

On the near horizon rotating black hole geometries with NUT charges

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Abstract The near horizon geometries are usually constructed by implementing a specific limit to a given extreme black hole configuration. Their salient feature is that the isometry group includes the conformal subgroup $SO(2, 1)$. In this work, we turn the logic around and use the conformal invariants for constructing Ricci-flat metrics in $d = 4$ and $d = 5$ where the vacuum Einstein equations reduce to a coupled set of ordinary differential equations. In four dimensions the analysis can be carried out in full generality and the resulting metric describes the $d = 4$ near horizon Kerr–NUT black hole. In five dimensions we choose a specific ansatz whose structure is similar to the $d = 5$ near horizon Myers–Perry black hole. A Ricci-flat metric involving five arbitrary parameters is constructed. A particular member of this family, which is characterized by three parameters, seems to be a natural candidate to describe the $d = 5$ near horizon Myers–Perry black hole with a NUT charge.

1 Introduction

Motivated by the Kerr/CFT-correspondence [1], the near horizon black hole geometries in various dimensions attracted recently considerable attention.¹ Following the original work of Bardeen and Horowitz on the $d = 4$ near horizon Kerr black hole [4], such geometries are usually constructed by implementing a specific limit to a given extreme black hole configuration (for a detailed discussion see, e.g., Ref. [5]). In general, the limit yields a metric which can be interpreted as describing a complete vacuum spacetime on its own. Its remarkable property is that the isometry group is extended to involve the conformal subgroup $SO(2, 1)$, which motivates the holographic applications. The near horizon conformal symmetry of rotating black holes also proved useful in the

study of superconformal mechanics [6] and superintegrable models [7].

Denoting the temporal, radial, and azimuthal coordinates by t, r , and ϕ_i , $i = 1, \dots, n$, one can write the $SO(2, 1)$ -transformations in the form

$$t' = t + \alpha; \quad t' = t + \beta t, \quad r' = r - \beta r;$$
$$t' = t + (t^2 + \frac{1}{r^2})\gamma, \quad r' = r - 2tr\gamma, \quad \phi'_i = \phi_i - \frac{2}{r}\gamma,$$

where the infinitesimal parameters α, β , and γ correspond to the time translations, dilatations and special conformal transformations, respectively. Note that latitudinal coordinates remain inert under the action of the conformal group. Focusing on axially symmetric metrics which do not explicitly depend on the azimuthal angular variables ϕ_i , the conformal invariants which typically enter the near horizon metrics read

$$r^2 dt^2 - \frac{dr^2}{r^2}, \quad r dt + d\phi_i, \quad d\phi_i - d\phi_j.$$

These are accompanied by coefficients which depend on latitudinal angular coordinates only.

One may wonder what happens if the logic is turned around and the conformal invariants are used to construct Ricci-flat metrics in diverse dimensions. It is well known that not any $SO(2, 1)$ -invariant geometry can be linked to a black hole predecessor (see, e.g., Ref. [3]). Yet, such geometries are definitely amenable to holographic applications and in some instances they may provide useful insights into the structure of a black hole progenitor.

The goal of this work is to address this issue for the cases of four and five dimensions for which the vacuum Einstein equations reduce to a coupled set of ordinary differential

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¹ By now there exists a very extensive literature on the subject. For reviews and further references see, e.g., Refs. [2, 3].

equations. In four dimensions the analysis can be carried out in full generality. In the next section we demonstrate that the $d = 4$, $SO(2, 1)$ -invariant configuration corresponds to the near horizon Kerr–NUT black hole. In five dimensions the number of conformally invariant terms to be included into a metric grows notably. So in Sect. 3 we choose a specific ansatz whose structure is similar to the $d = 5$ near horizon Myers–Perry black hole. A Ricci-flat metric which includes five arbitrary parameters is constructed. Setting two of them to vanish, one obtains a solution which seems to be a natural candidate to describe the $d = 5$ near horizon Myers–Perry black hole with a NUT charge. In Sect. 4 we discuss the results obtained in this work as well as possible further developments. Some technical issues related to the material presented in Sect. 3 are gathered in the appendix.

