

Integrability of geodesics and action-angle variables in Sasaki–Einstein space $T^{1,1}$

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Abstract We briefly describe the construction of Stäckel–Killing and Killing–Yano tensors on toric Sasaki–Einstein manifolds without working out intricate generalized Killing equations. The integrals of geodesic motions are expressed in terms of Killing vectors and Killing–Yano tensors of the homogeneous Sasaki–Einstein space $T^{1,1}$. We discuss the integrability of geodesics and construct explicitly the action-angle variables. Two pairs of frequencies of the geodesic motions are resonant giving way to chaotic behavior when the system is perturbed.

1 Introduction

The renewed interest in Sasaki geometries has arisen in connection with some recent developments in mathematics and theoretical physics [1]. Sasaki geometry represents the natural odd-dimensional counterpart of the Kähler geometry. The Sasaki structure in $(2n - 1)$ dimensions is sandwiched between the Kähler metric cone in n complex dimensions and the transverse Kähler structure of complex dimension $(n - 1)$. In particular the Kähler cone is Ricci flat, i.e. a Calabi–Yau manifold, if and only if the corresponding Sasaki manifold is Einstein.

The interest in physics for Sasaki–Einstein (SE) geometry is connected with its relevance in AdS/CFT duality. A new nontrivial infinite family of toric SE manifolds was constructed [2] and many new insights were obtained for the AdS/CFT correspondence. These SE spaces are denoted by $Y^{p,q}$ and characterized by the two coprime positive integers p and q with $q < p$. A vastly greater number of SE spaces was constructed in [3] and denoted by $L^{p,q,r}$ where p , q , and r are coprime positive integers with $0 < p \leq q$, $0 < r < p + q$, and with p and q each coprime to r and to $s = p + q - r$. These metrics have the $U(1) \times U(1) \times U(1)$ isometry. In the special case $p + q = 2r$ the isometry of these metrics

is enlarged to $SU(2) \times U(1) \times U(1)$, which is the isometry of the spaces $Y^{p,q} = L^{p-q,p+q,p}$. Another special limit is $p = q = r = 1$ and the metric becomes the homogeneous $T^{1,1}$ space with $SU(2) \times SU(2) \times U(1)$ isometry.

The symmetries of SE spaces play an important role in connection with the study of integrability properties of geodesic motions and separation of variables of the Hamilton–Jacobi or quantum Klein–Gordon, Dirac equations.

Higher order symmetries associated with Stäckel–Killing (SK) and Killing–Yano (KY) tensors generate conserved quantities polynomial in momenta describing the so called *dynamical* or *hidden symmetries*. In general it is a difficult task to solve straightly the generalized Killing equations satisfied by KY and SK tensors. Using the geometrical structure of a SE manifold, its connection with the complex structure of the Calabi–Yau metric cone it is possible to produce explicitly the complete set of Killing tensors and consequently the integrals of the geodesic motions.

One of the interesting aspect of the AdS/CFT correspondence is integrability which allows obtaining many new classical solutions of the theory (see the review [4]). By focusing on the integrability of the geodesics on SE spaces we gain a better understanding of the geometries produced by D -branes on non-flat bases.

The standard way to decide if a system is integrable is to find integrals of motion. A dynamical system is called integrable if the number of functionally independent integrals of motion is equal to the number of degrees of freedom.

In the context of AdS/CFT duality type IIB strings on $AdS_5 \times S^5$ with Ramond–Ramond fluxes is related to $\mathcal{N} = 4$ $SU(N)$ gauge theory. An interesting generalization of this duality between gauge theory and strings is to consider backgrounds of the form $AdS_5 \times X_5$ where X_5 is in a general class of five-dimensional Einstein spaces admitting $U(1)$ fibration. While the type IIB string theory on $AdS_5 \times S^5$ is classically integrable [5] there are many non-

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integrable AdS/CFT dualities in which the string world-sheet theory exhibits a chaotic behavior. This is the case when the internal space is a SE manifold like $T^{1,1}$ or $Y^{p,q}$ [6].

