Eur. Phys. J. C (2016) 76:431
DOI 10.1140/epjc/s10052-016-4284-5

THE EUROPEAN PHYSICAL JOURNAL C

CrossMark

Regular Article - Theoretical Physics

On stability of exponential cosmological solutions with non-static volume factor in the Einstein–Gauss–Bonnet model

V. D. Ivashchuk[1,2,a]

[1] Center for Gravitation and Fundamental Metrology, VNIIMS, 46 Ozyornaya ul., Moscow 119361, Russia
[2] Institute of Gravitation and Cosmology, Peoples' Friendship University of Russia (RUDN University), 6 Miklukho-Maklaya ul., Moscow 117198, Russia

Received: 6 July 2016 / Accepted: 23 July 2016 / Published online: 1 August 2016

Abstract A $(n+1)$-dimensional gravitational model with Gauss–Bonnet term and a cosmological constant term is considered. When ansatz with diagonal cosmological metrics is adopted, the solutions with an exponential dependence of the scale factors, $a_i \sim \exp(v^i t)$, $i = 1, \ldots, n$, are analyzed for $n > 3$. We study the stability of the solutions with non-static volume factor, i.e. if $K(v) = \sum_{k=1}^{n} v^k \neq 0$. We prove that under a certain restriction R imposed solutions with $K(v) > 0$ are stable, while solutions with $K(v) < 0$ are unstable. Certain examples of stable solutions are presented. We show that the solutions with $v^1 = v^2 = v^3 = H > 0$ and zero variation of the effective gravitational constant are stable if the restriction R is obeyed.

1 Introduction

This paper is devoted to a D-dimensional gravitational model with the so-called Gauss–Bonnet term. It is governed by the action

$$S = \int_M d^D z \sqrt{|g|} \{\alpha_1 (R[g] - 2\Lambda) + \alpha_2 \mathcal{L}_2[g]\}, \tag{1.1}$$

where $g = g_{MN} dz^M \otimes dz^N$ is the metric defined on the manifold M, $\dim M = D$, $|g| = |\det(g_{MN})|$ and

$$\mathcal{L}_2 = R_{MNPQ} R^{MNPQ} - 4 R_{MN} R^{MN} + R^2 \tag{1.2}$$

is the quadratic "Gauss–Bonnet term" and Λ is the cosmological term. Here α_1 and α_2 are non-zero constants. The appearance of the Gauss–Bonnet term was motivated by string theory [1–3].

At present, the so-called Einstein–Gauss–Bonnet (EGB) gravitational model which is governed by the action (1.1) and its modifications are intensively used in cosmology; see [4–23] and references therein, e.g. for explanation of accelerating expansion of the Universe following from supernovae (type Ia) observational data [24–26].

Here we consider the cosmological solutions with diagonal metrics governed by n scale factors depending upon one variable, where $n > 3$; $D = n + 1$. We study the stability of solutions with an exponential dependence of the scale factors with respect to the synchronous time variable t,

$$a_i(t) \sim \exp(v^i t), \tag{1.3}$$

$i = 1, \ldots, n$. In our analysis we restrict ourselves to a class of perturbations which depend on t and do not touch the diagonal form of the metric.

For possible physical applications solutions describing an exponential isotropic expansion of 3-dimensional flat factor-space, i.e. with

$$v^1 = v^2 = v^3 = H > 0, \tag{1.4}$$

and small enough variation of the effective gravitational constant G are of interest. We recall that G (for $4d$ metric in Jordan frame; see Remark 4 in Sect. 4) is proportional to the inverse volume scale factor of the internal space; see [27–29] and the references therein. Due to experimental data, the variation of G is allowed at the level of 10^{-13} per year and less. The most stringent limitation on G-dot (coming from the set of ephemerides) was obtained in Ref. [30],

$$\dot{G}/G = (0.16 \pm 0.6) \cdot 10^{-13} \text{ year}^{-1}, \tag{1.5}$$

allowed at 95 % confidence (2σ).

Here we reduce the set of cosmological equations to the (mixed) set of algebraic and differential equations

$$f_0(h) = 0, \tag{1.6}$$

$$f_i(\dot{h}, h) = 0, \tag{1.7}$$

[a] e-mail: ivashchuk@mail.ru

where $h = h(t) = (h^i(t) = \dot{a}_i(t)/a_i(t))$ is the set of so-called "Hubble-like" parameters, corresponding to scale factors $a_i(t)$; $f_0(h)$ and $f_i(\dot{h}, h)$ are polynomials of the fourth order in h^i; $f_i(\dot{h}, h)$ are polynomials of the first order in $\dot{h}^i$. The fixed point solutions $h^i(t) = v^i$ $(i = 1, \ldots, n)$ correspond to exponential solutions of (1.3). They obey a set of $n + 1$ polynomial equations of the fourth order. We analyze the stability of the fixed point solutions by imposing the following restriction:

$$\text{(R)} \quad \det\left(\frac{\partial f_i}{\partial \dot{h}^j}(\mathbf{0}, v)\right) \neq 0, \tag{1.8}$$

which guarantees the local resolution of Eq. (1.7) in the vicinity of the point $(\mathbf{0}, v) \in \mathbb{R}^{2n}$: $\dot{h}^i = \varphi^i(h)$ with $\varphi^i(v) = 0$, $i = 1, \ldots, n$. Here $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$.

We also impose another restriction on v:

$$\sum_{k=1}^{n} v^k \neq 0, \tag{1.9}$$

which means that the solutions with constant volume scale factor are not considered here. We note that a solution with $\sum_{k=1}^{n} v^k = 0$ obeying (1.4) gives an enormously big value of the variation of G: $\dot{G}/G = 3H$, where H is the Hubble parameter; see Remark 5 in Sect. 4 below. This value of G-dot contradicts the observational restrictions, e.g. (1.5). We recall that the present value of H is $(6.929 \pm 0.157) \cdot 10^{-11}$ year^{-1} [31] (with 95 % confidence level).

The main result of the paper is the following one: fixed point solutions $h(t) = v$ to Eqs. (1.6) and (1.7), which obey restrictions (1.8) and (1.9), are stable if and only if $\sum_{k=1}^{n} v^k > 0$. This result is in agreement with the approach of Pavluchenko from Ref. [22]; see Remark 2 in Sect. 3 below.

The paper is organized as follows. In Sect. 2 the equations of motion for a D-dimensional EGB model are considered. For diagonal cosmological metrics the equations of motion are equivalent to a set of Lagrange equations corresponding to a certain "effective" Lagrangian. The Lagrange equations for a certain choice of the lapse function (corresponding to the synchronous time variable) are reduced to the set of equations (1.6) and (1.7). Section 3 is devoted to an analysis of the stability of the exponential solutions with constant Hubble-like parameters: here a set of equations for perturbations $\delta h^i(t)$ (obtained in linear approximation) is studied and a general solution to these equations is found. The main proposition on the stability of the exponential solutions (Proposition 2) is proved. In Sect. 4 some examples of stable cosmological solutions with exponential behavior of the scale factors are presented.

