

A model of the two-dimensional quantum harmonic oscillator in an AdS_3 background

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Abstract In this paper we study a model of the two-dimensional quantum harmonic oscillator in a three-dimensional anti-de Sitter background. We use a generalized Schrödinger picture in which the analogs of the Schrödinger operators of the particle are independent of both the time and the space coordinates in different representations. The spacetime independent operators of the particle induce the Lie algebra of Killing vector fields of the AdS_3 spacetime. In this picture, we have a metamorphosis of the Heisenberg uncertainty relations.

1 Introduction

In [1] it was proposed to classify the states of a relativistic particle by means of the invariant operators (\mathbf{p} = momentum, $p_0 = \sqrt{m^2c^4 + c^2\mathbf{p}^2}$, m = mass,)

$$C_1(\mathbf{p}) = \mathbf{N}^2 - \mathbf{L}^2, \quad C_2(\mathbf{p}) = \mathbf{N} \cdot \mathbf{L} \quad (1)$$

characterizing the infinite-dimensional unitary representations of the Lorentz group, and to carry out the expansion of the wave function in the momentum space representation over the functions ($0 \leq \alpha < \infty$, $\mathbf{n}^2(\theta, \varphi) = 1$)

$$\xi^{(0)}(\mathbf{p}, \alpha, \mathbf{n}) := [(p_0 - c\mathbf{p} \cdot \mathbf{n})/mc^2]^{-1+i\alpha}. \quad (2)$$

The functions $\xi^{(0)}(\mathbf{p}, \alpha, \mathbf{n})$ are the eigenfunctions of the operator $C_1(\mathbf{p})$, ($C_1(\mathbf{p}) \implies 1 + \alpha^2$). The boost and rotation generators of the Lorentz group have the form (spin = 0)

$$N_i = ip_0 \frac{\partial}{\partial cp_i}, \quad L_i = i\epsilon_{ijk} p_k \frac{\partial}{\partial p_j}. \quad (3)$$

The operator $C_2(\mathbf{p})$ vanishes for a spinless particle.

The expansion proposed in [1] does not include any dependence on the time t and space coordinates \mathbf{x} , i.e. it is "spacetime independent". In [2], in the framework of a two-particle equation of the quasipotential type, the expansion over the functions $\xi^*(\mathbf{p}, \alpha, \mathbf{n})$ was used to introduce the "relativistic configurational" representation (in following the $\rho\mathbf{n}$ -representation, $\rho = \alpha\hbar/mc$). In this approach the variable ρ was interpreted as the relativistic generalization of a relative coordinate. It was shown that the corresponding operators of the Hamiltonian $H(\rho, \mathbf{n})$ and the 3-momentum $\mathbf{P}(\rho, \mathbf{n})$, defined on the functions $\xi^*(\mathbf{p}, \rho, \mathbf{n})$, has a form of the differential-difference operators.

In Refs. [3,4] it has been shown that the $\rho\mathbf{n}$ -representation may also be used in a so-called generalized Schrödinger picture in which the analogs of the Schrödinger operators of a particle are independent of both the time and the space coordinates in different representations. It was found that the operators $H(\rho, \mathbf{n})$, $\mathbf{P}(\rho, \mathbf{n})$, $\mathbf{L}(\mathbf{n})$, and $\mathbf{N}(\rho, \mathbf{n}) = \rho\mathbf{n} + (\mathbf{n} \times \mathbf{L} - \mathbf{L} \times \mathbf{n})/2mc$ satisfy the commutations relations of the Poincaré algebra in the $\rho\mathbf{n}$ -representation. We have two spacetime independent representations of the Poincaré algebra; the \mathbf{p} and the $\rho\mathbf{n}$ -representation. In the GS-picture the $\rho\mathbf{n}$ -representation may be used to describe extended objects like strings.