The purpose of this paper is to analyze the integrability of geodesics of the homogeneous regular SE 5-dimensional space $T^{1,1}$. In the light of the AdS/CFT correspondence, $AdS \times T^{1,1}$ represents the first example of a supersymmetric holographic theory based on a compact manifold which is not locally S^5 [7].

Using the multitude of Killing vector (Kv) fields and SK tensors it is possible to construct the conserved quantities for geodesic motions on $T^{1,1}$. However, the number of functionally independent constants of motion is only 5, implying the complete integrability of geodesics, but not superintegrability.

In order to understand the peculiarities of the geodesic motions in $T^{1,1}$ space we shall perform the *action-angle* formulation of the phase space. The description of the integrability of geodesics in $T^{1,1}$ in these variables gives us a comprehensive geometric description of the dynamics.

The existence of the action-angle variables is very important both for the theory of near-integrable systems (Kolmogorov–Arnold–Moser (KAM) theory) and for the quantization of integrable systems (Bohr–Sommerfeld rule). The action-angle variables define an n -dimensional surface which is a topological torus (KAM torus) [8]. The KAM theorem states that when an integrable Hamiltonian is perturbed by a small non-integrable piece most tori survive but suffer small deformations. On the other hand the resonant tori which have rational ratios of frequencies get destroyed and motion on them becomes chaotic. The use of action-angle variables provides a powerful technique to quickly obtain the frequencies of the periodic motions without finding a complete solution to the motion of the system.

The paper is organized as follows. In the next two sections we recall some definitions and well-known results concerning the Killing tensors and SE spaces. For the completeness and clarity of the exposition, in Sect. 4 we present the constants of motion on $T^{1,1}$ space proving the integrability of geodesics. In Sect. 5 we construct explicitly the action-angle variables and the frequencies of the motions. Finally in Sect. 6 we present some concluding remarks and discuss the issues regarding the presence of resonant frequencies. For the convenience of the reader, some details concerning the evaluation of some integrals from Sect. 5 are deferred to the appendix.

2 Stäckel–Killing and Killing–Yano tensors

SK and KY tensors stand as a natural extension of the Kv fields which are linked to the continuous isometries that leave the metric invariant.

On a Riemannian manifold (M, g) with local coordinates x^μ and metric $g_{\mu\nu}$ the geodesics can be obtained as the trajectories of test-particles with the proper-time Hamiltonian

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu. \tag{1}$$

In the presence of a Kv the system of a free particle with Hamiltonian (1) admits a conserved quantity linear in the momenta.

A SK tensor of rank r is a symmetric tensor defined on the manifold M

$$K = \frac{1}{r!} K^{(\mu_1 \dots \mu_r)} \frac{\partial}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_r}} \tag{2}$$

such that

$$\nabla_{(\mu} K_{\mu_1 \dots \mu_r)} = 0 \tag{3}$$

where ∇ is the Levi-Civita connection. If M admits a SK tensor, the system of a free particle possesses a conserved quantity of higher order in the momenta

$$C_{SK} = K^{(\mu_1 \dots \mu_r)} p_{\mu_1} \dots p_{\mu_r}. \tag{4}$$

A different generalization of the Kv’s is represented by the KY tensors. A KY tensor is a differential p -form defined on M ,

$$\Psi = \frac{1}{r!} \Psi_{[\mu_1 \dots \mu_r]} dx^{\mu_1} \dots dx^{\mu_r}, \tag{5}$$

satisfying the equation

$$\nabla_{(\mu} \Psi_{\mu_1) \dots \mu_r} = 0. \tag{6}$$

Giving two KY tensors $\Psi^{\mu_1 \dots \mu_r}$ and $\Sigma^{\mu_1 \dots \mu_r}$ the partial contracted product generates a SK tensor of rank 2:

$$K_{\mu\nu}^{(\Psi, \Sigma)} = \Psi_{\mu\lambda_2 \dots \lambda_r} \Sigma_{\nu}^{\lambda_2 \dots \lambda_r} + \Sigma_{\mu\lambda_2 \dots \lambda_r} \Psi_{\nu}^{\lambda_2 \dots \lambda_r}. \tag{7}$$

This property offers a method to generate higher order integrals of motion (4) by identifying the complete set of KY tensors.