2 The model

2.1 The set-up

Here we consider the manifold

$$M = (t_-, t_+) \times M_1 \times \cdots \times M_n, \tag{2.1}$$

with the metric

$$g = -e^{2\gamma(t)} \mathrm{d}t \otimes \mathrm{d}t + \sum_{i=1}^{n} e^{2\beta^i(t)} \mathrm{d}y^i \otimes \mathrm{d}y^i, \tag{2.2}$$

where $i = 1, \ldots, n$; $M_1, \ldots, M_n$ are 1-dimensional manifolds (either $\mathbb{R}$ or S^1) and $n > 3$. The functions $\gamma(t)$ and $\beta^i(t)$, $i = 1, \ldots, n$, are smooth on (t_-, t_+).

For physical applications we put $M_1 = M_2 = M_3 = \mathbb{R}$, while $M_4, \ldots, M_n$ may be considered to be compact ones (i.e. coinciding with S^1).

The integrand in (1.1), when the metric (2.2) is substituted, reads as follows:

$$\sqrt{|g|}\{\alpha_1 R[g] + \alpha_2 \mathcal{L}_2[g]\} = L + \frac{\mathrm{d}f}{\mathrm{d}t}, \tag{2.3}$$

where

$$\begin{aligned} L = {} & \alpha_1 (e^{-\gamma+\gamma_0} G_{ij} \dot{\beta}^i \dot{\beta}^j - 2\Lambda e^{\gamma+\gamma_0}) \\ & - \frac{1}{3} \alpha_2 e^{-3\gamma+\gamma_0} G_{ijkl} \dot{\beta}^i \dot{\beta}^j \dot{\beta}^k \dot{\beta}^l, \end{aligned} \tag{2.4}$$

$\gamma_0 = \sum_{i=1}^{n} \beta^i$ and

$$G_{ij} = \delta_{ij} - 1, \tag{2.5}$$

$$G_{ijkl} = G_{ij} G_{ik} G_{il} G_{jk} G_{jl} G_{kl} \tag{2.6}$$

are, respectively, the components of two metrics on $\mathbb{R}^n$ [15,16]. The first one is the "minisupermetric" – 2-metric of pseudo-Euclidean signature and the second one is the Finslerian 4-metric [15,16]. Here we denote $\dot{A} = \mathrm{d}A/\mathrm{d}t$ etc. The function $f(t)$ in (2.3) is irrelevant for our consideration (see [15,16]).

In the derivation of (2.4) the following identities [15,16] were used:

$$G_{ij} v^i v^j = \sum_{i=1}^{n} (v^i)^2 - \left(\sum_{i=1}^{n} v^i\right)^2 = S_2 - S_1^2, \tag{2.7}$$

$$G_{ijkl} v^i v^j v^k v^l = S_1^4 - 6 S_1^2 S_2 + 3 S_2^2 + 8 S_1 S_3 - 6 S_4. \tag{2.8}$$

Here and in the following $S_k = S_k(v) = \sum_{i=1}^{n} (v^i)^k$.

The definitions (2.5) and (2.6) imply

$$G_{ij} v^i v^j = -2 \sum_{i<j} v^i v^j, \tag{2.9}$$

$$G_{ijkl} v^i v^j v^k v^l = 24 \sum_{i<j<k<l} v^i v^j v^k v^l. \tag{2.10}$$

The equations of motion corresponding to the action (1.1) have the following form:

$$\mathcal{E}_{MN} = \alpha_1 \mathcal{E}^{(1)}_{MN} + \alpha_2 \mathcal{E}^{(2)}_{MN} = 0, \tag{2.11}$$

where

$$\mathcal{E}^{(1)}_{MN} = R_{MN} - \frac{1}{2} R g_{MN} + \Lambda g_{MN}, \tag{2.12}$$

$$\mathcal{E}^{(2)}_{MN} = 2(R_{MPQS} R_N{}^{PQS} - 2 R_{MP} R_N{}^P - 2 R_{MPNQ} R^{PQ} + R R_{MN}) - \frac{1}{2} \mathcal{L}_2 g_{MN}. \tag{2.13}$$

It may be shown (along lines as in [16] for the case $\Lambda = 0$) that the field equations (2.11) for the metric (2.2) are equivalent to the Lagrange equations corresponding to the Lagrangian L from (2.4).

Thus, Eqs. (2.11) read as follows:

$$\alpha_1 (G_{ij} \dot\beta^i \dot\beta^j + 2\Lambda e^{2\gamma}) - \alpha_2 e^{-2\gamma} G_{ijkl} \dot\beta^i \dot\beta^j \dot\beta^k \dot\beta^l = 0, \tag{2.14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[2\alpha_1 G_{ij} e^{-\gamma+\gamma_0} \dot\beta^j - \frac{4}{3}\alpha_2 e^{-3\gamma+\gamma_0} G_{ijkl} \dot\beta^j \dot\beta^k \dot\beta^l] - L = 0, \tag{2.15}$$

$i = 1, \ldots, n$; and L is defined in (2.4).

Now we put $\gamma = 0$. By introducing "Hubble-like" variables $h^i = \dot\beta^i$, Eqs. (2.14) and (2.15) may be rewritten as follows:

$$E = E(h) \equiv G_{ij} h^i h^j + 2\Lambda - \alpha G_{ijkl} h^i h^j h^k h^l = 0, \tag{2.16}$$

$$U_i = U_i(\dot h, h) \equiv \frac{\mathrm{d}L_i}{\mathrm{d}t} + \left(\sum_{j=1}^n h^j \right) L_i - L_0 = 0, \tag{2.17}$$

where $\alpha = \alpha_1/\alpha_2$,

$$L_0 = G_{ij} h^i h^j - 2\Lambda - \frac{1}{3} \alpha G_{ijkl} h^i h^j h^k h^l, \tag{2.18}$$

and

$$L_i = L_i(h) = 2 G_{ij} h^j - \frac{4}{3} \alpha G_{ijkl} h^j h^k h^l, \tag{2.19}$$

$i = 1, \ldots, n$. Thus, we are led to the autonomous system of first-order differential equations on $h^1(t), \ldots, h^n(t)$ (see [15,16] for $\Lambda = 0$).

Due to (2.16) we have

$$L_0 = \frac{2}{3}(G_{ij} h^i h^j - 4\Lambda). \tag{2.20}$$

In the following we will use instead of (2.16), (2.17) an equivalent set of equations: (2.16) and

$$Y_i = Y_i(\dot h, h) \equiv \frac{\mathrm{d}L_i}{\mathrm{d}t} + \left(\sum_{j=1}^n h^j \right) L_i - \frac{2}{3}(G_{ij} h^i h^j - 4\Lambda) = 0. \tag{2.21}$$

We note that the following identity is valid:

$$U_i(\dot h, h) = Y_i(\dot h, h) - \frac{1}{3} E(h), \tag{2.22}$$

$i = 1, \ldots, n$.

Equations (2.16) and (2.21) are dependent, since

$$h^i Y_i = \frac{\mathrm{d}E}{\mathrm{d}t} + \frac{4}{3} \left(\sum_{j=1}^n h^j \right) E. \tag{2.23}$$

This identity may be proved by using two relations:

$$h^i \frac{\mathrm{d}L_i}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t}, \tag{2.24}$$

$$h^i L_i = L_0 + \frac{4}{3} E, \tag{2.25}$$

following from (2.16) and (2.19).

