}a^2 - \frac{1}{6}a^2\Lambda(a). \tag{36}$$

The integration of the above function gives the form of the potential.

Of course, Eq. (25) can be rewritten as the Newtonian two-dimensional dynamical system

$$\dot{a} = p, \quad \dot{p} = -\frac{\partial V}{\partial a}, \tag{37}$$

where the first integral has the form of Eq. (27). The integral of energy (27) should be consistent with the first integral (32), i.e.,

$$\rho_{\text{m}} + \rho_\Lambda = 3H^2, \tag{38}$$

$$a^{-3}\int a^3\Lambda'(a)\text{d}a + 3\frac{\dot{a}^2}{a^2} = \Lambda_{\text{bare}} + \Lambda(a). \tag{39}$$

Because the system under consideration is a conservative system, centres or saddles can appear in the phase space. If the potential function $V(a)$ possesses a maximum, then in the phase space we obtain a saddle type critical point. If $V(a)$ has a minimum, this point is a centre.

As an example of adopting the method of the effective potential, which is presented here, let us consider the parametrization of $\Lambda(a)$ like in the Alcaniz–Lima model of decaying vacuum [9]. They assumed that energy density of vacuum is of the form (see Table 1)

$$\rho_\Lambda = \rho_{v,0} + \frac{\epsilon\rho_{\text{m},0}}{3-\epsilon}a^{-3+\epsilon}, \tag{40}$$

where $\rho_{v,0}$ is vacuum energy $\rho_{\text{m},0}$ is the energy density of matter at the present moment for which we choose $a = 1 = a_0$. Because $\dot{\rho}_{\text{vac}} < 0$, i.e., the energy of vacuum is decaying, from the conservation condition

$$\dot{\rho}_{\text{m}} = -3H\rho_{\text{m}} - \dot{\rho}_{\text{vac}} \tag{41}$$

we obtain

$$\dot{\rho}_{\text{vac}} = -\dot{\rho}_{\text{m}} - 3H\rho_{\text{m}} = -\rho_{\text{m}}\left(\frac{\dot{\rho}_{\text{m}}}{\rho_{\text{m}}} + 3H\right) \tag{42}$$

and the vacuum is decaying if

$$\frac{\text{d}\ln\rho_{\text{m}}}{\text{d}\ln a} > -3. \tag{43}$$

Let us notice that $\rho_{\text{m}} = 0$ is a solution of the system (41) only if Λ is constant. It is a source of some difficulties in the phase space because the trajectories can pass through the line $\rho_{\text{m}} = 0$. As a consequence of decaying vacuum energy density of matter will dilute more slowly compared to the corresponding canonical relation in the ΛCDM model, i.e., the energy density of matter is scaling following the rule

$$\rho_{\text{m}} = \rho_{\text{m},0}a^{-3+\epsilon}, \tag{44}$$

where $\epsilon > 0$.

The dynamical system obtained from Eqs. (30)–(31) with the parametrization (40) has the following form:

$$x' = -3x + (y - \Lambda)(3 - \epsilon), \tag{45}$$

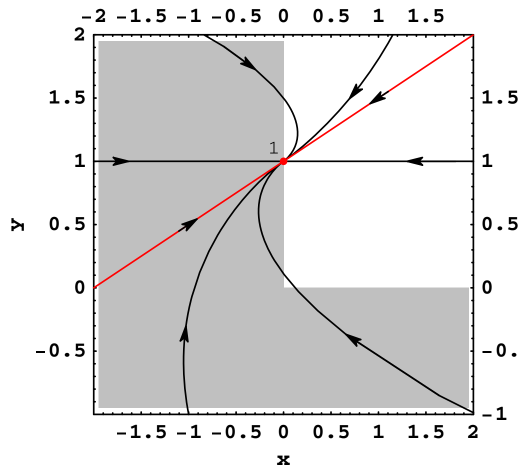


Fig. 3 A phase portrait for the dynamical system (45)–(46). The critical point (1) at $x = 0$, $y = \Lambda$ is a stable node. It represents a de Sitter universe. The *red line* represents the solutions of scaling type $y = \frac{\epsilon}{3-\epsilon}x + \Lambda$. The *grey region* represents a non-physical domain excluded by the condition $\rho_m = x > 0$, $\rho_\Lambda = y > 0$. Note that trajectories approach the attractor along a *straight line*. Let us note the existence of trajectories coming to the physical region from the non-physical one. We treated this type of behaviour as a difficulty related to an appearance of ghost trajectories, which emerges from the non-physical region

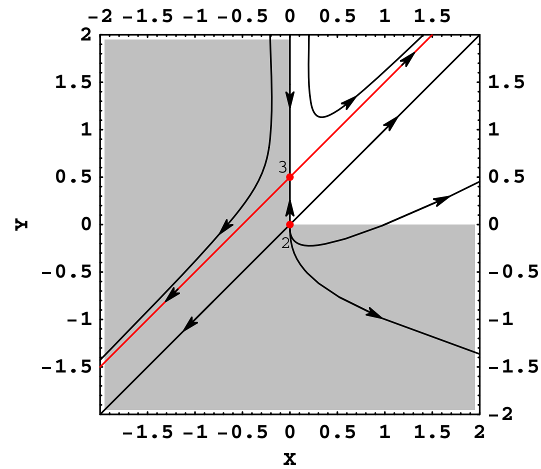


Fig. 4 A phase portrait for the dynamical system (48)–(49). Both the critical point (2) at the origin $X = 0$, $Y = 0$ and the critical point (3) at $X = 0$, $Y = \frac{\epsilon}{3-\epsilon}$ present nodes. The *red line* represents the solutions of a scaling type $Y = \frac{\epsilon}{3-\epsilon} + \Lambda X$. The *grey region* represents a non-physical domain excluded by the condition $X > 0$, $Y > 0$

$$y' = -(y - \Lambda)(3 - \epsilon), \quad (46)$$

$$z' = -z - \frac{x}{6} + \frac{y}{3}, \quad (47)$$

with the condition $y = \Lambda + \frac{\epsilon}{3-\epsilon}x$, where $x = \rho_m$, $y = \rho_\Lambda$, $z = H^2$ and $' \equiv \frac{d}{d\tau}$. The above dynamical system contains the autonomous two-dimensional dynamical system (45)–(46). Therefore this system has an invariant two-dimensional submanifold. A phase portrait with this invariant submanifold is demonstrated in Fig. 3.

For a deeper analysis of the system, the investigation of trajectories at the circle $x^2 + y^2 = \infty$ at infinity is required. For this aim the dynamical system (45)–(46) is rewritten in projective coordinates. Two maps (X, Y) and $(\tilde{X}, \tilde{Y})$ cover the circle at infinity. In the first map we use the following projective coordinates: $X = \frac{1}{x}$, $Y = \frac{y}{x}$ and in the second one $\tilde{X} = \frac{x}{y}$, $\tilde{Y} = \frac{1}{y}$. System (45)–(46) rewritten in coordinates X and Y has the following form:

$$X' = X\left((Y - \Lambda X)(-3 + \epsilon) + 3\right), \quad (48)$$

$$Y' = (Y + 1)(Y - \Lambda X)(-3 + \epsilon) + 3Y \quad (49)$$

and for variables $\tilde{X}$, $\tilde{Y}$, we obtain

Table 2 Critical points for autonomous dynamical systems (45)–(46), (48)–(49), (50)–(51), their eigenvalues and cosmological interpretation

No.	Critical point	Eigenvalues	Type of critical point	Type of universe
1	$x = 0, y = \Lambda$	$-3, -3 + \epsilon$	Stable node	de Sitter
2	$X = 0, Y = 0$	$3, \epsilon$	Unstable node	Einstein–de Sitter
3	$X = 0, Y = \frac{\epsilon}{3-\epsilon}$	$3 - \epsilon, -\epsilon$	Saddle	Scaling universe ρ_m Is proportional to ρ_Λ

$$\tilde{X}' = (1 + \tilde{X})(1 - \Lambda\tilde{Y})(3 - \epsilon) - 3\tilde{X}, \quad (50)$$

$$\tilde{Y}' = \tilde{Y}(1 - \Lambda\tilde{Y})(3 - \epsilon). \quad (51)$$

The phase portraits for dynamical systems (48)–(49) and (50)–(51) are demonstrated in Figs. 4 and 5. The critical points for the above dynamical system are presented in Table 2 (Fig. 3, 4).

The reduction of the dynamics to the particle-like description with the effective potential enables us to treat the evolution of the universe in manners of classical mechanics. One treats the scale factor as a positional variable and

$$V_{\text{eff}}(a) = -\frac{\rho_{\text{eff}}(a)a^2}{6}$$
$$= -\frac{1}{6}a^2\left(\rho_m a^{-3+\epsilon} + \rho_{v,0} + \frac{\epsilon\rho_{m,0}a^{-3+\epsilon}}{3-\epsilon}\right), \quad (52)$$

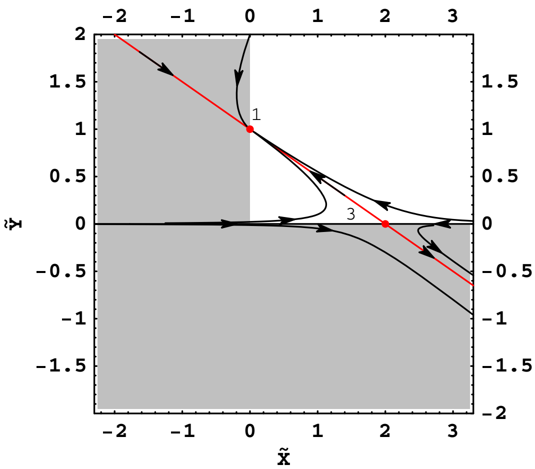


Fig. 5 A phase portrait for the dynamical system (50)–(51). The critical point (1) at $\tilde{X} = 0$, $\tilde{Y} = 1/\Lambda$ presents a stable node and the critical point (3) is at $\tilde{X} = \frac{3-\epsilon}{\epsilon}$, $\tilde{Y} = 0$ presents a saddle type point. The *red line* represents the solutions of a scaling type $\tilde{Y} = \left(1 - \frac{\epsilon}{3-\epsilon}\tilde{X}\right)/\Lambda$. The *grey region* represents non-physical domain excluded by the condition $\tilde{X} > 0$, $\tilde{Y} > 0$

where $\rho_{\text{eff}} = \rho_{\text{m}} + \rho_{\text{vac}}(a)$ and

$$\frac{\dot{a}^2}{2} + V_{\text{eff}} = -\frac{k}{2}. \tag{53}$$

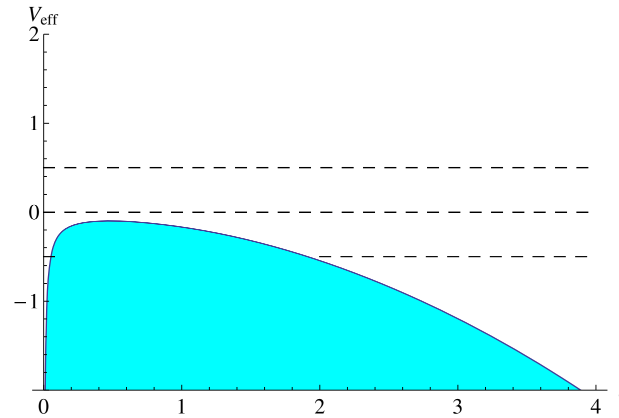
The motion of a particle (the universe, that is; it mimics a unit-mass particle in that description) is restricted to the zero energy level $E = 0$ (because we considered a flat model). The evolutionary paths of the model can be directly determined from the diagram of the effective potential $V_{\text{eff}}(a)$.