It is worth mentioning that in general it is a hard task to find the complete set of SK or KY tensors trying to solve directly Eqs. (3) and (6). In some cases it is possible to produce the complete set of KY tensors taking advantage of geometrical properties of the space. That is the case of toric SE spaces for which the explicit construction of KY tensors is possible [9].

3 Sasaki–Einstein spaces

Recall that a $(2n - 1)$ -dimensional manifold M is a *contact manifold* if there exists a 1-form η , called a contact 1-form, on M such that

$$\eta \wedge (d\eta)^{n-1} \neq 0 \tag{8}$$

everywhere on M [10]. For every choice of contact 1-form η there exists a unique vector field K_η , called a Reeb vector field, that satisfies

$$\eta(K_\eta) = 1 \quad \text{and} \quad K_\eta \lrcorner d\eta = 0. \tag{9}$$

The Reeb vector field K_η is a Kv field on M , it has unit length, and its integral curve is a geodesic.

A contact Riemannian manifold is Sasakian if its metric cone

$$(C(M), \bar{g}) = (\mathbb{R}_+ \times M, dr^2 + r^2g) \tag{10}$$

is Kähler. Here $r \in (0, \infty)$ may be considered as a coordinate on the positive real line \mathbb{R}_+ .

As part of the connection between Sasaki and Kähler geometries it is worth noting that in the case of a SE manifold, $\text{Ric}_g = 2(n - 1)g$, the metric cone is Ricci flat $\text{Ric}_{\bar{g}} = 0$, i.e. a Calabi–Yau manifold.

The KY tensors on a SE manifold are described by the Killing forms

$$\Psi_k = \eta \wedge (d\eta)^k, \quad k = 0, 1, \dots, n - 1. \tag{11}$$

These KY tensors do not exhaust the complete set of KY tensors on SE spaces. The Calabi–Yau metric cone has the holonomy $SU(n)$ and admits two additional parallel forms given by the real and imaginary parts of the complex volume form [9]. In order to complete the set of KY tensors on a SE manifold it is necessary to use the relation between the KY tensors on SE space and its metric cone. More precisely, for any p -form Ψ on the space M we can define an associated $(p + 1)$ -form Ψ^C on the cone $C(M)$ [9].

4 Integrability of geodesics in Sasaki–Einstein space $T^{1,1}$

Any complete homogeneous SE 5-dimensional manifold is a $U(1)$ -bundle over the complex projective plane $\mathbb{C}P^2$ or $\mathbb{C}P^1 \times \mathbb{C}P^1$ [1]. The well-known realizations are the round metric on S^5 and the homogeneous metric $T^{1,1}$ on $S^2 \times S^3$.

The metric on $T^{1,1}$ is [11, 12]

$$ds^2(T^{1,1}) = \frac{1}{6} \left(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2 \right) + \frac{1}{9} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2. \tag{12}$$

Here $\theta_i, \phi_i, i = 1, 2$, are the usual coordinates on two round S^2 spheres and $\psi \in [0, 4\pi)$ parametrizes the $U(1)$ fiber over $S^2 \times S^2$.

The globally defined contact 1-form η is

$$\eta = \frac{1}{3} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2) \tag{13}$$

and the corresponding Reeb vector is

$$K_\eta = 3 \frac{\partial}{\partial \psi}. \tag{14}$$

In the following we define $2\nu = \psi$ so that ν has canonical period 2π .