2 $d = 4$ near horizon Kerr–NUT geometry

Given the conformal invariants, the most general $d = 4$ metric invariant under the action of $SO(2, 1)$ reads²

$$ds^2 = a(\theta) \left(r^2 dt^2 - \frac{dr^2}{r^2} - d\theta^2 \right) - b(\theta)(r dt + d\phi)^2, \quad (1)$$

where θ is the latitudinal angular variable. The vacuum Einstein equations yield a coupled set of ordinary differential equations to determine the coefficients $a(\theta)$ and $b(\theta)$. A thorough investigation shows that they can be reduced to the nonlinear ordinary differential equation for $a(\theta)$

$$4a^2 + 2a''(a - a'') + 3a'(a' + a^{(3)}) = 0, \quad (2)$$

while $b(\theta)$ is fixed provided $a(\theta)$ is known,

$$b = \frac{4}{3} (a + a'') - \frac{a'^2}{a}. \quad (3)$$

It is worth mentioning that the near horizon extreme Kerr geometry which is characterized by the coefficients [4]

$$a(\theta) = L_1(1 + \cos^2 \theta), \quad b(\theta) = \frac{4L_1 \sin^2 \theta}{1 + \cos^2 \theta}, \quad (4)$$

where L_1 is a constant related to the rotation parameter, does provide a particular solution to Eqs. (2) and (3). As (2) is a third order ordinary differential equation, its general solution involves three constants of integration. It is easy to verify that shifting θ in (4) by a constant one obtains a new solution to (2) and (3). However, the new parameter is physically irrelevant as it does not alter the metric. It is then natural to expect

² Note that one more possible term of the type $p(\theta)d\theta(r dt + d\phi)$ can always be removed by redefining the variables θ and ϕ .

that the Bardeen–Horowitz solution (4) can be extended to include one more arbitrary parameter, the latter to be identified with a NUT charge.

In order to solve (2) in full generality, we first note that it is homogeneous in $a(\theta)$ and its derivatives. This justifies the substitution

$$a(\theta) = e^{q(\theta)}, \quad (5)$$

which is consistent with the signature of the metric chosen and gives a simpler third order differential equation for $q(\theta)$. As the latter does not involve θ and $q(\theta)$ explicitly, the two consecutive substitutions

$$q'(\theta) = p(\theta), \quad p'(\theta) = s(p(\theta)) \quad (6)$$

reduce it to a first order equation for $s(p)$. The latter can be further simplified by introducing the new variable

$$y = p^2, \quad (7)$$

which yields

$$(4 + 5y + y^2) + s(2 + 5y - 2s + 6ys') = 0, \quad (8)$$

where $s'(y) = \frac{ds(y)}{dy}$. This is a variant of the Abel equation which can be explicitly solved in some exceptional cases only. Representing $s(y)$ in the form

$$s(y) = -(1 + y) + u(y) \quad (9)$$

where $-(1 + y)$ is a particular solution to (8), one converts (8) into the equation for $u(y)$

$$-2u^2 - 6y(1 + y)u' + 3u(2 + y + 2yu') = 0 \quad (10)$$

with the coefficients in front of $u(y)$ and $u'(y)$ being quadratic polynomials in y . The latter fact prompts one to search for the general solution to Eq. (10) in the parametric form

$$y = w(z), \quad u(y) = zw(z), \quad (11)$$

which ultimately yields

$$w(z) = \frac{-9 + 6z + 2c_1\sqrt{-3 + 4z}}{2z^2}, \quad (12)$$

where c_1 is a constant of integration.

When returning back to $a(\theta)$, it proves technically convenient to keep the variable z explicit until the very last step

$$a(z) = \frac{c_3 z}{-3 + 4z},$$

$$\theta(z) = c_2 - \arctan \frac{-9 + 2c_1\sqrt{-3 + 4z}}{3\sqrt{-9 + 4c_1\sqrt{-3 + 4z} + 3(-3 + 4z)}}, \quad (13)$$

where c_2 and c_3 are constants of integration. Solving the rightmost equation for z and removing c_2 by redefining θ , one finally gets

$$a(\theta) = L_1(1 + \cos^2 \theta) + L_2 \cos \theta, \tag{14}$$

where L_1 and L_2 are arbitrary parameters. The form of the function $b(\theta)$ follows from (3)

$$b(\theta) = \frac{(4L_1^2 - L_2^2) \sin^2 \theta}{a(\theta)}. \tag{15}$$

Note that the resulting metric (1) has the Lorentzian signature provided

$$4L_1^2 > L_2^2, \tag{16}$$

while for $L_1 = 0$ the solution is of the ultrahyperbolic signature (2, 2).