Figure 6 demonstrates the diagram $V_{\text{eff}}(a)$ for values $\epsilon = 0.1$ and 1. In general, for the phase portrait in the plane $(a, \dot{a})$ the maximum of $V(a)$ corresponds to the static Einstein universe. This critical point is situated on the a-axis and it is always of the saddle type. Of course, it is only admissible for closed universes. In that case a minimum corresponds to a critical point of a centre type. If we include the curvature in the dynamical analysis then we get new solutions for the positive curvature such as the oscillating universe with an initial singularity and the universe with a bounce. But if we perturb solutions for the flat universe by a small spatial curvature then these solutions do not change qualitatively (see Fig. 6).

The Alcaniz–Lima model behaves in the phase space $(a, \dot{a})$ like the ΛCDM one [9]. Trajectories start from $(a, \dot{a}) = (0, \infty)$ (corresponding to the big bang singularity), approach the static universe and then evolve to infinity. Note that if $0 < \epsilon < 1$ then dynamics is qualitatively equivalent to the ΛCDM model.

The Eq. (29) can be written as

$$\dot{\rho}_{\text{m}} = -3H\rho_{\text{m}} - 3H\rho_{\text{m}}\delta(t), \tag{54}$$



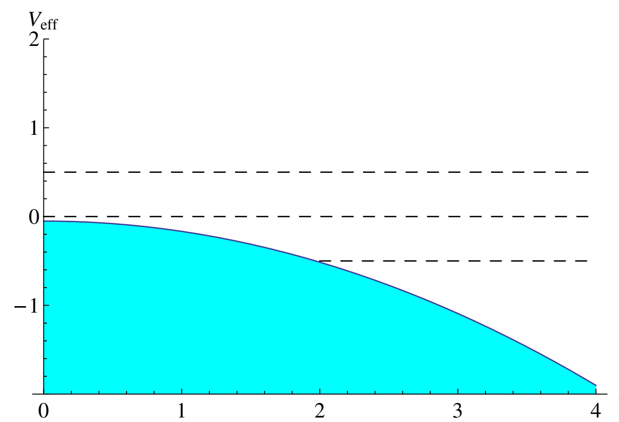


Fig. 6 The potential $V_{\text{eff}}(a)$ for $\epsilon = 0.1$ (*top diagram*) and for $\epsilon = 1$ (*bottom diagram*). The *top dashed lines* ($V_{\text{eff}} = 1/2$) represent the energy level, which corresponds with the negative curvature. The *bottom dashed lines* ($V_{\text{eff}} = -1/2$) represent the energy level, which corresponds with the positive curvature. The *middle dashed lines* ($V_{\text{eff}} = 0$) represent the energy level, which corresponds with the flat universe. The *colored region* represents a forbidden domain for the motion. The *shape of diagram* of the potential determines the phase space structure. The maximum of the potential is corresponding to a static Einstein universe in the phase space. Note that the universe with the positive curvature is an oscillating universe with the initial singularity (the *left bottom* part of the *top diagram*) or is a universe with a bounce (the *right bottom* part of *both diagrams*)

where $\delta(t) = -\frac{1}{3\rho_{\text{m}}}\frac{d\Lambda}{da}a$. Therefore,

$$\dot{\rho}_{\text{m}} = -3H\rho_{\text{m}}(1 + \delta(t)), \tag{55}$$

where $-3H\rho_{\text{m}}\delta(t) = \frac{\text{d}\Lambda}{\text{d}a}Ha$, i.e., $\delta(t) = -\frac{\frac{\text{d}\Lambda}{\text{d}a}a}{3\rho_{\text{m}}} \propto -\frac{\rho_{\Lambda}}{\rho_{\text{m}}}$. If $\delta(t)$ is a slowly changing function of time, i.e., $\delta(t) \simeq \delta$ then (55) has the solution $\rho_{\text{m}} = \rho_{\text{m},0}a^{-3+\delta}$.

4 $\Lambda(R)$CDM cosmologies as a dynamical system

The Ricci scalar dark energy idea has been recently considered in the context of the holographic principle [31]. In this case dark energy can depend on time t through the Ricci

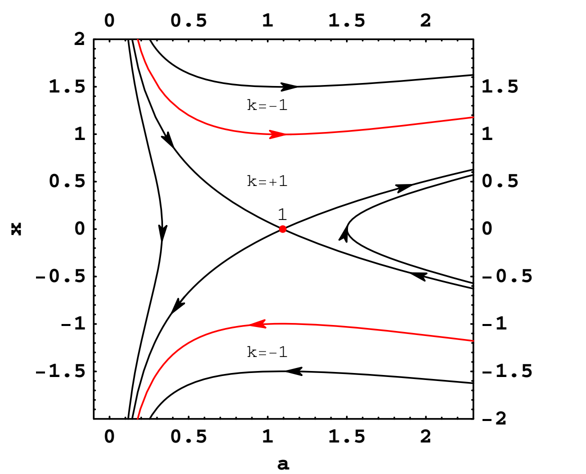


Fig. 7 A phase portrait for dynamical system (59)–(60) with $\alpha = 2/3$. The critical point (1), which is located on a-axis, ($a = \sqrt{\frac{3\rho_{m,0}}{2\rho_\Lambda - \rho_{m,0}}}$, $x = 0$), is a saddle point and represents a static Einstein universe. The *red lines* represent the trajectories of the flat universe. They separate the regions in which closed and open models lie. In the region, at the *right* from the critical point (1), bounded by the incoming separatrix from above and the outgoing separatrix from below, trajectories are going out from the contracting Milne solution, reaching the a_{min} and coming into the expanding Milne solution

scalar $R(t)$, i.e., $\Lambda(t) = \Lambda(R(t))$. Such a choice does not violate covariance of general relativity. A special case is the parametrization $\rho_\Lambda = -\frac{\alpha}{2}R = 3\alpha(\dot{H} + 2H^2 + \frac{k}{a^2})$ [27]. Then the cosmological equations are also formulated in the form of a two-dimensional dynamical system,

$$\dot{H} = -H^2 - \frac{1}{6}(\rho_m + \rho_\Lambda), \tag{56}$$

$$\dot{\rho} = -3H\rho_m \tag{57}$$

with the first integral of the form

$$H^2 = \frac{1}{3}\left(-\frac{3k}{a^2} + \frac{2}{2-\alpha}\rho_{m,0}a^{-3} + f_0 a^{2\frac{1-2\alpha}{\alpha}}\right), \tag{58}$$

where f_0 is an integration constant.

From the above equations, we can obtain a dynamical system in the state variables a, $x = \dot{a}$,

$$\dot{a} = x, \tag{59}$$

$$\dot{x} = -\Omega_{m,0}\frac{1}{2-\alpha}a^{-2} + \left(\frac{1}{\alpha} - 1\right)\left(\Omega_{\Lambda,0} - \Omega_{m,0}\frac{\alpha}{2-\alpha}\right)a^{\frac{2}{\alpha}-3}. \tag{60}$$

The phase portrait on the plane (a, x) is shown in Fig. 7.

In order to analyze the trajectories behaviour at infinity we use the following sets of projective coordinates: $A = \frac{1}{a}$, $X = \frac{x}{a}$.

The dynamical system for variables A and X is expressed by

$$\dot{A} = -XA, \tag{61}$$

$$\dot{X} = A^3\Big[-\Omega_{m,0}\frac{1}{2-\alpha} + \left(\frac{1-\alpha}{\alpha}\right)\left(\Omega_{\Lambda,0} - \Omega_{m,0}\frac{\alpha}{2-\alpha}\right)A^{\frac{\alpha-2}{\alpha}}\Big] - X^2. \tag{62}$$

We can use also the Poincaré sphere to identify the critical points at infinity. We introduce the following new variables: $B = \frac{a}{\sqrt{1+a^2+x^2}}$, $Y = \frac{x}{\sqrt{1+a^2+x^2}}$. In the variables (B, Y), we obtain a dynamical system of the form

$$B' = YB^2(1-B^2) - BY\Big[-\Omega_{m,0}\frac{1}{2-\alpha}(1-B^2-Y^2)^{3/2} + \left(\frac{1-\alpha}{\alpha}\right)\left(\Omega_{\Lambda,0} - \Omega_{m,0}\frac{\alpha}{2-\alpha}\right)B^{-1+2/\alpha} \times (1-B^2-Y^2)^{2-1/\alpha}\Big], \tag{63}$$

$$Y' = \Big[-\Omega_{m,0}\frac{1}{2-\alpha}(1-B^2-Y^2)^{3/2} + \left(\frac{1-\alpha}{\alpha}\right)\left(\Omega_{\Lambda,0} - \Omega_{m,0}\frac{\alpha}{2-\alpha}\right)B^{-1+2/\alpha} \times (1-B^2-Y^2)^{2-1/\alpha}\Big](1-Y^2) - Y^2B^3, \tag{64}$$

where $' \equiv B^2\frac{d}{dt}$.