On the manifold $T^{1,1}$ with the metric (12) the geodesics are described by the Hamiltonian (1) where the canonical momenta conjugate to the coordinates x^μ are $p_\mu = g_{\mu\nu} \dot{x}^\nu$ with an overdot denoting the proper-time derivative. In particular the conjugate momenta to the coordinates $(\theta_1, \theta_2, \phi_1, \phi_2, \nu)$ are:

$$\begin{aligned} p_{\theta_1} &= \frac{1}{6} \dot{\theta}_1 \\ p_{\theta_2} &= \frac{1}{6} \dot{\theta}_2 \\ p_{\phi_1} &= \frac{1}{6} \sin^2 \theta_1 \dot{\phi}_1 + \frac{1}{9} \cos^2 \theta_1 \dot{\phi}_1 + \frac{2}{9} \cos \theta_1 \dot{\nu} \\ &\quad + \frac{1}{9} \cos \theta_1 \cos \theta_2 \dot{\phi}_2 \\ p_{\phi_2} &= \frac{1}{6} \sin^2 \theta_2 \dot{\phi}_2 + \frac{1}{9} \cos^2 \theta_2 \dot{\phi}_2 + \frac{2}{9} \cos \theta_2 \dot{\nu} \\ &\quad + \frac{1}{9} \cos \theta_1 \cos \theta_2 \dot{\phi}_1 \\ p_\nu &= \frac{2}{9} \dot{\nu} + \frac{1}{9} \cos \theta_1 \dot{\phi}_1 + \frac{1}{9} \cos \theta_2 \dot{\phi}_2 \end{aligned} \tag{15}$$

and the conserved Hamiltonian (1) takes the form

$$H = 3 \left[p_{\theta_1}^2 + p_{\theta_2}^2 + \frac{1}{4 \sin^2 \theta_1} (2p_{\phi_1} - \cos \theta_1 p_\nu)^2 + \frac{1}{4 \sin^2 \theta_2} (2p_{\phi_2} - \cos \theta_2 p_\nu)^2 \right] + \frac{9}{8} p_\nu^2. \tag{16}$$

Taking into account the isometries of $T^{1,1}$, the momenta p_{ϕ_1}, p_{ϕ_2} , and p_ν are conserved. Since $T^{1,1}$ has $SU(2) \times SU(2) \times U(1)$ symmetry, two total $SU(2)$ angular momenta are also conserved:

$$\begin{aligned} \mathbf{J}_1^2 &= p_{\theta_1}^2 + \frac{1}{4 \sin^2 \theta_1} (2p_{\phi_1} - \cos \theta_1 p_\nu)^2 + \frac{1}{4} p_\nu^2 \\ \mathbf{J}_2^2 &= p_{\theta_2}^2 + \frac{1}{4 \sin^2 \theta_2} (2p_{\phi_2} - \cos \theta_2 p_\nu)^2 + \frac{1}{4} p_\nu^2. \end{aligned} \tag{17}$$

Other constants of motion are constructed according to (7) from the KY tensors of $T^{1,1}$. First of all, there are two KY tensors (11) for $k = 1, 2$ constructed from the contact form η (13). Finally there are two additional KY tensors related to the complex volume form of the metric cone $C(T^{1,1})$. All these constants of motion are explicitly given in [13, 14].

In spite of the existence of a multitude of constants of motion, the number of functionally independent is only 5 [13, 15], exactly the dimension of the SE space $T^{1,1}$. In the next section we shall use the complete integrability of geodesics to solve the Hamilton–Jacobi equation by separation of variables and construct the action-angle variables.

5 Action-angle variables

We start with the Hamilton–Jacobi equation

$$H\left(q_1, \dots, q_n; \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_n}; t\right) + \frac{\partial S}{\partial t} = 0 \tag{18}$$

with the *Hamilton principal function*

$$S = S(q_1, \dots, q_n; \alpha_1, \dots, \alpha_n; t) \tag{19}$$

depending on n constants of integration. Taking into account that the Hamiltonian (16) has no explicit time dependence, Hamilton’s principal function can be written in the form [16]

$$S(q, \alpha, t) = W(q, \alpha) - Et \tag{20}$$

where one of the constants of integration is equal to the constant value E of the Hamiltonian and $W(q, \alpha)$ is the *Hamilton characteristic function*

$$W = \sum \int p_i \dot{q}_i dt = \int p_i dq_i. \tag{21}$$

If the motion stays finite, each q_i will go through repeated cycles and the canonical *action variables* are defined as integrals over complete period of the orbit in the (q_i, p_i) plane

$$J_i = \oint p_i dq_i = \oint \frac{\partial W_i(q_i; \alpha)}{\partial q_i} dq_i \quad (\text{no summation}). \tag{22}$$

The J_i form n independent functions of the constants α_i and can be taken as a set of new constant momenta.