Remarkably enough, the metric which we constructed by providing the general solution to Eq. (2) precisely coincides with the near horizon limit of the extreme Kerr–NUT black hole [8]. L_1 can be linked to the rotation parameter, while L_2 represents a NUT charge. We thus conclude that in four dimensions the $SO(2, 1)$ -invariance allows one to unambiguously fix the NUT-extension of the near horizon Kerr geometry.

3 NUT-extension of $d = 5$ near horizon Myers–Perry geometry

In five dimensions the number of $SO(2, 1)$ -invariant terms to be included into a metric grows notably. So we choose a specific ansatz

$$ds^2 = a(\theta) \left(r^2 dt^2 - \frac{dr^2}{r^2} - d\theta^2 \right) - b(\theta)(rdt + d\phi_1)^2 - c(\theta)(rdt + d\phi_2)^2 + d(\theta)(d\phi_1 - d\phi_2)^2, \tag{17}$$

with $a(\theta), b(\theta), c(\theta), d(\theta)$ to be determined, whose structure is similar to the $d = 5$ near horizon Myers–Perry black hole [5]³

$$ds^2 = \alpha(\theta) \left(r^2 dt^2 - \frac{dr^2}{r^2} - d\theta^2 \right) - \frac{a_2(1 - \sin \theta)(2a_1 + \alpha(\theta))}{\alpha(\theta)}(rdt + d\phi_1)^2 - \frac{a_1(1 + \sin \theta)(2a_2 + \alpha(\theta))}{\alpha(\theta)}(rdt + d\phi_2)^2 + \frac{a_1 a_2 \cos^2 \theta}{\alpha(\theta)}(d\phi_1 - d\phi_2)^2, \tag{18}$$

$$\alpha(\theta) = a_1 + a_2 + (a_1 - a_2) \sin \theta,$$

³ As compared to the notation in [5], we redefined the latitudinal angular variable $2\theta \rightarrow \theta - \frac{\pi}{2}$ and omitted the overall factor $\frac{1}{2}(a_1 + a_2)$.

where a_1 and a_2 represent the rotation parameters. Throughout this work we consider the case of nonzero and unequal rotation parameters.

A careful analysis of components of the Ricci tensor constructed from the metric (17) shows that $R_{t\phi_1}, R_{t\phi_2}, R_{rr}, R_{\phi_1\phi_2}$ produce the coupled set of second order ordinary differential equations

$$a'' = \frac{(2a(b+c) - 4a^2 + a'^2)g - aa'g'}{2ag}, \tag{19}$$

$$b'' = \frac{-(2b(b+c) + a'b')g + ab'g' + 2ab(c'd' - b'c' + b'd')}{2ag}, \tag{20}$$

$$c'' = \frac{-(2c(b+c) + a'c')g + ac'g' + 2ac(c'd' - b'c' + b'd')}{2ag}, \tag{21}$$

$$d'' = \frac{-(2bc + a'd')g + ad'g' + 2ad(c'd' - b'c' + b'd')}{2ag}, \tag{22}$$

where we denoted⁴

$$g = bc - d(b+c), \tag{23}$$

while $R_{\theta\theta}$ yields the compatibility condition

$$c'd' - b'c' + b'd' = \frac{(4a^2 - a(b+c) + a'^2)g + 2aa'g'}{a^2}. \tag{24}$$

One more compatibility condition comes from the definition (23) and Eqs. (19)–(22)

$$g'' = \frac{-(2(b+c)g + a'g')g + ag'^2}{2ag}. \tag{25}$$

Other components of the Ricci tensor prove to vanish identically, provided Eqs. (19)–(22) and (24) hold.