The phase portraits for the dynamical systems (61)–(62) and (63)–(64) are demonstrated in Figs. 8 and 9, respectively.

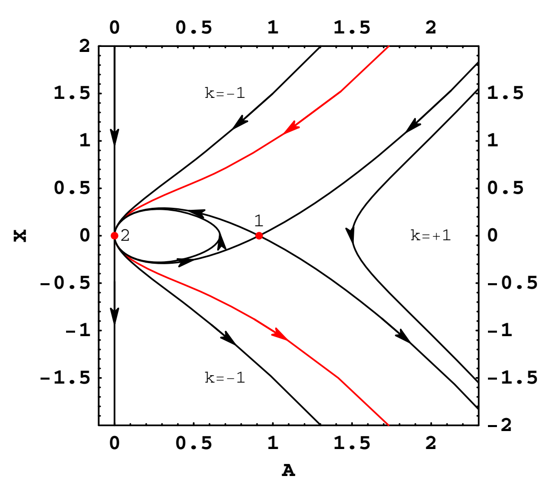


Fig. 8 A phase portrait for dynamical system (61)–(62) with $\alpha = 2/3$. The critical point (1) on the A-axis, ($A = \sqrt{\frac{2\rho_\Lambda - \rho_{m,0}}{3\rho_{m,0}}}$, $X = 0$), is a saddle and represents a static Einstein universe. The *red lines* represent the trajectories of a flat universe and they separate the regions in which closed and open models lie. The critical point (2) is a degenerate point at which the expanding and contracting Milne solutions are glued

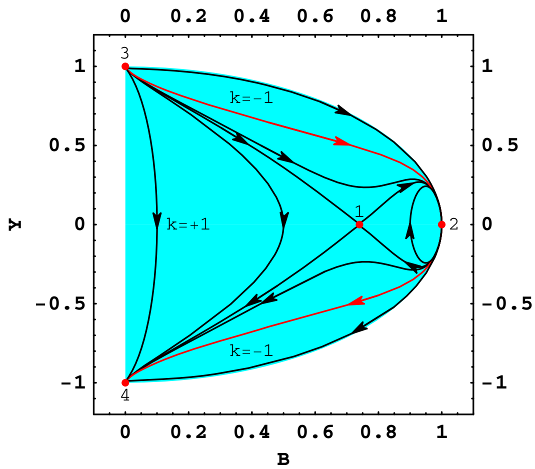


Fig. 9 A phase portrait for dynamical system (63)–(64) with $\alpha = 2/3$. The critical point (1) is at $(B = 1/\sqrt{\left(\frac{2\rho_\Lambda - \rho_{m,0}}{3\rho_{m,0}}\right)^2 + 1},\ Y = 0)$ is a saddle and represents a static Einstein universe. The critical point (2) at the B-axis, $Y = 0$ is a stable node and represents a Milne universe. The critical points (3) and (4) at $(B = 0, Y = 1)$ and $(B = 0, Y = -1)$ are nodes and represent Einstein–de Sitter universes. The *blue region* represents a physical domain restricted to $B^2 + Y^2 \leq 0$, $B \geq 0$. The *red lines* represent the flat universe and they separate the regions in which *closed and open* models lie

If we include the curvature in the dynamical analysis then we get new types of universes. In the phase space in the positive curvature domain, new trajectories appear which represent the oscillating universe with an initial singularity and the universe with a bounce. For this model, the universe with the bounce start from the Milne universe and is the universe without the initial singularity. A similar situation holds for many $f(R)$ models, where the de Sitter universe is at the infinite past. Non-singular solutions of this type were found by Starobinsky [32].

5 Cosmology with emergent $\Lambda(a)$ relation from exact dynamics

In order to illustrate the idea of an emergent $\Lambda(a)$ relation let us consider cosmology with a scalar field which is non-minimal coupled to gravity. For simplicity, without loss of generality of our consideration, we assume that the non-minimal coupling ξ is constant like the conformal coupling. It is also assumed that dust matter, present in the model, does not interact with the scalar field. Since we would like to nest the ΛCDM model in our model we postulate that the potential of the scalar field is constant. We also assume a flat geometry with the R-W metric. The action for our model assumes the following form:

$$S = S_g + S_\phi + S_m, \tag{65}$$

where

$$S_g + S_\phi = \frac{1}{2}\int \sqrt{g}\big(R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \xi R\phi^2 - 2V(\phi)\big)d^4x, \tag{66}$$

$$S_m = \int \sqrt{g}\mathcal{L}_m d^4x, \tag{67}$$

where the metric signature is $(-,+,+,+)$, $R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)$ is the Ricci scalar and the dot denotes the differentiation with respect to the cosmological time t, i.e., $\dot{} \equiv \frac{d}{dt}$ and $\mathcal{L}_m = -\rho_m\left(1 + \int \frac{p_m(\rho_m)}{\rho_m^2}d\rho_m\right)$.

After skipping the full derivatives with respect to the time, the equation of motion for the scalar field is obtained after the variation over the scalar field and metric,

$$\frac{\delta S}{\delta\phi} = 0 \quad \Leftrightarrow \quad \ddot{\phi} + 3H\dot{\phi} + \xi R\phi + V'(\phi) = 0, \tag{68}$$

where $' \equiv \frac{d}{d\phi}$ and

$$\frac{\delta S}{\delta g} = 0 \quad \Leftrightarrow \quad \mathcal{E} = \frac{1}{2}\dot{\phi}^2 + 3\xi H^2\phi^2 + 6\xi H\phi\dot{\phi} + V(\phi) - 3H^2 \equiv 0. \tag{69}$$

Additionally, from the conservation condition of the equation of state $p_m = p_m(\rho_m)$ for the barotropic matter we have

$$\dot{\rho}_m = -3H(\rho_m + p_m(\rho_m)). \tag{70}$$

Because we assume dust matter (p_m=0), Eq. (70) has a simple scaling solution of the form

$$\rho_m = \rho_{m,0}a^{-3}, \tag{71}$$

where $a = a(t)$ is the scale factor from the R-W metric $ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$.

Analogously, the effects of the homogeneous scalar field satisfy the conservation condition

$$\dot{\rho}_\phi = -3H(\rho_\phi + p_\phi), \tag{72}$$

where

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) + 6\xi H\phi\dot{\phi} + 3\xi H^2\phi^2, \tag{73}$$

$$p_\phi = \frac{1}{2}(1 - 4\xi)\dot{\phi}^2 - V(\phi) + 2\xi H\phi\dot{\phi} - 2\xi(1 - 6\xi)\dot{H}\phi^2 - 3\xi(1 - 8\xi)H^2\phi^2 + 2\xi\phi V'(\phi). \tag{74}$$

In the investigation of the dynamics it would be convenient to introduce the so-called energetic state variables [33]

$$x \equiv \frac{\dot{\phi}}{\sqrt{6}H}, \quad y \equiv \frac{\sqrt{V(\phi)}}{\sqrt{3}H}, \quad z \equiv \frac{\phi}{\sqrt{6}}. \tag{75}$$

The choice of such state variables (75) is suggested by the energy constraint $\mathcal{E} = 0$ (69).

The energy constraint condition can be rewritten in terms of dimensionless density parameters

$$\Omega_{\rm m} + \Omega_\phi = 1 \tag{76}$$

then

$$\Omega_\phi = 1 - \Omega_{\rm m} = (1-6\xi)x^2 + y^2 + 6\xi(x+z)^2 = 1 - \Omega_{{\rm m},0}a^{-3} \tag{77}$$

and the formula $H(x, y, z, a)$ rewritten in the terms of state variables x, y, z assumes the following form:

$$\left(\frac{H}{H_0}\right)^2 = \Omega_\phi + \Omega_{\rm m} = (1-6\xi)x^2 + y^2 + 6\xi(x+z)^2 + \Omega_{{\rm m},0}a^{-3}. \tag{78}$$

Equation (78) is crucial for the model testing and estimation of the model parameters using astronomical data.

Because we try to generalize the ΛCDM model it is natural to interpret the additional contribution beyond $\Lambda_{\rm bare}$ as a running Λ term in (78). In our further analysis we will called this term 'emergent Λ term'. Therefore,

$$\Omega_{\Lambda,\rm emergent} = (1-6\xi)x^2 + y^2 + 6\xi(x+z)^2. \tag{79}$$

Of course, state variables satisfy a set of the differential equations in the consequence of Einstein equations. We try to organize them in the form of autonomous differential equations, i.e., some dynamical system.

For this aim let us start from the acceleration equation,

$$\dot{H} = -\frac{1}{2}(\rho_{\rm eff} + p_{\rm eff}) = -\frac{3}{2}H^2(1 + w_{\rm eff}), \tag{80}$$

where $\rho_{\rm eff}$ and $p_{\rm eff}$ are the effective energy density and the pressure, while $w_{\rm eff} = \frac{p_{\rm eff}}{\rho_{\rm eff}}$ is an effective coefficient of equation of state. Moreover, $\rho_{\rm eff} = \rho_{\rm m} + \rho_\phi$ and $p_{\rm eff} = 0 + p_\phi$.

The coefficient equation of state $w_{\rm eff}$ is given by the formula

$$w_{\rm eff} = \frac{1}{1 - 6\xi(1-6\xi)z^2} \times \left[(1-4\xi)x^2 - y^2(1+2\xi\lambda z) + 4\xi xz + 12\xi^2 z^2\right], \tag{81}$$

where $\lambda \equiv -\sqrt{6}\frac{V'(\phi)}{V(\phi)}$ is related to geometry of the potential, where $' \equiv \frac{d}{d\phi}$.