2.2 Useful relations

In the following we use the definitions

$$B = B(v) = G_{ijks} v^i v^j v^k v^s, \quad A_i = A_i(v) = G_{ijkl} v^j v^k v^l. \tag{2.26}$$

For the isotropic case

$$v = (v^i) = (H, \ldots, H) \tag{2.27}$$

we get

$$B = n(n-1)(n-2)(n-3) H^4, \quad A_i = (n-1)(n-2)(n-3) H^3, \tag{2.28}$$

$i = 1, \ldots, n$.

Here we deal with the ansatz which contains two Hubble parameters,

$$v = (v^i) = (H, \ldots, H, h, \ldots, h), \tag{2.29}$$

where H appears m times and h appears l times, $n = m + l$. In the following we adopt the following for the indices: $\mu, \nu, \ldots = 1, \ldots, m$; $\alpha, \beta, \cdots = m+1, \ldots, n$. Thus, $v^\mu = H$ and $v^\alpha = h$.

We obtain

$$B = m(m-1)(m-2)(m-3)H^4 + 4m(m-1)(m-2)lH^3h + 6m(m-1)l(l-1)H^2h^2 + 4ml(l-1)(l-2)Hh^3 + l(l-1)(l-2)(l-3)h^4 \quad (2.30)$$

and

$$A_H \equiv A_\mu = (m-1)(m-2)(m-3)H^3 + 3(m-1)(m-2)lH^2h + 3(m-1)l(l-1)Hh^2 + l(l-1)(l-2)h^3, \quad (2.31)$$

$$A_h \equiv A_\alpha = m(m-1)(m-2)H^3 + 3m(m-1)(l-1)H^2h + 3m(l-1)(l-2)Hh^2 + (l-1)(l-2)(l-3)h^3. \quad (2.32)$$

We also denote

$$S_{ij} = G_{ijks}v^k v^s, \quad (2.33)$$

We note that $S_{ij} = S_{ji}$ and $S_{ii} = 0$. For the isotropic case (2.27) we obtain

$$S_{ij} = (n-2)(n-3)H^2, \qquad i \neq j. \quad (2.34)$$

For the ansatz (2.29) we obtain

$$S_{HH} = (m-2)(m-3)H^2 + 2(m-2)lHh + l(l-1)h^2, \quad (2.35)$$

$$S_{Hh} = (m-1)(m-2)H^2 + 2(m-1)(l-1)Hh + (l-1)(l-2)h^2, \quad (2.36)$$

$$S_{hh} = m(m-1)H^2 + 2m(l-2)Hh + (l-2)(l-3)h^2. \quad (2.37)$$

Here we denote $S_{\mu\nu} = S_{HH}$ for $\mu \neq \nu$; $S_{\mu\alpha} = S_{\alpha\mu} = S_{Hh}$; $S_{\alpha\beta} = S_{hh}$ for $\alpha \neq \beta$.

2.3 Polynomial equations for solutions with constant h^i

Let us consider the following solutions to Eqs. (2.16) and (2.21):

$$h^i(t) = v^i, \quad (2.38)$$

with constant v^i, which corresponds to the solutions

$$\beta^i = v^i t + \beta^i_0, \quad (2.39)$$

where β^i_0 are constants, $i = 1, \ldots, n$.

In this case we obtain the metric (2.2) with the exponential dependence of the scale factors

$$g = -\mathrm{d}t \otimes \mathrm{d}t + \sum_{i=1}^{n} B_i^2 e^{2v^i t} \mathrm{d}y^i \otimes \mathrm{d}y^i, \quad (2.40)$$

where the $B_i > 0$ are arbitrary constants.

For the fixed point $v = (v^i)$ we have the set of polynomial equations

$$E = E(v) = G_{ij}v^i v^j + 2\Lambda - \alpha G_{ijkl}v^i v^j v^k v^l = 0, \quad (2.41)$$

$$Y_i = Y_i(\mathbf{0}, v) = \left(\sum_{j=1}^{n} v^j\right) L_i(v) - \frac{2}{3}G_{kj}v^k v^j + \frac{8}{3}\Lambda = 0, \quad (2.42)$$

where L_i is defined in (2.19), $i = 1, \ldots, n$. For $n > 3$ this is the set of fourth-order polynomial equations.

Here and in the following we use Eqs. (2.7), (2.8), and the following formulas:

$$v_i = G_{ij}v^j = v^i - S_1, \quad (2.43)$$

$$A_i = G_{ijkl}v^j v^k v^l = S_1^3 + 2S_3 - 3S_1S_2 + 3(S_2 - S_1^2)v^i + 6S_1(v^i)^2 - 6(v^i)^3, \quad (2.44)$$

$i = 1, \ldots, n$ ($S_k = \sum_{i=1}^{n}(v^i)^k$).

Proposition 1 *For any solution $v = (v^1, \ldots, v^n)$ to the polynomial equations (2.41), (2.42) with $n > 3$ there are no more than three different numbers among $v^1, \ldots, v^n$, if $\sum_{i=1}^{n} v^i \neq 0$.*

Proof Let us suppose that there exists a non-trivial solution $v = (v^1, \ldots, v^n)$ with more than three different numbers among $v^1, \ldots, v^n$. Due to (2.44), (2.42), and $\sum_{i=1}^{n} v^i \neq 0$ we get $C_0 + C_1 v^i + C_2(v^i)^2 + C_3(v^i)^3 = 0$, $i = 1, \ldots, n$, with some real numbers C_0, C_1, C_2, and $C_3 \neq 0$. Let us consider the cubic equation $C_0 + C_1 x + C_2 x^2 + C_3 x^3 = 0$. Any number v^i obeys this equation and hence at most three numbers among the v^i may be different. Thus, we are led to a contradiction. The proposition is proved. The case $\Lambda = 0$ was considered earlier in [15,16]. □

Remark 1 In the pure Einstein case ($\alpha = 0$) with $\Lambda > 0$ we get two exponential solutions with $v^1 = \cdots = v^n = H$ and $n(n-1)H^2 = 2\Lambda > 0$; a solution with $H > 0$ is an attractor for cosmological solutions with diagonal metrics, as $t \to +\infty$, see [32,33] (for $\varphi = 0$). Thus in this case ($\alpha = 0$) we have an isotropization for $t \to +\infty$, while for $t \to +0$ we have a Kasner-like behavior of scale factors near the singularity: $a_i(t) \sim t^{p_i}$ with Kasner parameters $p_1, \ldots, p_n$ obeying $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} p_i^2 = 1$. In the case of the EGB model with Λ-term we have for certain Λ and α isotropic exponential solutions with $v^1 = \cdots = v^n = H$ (see Sect. 4 below), but we also may have partially anisotropic (PA) solutions, which obey $\sum_{i=1}^{n} v^i \neq 0$, with $v = (H, \dots, H, h, \dots, h)$ or $v = (H, \dots, H, h, \dots, h, z, \dots, z)$, and also solutions with $\sum_{i=1}^{n} v^i = 0$ may occur. For $\sum_{i=1}^{n} v^i = 0$ (and certain Λ and α) one may obtain examples of totally anisotropic exponential solutions with non-coinciding parameters among $v^1, \dots, v^n$. Some of the exponential PA solutions are stable (see below) and they are attractors of certain subclasses of general solutions. The appearance of three (or less) independent scale factors in the model under consideration is a feature of exponential (e.g. attractor) solutions, when the restriction $\sum_{i=1}^{n} v^i \neq 0$ is imposed. We also note that the metric (2.40) may be analyzed on symmetries (apparent or hidden) by using the results of Ref. [34], i.e. Killing vectors, isometry group, coset structure G/H etc., may be presented. Proposition 2 may also be generalized to the Lovelock case [35], which may be a subject of a separate publication.