In the case of the one-dimensional momentum space representation (p = momentum, m = mass, $p_0^2 - c^2p^2 = m^2c^4$) the eigenfunctions of the boost generator $N(p) = ip_0\partial_{cp}$, ($N \implies \frac{mc}{\hbar}\rho$) may be written in the form

$$\xi_1(p, \rho) = [(p_0 - cp)/mc^2]^{i\frac{mc}{\hbar}\rho}. \quad (4)$$

The expansion

$$\psi(\rho) = \frac{1}{(2\pi)^{1/2}} \int \frac{dp}{p_0} \psi(p) \xi^*_{-1}(p, \rho) \quad (5)$$

leads to the functions $\psi(\rho)$ in the ρ -representation. In the ρ -representation the Hamilton operator H and the momentum

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operator P of the particle have the form ($\tilde{\lambda} = \frac{\hbar}{mc}$)

$$H(\rho) = mc^2 \cosh(-i\tilde{\lambda}\partial_\rho), \quad P(\rho) = mc \sinh(-i\tilde{\lambda}\partial_\rho), \tag{6}$$

and they satisfy the commutation relations of the Poincaré algebra

$$[\rho, P] = i \frac{\hbar}{mc^2} H, \quad [P, H] = 0, \quad [H, \rho] = -i \frac{\hbar}{m} P. \tag{7}$$

For a free particle in the Minkowski spacetime of two dimensions ($d = 2$), the coordinates t, x may be introduced in the states with the help of the transformation

$$S(t, x) = \exp[-i(tH - xP)/\hbar]. \tag{8}$$

We obtain

$$S(t, x)\psi(\rho) = \psi(\rho, t, x) = \frac{1}{(2\pi)^{1/2}} \int \frac{dp}{p_0} \psi(p, t, x) \xi_1^*(p, \rho), \tag{9}$$

where $\psi(p, t, x) = \psi(p)\exp[-i(tp_0 - xp)/\hbar]$.

In the case of a point particle ($\rho = 0$) we have the Fourier transform in relativistic quantum mechanics,

$$\psi(t, x) = \frac{1}{(2\pi)^{1/2}} \int \frac{dp}{p_0} \psi(p, t) e^{\frac{ixp}{\hbar}}. \tag{10}$$

In (9), the spacetime coordinates appear in the states in the ρ - and in the p -representation. We have a metamorphosis of the Heisenberg uncertainty relation, $\Delta x \cdot \Delta p \geq \hbar/2$. From $[\rho, P] = i\hbar H/mc^2$ in (7) it follows that instead of $\Delta x \cdot \Delta p \geq \hbar/2$, we have

$$\Delta\rho \cdot \Delta p \geq \hbar/2. \tag{11}$$

The GS-picture may be used in a quantum theory of gravity in which objects need a sharply defined frame. In Ref. [4], this picture was used to describe the motion of a relativistic particle in anti-de Sitter spacetime ($d = 2, d = 4$). It was found that the spacetime independent operators of the particle in an external field (like in the case of a harmonic oscillator) induce the Lie algebra of Killing vector fields of the AdS_4 spacetime ($d = 4; a = 1, 2, \dots, 10; \{x^i\}, i = 1, 2, 3.$)

$$K_a(t, x^i)\tilde{\Phi}(\rho, \mathbf{n}, t, x^i) = B_a(\rho, \mathbf{n})\tilde{\Phi}(\rho, \mathbf{n}, t, x^i). \tag{12}$$

Here $\tilde{\Phi}$ denotes the wave function of the particle. The operators of the Killing vector field $K_a(t, x^i)$ satisfy the same commutation rules as the spacetime independent operators $B_a(\rho, \mathbf{n})$, except for the minus signs on the right-hand sides. Equations (12) are valid for any d . In the present paper we

use these equations to describe the motion of a particle in AdS_3 spacetime. In the case of $d = 3$ we need six spacetime independent operators of the particle. In Sect. 2 we will now show that the operators of a relativistic model of the two-dimensional quantum harmonic oscillator in the $\rho\mathbf{n}$ -representation can be used in Eq. (12). This will allow us to obtain an exact expression for the energy levels of the particle and an expression for the spectrum of the AdS_3 radius.