The system of ordinary differential equations exposed above can be solved in full generality. Gathering technical details in the appendix, we display below a solution which, in our opinion, seems to be a natural candidate to describe the $d = 5$ near horizon Myers–Perry black hole with a NUT charge

$$a(\theta) = L_1 + L_2 \sin \theta + L_3 \sin^2 \theta, \tag{26}$$

$$d(\theta) = \frac{a(\theta)b(\theta)c(\theta) - N \cos^2 \theta}{a(\theta)(b(\theta) + c(\theta))}, \tag{27}$$

⁴ It is worth mentioning that g is proportional to the determinant of the metric $\det(g_{ij}) = 4a^3g$.

$$\begin{aligned}
 b(\theta) &= \frac{(L_1 - L_2)(2L_1 + L_2 - 2(L_1 - L_3(1 + 2L_1/L_2))) \sin \theta - L_2 \sin^2 \theta}{2a(\theta)}, \\
 c(\theta) &= \frac{2L_1(L_1 + L_3) - L_2^2 + 2L_2(L_1 - L_3) \sin \theta + (L_2^2 - 2L_3(L_1 + L_3)) \sin^2 \theta}{a(\theta)} - b(\theta), \\
 N &= \frac{(L_1 - L_2)^2(L_1 L_2(L_1 + L_2) - 2L_3 L_1^2 - L_3^2(2L_1 + L_2))^2}{2L_2^2(L_1 - L_3)(L_1^2 - L_2^2 + L_3(2L_1 + L_3))},
 \end{aligned} \tag{26}$$

where L_1 , L_2 , and L_3 are constants. A more detailed form of the function $d(\theta)$ is given in the appendix.

As follows from Eq. (18), L_1 and L_2 can be linked to the rotation parameters via

$$L_1 = a_1 + a_2, \quad L_2 = a_1 - a_2. \tag{27}$$

By analogy with the $d = 4$ case, it seems natural to interpret L_3 as a NUT charge. In particular, in the limit $L_3 \rightarrow 0$ the solution (26) reduces to (18). Note that the metric has the Lorentzian signature provided

$$(L_1 - L_3)(L_1^2 - L_2^2 + 2L_1 L_3 + L_3^2) > 0. \tag{28}$$

As is shown in the appendix, the functions b , c , and d in (26) can be deformed to include two more arbitrary parameters in such a way that the resulting metric (17) still provides a solution to the vacuum Einstein equations. A geometrical or physical interpretation of the extra parameters remains a challenge.

4 Discussion

To summarize, in this work we employed the invariants of the conformal group $SO(2, 1)$ so as to construct Ricci-flat metrics in four and five dimensions. Our consideration was primarily concerned with $d = 4$ and $d = 5$ because in these cases the vacuum Einstein equations reduced to a coupled set of ordinary differential equations which could be analyzed in full generality. In four dimensions the resulting metric reproduced the near horizon Kerr–NUT black hole [8]. To the best of our knowledge, the five-dimensional metric presented in Sect. 3 is new. It involves five arbitrary parameters. Setting two of them to vanish, one obtains a natural candidate to describe the $d = 5$ near horizon Myers–Perry black hole with a NUT charge.

The NUT-charged rotating black hole geometries in arbitrary dimension have been constructed in Refs. [9, 10]. They were built by appropriately equating rotation parameters and making use of a special coordinate system. In particular, according to the results in [9, 10], in five dimensions a NUT charge is bogus as it can be removed by redefining the variables. This implies that our metric in Sect. 3 cannot be obtained as the near horizon limit of that in [9, 10].

In odd dimensions the NUT charges enter the metric in [10] as additive constants. Note, however, that in the absence of rotation parameters NUT charges typically accompany terms involving the latitudinal angular variables and they are not just additive constants (see, e.g., the construction in Ref. [11]). In this regard the $d = 4$ and $d = 5$ metrics constructed above are universal and involve a NUT charge in a uniform way.

Turning to possible further developments, it is worth mentioning that in Ref. [12] yet even more general five-dimensional solution has been constructed which involves one extra parameter over and above the rotation parameters characterizing the Myers–Perry black hole. It would be interesting to study the near horizon limit of the metric in [12] and confront it with that in Sect. 3. In this regard the important thing to understand is how the coordinate systems used in [12] is related to that in this work. A generalization to $d > 5$, including the case of a nonvanishing cosmological constant, is an important open problem. In this case, in order to solve the vacuum Einstein equations, one has to deal with a coupled set of partial differential equations which are technically much more difficult. Finally, it would be interesting to construct integrable systems associated with NUT-charged near horizon black hole geometries in the spirit of [7].

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Appendix

In this appendix we discuss some technical issues involved in the construction of the solution (26) to the system of ordinary differential equations (19)–(22) and the compatibility conditions (24) and (25).