The conjugate *angle variables* w_i are defined by the equations:

$$w_i = \frac{\partial W}{\partial J_i} = \sum_{j=1}^n \frac{\partial W_j(q_j; J_1, \dots, J_n)}{\partial J_i} \tag{23}$$

having a linear evolution in time

$$w_i = \omega_i t + \beta_i \tag{24}$$

with β_i the other constants of integration, and the ω_i are frequencies associated with the periodic motion of q_i .

In the case of geodesic motions in $T^{1,1}$ space the variables in the Hamilton–Jacobi equation are separable and we seek a solution with the characteristic function

$$W(\theta_1, \theta_2, \phi_1, \phi_2, \nu) = W_{\theta_1}(\theta_1) + W_{\theta_2}(\theta_2) + W_{\phi_1}(\phi_1) + W_{\phi_2}(\phi_2) + W_\nu(\nu). \tag{25}$$

Since ϕ_1, ϕ_2, ν are cyclic variables we have

$$\begin{aligned} W_{\phi_1} &= p_{\phi_1} \phi_1 = \alpha_{\phi_1} \phi_1, \\ W_{\phi_2} &= p_{\phi_2} \phi_2 = \alpha_{\phi_2} \phi_2, \\ W_\nu &= p_\nu \nu = \alpha_\nu \nu, \end{aligned} \tag{26}$$

where $\alpha_{\phi_1}, \alpha_{\phi_2}, \alpha_\nu$ are constants of integration.

The action variables corresponding to the cyclic coordinates can be straightforwardly obtained from (26),

$$\begin{aligned} J_{\phi_1} &= 2\pi \alpha_{\phi_1}, \\ J_{\phi_2} &= 2\pi \alpha_{\phi_2}, \\ J_\nu &= 2\pi \alpha_\nu. \end{aligned} \tag{27}$$

Using (26) the Hamilton–Jacobi equation reduces to

$$\begin{aligned} E &= 3 \left[\left(\frac{\partial W_{\theta_1}}{\partial \theta_1} \right)^2 + \frac{1}{4 \sin^2 \theta_1} (2\alpha_{\phi_1} - \cos \theta_1 \alpha_\nu)^2 \right] \\ &+ 3 \left[\left(\frac{\partial W_{\theta_2}}{\partial \theta_2} \right)^2 + \frac{1}{4 \sin^2 \theta_2} (2\alpha_{\phi_2} - \cos \theta_2 \alpha_\nu)^2 \right] \\ &+ \frac{9}{8} \alpha_\nu^2. \end{aligned} \tag{28}$$

We observe that the dependences on θ_1 and θ_2 has been separated into the expressions within the square brackets. The quantities in the square brackets must be constants which will be denoted by $\alpha_{\theta_1}^2$ and $\alpha_{\theta_2}^2$.

To find the action variables J_{θ_i} ($i = 1, 2$) we infer from (28)

$$\frac{\partial W_{\theta_i}}{\partial \theta_i} = \sqrt{\alpha_{\theta_i}^2 - \frac{(2\alpha_{\phi_i} - \cos \theta_i \alpha_\nu)^2}{4 \sin^2 \theta_i}}, \quad i = 1, 2, \tag{29}$$

and the corresponding action variables are

$$J_{\theta_i} = \oint d\theta_i \sqrt{\alpha_{\theta_i}^2 - \frac{(2\alpha_{\phi_i} - \cos \theta_i \alpha_\nu)^2}{4 \sin^2 \theta_i}}, \quad i = 1, 2. \tag{30}$$

The limits of integrations are defined by the roots θ_{i-} and θ_{i+} of the expressions in the square root signs and a complete cycle of θ_i involves going from θ_{i-} to θ_{i+} and back to θ_{i-} .