The dynamical system which describes the evolution in the phase space is in the form

$$\frac{{\rm d}x}{{\rm d}(\ln a)} = \frac{{\rm d}x}{{\rm d}\tau} = -3x - 12\xi z + \frac{1}{2}\lambda y^2 - (x + 6\xi z)\frac{\dot{H}}{H^2}, \tag{82}$$

$$\frac{{\rm d}y}{{\rm d}(\ln a)} = \frac{{\rm d}y}{{\rm d}\tau} = -\frac{1}{2}\lambda xy - y\frac{\dot{H}}{H^2}, \tag{83}$$

$$\frac{{\rm d}z}{{\rm d}(\ln a)} = \frac{{\rm d}z}{{\rm d}\tau} = x, \tag{84}$$

$$\frac{{\rm d}\lambda}{{\rm d}(\ln a)} = \frac{{\rm d}\lambda}{{\rm d}\tau} = -\lambda^2(\Gamma(\lambda) - 1)x, \tag{85}$$

where $\Gamma = \frac{V''(\phi)V(\phi)}{V'^2(\phi)}$ and

$$\frac{\dot{H}}{H^2} = \frac{1}{H^2}\left[-\frac{1}{2}(\rho_\phi + p_\phi) - \frac{1}{2}\rho_{{\rm m},0}a^{-3}\right] = \frac{1}{6\xi z^2(1-6\xi) - 1}\left[-12\xi(1-6\xi)z^2 - 3\xi\lambda y^2 z + \frac{3}{2}(1-6\xi)x^2 + 3\xi(x+z)^2 + \frac{3}{2} - \frac{3}{2}y^2\right]. \tag{86}$$

Let us notice that the dynamical system (82)–(85) is closed if we only we assume that $\Gamma = \Gamma(\lambda)$.

From the form of system (82)–(85) one can observe that it admits the invariant submanifold $\left\{\frac{\dot{H}}{H^2} = 0\right\}$ for which the equation in the phase space is of the form

$$-12\xi(1-6\xi)z^2 - 3\xi\lambda y^2 z + \frac{3}{2}(1-6\xi)x^2 + 3\xi(x+z)^2 + \frac{3}{2} - \frac{3}{2}y^2 = 0. \tag{87}$$

Therefore, there are no trajectories which intersect this invariant surface in the phase space. From the physical point of view the trajectories are stationary solutions and on this invariant submanifold they satisfy the condition

$$\frac{\dot{H}}{H^2} = 0 \quad \Leftrightarrow \quad -\frac{1}{2}(\rho_\phi + p_\phi) - \frac{1}{2}\rho_{{\rm m},0}a^{-3} = 0. \tag{88}$$

If we look at the trajectories in the whole phase in the neighbourhood of this invariant submanifold, then we can observe that they will be asymptotically reached at an infinite value of time $\tau = \ln a$. They are tangent asymptotically to this surface. Note that in many cases the system on this invariant submanifolds can be solved and the exact solutions can be obtained.

As an illustration of the idea of the emergent $\Lambda(a)$ relation we consider two cases of cosmologies for which we derive $\Lambda = \Lambda(a)$ formulae. Such parametrizations of $\Lambda(a)$ arise if we consider the behaviour of trajectories near the invariant submanifold of dynamical systems

1. $V = {\rm const}$ or $\lambda = 0$, the case of minimal coupling, $\xi = 0$;
2. $V = {\rm const}$, the case of conformal coupling, $\xi = \frac{1}{6}$.

In these cases the dynamical system (82)–(85) reduces to

$$\frac{\mathrm{d}x}{\mathrm{d}(\ln a)} = \frac{\mathrm{d}x}{\mathrm{d}\tau} = -3x - x\frac{\dot{H}}{H^2}, \tag{89}$$
$$\frac{\mathrm{d}y}{\mathrm{d}(\ln a)} = \frac{\mathrm{d}y}{\mathrm{d}\tau} = -y\frac{\dot{H}}{H^2}, \tag{90}$$
$$\frac{\mathrm{d}z}{\mathrm{d}(\ln a)} = \frac{\mathrm{d}z}{\mathrm{d}\tau} = x, \tag{91}$$

where

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}x^2 - \frac{3}{2} + \frac{3}{2}y^2. \tag{92}$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = -3x - 2z - \frac{\dot{H}}{H^2}(x+z), \tag{93}$$
$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = -y\frac{\dot{H}}{H^2}, \tag{94}$$
$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = x, \tag{95}$$

where

$$\frac{\dot{H}}{H^2} = -\frac{1}{2}(x+z)^2 - \frac{3}{2} + \frac{3}{2}y^2. \tag{96}$$

The dynamical system (93)–(95) can be rewritten using the variables $X = x + z$, $Y = y$ and $Z = z$. Then we get

$$\frac{\mathrm{d}X}{\mathrm{d}\tau} = -2X - \frac{\dot{H}}{H^2}X, \tag{97}$$
$$\frac{\mathrm{d}Y}{\mathrm{d}\tau} = -Y\frac{\dot{H}}{H^2}, \tag{98}$$
$$\frac{\mathrm{d}Z}{\mathrm{d}\tau} = X - Z, \tag{99}$$

where

$$\frac{\dot{H}}{H^2} = -\frac{1}{2}X^2 - \frac{3}{2} + \frac{3}{2}Y^2. \tag{100}$$

The next step in a realization of our idea of the emergent Λ is to solve the dynamical system on invariant submanifold and then to substitute this solution into Eq. (79).

For the first case ($\xi = 0$, $V = \text{const}$), the dynamical system (89)–(91) has the following form:

$$\frac{\mathrm{d}x}{\mathrm{d}(\ln a)} = \frac{\mathrm{d}x}{\mathrm{d}\tau} = -3x, \tag{101}$$
$$\frac{\mathrm{d}y}{\mathrm{d}(\ln a)} = \frac{\mathrm{d}y}{\mathrm{d}\tau} = 0, \tag{102}$$
$$\frac{\mathrm{d}z}{\mathrm{d}(\ln a)} = \frac{\mathrm{d}z}{\mathrm{d}\tau} = x, \tag{103}$$

with the condition

$$0 = x^2 - y^2 + 1. \tag{104}$$

The solution of the dynamical system (101)–(103) is $x = C_1 a^{-3}$, $y = \text{const}$ and $z = -\frac{1}{3}C_1 a^{-3} + C_2$.

The phase portraits and a list of critical points for the dynamical system (89)–(91) is presented in Figs. 10, 11 and Table 3, respectively. The critical point (1) represents the matter dominating universe – an Einstein–de Sitter universe.

Finally, for first case $\Omega_{\Lambda,\text{emergent}}$ is given as

$$\Omega_{\Lambda,\text{emergent}} = \Omega_{\Lambda,\text{emergent},0}a^{-6} + \Omega_{\Lambda,0}. \tag{105}$$

Now, let us concentrate on the second case ($\xi = 1/6$, $V = \text{const}$). The system (93)–(95) assumes the following form:

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = -3x - 2z, \tag{106}$$
$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = 0 \quad \Rightarrow \quad y = \text{const}, \tag{107}$$
$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = x \tag{108}$$

with the condition

$$0 = (x+z)^2 - 3y^2 + 3. \tag{109}$$

The dynamical system (106)–(108) is linear and can be simply integrated. The solution of the above equations are $x = -2C_1a^{-2} - C_2a^{-1}$, $y = \text{const}$ and $z = C_1a^{-2} + C_2a^{-1}$.

The phase portrait and critical points for the dynamical system (93)–(95) are presented in Figs. 12, 13 and Table 4.

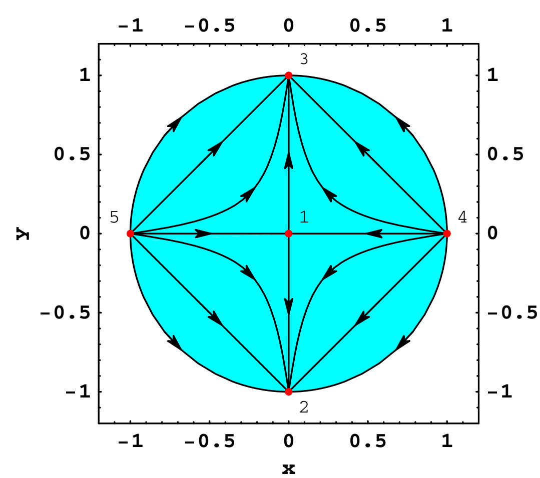


Fig. 10 The phase portrait for autonomous dynamical system (89)–(90). The critical point *(1)* represents a Einstein–de Sitter universe. The critical points *(4)* and *(5)* represent a Zeldovich stiff matter universe. The critical point *(2)* represents a contracting de Sitter universe. The critical point *(3)* represents stable de Sitter universe. The de Sitter universe is located on the invariant submanifold $\frac{\dot{H}}{H^2} = 0$. The *blue region* presents the physical region restricted by the condition $x^2 + y^2 \leq 1$, which is a consequence of $\Omega_m \geq 0$

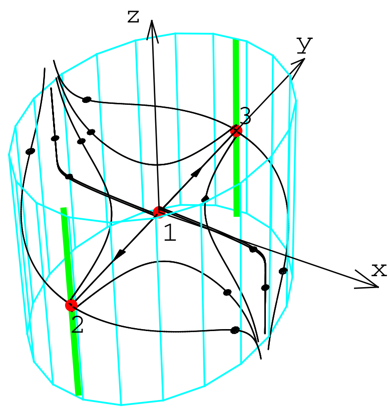


Fig. 11 The phase portrait for dynamical system (89)–(91). The critical point (*1*) represents the Einstein–de Sitter universe. Note that time $dt = Hd\tau$ is measured along trajectories, therefore in the region $H < 0$ (contracting model) time τ is reversed to the original time t. Hence, the critical point (*2*) represents an unstable de Sitter universe. Point (*3*) is opposite to the critical point (*2*) which represents a contracting de Sitter universe. The de Sitter universe is located on the invariant submanifold $\frac{\dot{H}}{H^2} = 0$, which is an element of a cylinder and is presented by *green lines*. The surface of the cylinder presents a boundary of the physical region restricted by the condition $x^2 + y^2 \leq 1$, which is a consequence of $\Omega_m \geq 0$

To illustrate the trajectories' behaviour close to the invariant submanifold (represented by the green lines) in the phase portrait (13) we construct two-dimensional phase portraits; see Fig. 14. In the latter trajectories reach the stationary states along tangential vertical lines (green lines).

On invariant submanifold (109) the dynamical system (106)–(108) reduces to

$$\frac{dx}{d\tau} = -x, \tag{110}$$

$$\frac{dz}{d\tau} = -z. \tag{111}$$

The solutions of (110)–(111) are $x = C_1 a^{-1}$ and $z = C_2 a^{-1}$.