Multiplying Eqs. (19) and (25) by g and a , respectively, and taking the sum, one gets the simple differential equation

$$(ag)'' = \frac{(ag)'^2}{2ag} - 2ag,$$

whose general solution reads

$$ag = c_1 \cos^2(\theta + c_2),$$

where c_1 and c_2 are constants of integration. In the following we disregard c_2 as it can be eliminated by redefining θ . Taking into account the definition (23), one concludes that one of the functions a, b, c , or d can be algebraically expressed in terms of the others. For definiteness, we choose

$$d = \frac{abc - c_1 \cos^2 \theta}{a(b + c)}.$$

One more algebraic relation is obtained by substituting the solution for the product ag into Eq. (19), which yields

$$b + c = a'' + 2a - a' \left(\tan \theta + \frac{a'}{a} \right).$$

Taking into account this expression and computing the sum of Eqs. (20) and (21), one gets the fourth order ordinary differential equation to fix $a(\theta)$

$$a \left(3a' \cos 2\theta \sec^2 \theta \tan \theta + a''(7 - 3 \sec^2 \theta) + a^{(4)} \right) - a' \left(a'(-2 + 3 \sec^2 \theta) + 3a'' \tan \theta + a^{(3)} \right) = 0.$$

In order to simplify it, we introduce the new variable

$$y = \sin \theta,$$

represent a in the form

$$a(\theta) = e^{q(y(\theta))},$$

and implement two consecutive substitutions

$$q' = p, \quad p^3 + 3pp' + p'' = u,$$

where the prime denotes the derivative with respect to y . Then the fourth order equation for a reduces to the first order equation for u

$$(1 - y^2)u' - 6yu = 0,$$

which has the simple solution

$$u(y) = \frac{u_0}{(y^2 - 1)^3},$$

where u_0 is a constant of integration. The general solution to the equation $p^3 + 3pp' + p'' = u$ is obtained by making use of the substitution

$$p = \frac{w'}{w},$$

which raises the order of the equation by one

$$\frac{w^{(3)}}{w} = \frac{u_0}{(y^2 - 1)^3}.$$

Taking into account the relations $q' = p$ and $p = \frac{w'}{w}$, one concludes that $a(\theta)$ coincides with $w(y(\theta))$ up to a constant factor. The integration of the equation for w then gives

$$a(y) = (y^2 - 1) \left(a_1 \left(\frac{y - 1}{y + 1} \right)^{a_4} + a_2 \left(\frac{y + 1}{y - 1} \right)^{\frac{a_4 + \sqrt{4 - 3a_4^2}}{2}} + a_3 \left(\frac{y + 1}{y - 1} \right)^{\frac{a_4 - \sqrt{4 - 3a_4^2}}{2}} \right),$$

where $y = \sin \theta$ and a_1, a_2, a_3, a_4 are constants of integration.

Since in this work we are primarily concerned with the construction of a NUT-extension of the $d = 5$ near horizon Myers–Perry geometry and $a(y)$ in the preceding formula is a transcendental function, in the following we choose a_4 to take one of the three integer values $-1, 0$, or 1 , which yield⁵

$$a(\theta) = L_1 + L_2 \sin \theta + L_3 \sin^2 \theta,$$

where L_1, L_2 , and L_3 are arbitrary parameters. A comparison with (18) shows that the first two constants can be linked to the rotation parameters via $L_1 = a_1 + a_2$ and $L_2 = a_1 - a_2$, while L_3 can be interpreted as a NUT charge.

Given the explicit form of the function a , one can immediately compute $g, (b + c)$, and $c'd' - b'c' + b'd'$. At this stage, the linear ordinary differential equation (20) can be integrated to yield

$$b(\theta) = \frac{b_1(L_2 + 2L_3 \sin \theta) + b_2(2L_1 \sin \theta + L_2 \sin^2 \theta)}{a(\theta)},$$

where b_1, b_2 are constants of integration and the functions they accompany represent the two linearly independent solutions to Eq. (20). Because in this work we are concerned with the construction of a NUT-deformation of the $d = 5$ near horizon Myers–Perry geometry, we choose to fix b_1 and b_2 in such a way that the resulting metric reduces to (18) in the limit $L_3 \rightarrow 0$. This gives

$$b_1 = \frac{1}{2L_2} \left(2L_1^2 - L_2(L_1 + L_2) \right), \quad b_2 = \frac{1}{2}(L_2 - L_1),$$