2 One particle quantum equation in AdS_3 spacetime

In the two-dimensional momentum space representation, the first Casimir operator of the Lorentz group $C_1(\mathbf{p})$ has the eigenfunctions ($\rho_0^2 - c^2 p_1^2 - c^2 p_2^2 = m^2 c^4$)

$$\xi_2(\mathbf{p}, \rho, \mathbf{n}) := [(p_0 - c\mathbf{p} \cdot \mathbf{n})/mc^2]^{-\frac{1}{2} + i\frac{mc}{\hbar}\rho}, \tag{13}$$

where ($n_1 = \cos\varphi, n_2 = \sin\varphi$).

The Hamilton operator and the momentum operators of the particle defined on the functions $\xi_2^*(\mathbf{p}, \rho, \mathbf{n})$ have the form [5]

$$H(\rho, \mathbf{n}) = mc^2 \cosh(i\tilde{\lambda}\partial_\rho) + \frac{i\hbar c}{2\rho} \sinh(i\tilde{\lambda}\partial_\rho) - \frac{(-i\hbar\partial_\varphi)^2}{m\rho(2\rho + i\tilde{\lambda})} e^{i\tilde{\lambda}\partial_\rho}, \tag{14}$$

$$P_1(\rho, \mathbf{n}) = mc n_1 (H/mc^2 - e^{i\tilde{\lambda}\partial_\rho}) + \frac{i\hbar n_2 \cdot \partial_\varphi}{\rho + 2i\tilde{\lambda}} e^{i\tilde{\lambda}\partial_\rho}, \tag{15}$$

$$P_2(\rho, \mathbf{n}) = mc n_2 (H/mc^2 - e^{i\tilde{\lambda}\partial_\rho}) - \frac{i\hbar n_1 \cdot \partial_\varphi}{\rho + 2i\tilde{\lambda}} e^{i\tilde{\lambda}\partial_\rho}. \tag{16}$$

The operators H, \mathbf{P} , and the three operators of the Lorentz algebra in the $\rho\mathbf{n}$ -representation

$$N_1(\rho, \mathbf{n}) = n_1 \left(\rho - \frac{i}{2}\tilde{\lambda} \right) - i\tilde{\lambda} n_2 \partial_\varphi, \tag{17}$$

$$N_2(\rho, \mathbf{n}) = n_2 \left(\rho - \frac{i}{2}\tilde{\lambda} \right) + i\tilde{\lambda} n_1 \partial_\varphi, \quad L = -i\hbar \partial_\varphi, \tag{18}$$

satisfy the commutation relations of the Poincaré algebra.

For the particle in an external field like the two-dimensional harmonic oscillator potential we use the following operators:

$$\hat{P}_0(\rho, \mathbf{n}) = H(\rho, \mathbf{n}) + H'_0(\rho), \tag{19}$$

$$\hat{P}_i(\rho, \mathbf{n}) = P_i(\rho, \mathbf{n}) + P'_i(\rho, \mathbf{n}), \tag{20}$$

where ($\omega =$ frequency, $i = 1, 2$)

$$H'_0 = \frac{m\omega^2}{2} \left(\rho - \frac{i}{2}\tilde{\lambda} \right) \left(\rho - i\tilde{\lambda} \right) e^{-i\tilde{\lambda}\partial_\rho}, \tag{21}$$

$$P'_i = n_i \frac{m\omega^2}{2c} \left(\rho - \frac{i}{2}\tilde{\lambda} \right) \left(\rho - i\tilde{\lambda} \right) e^{-i\tilde{\lambda}\partial_\rho}. \tag{22}$$