This integral can be evaluated by elementary means or using the complex integration method of residues. In the latter case, putting $\cos \theta_i = t_i$, for the evaluation of the action variables J_{θ_i} , we extend t_i to a complex variable z and interpret the integral as a closed contour integral in the complex z -plane. The turning points of the t_i -motion are

$$t_{i\pm} = 2 \frac{\alpha_{\phi_i} \alpha_v \pm \alpha_{\theta_i} \sqrt{4\alpha_{\theta_i}^2 + \alpha_v^2 - 4\alpha_{\phi_i}^2}}{4\alpha_{\theta_i}^2 + \alpha_v^2}, \tag{31}$$

which are real for

$$4\alpha_{\theta_i}^2 + \alpha_v^2 - 4\alpha_{\phi_i}^2 \geq 0 \tag{32}$$

and situated in the interval $(-1, +1)$. We cut the complex z -plane from t_{i-} to t_{i+} and the closed contour integral of the integrand is a loop enclosing the cut in a clockwise sense. The contour can be deformed to a large circular contour plus two contour integrals about the poles at $z = \pm 1$. After a simple evaluation of the residues and the contribution of the large contour integral we finally get

$$J_{\theta_i} = 2\pi \left[\frac{1}{2} \sqrt{4\alpha_{\theta_i}^2 + \alpha_v^2} - \alpha_{\phi_i} \right], \quad i = 1, 2. \tag{33}$$

We notice that the constants J_{θ_i} , J_{ϕ_i} , J_v , and E are connected by the relation

$$H = E = \frac{3}{4\pi^2} \left[(J_{\theta_1} + J_{\phi_1})^2 + (J_{\theta_2} + J_{\phi_2})^2 - \frac{1}{8} J_v^2 \right], \tag{34}$$

making manifest that the Hamiltonian depends only on the action variables. The number of independent constants of motion is *five* implying the complete integrability of geodesics on $T^{1,1}$.

A particular advantage of the change to the action-angle variables is that one can identify the fundamental frequencies of the system. In this regard let us observe that the Hamiltonian (34) depends on J_{θ_i} and J_{ϕ_i} in the combination $J_{\theta_i} + J_{\phi_i}$ meaning that the frequencies of motion in θ_i and ϕ_i are identical:

$$\omega_{\theta_i} = \omega_{\phi_i} = \frac{\partial H}{\partial J_{\theta_i}} = \frac{\partial H}{\partial J_{\phi_i}} = \frac{3}{2\pi^2} (J_{\theta_i} + J_{\phi_i}). \tag{35}$$

Finally, using the action variables from (23) and (25) we have the angle variables (see the appendix):

$$\begin{aligned} w_{\theta_i} &= \frac{\partial W}{\partial J_{\theta_i}} = \frac{\partial W_{\theta_i}}{\partial J_{\theta_i}} \\ &= -\frac{J_{\theta_i} + J_{\phi_i}}{2\pi} I_1(a_i, b_i, c_i; \cos \theta_i), \end{aligned} \tag{36}$$

$$\begin{aligned} w_{\phi_i} &= \frac{\partial W}{\partial J_{\phi_i}} = \frac{\partial W_{\theta_i}}{\partial J_{\phi_i}} + \frac{\partial W_{\phi_i}}{\partial J_{\phi_i}} = \frac{\partial W_{\theta_i}}{\partial J_{\phi_i}} + \frac{1}{2\pi} \phi_i \\ &= -\frac{J_{\theta_i} + J_{\phi_i}}{2\pi} I_1(a_i, b_i, c_i; \cos \theta_i) \\ &\quad + \frac{-2J_{\phi_i} + J_v}{8\pi} \\ &\quad \times I_2(a_i + b_i + c_i, b_i + 2c_i, c_i; \cos \theta_i - 1) \\ &\quad + \frac{2J_{\phi_i} + J_v}{8\pi} \\ &\quad \times I_2(a_i - b_i + c_i, b_i - 2c_i, c_i; \cos \theta_i + 1) \\ &\quad + \frac{1}{2\pi} \phi_i, \end{aligned} \tag{37}$$