⁵ It is likely that irrational values of a_4 lead to trivial solutions. For example, choosing $a_1 = a_3 = 0, a_4 = \frac{1}{4}(1 - \sqrt{13})$, one finds $a(\theta) = a_2 \cos \theta(1 + \sin \theta)$. It is straightforward to verify that this form of $a(\theta)$ implies $b(\theta) = c(\theta) = 0$, which lead to the divergent metric.

which ultimately yield $b(\theta)$ of the form

$$b(\theta) = \frac{(L_1 - L_2)(2L_1 + L_2 - 2(L_1 - L_3(1 + 2L_1/L_2)) \sin \theta - L_2 \sin^2 \theta)}{2a(\theta)}$$

Because Eq. (21) for c has exactly the same form as that for b , its solution has a similar structure,

$$c(\theta) = \frac{C_1(L_2 + 2L_3 \sin \theta) + C_2(2L_1 \sin \theta + L_2 \sin^2 \theta)}{a(\theta)},$$

where C_1 and C_2 are constants of integration. Taking into account the relation $b + c = a'' + 2a - a' \left(\tan \theta + \frac{a'}{a} \right)$, one can relate C_1 and C_2 to b_1, b_2, L_1, L_2 , and L_3 . The ultimate result reads

$$c(\theta) = \frac{L_1 L_2 - L_2^2 + 2L_1^2 + 4L_3 L_1 + 2(L_1^2 + L_1 L_2 + L_3(L_1 - L_2 - 2L_1^2/L_2)) \sin \theta}{2(L_1 + L_2 \sin \theta + L_3 \sin^2 \theta)} + \frac{(L_2(L_1 + L_2) - 4L_3(L_1 + L_3)) \sin^2 \theta}{2(L_1 + L_2 \sin \theta + L_3 \sin^2 \theta)}$$

It remains to fix the function d . The simplest way to solve Eq. (22) is to start with the ansatz

$$d(\theta) = \frac{d_1 + d_2 \sin \theta + d_3 \sin^2 \theta}{a(\theta)} \tag{29}$$

and then determine the constants d_1, d_2 , and d_3 from Eq. (22) and the compatibility conditions (23) and (24). This fixes d unambiguously. In order to facilitate the comparison with (18), below we represent the constants as power series in L_3 ,

$$d_1 = (L_1 - L_2) \left(L_1(L_1 - L_2)L_2^2(L_1 + L_2)^2 + L_3 L_2(L_1 + L_2)(4L_1^3 + L_1^2 L_2 + L_2^3) - L_3^2 L_1(2L_1 + L_2)(2L_1^2 - 3L_1 L_2 + 3L_2^2) - L_3^3(2L_1 + L_2)(2L_1^2 - L_1 L_2 + L_2^2) \right) / d_4,$$

$$d_2 = L_3(L_1 - L_2) \left(-4L_1 L_2^3(L_1 + L_2) + 2L_3 L_2(L_1 + L_2) \times (4L_1^2 + 2L_1 L_2 + L_2^2) - 8L_3^2 L_1^2 L_2 - 4L_3^3 L_2(2L_1 + L_2) \right) / d_4,$$

$$d_3 = -(L_1 - L_2) \left(L_1(L_1 - L_2)L_2^2(L_1 + L_2)^2 - L_3 L_2(L_1 + L_2)(4L_1^3 - 5L_1^2 L_2 - L_2^3) + L_3^2 L_1(4L_1^3 - 4L_1^2 L_2 - L_1 L_2^2 - L_2^3) + L_3^3(4L_1^3 - 3L_1 L_2^2 - 3L_2^3) \right) / d_4,$$

where d_4 reads

$$d_4 = 4L_2^2 \left(L_1(L_1^2 - L_2^2) + L_3(L_1^2 + L_2^2) - L_3^2 L_1 - L_3^3 \right).$$

In the process one also determines the constant c_1 which enters the expression for d exposed above,

$$c_1 = \frac{(L_1 - L_2)^2(L_1 L_2(L_1 + L_2) - 2L_3 L_1^2 - L_3^2(2L_1 + L_2))^2}{2L_2^2(L_1 - L_3)(L_1^2 - L_2^2 + L_3(2L_1 + L_3))}$$