Now let us consider the ansatz (2.29) with two Hubble parameters H and h with two restrictions imposed,

$$mH + lh \neq 0, \qquad H \neq h. \tag{2.45}$$

In this case the set of $n + 1$ equations (2.16), (2.17) is equivalent to the set of three equations

$$E = 0, \qquad Y_H = 0, \qquad Y_h = 0, \tag{2.46}$$

where $Y_H = Y_\mu$, $Y_h = Y_\alpha$ ($\mu = 1, \dots, m$, $\alpha = m + 1, \dots, n$).

Due to (2.44) we have

$$\begin{aligned} A_H - A_h = (H - h)[3(S_2 - S_1^2) + 6S_1(H + h) \\ - 6(H^2 + Hh + h^2)], \end{aligned} \tag{2.47}$$

and hence, by using (2.19), (2.43), we obtain

$$Y_H - Y_h = (H - h)(mH + lh)[2 + 4\alpha Q(H, h)], \tag{2.48}$$

where

$$\begin{aligned} Q(H, h) = (m - 1)(m - 2)H^2 \\ +2(m - 1)(l - 1)Hh + (l - 1)(l - 2)h^2. \end{aligned} \tag{2.49}$$

For $m > 1$ and $l > 1$ the quadratic form has the signature $(-, +)$. Due to $mH + lh \neq 0$ the set of equations (2.46) is equivalent to another set of equations

$$E = 0, \qquad Y_H - Y_h = 0, \qquad mHY_H + lhY_h = 0, \tag{2.50}$$

According to (2.23) $E = 0$ implies $h^i Y_i = mHY_H + lhY_h = 0$ and hence the third equation in (2.50) may be omitted. Using the restrictions (2.45), Eqs. (2.30) and (2.48), we reduce the set of equations (2.50) to the following set of equations:

$$\begin{aligned} E = mH^2 + lh^2 - (mH + lh)^2 \\ + 2\Lambda - \alpha[m(m - 1)(m - 2)(m - 3)H^4 \\ + 4m(m - 1)(m - 2)lH^3h + 6m(m - 1)l(l - 1)H^2h^2 \\ + 4ml(l - 1)(l - 2)Hh^3 \\ + l(l - 1)(l - 2)(l - 3)h^4] = 0, \end{aligned} \tag{2.51}$$

$$1 + 2\alpha Q(H, h) = 0, \tag{2.52}$$

where $Q(H, h)$ is defined in (2.49).[1] Thus, for the anisotropic solutions with two different Hubble parameters H and h and non-static volume factor (see (2.29) and (2.45)) the set $(n+1)$ polynomial equations of fourth order (2.41) and (2.42) is equivalent to the set of the two equations (2.51) and (2.52) of fourth and second order, respectively.

3 Stability of fixed point solutions $h^i(t) = v^i$

Here we study the stability of static solutions $h^i(t) = v^i$ to Eqs. (2.16) and (2.17) in a linear approximation in the perturbations. We put

$$h^i(t) = v^i + \delta h^i(t), \tag{3.1}$$

$i = 1, \dots, n$. By substitution of (3.1) into Eqs. (2.16) and (2.17) we obtain in linear approximation the following relations for the perturbations δh^i:

$$C_i(v)\delta h^i = 0, \tag{3.2}$$

$$L_{ij}(v)\delta \dot{h}^j = B_{ij}(v)\delta h^j, \tag{3.3}$$

where

$$C_i = C_i(v) = 2v_i - 4\alpha G_{ijks} v^j v^k v^s, \tag{3.4}$$

$$L_{ij} = L_{ij}(v) = 2G_{ij} - 4\alpha G_{ijks} v^k v^s, \tag{3.5}$$

$$B_{ij} = B_{ij}(v) = -\left(\sum_{k=1}^{n} v^k\right) L_{ij}(v) - L_i(v) + \frac{4}{3} v_j. \tag{3.6}$$

We recall that $v_i = G_{ij} v^j$, $L_i(v) = 2v_i - \frac{4}{3}\alpha G_{ijks} v^j v^k v^s$ and $i, j, k, s = 1, \dots, n$.

We put the following restriction on the matrix $L = (L_{ij}(v))$

$$\text{(R)} \quad \det(L_{ij}(v)) \neq 0, \tag{3.7}$$

i.e. the matrix L should be invertible.

Here we restrict ourselves to exponential solutions (2.40) with non-static volume factor, proportional to $\exp(\sum_{i=1}^{n} v^i t)$,

[1] For the general reduction scheme see [20].

i.e. we put

$$K = K(v) = \sum_{i=1}^{n} v^i \neq 0. \tag{3.8}$$

Then we get from Eq. (2.42)

$$L_i(v) = L_1 = \frac{2}{3}\left(\sum_{k=1}^{n} v^k\right)^{-1} (G_{ij}v^i v^j - 4\Lambda). \tag{3.9}$$

Due to definition (2.19) we have

$$\alpha A_i = \alpha G_{ijks} v^j v^k v^s = \frac{3}{4}(2v_i - L_1), \tag{3.10}$$

and hence

$$C_i(v) = 2v_i - 4\alpha A_i = -4v_i + 3L_1. \tag{3.11}$$

We rewrite Eq. (3.6) as

$$B_{ij} = -\left(\sum_{k=1}^{n} v^k\right) L_{ij}(v) + \hat{B}_{ij}, \quad \hat{B}_{ij} = -L_i(v) + \frac{4}{3}v_j. \tag{3.12}$$

Due to $L_i(v) = L_1$ and (3.2) we get

$$\hat{B}_{ij}\delta h^j = -L_1 \sum_{j=1}^{n} \delta h^j + \frac{4}{3} v_j \delta h^j = -\frac{1}{3} C_j(v)\delta h^j = 0. \tag{3.13}$$

Hence Eq. (3.3) reads

$$L_{ij}(v)\delta\dot{h}^j = -\left(\sum_{k=1}^{n} v^k\right) L_{ij}\delta h^j, \tag{3.14}$$

or, equivalently,

$$\delta\dot{h}^i = -\left(\sum_{k=1}^{n} v^k\right)\delta h^i, \tag{3.15}$$

$i = 1, \ldots, n$. Here we used the restriction (3.7).

Thus, the set of linear equations on perturbations (3.2), (3.3) is equivalent to the set of linear equations (3.2), (3.15), which has the following solution:

$$\delta h^i = A^i \exp(-K(v)t), \tag{3.16}$$

$$\sum_{i=1}^{n} C_i(v)A^i = 0, \tag{3.17}$$

$i = 1, \ldots, n$. We recall that $K(v) = \sum_{k=1}^{n} v^k$.

Due to (3.16) the following proposition is valid.