Finally, we have

$$\Omega_{\Lambda,\text{emergent}} = \Omega_{\Lambda,0} + \Omega_{\Lambda,\text{emergent},0} a^{-4}, \tag{112}$$

i.e., the relation $\Lambda(a) \propto a^{-4}$ appears if we consider the behaviour of trajectories in the neighbourhood of an unstable de Sitter state $\frac{\dot{H}}{H^2} = 0$. Therefore, the emergent term is of the type 'radiation'. In the scalar field cosmology there is a phase of evolution during each effective coefficient e.o.s. is 1/3 like for radiation. If we find a trajectory in a neighbourhood of a saddle point then such a type of behaviour appears [33] (Fig. 14).

We can rewrite Eq. (86) as the Newtonian equation of motion for a particle of unit mass moving in the potential $V(a)$ (Eq. (25)). On the invariant submanifold $\{\frac{\dot{H}}{H^2} = 0\}$ the above equation gives the following form of the potential:

$$V(a) = -\frac{1}{2} H_0^2 a^2. \tag{113}$$

Figure 15 presents the evolution of $V(a)$. For the positive curvature we get new solution which is the universe with the bounce. If we perturb solutions for the flat universe by a small negative spatial curvature then these solutions do not change qualitatively (see Fig. 15). But for the positive curvature, we always get the solutions, which represents the universe with bounce.

6 How to constrain emergent running $\Lambda(a)$ cosmologies?

Dark energy can be divided into two classes: with or without early dark energy [34]. Models without early dark energy behave like the ΛCDM model in the early time universe. For models with early dark energy, dark energy plays an important role in evolution of early universe. The second type models should have a scaling or attractor solution where the fraction of dark energy follows the fraction of the dominant matter or radiation component. In this case, we use the fractional early dark energy parameter Ω_d^e to measure a ratio of dark energy to matter or radiation.

The model with $\xi = 1/6$ (conformal coupling) and $V = \text{const}$ belongs to a class of models with early constant ratio dark energy in which Ω_{de}=const during the radiation dominated stage. In this case we can use the fractional

Table 3 The complete list of critical points of the autonomous dynamical system (89)–(90) which are shown in Figs. 10 and 11

Critical point	Coordinates	Eigenvalues	Type of critical point	Type of universe
1	$x = 0, y = 0$	$3 -3$	Saddle	Einstein–de Sitter
2	$x = 0, y = -1$	$-3, -3$	Stable node	Contracting de Sitter
3	$x = 0, y = 1$	$-3, -3$	Stable node	de Sitter
4	$x = 1, y = 0$	$3, 3$	Unstable node	Zeldovich stiff Matter dominating
5	$x = -1, y = 0$	$3, 3$	Unstable node	Zeldovich stiff Matter dominating

Coordinates, eigenvalues of the critical point as well as its type and cosmological interpretation are given

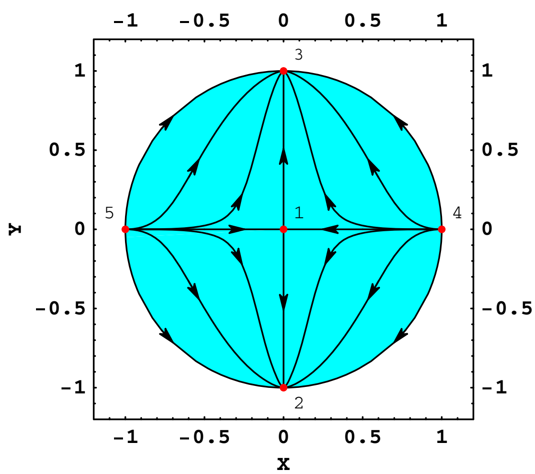


Fig. 12 The phase portrait for dynamical system (97)–(98). The critical point (*1*) represents a Einstein–de Sitter universe. The critical point (*4*) and (*5*) represent Zeldovich stiff matter universes. The critical points (*2*) represents a contracting de Sitter universe. The critical point (*3*) represents a stable de Sitter universe. The de Sitter universe is located on the invariant submanifold $\frac{\dot{H}}{H^2} = 0$. The *blue region* presents the physical region restricted by the condition $X^2 + Y^2 \leq 1$, which is a consequence of $\Omega_m \geq 0$

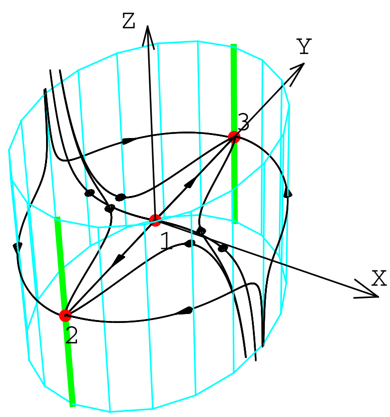


Fig. 13 The phase portrait for dynamical system (97)–(99). The critical point (*1*) represents an Einstein–de Sitter universe. Note that time $d\tau = H dt$ is measured along trajectories, therefore in the region $H < 0$ (contracting model) time τ is reversed to the original time t. Hence, the critical point (*2*) represents an unstable de Sitter universe. Point (*3*) is opposite to critical points (*2*) which represents a contracting de Sitter universe. The de Sitter universe is located on the invariant submanifold $\frac{\dot{H}}{H^2} = 0$, which is the element of a cylinder and is presented by *green lines*. The surface of the cylinder presents a boundary of the physical region restricted by the condition $X^2 + Y^2 \leq 1$, which is a consequence of $\Omega_m \geq 0$

early dark energy parameter Ω_d^e [34,35] which is constant for models with constant dark energy in the early universe. The fractional density of early dark energy is defined by the expression $\Omega_d^e = 1 - \frac{\Omega_m}{\Omega_{tot}}$, where Ω_{tot} is the sum of dimensionless density of matter and dark energy. In this case, there exist strong observational upper limits on this quantity [14].

For this aim let us notice that during the 'radiation' epoch we can apply this limit $\Omega_d^e < 0.0036$ [14] and

$$1 - \Omega_d^e = \frac{\Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4}}{\Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4} + \Omega_{\Lambda,0} + \Omega_{\text{emergent},0}a^{-4}}. \quad (114)$$

Let us consider a radiation dominating phase $a(t) \propto t^{\frac{1}{2}}$ ($p_{\text{eff}} = \frac{1}{3}\rho_{\text{eff}}$) [33],

$$1 - \Omega_d^e = \frac{\Omega_{m,0}t^{-\frac{3}{2}} + \Omega_{r,0}t^{-2}}{\Omega_{m,0}t^{-\frac{3}{2}} + \Omega_{r,0}t^{-2} + \Omega_{\Lambda,0} + \Omega_{\text{emergent},0}t^{-2}} \overset{\text{at early universe}}{\simeq} \frac{\Omega_{r,0}}{\Omega_{r,0} + \Omega_{\text{emergent},0}}. \quad (115)$$

Ω_d^e at the early universe is constant and

$$\Omega_d^e = 1 - \frac{\Omega_{r,0}}{\Omega_{r,0} + \Omega_{\text{emergent},0}} < 0.0036. \quad (116)$$

From the above formula we get $\frac{\Omega_{\text{emergent},0}}{\Omega_{r,0}} < 0.003613$. In consequence we have a strict limit on a strength of the running Λ parameter in the present epoch, $\Omega_{\text{emergent},0} < 3.19 \times 10^{-7}$.

7 Cosmology with non-canonical scalar field

The dark energy can also be parameterized in a covariant way by a non-canonical scalar field ϕ [36]. The main difference between canonical and non-canonical description of the scalar field is in the generalized form of the pressure p_ϕ of the scalar field. For the canonical scalar field, the pressure p_ϕ is expressed by the formula $p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi)$, where $\dot{} \equiv \frac{d}{dt}$ and $V(\phi)$ is the potential of the scalar field. In the non-canonical case, the pressure is described by the expression $p_\phi = \left(\frac{\dot{\phi}^2}{2}\right)^\alpha - V(\phi)$, where α is an additional parameter. If α is equal 1 then the pressure of the non-canonical scalar field represents the canonical case.

The theory of the non-canonical scalar field is of course a covariant formulation because this theory can be obtained from the action, which is described by the following formula:

$$S = \int \sqrt{-g}\left(R + \left(\frac{\dot{\phi}^2}{2}\right)^\alpha - V(\phi) + \mathcal{L}_m\right) d^4x, \quad (117)$$

where $\mathcal{L}_m$ is the Lagrangian for the matter. Note that if $V(\phi)$ is constant then the model is equivalent to the model which is filled with an ideal fluid with the equation of state $p = w\rho$ (where w is determined by α) and the cosmological constant. After variation of the Lagrangian $\mathcal{L}$ with respect to the metric we get the Friedmann equations in the following form:

$$3H^2 = \rho_m + (2\alpha - 1)\left(\frac{\dot{\phi}^2}{2}\right)^\alpha + V(\phi) - \frac{3k}{a^2}, \quad (118)$$

Table 4 The list of critical points for the autonomous dynamical system (97)–(98) which are shown in Fig. 12 and 13

Critical point	Coordinates	Eigenvalues	Type of critical point	Type of universe
1	$X = 0, Y = 0$	$3/2, -1/2$	Saddle	Einstein–de Sitter
2	$X = 0, Y = -1$	$-3, -2$	Stable node	Contracting de Sitter
3	$X = 0, Y = 1$	$-3, -2$	Stable node	de Sitter
4	$X = 1, Y = 0$	$1, 2$	Unstable node	Zeldovich stiff Matter dominating
5	$X = -1, Y = 0$	$1, 2$	Unstable node	Zeldovich stiff Matter dominating

Coordinates, eigenvalues of the critical point as well as its type and cosmological interpretation are given