$$\begin{aligned} w_v &= \frac{\partial W}{\partial J_v} = \frac{\partial W_{\theta_1}}{\partial J_v} + \frac{\partial W_{\theta_2}}{\partial J_v} + \frac{\partial W_v}{\partial J_v} \\ &= \frac{\partial W_{\theta_1}}{\partial J_v} + \frac{\partial W_{\theta_2}}{\partial J_v} + \frac{1}{2\pi} v \\ &= \sum_{i=1,2} \frac{2J_{\phi_i} - J_v}{16\pi} \\ &\quad \times I_2(a_i + b_i + c_i, b_i + 2c_i, c_i; \cos \theta_i - 1) \\ &\quad + \sum_{i=1,2} \frac{2J_{\phi_i} + J_v}{16\pi} \\ &\quad \times I_2(a_i - b_i + c_i, b_i - 2c_i, c_i; \cos \theta_i + 1) \\ &\quad + \frac{1}{2\pi} v. \end{aligned} \tag{38}$$

6 Conclusions

Motivated by the great interest of higher order symmetries and their applications in various field theories, in this paper we describe the construction of conserved quantities for geodesic motions on toric SE manifold $T^{1,1}$. The integrability of geodesics permits us to give an action-angle formulation of the phase space variables. Such a formulation represents a useful tool for developing perturbation theory.

The KAM theorem describes how an integrable system reacts to small perturbations. The loss of integrability is characterized by the presence of resonant tori which have rational ratios of frequencies,

$$m_i \omega_i = 0 \quad \text{with} \quad m_i \in \mathbb{Q}. \tag{39}$$

That is the case of the frequencies (35) which evinces that the frequencies corresponding to θ_i and ϕ_i coordinates are equal.

This result is in accord with the numerical simulations [17], which show that certain classical string configurations in $AdS \times T^{1,1}$ are chaotic. One considered a string localized on the center of AdS_5 that wraps the directions ϕ_1 and ϕ_2 in $T^{1,1}$. To numerically investigate the perturbation of the integrable Hamiltonian by a small non-integrable piece, Ref. [18]

considered Poincaré sections. In [6] an analytical approach is presented of the loss of the integrability using the techniques of differential Galois theory for *normal variational equation*. It is quite remarkable that, while the point-like string (geodesic) equations are integrable in some backgrounds, the corresponding extended classical string motion is not integrable in general [6, 19]. A similar situation encountered in [20] in the study of (non)-integrability of geodesics in *D*-brane background.

Since the Reeb vector field K_η (9) is nowhere vanishing, its orbits define a foliation of the SE space M . If all the orbits close, K_η generates a circle action on M . If, moreover, the action is free the SE manifold is said to be *regular* and is the total space of a $U(1)$ principal bundle over a Kähler–Einstein space. That is the case of the $T^{1,1}$ space with the Reeb vector field (14). More generally the $U(1)$ action on M is only locally free and such structures are referred to as *quasi-regular*. The geometries $Y^{p,q}$ with $4p^2 - 3q^2$ a square are examples of such manifolds. If the orbits of the Reeb vector field do not close, the K_v generates an action \mathbb{R} on M , with the orbits densely filling the orbits of a torus and the SE space is said to be *irregular*. The manifolds $Y^{p,q}$ with $4p^2 - 3q^2$ not a square represent examples of such geometries [12].

From these considerations it is obvious that there are significant differences between $T^{1,1}$ and $Y^{p,q}$ SE spaces. The construction of the action-angle variables for integrable geodesic motions in quasi-regular and irregular spaces is more intricate and deserves a special study.

Concluding, KY tensors provide a powerful tool for studying symmetries of black holes in higher dimensions and stringy geometries. It would be interesting to extend the action-angle formulation to quasi-regular and irregular 5-dimensional SE metrics and their higher dimensional generalizations relevant for the predictions of the AdS/CFT correspondence.