Proposition 2 *The fixed point solution $(h^i(t)) = (v^i)$ ($i = 1, \ldots, n$; $n > 3$) to Eqs.* (2.16), (2.17) *obeying the restrictions* (3.7), (3.8) *is stable under perturbations (3.1) (as $t \to +\infty$) if $K(v) = \sum_{k=1}^{n} v^k > 0$ and it is unstable (as $t \to +\infty$) if $K(v) = \sum_{k=1}^{n} v^k < 0$.*

It follows from (2.34) that in the isotropic case the matrix (3.5) reads

$$L_{ij} = \varphi(H)G_{ij}, \quad \varphi(H) = 2 + 4\alpha(n-2)(n-3)H^2. \tag{3.18}$$

Since the matrix $(G_{ij}) = (\delta_{ij} - 1)$ is invertible (or non-degenerate) for $n > 1$ (its inverse is $(G^{ij}) = (\delta^{ij} - \frac{1}{n-1})$), the matrix (L_{ij}) is invertible if and only if $\varphi(H) \neq 0$.

Now let us consider the matrix (3.5) for the anisotropic case (2.29) with the two Hubble parameters obeying (2.45).

For the ansatz (2.29) we obtain

$$L_{\mu\nu} = G_{\mu\nu}(2 + 4\alpha S_{HH}), \tag{3.19}$$

$$L_{\mu\alpha} = L_{\alpha\mu} = -2 - 4\alpha S_{Hh}, \tag{3.20}$$

$$L_{\alpha\beta} = G_{\alpha\beta}(2 + 4\alpha S_{hh}). \tag{3.21}$$

Here S_{HH}, S_{Hh} and S_{hh} are defined in (2.35), (2.36), and (2.37), respectively. But here we have a remarkable coincidence (see (2.49)):

$$Q(H, h) = S_{Hh}, \tag{3.22}$$

which implies $L_{\mu\alpha} = L_{\alpha\mu} = 0$ due to Eq. (2.52). Thus under the assumed restrictions (2.45) the matrix (L_{ij}) has a block-diagonal form

$$(L_{ij}) = \mathrm{diag}(L_{\mu\nu}, L_{\alpha\beta}). \tag{3.23}$$

This matrix is invertible if and only if $m > 1$, $l > 1$, and

$$S_{HH} \neq -\frac{1}{2\alpha}, \quad S_{hh} \neq -\frac{1}{2\alpha}. \tag{3.24}$$

We recall that the $m \times m$ matrix $(G_{\mu\nu})$ and $l \times l$ matrix $(G_{\alpha\beta})$ are invertible only for $m > 1$ and $l > 1$, respectively.

Remark 2 Recently, in Ref. [22] a criterion for the stability of fixed point solutions in the model under consideration (and its extension to the Lovelock case) was used. In our notations (see Sect. 1) it reads

$$\frac{\partial \dot{h}^i}{\partial h^i}(v) = \frac{\partial \varphi^i}{\partial h^i}(v) < 0, \tag{3.25}$$

$i = 1, \ldots, n$. It can readily be verified that for generic functions f_0, f_i in Eqs. (1.6), (1.7) the criterion (3.25) is not a necessary and/or a sufficient condition for the stability of the fixed point solutions. Fortunately, for a special choice of functions, e.g. for $f_0(h) = E(h)$, $f_i(\dot{h}, h) = Y_i(\dot{h}, h) + \frac{1}{3}E(h) = U_i(\dot{h}, h)$ (see (2.22) and (3.13)), it gives a correct result since in this case

$$\frac{\partial \dot{h}^i}{\partial h^i}(v) = -\sum_{k=1}^{n} v^k, \tag{3.26}$$

$i = 1, \ldots, n$. Equation (3.26) is also valid for $f_i(\dot{h}, h) = \lambda U_i(\dot{h}, h)$ with $\lambda \neq 0$, e.g. for the choice $\lambda = -1$ used in [22]. We also note that in our notations $2\Lambda = \Lambda_P$, where Λ_P is the Λ-term from Ref. [22].

4 Examples

Here we consider several examples of exponential solutions and analyze their stability.

4.1 Isotropic solution

Let us consider the isotropic solution $v = (v^i) = (H, \dots, H)$ to Eqs. (2.41), (2.42) for $n > 3$. Due to $G_{ij} v^i v^j = n(1-n)H^2$ and (2.28), Eq. (2.41) reads as follows:

$$2F(H^2) = n(n-1)H^2 + \alpha n(n-1)(n-2)(n-3)H^4 = 2\Lambda. \tag{4.1}$$

Equation (2.42) is also equivalent to (4.1) due to the relation

$$L_i = -2(n-1)H + \frac{4}{3}\alpha(n-1)(n-2)(n-3)H^3, \tag{4.2}$$

$i = 1, \dots, n$, which follows from (2.19), (2.28), and (2.43).

Let $\Lambda = 0$. The trivial solution $H = 0$ is valid for any α. This is the unique solution for $\alpha > 0$. For $\alpha < 0$ we have two non-trivial solutions [15,16] with

$$H^2 = \frac{1}{|\alpha|(n-2)(n-3)}. \tag{4.3}$$

This solution was generalized in [19] to the case $\Lambda \neq 0$.

Let us consider the case of generic Λ in detail. First, we put $\alpha > 0$. Then a solution to Eq. (4.1) does exist if and only if $\Lambda \geq 0$. For $\Lambda = 0$ we get $H = 0$, while for $\Lambda > 0$ we have two non-zero solutions for H with $H^2 > 0$:

$$H^2 = \frac{-n(n-1) + \sqrt{\Delta}}{2\alpha n(n-1)(n-2)(n-3)}, \tag{4.4}$$

where

$$\Delta = n^2(n-1)^2 + 8\Lambda\alpha n(n-1)(n-2)(n-3). \tag{4.5}$$

Now we put $\alpha < 0$. A solution to Eq. (4.1) exists only if $\Lambda \leq \Lambda_{cr}$, where

$$\Lambda_{cr} = -\frac{n(n-1)}{8\alpha(n-2)(n-3)} \tag{4.6}$$

is the maximum value of the function $F(H^2)$ from (4.1). For $0 < \Lambda < \Lambda_{cr}$ (and $\alpha < 0$) we have two solutions for H^2 (or four solutions for H), which are given by the relation

$$H^2 = \frac{-n(n-1) \pm \sqrt{\Delta}}{2\alpha n(n-1)(n-2)(n-3)}. \tag{4.7}$$

For $\Lambda = \Lambda_{cr}$ and $\alpha < 0$ we get one solution for H^2 (or two solutions for H):

$$H^2 = H_{cr}^2 = -\frac{1}{2\alpha(n-2)(n-3)}. \tag{4.8}$$

The case $\Lambda = 0$ (and $\alpha < 0$) was mentioned above (two solutions for H^2, or three for H). For $\Lambda < 0$ (and $\alpha < 0$) we obtain one solution for H^2 (or two solutions for H):

$$H^2 = \frac{-n(n-1) - \sqrt{\Delta}}{2\alpha n(n-1)(n-2)(n-3)}. \tag{4.9}$$

Due to (3.18) the matrix (L_{ij}) is invertible for all solutions but $H = H_{cr}$ from (4.8) for $\alpha < 0$, since only in this case $\varphi(H) = 0$. The relation $H = H_{cr}$ takes place only for $\Lambda = \Lambda_{cr}$ and $\alpha < 0$ and hence this case will be excluded from our analysis. Since $K(v) = nH$, the trivial solution $H = 0$ for $\Lambda = 0$ should also be excluded from our consideration. It follows from Proposition 2 that all isotropic solutions $v = (v^i) = (H, \dots, H)$ obeying $H > 0$ and $H \neq H_{cr}$ for $\alpha < 0$ are stable while all isotropic solutions obeying $H < 0$ and $H \neq H_{cr}$ for $\alpha < 0$ are unstable.