In the nonrelativistic limit the operator $\hat{P}_0(\rho) - mc^2$ assumes the form

$$\hat{P}_{0nr} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \rho^2} + \frac{\hbar^2 \partial_\varphi^2}{2m\rho^2} + \frac{m\omega^2}{2} \rho^2. \tag{23}$$

The operators

$$\hat{P}_0(\rho, \mathbf{n}) = H(\rho, \mathbf{n}) + H'_0(\rho, \mathbf{n}), \tag{24}$$

$$\hat{P}_i(\rho, \mathbf{n}) = P_i(\rho, \mathbf{n}) + P'_i(\rho, \mathbf{n}), \tag{25}$$

and $L, N_i(\rho, \mathbf{n})$ satisfy the commutations rules of the Lie algebra $so(2, 2)$

$$\begin{aligned} [N_i, \hat{P}_j] &= \frac{i\hbar}{mc^2} \delta_{ij} \hat{P}_0, \quad [\hat{P}_i, \hat{P}_0] = -i\hbar m\omega^2 N_i \quad [\hat{P}_0, N_i] \\ &= -\frac{i\hbar}{m} \hat{P}_i, \end{aligned} \tag{26}$$

$$[\hat{P}_1, \hat{P}_2] = -i\hbar \frac{\omega^2}{c^2} L, \quad [L, \hat{P}_0] = 0, \quad [\hat{P}_1, L] = -i\hbar \hat{P}_2, \tag{27}$$

$$[N_1, N_2] = -\frac{i\hbar}{m^2 c^2} L, \quad [N_1, L] = -i\hbar N_2. \tag{28}$$

For the Casimir operator

$$C(\rho, \mathbf{n}) = \frac{1}{(\hbar\omega)^2} \left\{ \hat{P}_0^2 - c^2 \mathbf{P}^2 \right\} - \frac{m^2 c^2}{\hbar^2} \mathbf{N}^2 + \frac{1}{\hbar^2} L^2, \tag{29}$$

we have

$$C(\rho, \mathbf{n}) = \left(\frac{m^2 c^4}{\hbar^2 \omega^2} - 3/4 \right) I. \tag{30}$$

The explicit forms of the six operators $K_a(t, x^i)$ depend on the realization in terms of the spacetime coordinates. We have the problem of determining observables in the GS-picture. In order to interpret the operator \hat{P}_0 as a Hamilton operator, we choose the following realization $(t, x_1, x_2 (x_1 = r \cos \tilde{\varphi}, x_2 = r \sin \tilde{\varphi}, i = 1, 2))$:

$$K_{03} = i\hbar \frac{\partial}{\partial t}, \tag{31}$$

$$\begin{aligned} K_{i3} &= \sqrt{1 + (\omega r/c)^2} \cos \omega t (i\hbar \partial_{x_i}) \\ &\quad - \frac{(\omega x_i/c^2) \sin \omega t}{\sqrt{1 + (\omega r/c)^2}} i\hbar \partial_t, \end{aligned} \tag{32}$$

$$\begin{aligned} K_{i0} &= \frac{1}{m\omega} \sqrt{1 + (\omega r/c)^2} \sin \omega t (i\hbar \partial_{x_i}) \\ &\quad + \frac{x_i \cos \omega t}{mc^2 \sqrt{1 + (\omega r/c)^2}} i\hbar \partial_t, \end{aligned} \tag{33}$$

$$K_{12} = i\hbar \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) = i\hbar \partial_{\tilde{\varphi}}. \tag{34}$$