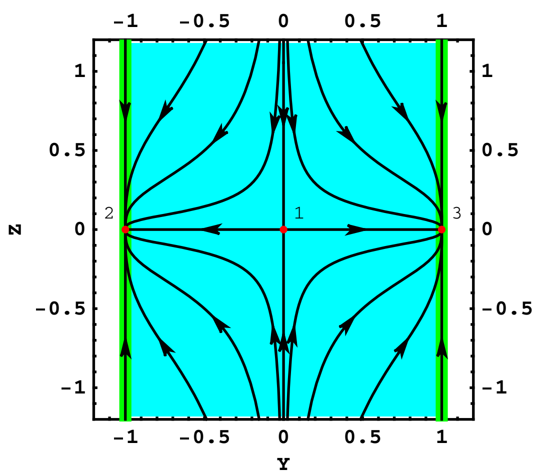


Fig. 14 The phase portrait of the invariant submanifold $X = 0$ of the dynamical system (97)–(99). The critical point (*1*) represents a Einstein–de Sitter universe. The critical points (*3*) represents a stable de Sitter universe. The critical point (2) represents a contracting de Sitter universe. Note that because of time parametrization $dt = Hd\tau$ in the region $X < 0$, the cosmological time t is reversed. In consequence, the critical point (2) is unstable. The de Sitter universe is located on the invariant submanifold $\{\frac{\dot{H}}{H^2} = 0\}$, which is represented by *green vertical lines*. By identification of *green lines* of the phase portrait one can represent the dynamics on the cylinder. The boundary of the physical region is restricted by the condition $Y^2 \leq 1$, which is a consequence $\Omega_m \geq 0$. Note that trajectories reach the de Sitter states along tangential *vertical lines*

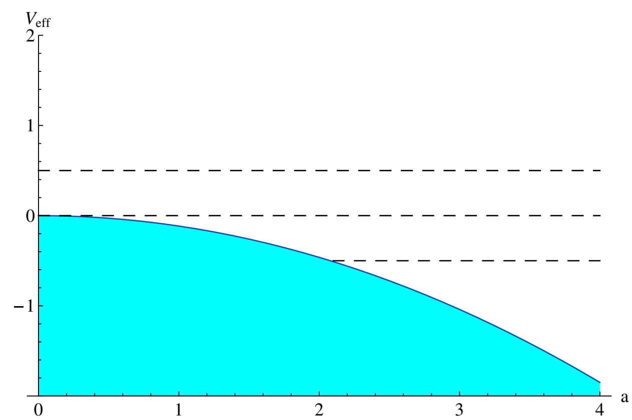


Fig. 15 The figure presents a potential $V_{eff}(a)$. The*top dashed lines* ($V_{eff} = 1/2$) represent the energy level, which corresponds with the negative curvature. The *bottom dashed lines* ($V_{eff} = -1/2$) represent the energy level, which corresponds with the positive curvature. The *middle dashed lines* ($V_{eff} = 0$) represent the energy level, which corresponds with the flat universe. The forbidden domain for the motion is colored. Note that, for the case of the positive curvature, the universe is with the bounce (the *right bottom* part of the diagram)

$$-3\frac{\ddot{a}}{a} = \frac{\rho_m}{2} + (\alpha + 1)\left(\frac{\dot{\phi}^2}{2}\right)^{\alpha} - V(\phi). \tag{119}$$

We obtain an additional equation of motion for a scalar field after the variation of the Lagrangian $\mathcal{L}$ with respect to the scalar field ϕ,

$$\ddot{\phi} + \frac{3H\dot{\phi}}{2\alpha - 1} + \left(\frac{V'(\phi)}{\alpha(2\alpha - 1)}\right)\left(\frac{2}{\dot{\phi}^2}\right)^{\alpha - 1} = 0, \tag{120}$$

where $' \equiv \frac{d}{d\phi}$.

For $\alpha = 1$, Eqs. (118), (119) and (120) reduce to the case of the canonical scalar field. For $\alpha = 0$ we have the case with the constant scalar field. The case $\alpha = 2$ with the constant potential V is interesting since the scalar field imitates radiation because $\phi^{2\alpha} \propto a^{-4}$ in the Friedmann equation.

For the constant potential $V = \Lambda$, Eq. (120) reduces to

$$\ddot{\phi} + \frac{3H\dot{\phi}}{2\alpha - 1} = 0. \tag{121}$$

Equation (121) has the following solution:

$$\dot{\phi} = \phi_0 a^{\frac{-3}{2\alpha-1}}. \tag{122}$$

We can obtain from (118), (119) and (120) the dynamical system for the non-canonical scalar field with the constant potential in the variables a and $x = \dot{a}$,

$$a' = xa^2, \tag{123}$$

$$x' = -\frac{\rho_{m,0}}{6} - \frac{\alpha + 1}{3}a^{\frac{3}{1-2\alpha}} + \frac{\Lambda}{3}a^3, \tag{124}$$

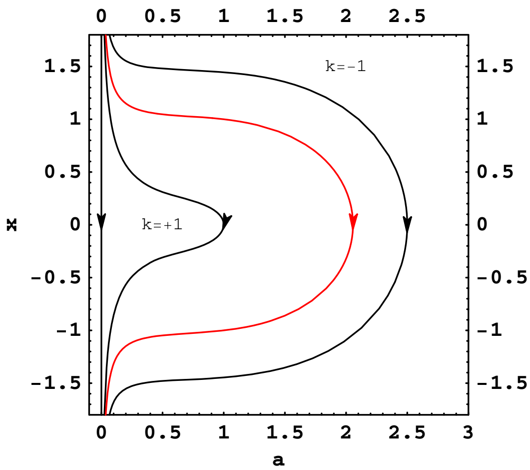


Fig. 16 A phase portrait for the dynamical system (123)–(124) for example with $\alpha = 1/8$. The *red lines* represent the flat universe and these trajectories separates the regions in which closed and open models lie. Note that all models independence on curvature are oscillating

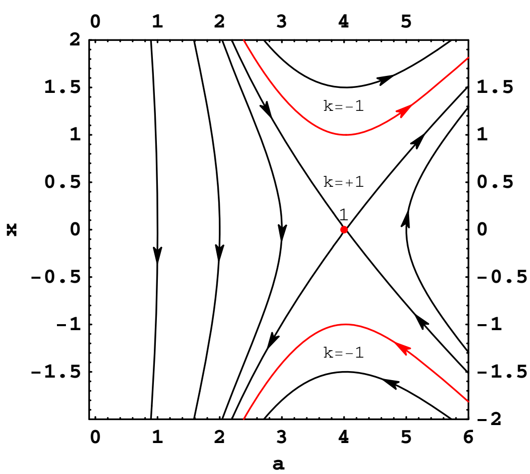


Fig. 17 A phase portrait for the dynamical system (125)–(126) with $\alpha = 1/8$ as an example. The *red lines* represent the flat universe and these trajectories separates the regions in which closed and open models lie

where $' \equiv a^2 \frac{\mathrm{d}}{\mathrm{d}t}$. The phase portrait for the dynamical system (123)–(124) is presented in Figs. 16 and 17.

The system (123)–(124) possesses critical points which belong to two types:

1. static critical points $x_0 = 0$,
2. non-static critical points $a_0 = 0$ (Big Bang singularity).

If we assume the matter in the form of dust ($p = 0$) then non-static critical points cannot exist at a finite domain of the phase space. The Big Bang singularity corresponds to a critical point at infinity.

Note that, if $\alpha > \frac{1}{2}$, then the eigenvalues for the critical point $(a_0, 0)$ are real and correspond to a saddle type of critical point. Therefore, for $\alpha > \frac{1}{2}$ the qualitative structure of the phase space is topologically equivalent (by homeomorphism) to the ΛCDM model. Hence, the phase space portrait is structurally stable, i.e., it is not disturbed under small changes of the right-hand side of the system.

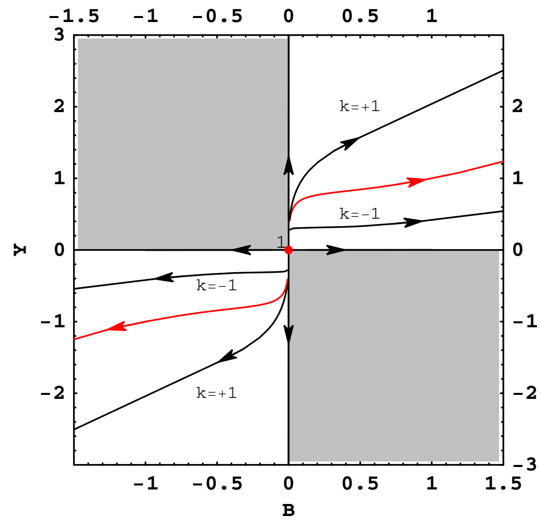


Fig. 18 A phase portrait for the dynamical system (127)–(128) with $\alpha = 1/8$ as an example. The critical point (*1*) at the origin $B = 0, Y = 0$ presents a stable node and Einstein–de Sitter universe. The *grey region* represents a non-physical domain excluded by the condition $\tilde{X}\tilde{Y} > 0$. The *red lines* represent the flat universe and these trajectories separate the regions in which closed and open models lie

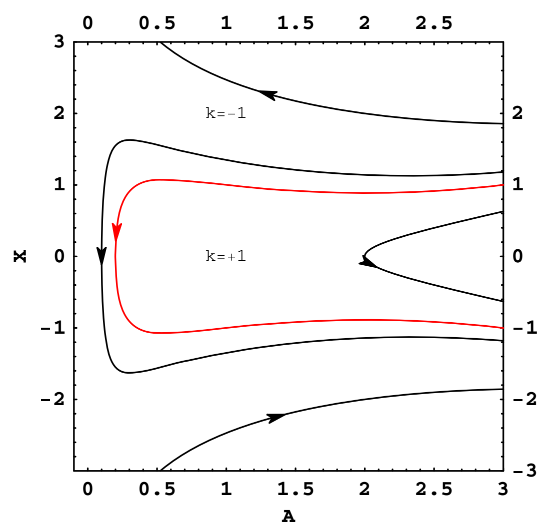


Fig. 19 A phase portrait for the dynamical system (123)–(124) with $\alpha = 50$ as an example. The critical point (*1*) is a saddle and represents a static Einstein universe. The *red lines* represent the flat universe and these trajectories separates the regions in which closed and open models lie. Note that all models have independence on curvature and are oscillating

For the analysis of the behaviour of trajectories at infinity we use the following sets of projective coordinates:

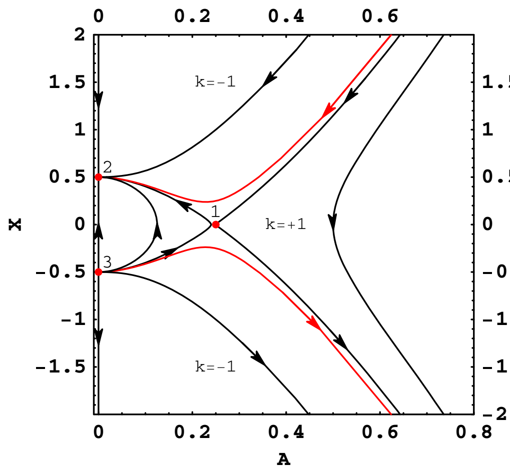


Fig. 20 A phase portrait for the dynamical system (125)–(126) with $\alpha = 50$ as an example. The critical point (*1*) is a saddle and represents a static Einstein universe. The critical point (*2*) represents a stable de Sitter universe. The critical point (*3*) represents a contracting de Sitter universe. The *red lines* represent the flat universe and these trajectories separate the regions in which closed and open models lie

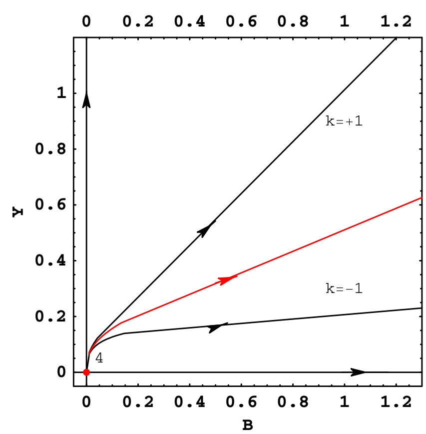


Fig. 21 A phase portrait for dynamical system (127)–(128) for example with $\alpha = 50$. The critical point (*4*) at the origin ($B = 0$, $Y = 0$) is an unstable node and represents an Einstein–de Sitter universe. The *red lines* represent the flat universe and these trajectories separate the regions in which closed and open models lie

1. $A = \frac{1}{a}$, $X = \frac{x}{a}$,
2. $B = \frac{a}{x}$, $Y = \frac{1}{x}$.

Two maps cover the behaviour of trajectories at the circle at infinity.

The dynamical system for variables A and X is expressed by

$$A' = -XA^2, \tag{125}$$

$$X' = A^4\left(-\frac{\rho_{\mathrm{m},0}}{6} - \frac{\alpha+1}{3}A^{\frac{3}{2\alpha-1}}\right) + A\left(\frac{\Lambda}{3} - X^2\right), \tag{126}$$

where $' \equiv A\frac{\mathrm{d}}{\mathrm{d}t}$. The dynamical system for variables B and Y is given by

$$\dot{B} = BY\left[B + \left(\frac{\rho}{6}Y^3 + \frac{\alpha+1}{3}B^{\frac{3}{1-2\alpha}}Y^{\frac{6\alpha}{2\alpha-1}} - \frac{\Lambda}{3}B^3\right)\right], \tag{127}$$

$$\dot{Y} = Y^2\left(\frac{\rho}{6}Y^3 + \frac{\alpha+1}{3}B^{\frac{3}{1-2\alpha}}Y^{\frac{6\alpha}{2\alpha-1}} - \frac{\Lambda}{3}B^3\right), \tag{128}$$

where $^{\dot{}} \equiv B^2Y\frac{d}{dt}$.

From the analysis of the dynamical system (127)–(128) we find one critical point ($B = 0$, $Y = 0$) which represents the Einstein–de Sitter universe. The phase portraits for dynamical system (125)–(126) and (127)–(128) are depicted in Figs. 18, 19, 20, and 21.

Let us consider the curvature in the dynamical analysis. Then in the phase space in the positive curvature domain, we find new trajectories which represent an oscillating universe with the initial singularity and a universe with a bounce (Figs. 17, 20, 21).

8 Cosmology with diffusion

The parametrization of dark energy can also be described in terms of the scalar field ϕ [37,38]. As an example of such a covariant parametrization of Λ let us consider cosmological models with diffusion. In this case the Einstein equations and equations of the current density J^μ are the following:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \phi g_{\mu\nu} = T_{\mu\nu}, \tag{129}$$

$$\nabla_\mu T^{\mu\nu} = \sigma J^\nu, \tag{130}$$

$$\nabla_\mu J^\mu = 0, \tag{131}$$

where σ is a positive parameter.

From the Bianchi identity, $\nabla^\mu\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 0$, and Eqs. (129) and (130) we get the following expression for $\Lambda(a(t))$:

$$\nabla_\mu\phi = \sigma J_\mu. \tag{132}$$

We assume also that matter is a perfect fluid. Then the energy-momentum tensor is expressed in the following form:

$$T_{\mu\nu} = \rho u_\mu u_\nu + p\left(g_{\mu\nu} + u_\mu u_\nu\right), \tag{133}$$

where u_μ is the 4-velocity and the current density is expressed by

$$J^\mu = nu^\mu. \tag{134}$$

Under these considerations Eqs. (130), (132) and (131) are described by the following expressions:

$$\nabla_\mu(\rho u^\mu) + p\nabla_\mu u^\mu = \sigma n, \tag{135}$$

$$\nabla_\mu(nu^\mu) = 0, \tag{136}$$

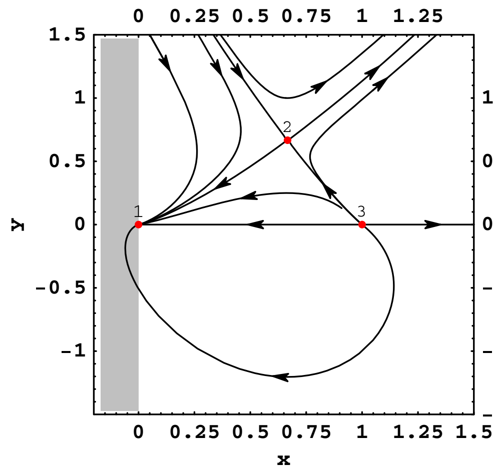


Fig. 22 A phase portrait for the dynamical system (142)–(142). The critical point *(1)* ($x = 0$, $y = 0$) is a stable node and represents the de Sitter universe. The critical point *(2)* ($x = 2/3$, $y = 2/3$) is a saddle and represents the Milne universe. The critical point *(3)* ($x = 1$, $y = 0$) is an unstable node type and represents the Einstein–de Sitter universe. Note the existence of trajectories crossing the boundary $x = \rho_m = 0$ in a non-physical region

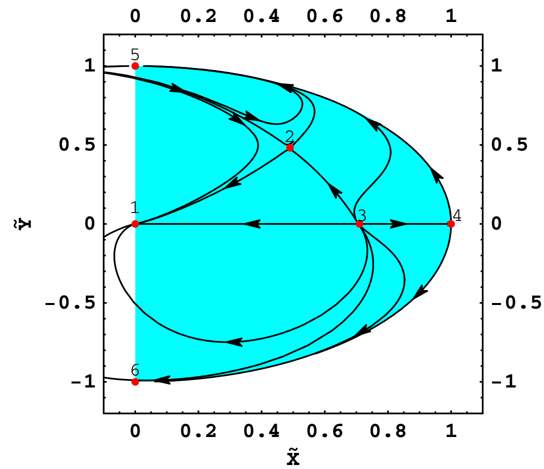


Fig. 24 A phase portrait for the dynamical system (146)–(147). The blue region represents the physical domain. The critical points *(5)* and *(6)* ($\tilde{X} = 0$, $\tilde{Y} = 1$) and ($\tilde{X} = 0$, $\tilde{Y} = -1$) represent the de Sitter universe with diffusion. The *blue region* represents a physical domain restricted by $B^2 + Y^2 \leq 0$, $B \geq 0$

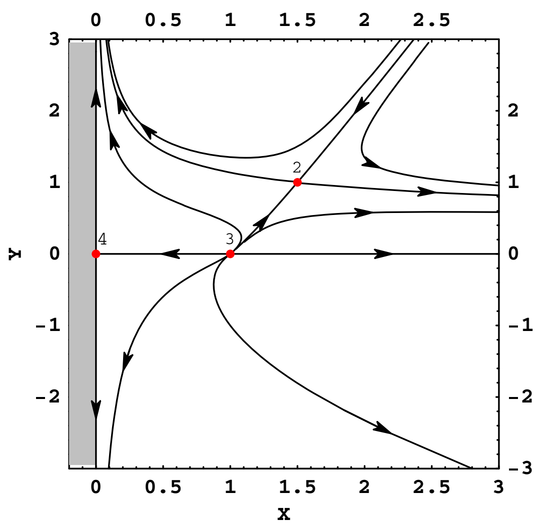


Fig. 23 A phase portrait for the dynamical system (144)–(145). The critical point *(4)* ($X = 0$, $Y = 0$) is a saddle and represents the static universe. The critical point *(2)* ($X = 3/2$, $Y = 1$) is a saddle and represents the Milne universe

and

$$\nabla_\mu \phi = \sigma n u_\mu. \tag{137}$$

We consider for simplicity the case of cosmological equations with the zero curvature. Equation (136) is now

$$n = n_0 a^{-3}. \tag{138}$$

In this case we have the following cosmological equations:

$$3H^2 = \rho_m + \Lambda(a(t)), \tag{139}$$

$$\dot{\rho}_m = -3H\rho_m + \sigma n_0 a^{-3}, \tag{140}$$

$$\frac{d\phi}{dt} = -\sigma n_0 a^{-3}. \tag{141}$$

If we choose the dimensionless state variables $x = \frac{\rho_m}{3H^2}$ and $y = \frac{\sigma n_0 a^{-3}}{3H^3}$ and the parametrization of time as $' \equiv \frac{d}{d\ln a}$ then we get the following dynamical system:

$$x' = 3x(x-1) + y, \tag{142}$$

$$y' = 3y(\frac{3}{2}x - 1). \tag{143}$$

The phase portrait for (142)–(143) is demonstrated in Fig. 22.

The dynamical system (142)–(143) can be rewritten in the projective variables for the analysis of critical points in infinity. In this case we use the following projective coordinates: $X = \frac{1}{x}$, $Y = \frac{y}{x}$. For the new variables X and Y, we obtain

$$X' = X\left(X(3-Y) - 3\right), \tag{144}$$

$$Y' = Y\left(\frac{3}{2} - XY\right) \tag{145}$$

where $' \equiv X\frac{d}{d\ln a}$.