Using (2.28), (2.43), and (3.4) we obtain $C_i(v) = -(n-1)H\varphi(H) \neq 0$, $i = 1, \dots, n$, for $H \neq 0$ and $H \neq H_{cr}$ for $\alpha < 0$. Under these restrictions on H, the solution for perturbations (3.16), (3.17) reads as follows:

$$\delta h^i = A^i \exp(-nHt), \tag{4.10}$$

$$\sum_{i=1}^{n} A^i = 0, \tag{4.11}$$

$i = 1, \dots, n$. Equation (4.10) was obtained earlier in [22].

4.2 Anisotropic solutions with two Hubble parameters

In this subsection we consider several examples of anisotropic solutions to Eqs. (2.41), (2.42) of the form $v = (H, \dots, H, h, \dots, h)$, where H is the Hubble-like parameter corresponding to the m-dimensional isotropic subspace with $m \geq 3$ and h is the Hubble-like parameter corresponding to the l-dimensional isotropic subspace with $l > 1$. Here we put $H > 0$.

4.2.1 *Solution for $m = 3$, $l = 2$, and $\Lambda = 0$.*

Let us consider the case $m = 3$, $l = 2$, $\Lambda = 0$. We have the following solution to the set of polynomial equations (2.51), (2.52) with $H > 0$:

$$H = \frac{1}{6}(7 + 4 \cdot 10^{1/3} + 10^{2/3})^{1/2}\alpha^{-1/2} \approx 0.750173\alpha^{-1/2}, \tag{4.12}$$

$$h = -\frac{1}{6}(7 - 0.5 \cdot 10^{1/3} + 10^{2/3})^{1/2}\alpha^{-1/2} \approx -0.541715\alpha^{-1/2}. \tag{4.13}$$

In the approximate form this solution was found earlier in [17], in analytic form (different from (4.12), (4.13)) it was obtained in [19].

Using (2.35) and (2.37) we get

$$S_{HH} = 2h(2H + h) \approx -1.038610\alpha^{-1},$$
$$S_{hh} = 6H^2 \approx 3.376557\alpha^{-1}. \tag{4.14}$$

Equations (3.24) are valid and hence the first restriction (3.7) is satisfied. The second restriction (3.8) is also satisfied since $K(v) = 3H + 2h > 0$. Thus, due to Proposition 2, the solution is stable, in agreement with [22].

4.2.2 *Solution for $m = l = 3$ and $\Lambda = 0$*

Now we consider solutions with $m = 3$, $l = 3$, and $\Lambda = 0$. There are two solutions to Eqs. (2.51), (2.52) with $H > 0$:

$$H_1 = \frac{1}{4}(\sqrt{5} - 1)\alpha^{-1/2}, \quad h_1 = \frac{1}{4}(-\sqrt{5} - 1)\alpha^{-1/2} \tag{4.15}$$

and

$$H_2 = \frac{1}{4}(\sqrt{5} + 1)\alpha^{-1/2}, \quad h_2 = \frac{1}{4}(-\sqrt{5} + 1)\alpha^{-1/2}. \tag{4.16}$$

For the first solution we get

$$S_{HH} = \frac{3}{4}(\sqrt{5} + 1)\alpha^{-1}, \quad S_{hh} = \frac{3}{4}(-\sqrt{5} + 1)\alpha^{-1}, \tag{4.17}$$

while for the second one we obtain

$$S_{HH} = \frac{3}{4}(-\sqrt{5} + 1)\alpha^{-1}, \quad S_{hh} = \frac{3}{4}(\sqrt{5} + 1)\alpha^{-1}. \tag{4.18}$$

In both cases Eqs. (3.24) are satisfied and hence the first restriction (3.7) is valid. The second restriction (3.8) is also valid for any of these solutions since $K(v_1) = 3H_1 + 3h_1 = -\frac{3}{2}\alpha^{-1/2} < 0$ and $K(v_2) = 3H_2 + 3h_2 = \frac{3}{2}\alpha^{-1/2} > 0$. According to Proposition 2 the first solution (4.15) is unstable, while the second one (4.16) is stable.

4.2.3 *Solution for $m = 11$, $l = 16$ and $\Lambda = 0$*

For $\Lambda = 0$ the solution (2.40) with $v = (v^i)$ from (2.29), $m = 11$, $l = 16$, and

$$H = \frac{1}{\sqrt{15\alpha}}, \quad h = -\frac{1}{2\sqrt{15\alpha}} \tag{4.19}$$

was found in [21]. This solution describes the zero variation of the effective cosmological constant G.

The calculations give us

$$S_{HH} = -\frac{4}{5}\alpha^{-1}, \quad S_{hh} = \frac{1}{10}\alpha^{-1}. \tag{4.20}$$

Due to (3.24) the symmetric matrix (L_{ij}), which has a block-diagonal form, is invertible, i.e. the condition (3.7) is satisfied.

Using (3.9) and (3.11) we find $(C_i) = (C_\mu = 12H, C_\alpha = 18H)$. From (3.16) we get the following solution for perturbations:

$$\delta h^i = A^i \exp(-3Ht), \tag{4.21}$$
$$2\sum_{\mu=1}^{11} A^\mu + 3\sum_{\alpha=12}^{27} A^\alpha = 0, \tag{4.22}$$

where $H = \frac{1}{\sqrt{15\alpha}}$, $i = 1, \ldots, 27$. Thus, the solution (4.19) is stable, as $t \to +\infty$.

4.2.4 *Solution for $m = 15$, $l = 6$, and $\Lambda = 0$*

Now we consider another exponential solution (2.40) from [21] with $v = (v^i)$ from (2.29), $m = 15$, $l = 6$, $\Lambda = 0$, and

$$H = \frac{1}{6}\alpha^{-1/2}, \qquad h = -\frac{1}{3}\alpha^{-1/2}. \tag{4.23}$$

We get

$$S_{HH} = -\alpha^{-1}, \qquad S_{hh} = \frac{1}{2}\alpha^{-1}. \tag{4.24}$$

According to (3.24) the symmetric block-diagonal matrix (L_{ij}) is non-degenerate.

By using (3.9) and (3.11) we get $(C_i) = (C_\mu = \frac{14}{3}\alpha^{-1/2}, C_\alpha = \frac{20}{3}\alpha^{-1/2})$. Due to (3.16) the solution for perturbations reads

$$\delta h^i = A^i \exp(-3Ht) = A^i \exp\left(-\frac{1}{2}\alpha^{-1/2}t\right), \tag{4.25}$$
$$7\sum_{\mu=1}^{15} A^\mu + 10\sum_{\alpha=16}^{21} A^\alpha = 0, \tag{4.26}$$

$i = 1, \ldots, 21$. Hence, the solution (4.23) is stable as $t \to +\infty$.