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Appendix A: Calculation of the integrals from Eqs. (36)–(38)

Putting $\cos \theta_i = t_i$ in (29) the characteristic function W_{θ_i} is given by the integral

$$W_{\theta_i} = \frac{1}{2\pi} \int \frac{dt_i}{t_i^2 - 1} \sqrt{a_i + b_i t_i + c_i t_i^2} \tag{A.1}$$

with

$$\begin{aligned} a_i &= J_{\theta_i}^2 + 2J_{\theta_i} J_{\phi_i} - \frac{1}{4} J_v^2, \\ b_i &= J_{\phi_i} J_v, \\ c_i &= -(J_{\theta_i} + J_{\phi_i})^2. \end{aligned} \tag{A.2}$$

For the evaluation of angle variables we need the partial derivatives of W_{θ_i} (A.1) with respect to the action variables J_{θ_i} , J_{ϕ_i} , and J_v .

First we evaluate

$$\frac{\partial W_{\theta_i}}{\partial J_{\theta_i}} = -\frac{J_{\theta_i} + J_{\phi_i}}{2\pi} \int \frac{dt_i}{\sqrt{a_i + b_i t_i + c_i t_i^2}}. \tag{A.3}$$

Taking into account that in (A.2) $c_i < 0$, the last integral is [21]

$$I_1(a_i, b_i, c_i; t_i) = \frac{-1}{\sqrt{-c_i}} \arcsin\left(\frac{2c_i t_i + b_i}{\sqrt{-\Delta_i}}\right) \tag{A.4}$$

where $\Delta_i = 4a_i c_i - b_i^2 < 0$ taking for granted (32).

For the partial derivative of W_{θ_i} (A.1) with respect to the action variable J_{ϕ_i} we have

$$\begin{aligned} \frac{\partial W_{\theta_i}}{\partial J_{\phi_i}} &= -\frac{J_{\theta_i} + J_{\phi_i}}{2\pi} \int \frac{dt_i}{\sqrt{a_i + b_i t_i + c_i t_i^2}} \\ &\quad - \frac{J_{\phi_i}}{2\pi} \int \frac{dt_i}{t_i^2 - 1} \frac{1}{\sqrt{a_i + b_i t_i + c_i t_i^2}} \\ &\quad + \frac{J_v}{4\pi} \int dt_i \frac{t_i}{t_i^2 - 1} \frac{1}{\sqrt{a_i + b_i t_i + c_i t_i^2}} \\ &= \frac{J_{\theta_i} + J_{\phi_i}}{2\pi} I_1(a_i, b_i, c_i; t_i) \\ &\quad + \frac{-2J_{\phi_i} + J_v}{8\pi} \\ &\quad \times I_2(a_i + b_i + c_i, b_i + 2c_i, c_i; t_i - 1) \\ &\quad + \frac{2J_{\phi_i} + J_v}{8\pi} \\ &\quad \times I_2(a_i - b_i + c_i, b_i - 2c_i, c_i; t_i + 1) \end{aligned} \tag{A.5}$$

where [21]

$$\begin{aligned} I_2(a, b, c; t) &= \int \frac{dt}{t\sqrt{a + bt + ct^2}} \\ &= \frac{1}{\sqrt{-a}} \arctan\left(\frac{2a + bt}{2\sqrt{-a}\sqrt{a + bt + ct^2}}\right) \end{aligned} \tag{A.6}$$

for $a < 0$.

Finally for the partial derivative of W_{θ_i} (A.1) with respect to the action variable J_ν we have

$$\begin{aligned} \frac{\partial W_{\theta_i}}{\partial J_\nu} &= \frac{J_{\phi_i}}{4\pi} \int dt_i \frac{t_i}{t_i^2 - 1} \frac{1}{\sqrt{a_i + b_i t_i + c_i t_i^2}} \\ &\quad - \frac{J_\nu}{8\pi} \int \frac{dt_i}{t_i^2 - 1} \frac{1}{\sqrt{a_i + b_i t_i + c_i t_i^2}} \\ &= \frac{2J_{\phi_i} - J_\nu}{16\pi} I_2(a_i + b_i + c_i, b_i + 2c_i, c_i; t_i - 1) \\ &\quad + \frac{2J_{\phi_i} + J_\nu}{16\pi} \\ &\quad \times I_2(a_i - b_i + c_i, b_i - 2c_i, c_i; t_i + 1). \end{aligned} \quad (\text{A.7})$$

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