The set of the operators $\{K_{03}, K_{i3}, K_{i0}, K_{ij}\}$ determine the same Lie algebra as the operators $\{\hat{P}_0, \hat{P}_i, N_i, L\}$ except for the minus signs on the right-hand sides,

$$\begin{aligned} [K_{i0}, K_{j3}] &= -\frac{i\hbar}{mc^2} \delta_{ij} K_{03}, \quad [K_{i3}, K_{03}] = i\hbar m\omega^2 K_{i0}, \\ [K_{03}, K_{i0}] &= \frac{i\hbar}{m} K_{i3}, \end{aligned} \tag{35}$$

$$\begin{aligned} [K_{i3}, K_{j3}] &= i\hbar \frac{\omega^2}{c^2} K_{ij}, \quad [K_{ij}, K_{03}] = 0, \\ [K_{i3}, K_{ik}] &= i\hbar K_{k3}, \end{aligned} \tag{36}$$

$$[K_{i0}, K_{j0}] = \frac{i\hbar}{m^2 c^2} K_{ij}, \quad [K_{i0}, K_{ik}] = i\hbar K_{k0}. \tag{37}$$

The operators $\{K_{03}, K_{i3}, K_{i0}, K_{ij}\}$ form a basis for the $SO(2, 2)$ group generators and are related to the Killing vectors of the AdS_3 spacetime with metric

$$ds^2 = \left(1 + \frac{\omega^2 r^2}{c^2} \right) c^2 dt^2 - \frac{1}{1 + \frac{\omega^2 r^2}{c^2}} dr^2 - r^2 d\tilde{\varphi}^2. \tag{38}$$

Here, the constant ω/c is related to the radius κ of the AdS_3 spacetime ($\kappa = c/\omega$).

We can introduce the equation

$$i\hbar \frac{\partial}{\partial t} \Phi(\rho, \mathbf{n}; t, x_1, x_2) = \hat{P}_0(\rho, \mathbf{n}) \Phi(\rho, \mathbf{n}; t, x_1, x_2), \tag{39}$$

which defines the operator $\hat{P}_0(\rho, \mathbf{n})$ as the Hamilton operator of the particle.

A general solution of $\Phi(\rho, \mathbf{n}; t, x_1, x_2)$ can be written as a sum of separated solutions or the eigenfunctions of the operators $\hat{P}_0(\rho, \mathbf{n})$ and the Casimir operator ($\tau = \omega t, \tan \sigma = \omega r/c$),

$$\begin{aligned} C(\tau, \sigma, \tilde{\varphi}) &= -\cos^2 \sigma \frac{\partial^2}{\partial \tau^2} + \cot \sigma \frac{\partial}{\partial \sigma} + \cos^2 \sigma \frac{\partial^2}{\partial \sigma^2} \\ &\quad + \cot^2 \sigma \frac{\partial^2}{\partial \tilde{\varphi}^2}. \end{aligned} \tag{40}$$

The eigenfunctions of $C(\tau, \sigma, \tilde{\varphi})$ are $(n = 0, 1, 2, \dots, |\tilde{m}| = 0, 1, 2, \dots, M = 2, 3, 4, \dots, \lambda = 2n + |\tilde{m}| + M)$

$$\psi_{n|\tilde{m}|\tilde{m}}^M = N_{n|\tilde{m}|\tilde{m}}^M e^{-i\lambda\tau} (\cos \sigma)^M (\sin \sigma)^{|\tilde{m}|} {}_2F_1(a, b, c; z) e^{i\tilde{m}\tilde{\varphi}}, \tag{41}$$

where $N_{n|\tilde{m}|\tilde{m}}^M$ are the normalization constants and

$${}_2F_1 \left(\frac{1}{2} (|\tilde{m}| + M - \omega), \frac{1}{2} (|\tilde{m}| + M + \omega), 3/2 + |\tilde{m}|; \sin^2 \sigma \right) \tag{42}$$

are the hypergeometric functions. Thus we find

$$C(\tau, \sigma, \tilde{\varphi}) \Rightarrow M(M - 2). \tag{43}$$

For the spectrum of $i\hbar \frac{\partial}{\partial t}$ we have

$$E = \hbar\omega(2n + |\tilde{m}| + M). \tag{44}$$

The eigensolutions of the Hamilton operator $\hat{P}_0(\rho, \mathbf{n})$

$$\hat{P}_0(\rho, \mathbf{n})\xi = \hbar\omega(2n + |\tilde{m}| + M)\xi \tag{45}$$