Remark 3 The stability of this solution as well as the previous one was also proved in Ref. [23] by using rather tedious calculations based on Eqs. (3.3) and (3.6) without using the identity (3.13).

4.2.5 *Solutions with $m \geq 3$, $l > 1$ and certain $\Lambda > 0$*

Here we consider the following solution to Eqs. (2.51), (2.52) for $m > 2$, $l > 1$ and $\alpha < 0$:

$$H^2 = -\frac{1}{2\alpha(m-1)(m-2)}, \quad h = 0, \tag{4.27}$$

which is valid for

$$\Lambda = -\frac{m(m+1)}{8\alpha(m-1)(m-2)} > 0. \tag{4.28}$$

We get from (2.35) and (2.37)

$$S_{HH} = (m-2)(m-3)H^2 = -\frac{m-3}{2\alpha(m-1)} \neq -\frac{1}{2\alpha} \tag{4.29}$$

and

$$S_{hh} = m(m-1)H^2 = -\frac{m}{2\alpha(m-2)} \neq -\frac{1}{2\alpha}, \tag{4.30}$$

which implies the fulfilment of the restriction (3.7) (here $m > 2$ and $l > 1$). Since $K(v) = mH$ we see from Proposition 2 that the cosmological solution (2.40) with H, h from (4.27) is stable for $H > 0$ and unstable for $H < 0$.

4.3 A subclass of solutions with zero variation of G

The 4d effective gravitational constant is proportional to the inverse volume scale factor of the internal space (see [27–29]), i.e.

$$G \sim \prod_{i=4}^{n} [a_i(t)]^{-1}, \tag{4.31}$$

where $a_i(t) = \exp(\beta^i(t))$.

Remark 4 Here $G = G^J_{\text{eff}}(t)$ is the 4-dimensional effective gravitational constant which appears in (the multi-dimensional analog of) the so-called Brans–Dicke–Jordan (or simply Jordan) frame [36]. In this case the physical 4-dimensional metric $g^{(4)}$ is defined as a 4-dimensional section of the multi-dimensional metric g, i.e. $g^{(4)} = g^{(4,J)}$, where $g = g^{(4,J)} + \sum_{i=4}^{n} a_i^2(t) dy^i \otimes dy^i$. When the Einstein–Pauli (or simply Einstein) frame is used, we put $g^{(4)} = g^{(4,E)} = (\prod_{i=4}^{n} a_i(t)) g^{(4,J)}$ [36,37] and hence we get the effective gravitational constant to be an exact constant: $G^E_{\text{eff}} = G^J_{\text{eff}}(t) \prod_{i=4}^{n} a_i(t) = \text{const}$ [36].

For the solutions (2.40) we obtain the following relations:

$$G(t) = G(0)\exp(-K_{\text{int}} t), \quad K_{\text{int}}(v) = \sum_{i=4}^{n} v^i, \tag{4.32}$$

which imply

$$\frac{\dot{G}}{G} = -K_{\text{int}}(v). \tag{4.33}$$

Now, let us consider a subclass of cosmological solutions (2.40) which obey restriction (3.7) and describe an exponential isotropic expansion of a 3-dimensional flat factor-space with $v^1 = v^2 = v^3 = H > 0$ with zero variation of G. Then we get from (4.33) $K_{\text{int}}(v) = 0$ and hence $K(v) = \sum_{i=1}^{n} v^i = 3H + K_{\text{int}}(v) = 3H > 0$. According to Proposition 2 any solution from this subclass is stable. Three solutions from the previous subsection: (4.19), (4.23), and (4.27) with $m = 3$ (and $l > 1$) belong to this subclass.

Remark 5 It should be noted that for $K(v) = 0$ and $v^1 = v^2 = v^3 = H > 0$ we obtain $K_{\text{int}}(v) = -3H$ and hence $\frac{\dot{G}}{G} = 3H > 0$.

5 Conclusions

We have considered the $(n+1)$-dimensional Einstein–Gauss–Bonnet (EGB) model with the Λ-term. By using the ansatz with diagonal cosmological metrics, we have studied the stability of solutions with exponential dependence of the scale factors $a_i \sim \exp(v^i t)$, $i = 1, \ldots, n$, with respect to the synchronous time variable t in dimensions $D > 4$.

The problem was reduced to the analysis of the stability of the fixed point solutions $h^i(t) = v^i$ to Eqs. (2.16) and (2.21), where $h^i(t)$ are Hubble-like parameters.

In this paper a set of equations for perturbations δh^i was considered (in linear approximation) and the general solution to these equations was found. We have proved (in Proposition 2) that the solutions with non-static volume factor, i.e. with $K(v) = \sum_{k=1}^{n} v^k \neq 0$, which obey restriction (3.7), are stable if $K(v) > 0$, while they are unstable if $K(v) < 0$.

We have also proved (in Proposition 1) that for any exponential solution with $v = (v^1, \ldots, v^n)$ there are no more than three different numbers among $v^1, \ldots, v^n$, if $\sum_{i=1}^{n} v^i \neq 0$.

Here we have presented several examples of stable cosmological solutions with exponential behavior of the scale factors. Among them the isotropic solution $v = (H, \ldots, H)$ and several anisotropic solutions with two Hubble parameters $v = (H, \ldots, H, h, \ldots, h)$ were considered. The isotropic solution is stable if $H > 0$ and $H \neq H_{cr}$ for $\alpha < 0$ (see (4.8)). For the anisotropic case our examples deal with the Hubble-like parameter $H > 0$ corresponding to m-dimensional flat subspace with $m \geq 3$ and the Hubble-like parameter h corresponding to l-dimensional flat subspace with $l > 1$. This subclass of (anisotropic) solutions contains the following cases: (i) $m = 3, l = 2, \Lambda = 0$; (ii) $m = l = 3$, $\Lambda = 0$; (iii) $m = 11, l = 16, \Lambda = 0$; (iv) $m = 15, l = 6$, $\Lambda = 0$; (v) $m \geq 3, l > 1, \Lambda > 0$. We have also shown that general solutions with $v^1 = v^2 = v^3 = H > 0$ and zero variation of the effective gravitational constant are stable if the restriction (3.7) is obeyed.

Acknowledgments This paper was funded by the Ministry of Education and Science of the Russian Federation in the Program to increase the competitiveness of Peoples' Friendship University (RUDN University) among the world's leading research and education centers in the 2016-2020 and by the Russian Foundation for Basic Research, grant Nr. 16-02-00602.