In our previous discussion we have fixed the arbitrary constants b_1 and b_2 entering the function $b(\theta)$ so as to conform

to the $d = 5$ near horizon Meyers–Perry metric which shows up in the limit $L_3 \rightarrow 0$. Leaving them arbitrary, one can construct a two-parametric deformation of the solution (26)

$$\tilde{b}(\theta) = b(\theta) + \frac{P_1 L_2 + 2(P_2 L_1 + P_1 L_3) \sin \theta + P_2 L_2 \sin^2 \theta}{a(\theta)},$$

$$\tilde{c}(\theta) = c(\theta) - \frac{P_1 L_2 + 2(P_2 L_1 + P_1 L_3) \sin \theta + P_2 L_2 \sin^2 \theta}{a(\theta)},$$

$$\tilde{d}(\theta) = d(\theta) + \frac{\tilde{d}_1 + \tilde{d}_2 \sin \theta + \tilde{d}_3 \sin^2 \theta}{a(\theta)},$$

where P_1 and P_2 are the new parameters. The function $a(\theta)$ maintains its form (26), while the constants $\tilde{d}_1, \tilde{d}_2, \tilde{d}_3$ are expressed via L_1, L_2, L_3, P_1 , and P_2 as follows:

$$\tilde{d}_1 = (-2P_2 L_1(-L_1^3 L_2 + (L_1 - P_1)L_2^3 + 2L_1(L_1^2 + P_1 L_2 - L_1 L_2)L_3 + (2P_1 L_2 + (L_1 - L_2)(2L_1 + L_2))L_3^2) - P_1(P_1 L_1 L_2^3 - 2L_2(L_1^3 + 2L_1^2 L_2 + P_1 L_2^2 - L_1 L_2^2)L_3 + 2L_1(2L_1^2 - 2L_1 L_2 + L_2(P_1 + L_2))L_3^2 + 2(2L_1^2 + P_1 L_2 - L_1 L_2)L_3^3) - P_2^2 L_1 L_2(-L_2^2 + 2L_1(L_1 + L_3))) / \tilde{d}_4,$$

$$\tilde{d}_2 = L_2(P_1^2 L_2(L_2^2 - 4L_1 L_3) + 2P_1(L_1^2(L_1 - 2P_2)L_2 + (P_2 - L_1)L_2^3 + 2L_1^2(L_2 - L_1)L_3 + (2L_1^2 + L_1 L_2 + L_2(L_2 - 2P_2)) \times L_3^2 - 2L_1 L_3^3 - 2L_3^4) + P_2(4L_1^2(L_2 - 2L_3)$$

$$\begin{aligned} & \times L_3 + 2L_1^3(L_2 + 2L_3) + L_2^2(P_2L_2 + 2L_2^2) \\ & - 2L_1(L_2^3 + L_2(2P_2 - L_3)L_3 + 2L_3^3))/\tilde{d}_4, \\ \tilde{d}_3 = & (-P_2^2L_2(2L_1^3 - 2L_1L_2^2 + 2L_1^2L_3 + L_2^2L_3) \\ & - P_1L_3(-P_1L_2^3 - 2L_1^3(L_2 - 2L_3) \\ & + 2(P_1 - L_2)L_2L_3^2 + 4L_1^2L_3(L_3 - L_2) \\ & + 2L_1L_2(L_2^2 + (P_1 - L_3)L_3)) \\ & - 2P_2(-L_1^4(L_2 - 2L_3) + 2P_1L_1L_2L_3^2 \\ & + 2L_1^3L_3(L_3 - L_2) + L_1^2L_2(L_2^2 - 2L_2L_3 + \\ & + (2P_1 - L_3)L_3) + L_2^2L_3(-P_1L_2 + L_3^2)))/\tilde{d}_4, \\ \tilde{d}_4 = & 2L_2(L_1 - L_3)(L_1 - L_2 + L_3)(L_1 + L_2 + L_3). \end{aligned}$$

It is straightforward to verify that the modified functions do provide a solution to the vacuum Einstein equations which reduces to (26) in the limit $P_1 \rightarrow 0$, $P_2 \rightarrow 0$. A geometrical or physical interpretation of the extra parameters remains a challenge.

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