are $(\tilde{\rho} = \frac{mc\rho}{\hbar}, M = 1 + \sqrt{1/4 + (\frac{mc^2}{\hbar\omega})^2}, l = |m| = 0, 1, 2, \dots)$

$$\xi_{nlm}^M = c_{nl}^M \left(\frac{\hbar\omega}{mc^2}\right)^{-i\tilde{\rho}} \Gamma(M-1/2-i\tilde{\rho}) \frac{\Gamma[l+1/2+i\tilde{\rho}]}{\tilde{\Gamma}(i\tilde{\rho}+1/2)} S_n(\tilde{\rho}) e^{im\varphi}, \tag{46}$$

where c_{nl}^M are normalization constants and $S_n(\tilde{\rho})$ are the Hahn polynomials,

$$S_n(\tilde{\rho}^2; l+1/2, M-1/2, 1/2) = \frac{\Gamma(l+M+n)\Gamma(l+M)}{\Gamma(l+1+n)\Gamma(l+1)} \times {}_3F_2(-n, l+1/2+i\tilde{\rho}, l+1/2-i\tilde{\rho}; l+M, l+1; 1). \tag{47}$$

For the function $\Phi(\rho, \mathbf{n}; t, x_1, x_2)$, we have $(m = \tilde{m}, l = |\tilde{m}|)$

$$\Phi^M = \sum_{n=0}^{\infty} \sum_{l=0}^{m=l} \psi_{n|\tilde{m}|\tilde{m}}^M \xi_{nlm}^M. \tag{48}$$

From

$$C(\rho, \mathbf{n})\Phi^M = C(\tau, \sigma, \tilde{\varphi})\Phi^M \tag{49}$$

it follows that the oscillator frequency is discrete and for higher M decreases according to

$$\omega_M = \frac{mc^2}{\hbar\sqrt{(M^2 - 2M + 3/4)}}. \tag{50}$$

The energy spectrum of the particle can be written as

$$E_{n|\tilde{m}|M} = \frac{mc^2}{\sqrt{(M^2 - 2M + 3/4)}}(2n + |\tilde{m}| + M). \tag{51}$$

For the AdS_3 radius $\kappa = c/\omega$, we have $\kappa_M = \sqrt{(M^2 - 2M + 3/4)} \hbar/mc$.

3 Conclusion

In this paper we have shown that a generalized Schrödinger picture may be used to describe a relativistic particle in a three-dimensional anti-de Sitter spacetime. A specific feature of this picture is that the frame itself becomes dynamical. It was found that in this picture we have a metamorphosis of the Heisenberg uncertainty relations. We have shown that the energy of the particle and the anti-de Sitter radius are discrete.

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References

1. I.S. Shapiro, Expansion of a wave function in irreducible representations of the Lorentz group. *Sov. Phys. Doklady.* **1**, 91–93 (1956)
2. V.G. Kadyshvsky, R.M. Mir-Kasimov, N.B. Skachkov, Quasipotential approach and the expansion in relativistic spherical functions. *Nuovo Cimento.* **55**, 233–257 (1968)
3. R.A. Frick, On a Heisenberg picture and Fourier transform on the Lorentz group. *Sov. J. Nucl. Phys.* **38**, 481–483 (1983)
4. R.A. Frick, Model of a relativistic oscillator in a generalized Schrödinger picture. *Ann. Phys. (Berlin)* **523**(11), 871–882 (2011)
5. S.M. Nagiyev, E.I. Jafarov, M.Y. Efendiyev, The relativistic two-dimensional harmonic oscillator. *Il Nuovo Cimento.* **124B**, 395–403 (2009)