References

1. B. Zwiebach, Curvature squared terms and string theories. Phys. Lett. B **156**, 315 (1985)
2. D. Gross, E. Witten, Superstrings modifications of Einstein's equations. Nucl. Phys. B **277**, 1 (1986)
3. R.R. Metsaev, A.A. Tseytlin, Two loop beta function for the generalized bosonic sigma model. Phys. Lett. B **191**, 354 (1987)
4. H. Ishihara, Cosmological solutions of the extended Einstein gravity with the Gauss-Bonnet term. Phys. Lett. B **179**, 217 (1986)
5. N. Deruelle, On the approach to the cosmological singularity in quadratic theories of gravity: the Kasner regimes. Nucl. Phys. B **327**, 253–266 (1989)
6. S. Nojiri, S.D. Odintsov, Introduction to modified gravity and gravitational alternative for Dark Energy. Int. J. Geom. Meth. Mod. Phys. **4**, 115–146 (2007). arxiv:hep-th/0601213
7. G. Cognola, E. Elizalde, S. Nojiri, S.D. Odintsov, S. Zerbini, One-loop effective action for non-local modified Gauss-Bonnet gravity in de Sitter space. Eur. Phys. J. C **64**(3), 483–494 (2009). arXiv:0905.0543
8. E. Elizalde, A.N. Makarenko, V.V. Obukhov, K.E. Osetrin, A.E. Filippov, Stationary vs. singular points in an accelerating FRW cosmology derived from six-dimensional Einstein-Gauss-Bonnet gravity. Phys. Lett. B **644**, 1–6 (2007). arxiv:hep-th/0611213
9. K. Bamba, Z.-K. Guo, N. Ohta, Accelerating cosmologies in the Einstein-Gauss-Bonnet theory with dilaton. Prog. Theor. Phys. **118**, 879–892 (2007). arXiv: 0707.4334
10. A. Toporensky, P. Tretyakov, Power-law anisotropic cosmological solution in 5+1 dimensional Gauss-Bonnet gravity. Grav. Cosmol. **13**, 207–210 (2007). arXiv: 0705.1346
11. S.A. Pavluchenko, A.V. Toporensky, A note on differences between $(4 + 1)$- and $(5 + 1)$-dimensional anisotropic cosmology in the presence of the Gauss-Bonnet term. Mod. Phys. Lett. A **24**, 513–521 (2009)
12. I.V. Kirnos, A.N. Makarenko, Accelerating cosmologies in Lovelock gravity with dilaton. Open Astron. J. **3**, 37–48 (2010). arXiv:0903.0083
13. S.A. Pavluchenko, On the general features of Bianchi-I cosmological models in Lovelock gravity. Phys. Rev. D **80**, 107501 (2009). arXiv: 0906.0141
14. I.V. Kirnos, A.N. Makarenko, S.A. Pavluchenko, A.V. Toporensky, The nature of singularity in multidimensional anisotropic Gauss-Bonnet cosmology with a perfect fluid. Gen. Relativ. Gravit. **42**, 2633–2641 (2010). arXiv: 0906.0140
15. V.D. Ivashchuk, On anisotropic Gauss-Bonnet cosmologies in (n + 1) dimensions, governed by an n-dimensional Finslerian 4-metric. Grav. Cosmol. **16**(2), 118–125 (2010). arXiv: 0909.5462
16. V.D. Ivashchuk, On cosmological-type solutions in multidimensional model with Gauss-Bonnet term. Int. J. Geom. Meth. Mod. Phys. **7**(5), 797–819 (2010). arXiv: 0910.3426
17. D.A. Ratanov, Multidimensional cosmology in the model of gravitation with Gauss-Bonnet adding, Magister dissertation (under the supervision of V.D. Ivashchuk), Inst. of Gravit. and Cosmol., RUDN, Moscow, pp. 1–30 (2011) **(in Russian)**
18. K.-I. Maeda, N. Ohta, Cosmic acceleration with a negative cosmological constant in higher dimensions. JHEP **1406**, 095 (2014). arXiv:1404.0561
19. D. Chirkov, S. Pavluchenko, A. Toporensky, Exact exponential solutions in Einstein-Gauss-Bonnet flat anisotropic cosmology. Mod. Phys. Lett. A **29**, 1450093 (11 pages) (2014). arXiv:1401.2962
20. D. Chirkov, S.A. Pavluchenko, A. Toporensky, Non-constant volume exponential solutions in higher-dimensional Lovelock cosmologies. Gen. Relativ. Gravit. **47**, 137 (33 pages) (2015). arXiv:1501.04360
21. V.D. Ivashchuk, A.A. Kobtsev, On exponential cosmological type solutions in the model with Gauss-Bonnet term and variation of gravitational constant. Eur. Phys. J. C **75**, 177 (12 pages) (2015). arXiv:1503.00860
22. S.A. Pavluchenko, Stability analysis of exponential solutions in Lovelock cosmologies, Phys. Rev. D **92**, 104017 (2015). arXiv:1507.01871
23. K.K. Ernazarov, V.D. Ivashchuk, A.A. Kobtsev, On exponential solutions in the Einstein–Gauss–Bonnet cosmology, stability and variation of G. Grav. Cosmol. **22**(3), 245–250 (2016)
24. A.G. Riess et al., Observational evidence from supernovae for an accelerating universe and a cosmological constant. Astron. J. **116**, 1009–1038 (1998)
25. S. Perlmutter et al., Measurements of Omega and Lambda from 42 High-Redshift Supernovae. Astrophys. J. **517**, 565–586 (1999)
26. M. Kowalski, D. Rubin et al., Improved cosmological constraints from new, old and combined supernova datasets. ApJ **686**(2), 749–778. arXiv: 0804.4142
27. K.A. Bronnikov, V.D. Ivashchuk, V.N. Melnikov, Time variation of gravitational constant in multidimensional cosmology. Nuovo Cimento B **102**, 209–215 (1998)
28. V.N. Melnikov, Models of G time variations in diverse dimensions. Front. Phys. China **4**, 75–93 (2009)
29. V.D. Ivashchuk, V.N. Melnikov, On time variations of gravitational and Yang-Mills constants in a cosmological model of superstring origin. Grav. Cosmol. **20**(1), 26–29 (2014). arxiv: 1401.5491
30. E.V. Pitjeva, Updated IAA RAS planetary ephemerides-EPM2011 and their use in scientific research. Astron. Vestnik **47**(5), 419–435 (2013). arXiv: 1308.6416
31. P.A.R. Ade et al., [Planck Collaboration], Planck 2013 results. I. Overview of products and scientific results. Astron. Astrophys. **571**, A1 (2014). arXiv: 1303.5076
32. V.D. Ivashchuk, V.N. Melnikov, Exact solutions in multidimensional cosmology with cosmological constant. Theor. Math. Phys. **98**(2), 212–217 (1994)
33. U. Bleyer, V.D. Ivashchuk, V.N. Melnikov, A.I. Zhuk, Multidimensional classical and quantum wormholes in models with cosmological constant. Nucl. Phys. B **429**, 177–204 (1994)
34. V.D. Ivashchuk, On symmetries of target space for σ-model of p-brane origin. Grav. Cosmol. **4**(3), 217–220 (1998). arXiv:hep-th/9804102
35. D. Lowelock, The Einstein tensor and its generalizations. J. Math. Phys. **12**(3), 498–501 (1971)
36. M. Rainer, A. Zhuk, Einstein and Brans-Dicke frames in multidimensional cosmology. Gen. Relativ. Gravit. **32**, 79–104 (2000). arXiv:gr-qc/9808073
37. V.D. Ivashchuk, V.N. Melnikov, Multidimensional gravity with einstein internal spaces. Grav. Cosmol. **2**(3), 211–220 (1996). arXiv:hep-th/9612054