We can use also the Poincaré sphere to search critical points in infinity. We introduce the following new variables: $\tilde{X} = \frac{x}{\sqrt{1+x^2+y^2}}$, $\tilde{Y} = \frac{y}{\sqrt{1+x^2+y^2}}$. In the variables $\tilde{X}$, $\tilde{Y}$, we obtain the dynamical system of the form

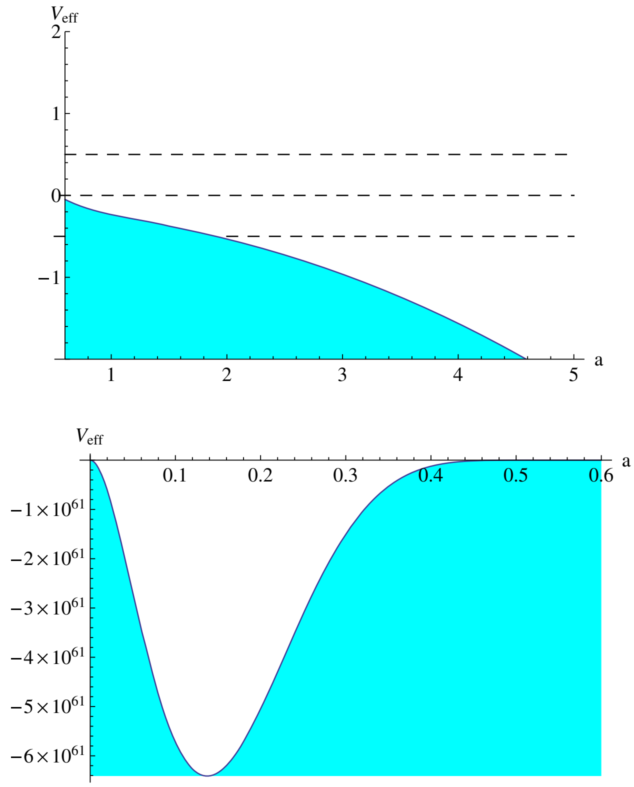


Fig. 25 The potential $V_{\rm eff}(a)$ for $a > 0.6$ (*top diagram*) and for $a < 0.6$ (*bottom diagram*). The *top dashed lines* ($V_{\rm eff} = 1/2$) represent the energy level, which corresponds with the negative curvature. The *bottom dashed lines* ($V_{\rm eff} = -1/2$) represent the energy level, which corresponds with the positive curvature. The *middle dashed lines* ($V_{\rm eff} = 0$) represent the energy level, which corresponds with the flat universe. The forbidden domain for the motion is *colored*. Note that, for the case of the positive curvature, the universe is oscillating (the *left bottom* part of the *top diagram*) or is the universe with the bounce (the *right bottom* part of *both diagrams*)

$$\tilde{X}' = (1-\tilde{X}^2)(3\tilde{X}^2 + (\tilde{Y}-3\tilde{X})\sqrt{1-\tilde{X}^2-\tilde{Y}^2}) - 3\tilde{X}\tilde{Y}^2\left(\frac{3}{2}\tilde{X} - \sqrt{1-\tilde{X}^2-\tilde{Y}^2}\right), \quad (146)$$

$$\tilde{Y}' = -\tilde{X}\tilde{Y}(3\tilde{X}^2 + (\tilde{Y}-3\tilde{X})\sqrt{1-\tilde{X}^2-\tilde{Y}^2}) + 3(1-\tilde{Y}^2)\tilde{Y}\left(\frac{3}{2}\tilde{X} - \sqrt{1-\tilde{X}^2-\tilde{Y}^2}\right), \quad (147)$$

where $' \equiv \sqrt{1-\tilde{X}^2-\tilde{Y}^2}\frac{\rm d}{{\rm d}\ln a}$. The phase portraits for (144)–(145) and (146)–(147) are demonstrated in Figs. 23 and 24.

We can rewrite Eqs. (139–141) as the Newtonian equation of motion for a particle of unit mass moving in the potential $V(a)$ (Eq. (25)) for finding the potential $V(a)$. The potential $V(a)$ has the following form:

$$V(a) = -\frac{1}{2}H(a)^2a^2. \quad (148)$$

Figure 25 presents the evolution of $V(a)$. For the curvature we get new solutions which are for a universe with a bounce and an oscillating universe without the initial singularity.

9 Conclusion

In this paper we have studied the dynamics of cosmological models with the running cosmological constant term using the dynamical system methods. We considered different parametrizations of the Λ term which are used in the cosmological applications. The most popular approaches are to parametrize the Λ term through the scale factor a or the Hubble parameter H. We considered cosmological models for which the energy-momentum tensor of matter (we assume dust matter) is not conserved. In this case there is an interaction between both dark matter and dark energy sectors.

There is a class of parameterizations of the Λ term through the Ricci scalar (or the trace of the energy-momentum tensor), the energy density of the scalar field or their kinetic part, and a scalar field ϕ minimally or non-minimally coupled to gravity. These choices are consistent with the covariance of general relativity.

We have discovered a new class of the emergent Λ parameterizations (in the case of $\Lambda(a)$) obtained directly from the exact dynamics, which does not violate the covariance of general relativity.

In consequence, the energy density deviates from the standard dilution. Due to decaying vacuum energy the standard relation $\rho_{\rm m} \propto a^{-3}$ is modified. From the cosmological point of view this class of models is a special case of cosmology with the interacting term $Q = -\frac{{\rm d}\Lambda}{{\rm d}t}$.

The main motivation for studying such models comes from the solution of the cosmological constant problem, i.e., explanations why the cosmological upper bound ($\rho_\Lambda \leqslant 10^{-47}$ GeV) dramatically differs from theoretical expectations ($\rho_\Lambda \sim 10^{71}$GeV) by more than 100 orders of magnitude [39]. In this context the running Λ cosmology is some phenomenological approach toward finding the relation $\Lambda(t)$ lowering the value of cosmological constant during the cosmic evolution.

In the study of the $\Lambda(t)$CDM cosmology different parametrizations of the Λ term are postulated. Some of them like $\Lambda(\phi)$, $\Lambda(R)$ or $\Lambda(\mathrm{tr}\, T^{\mu}_{\nu})$, $\Lambda(T)$, where $T = \frac{1}{2}\dot{\phi}^2$ are consistent with the principle of covariance of general relativity. Others, like $\Lambda = \Lambda(H)$, are motivated by the quantum field theory.

We demonstrated that the parameterization $\Lambda = \Lambda(a)$ can be obtain from the exact dynamics of the cosmological models with scalar field and the potential by taking approximation of trajectories in a neighbourhood of the invariant submanifold $\frac{\dot{H}}{H^2}$ of the original system. The trajectories approaching a stable de Sitter state are mimicking the effects of the running $\Lambda(a)$ term. The arbitrary parametrizations of $\Lambda(a)$, in general, violate the covariance of general relativity. However, some of them which emerge from the covariant theory are an effective description of the behaviour of trajectories in the neighbourhood of a stable de Sitter state.

In the paper we have studied the dynamics of these cosmological models in detail. We have examined the structure of the phase space which is organized by critical points representing stationary states, invariant manifolds, etc. We have explored the dynamics at finite domains of the phase space as well as at infinity using the projective coordinates.

The detailed results obtained from the dynamical system analysis are as follows:

- We have found that Alcaniz and Lima's solution in the exploration of the conception of $\Lambda(H)$ cosmology represents the scaling solution $\rho_\Lambda(a) \sim \rho_{\rm m}(a)$. For this trajectory deS_+ is a global attractor.
- The non-covariant $\Lambda(a)$ parametrization can be obtained from the covariant action for the scalar field as an emergent parameterization.
- We have found strong evidence for the tuned-in Λ term in the $\Lambda(a)$ cosmology: $\Omega_{\Lambda,0} < 3.19 \times 10^{-7}$. This limit was obtained on the base of Ade et al.'s estimation of the constant early dark energy fraction [14].
- We have shown that trajectories in the phase space for which $\rho_\Lambda \sim \rho_{\rm m}$ represent scaling solutions.

Due to the dynamical system analysis we can reveal the physical status of the Alcaniz–Lima ansatz in the $\Lambda(H)$ approach. From the point of view of dynamical system theory this solution is a universal asymptote for trajectories which go toward a global attractor, i.e. a de Sitter state. In this regime both $\rho_\Lambda - \Lambda_{\rm bare}$ and $\rho_{\rm m}$ are proportional, i.e., it is a scaling solution.

The detailed studies of the dynamics on the phase portraits showed how 'large' is the class of running Λ cosmological models for which the concordance ΛCDM model is a global attractor.

We also demonstrated on the example of cosmological models with non-minimal coupling constant and constant potential that a running part of the Λ term can be constrained by the Planck data. Applying the idea of constant early dark energy fraction and Ade et al.'s bound we have found a convincing constraint on the value of the running Λ term.

In the paper we considered some parametrization of the Λ term, which violates the covariance of the Lagrangian like $\Lambda(H)$, $\Lambda(a)$ parameterization but it is used as a some kind of an effective description. In the phase space of cosmological models with such a parametrization we observe some difficulties which are manifested by trajectories crossing the boundary line of zero energy density invariant submanifold. It is a consequence of the fact that $\rho_{\rm m} = 0$ is not a trajectory of the dynamical system. On the other hand the $\Lambda(a)$ parametrization can emerge from the basic covariant theory as some approximation of the true dynamics.

We illustrated such a possibility for the scalar field cosmology with a minimal and non-minimal coupling to gravity. In the phase space of evolutionary scenarios the difficulties disappear. Trajectories depart from the invariant submanifold $\frac{\dot{H}}{H^2} = 0$ of the corresponding dynamical system and this behaviour can be approximated by a running cosmological term such as a slow roll parameter $\epsilon_1 = \frac{\dot{H}}{H^2} \ll 1$.

We included the curvature in the dynamical analysis. In the phase space in the positive curvature domain, we found new trajectories which represent an oscillating universe with the initial singularity and without the initial singularity and a universe with a bounce. For models in this paper, perturbations of the flat model, by the negative curvature, do not change qualitatively this model in contrast to a closed model.

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