

Disformal transformation in Newton–Cartan geometry

Peng Huang^{1,2,a}, Fang-Fang Yuan^{3,b}

¹ Present address: Department of Information, Zhejiang Chinese Medical University, Hangzhou 310053, China

² School of Physics and Astronomy, Sun Yat-Sen University, Guangzhou 510275, China

³ School of Physics, Nankai University, Tianjin 300071, China

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Abstract Newton–Cartan geometry has played a central role in recent discussions of the non-relativistic holography and condensed matter systems. Although the conformal transformation in non-relativistic holography can easily be rephrased in terms of Newton–Cartan geometry, we show that it requires a nontrivial procedure to arrive at the consistent form of anisotropic disformal transformation in this geometry. Furthermore, as an application of the newly obtained transformation, we use it to induce a geometric structure which may be seen as a particular non-relativistic version of the Weyl integrable geometry.

1 Introduction

Newton–Cartan geometry (NCG) was proposed by Élie Cartan as a geometrical description for Newtonian gravity in the spirit of general relativity [1, 2] (see also [3]). The renewed interest in NCG is well motivated in view of its usefulness in the investigations of condensed matter systems and non-relativistic holography. Concretely speaking, it has been found that, together with the non-relativistic diffeomorphism invariance, NCG provides a natural geometrical background for the effective field theory description of fractional quantum Hall effect [4].

Moreover, the progress in gauge/gravity duality attaches increasing importance to non-Riemannian geometries while, in particular, NCG acts an important role in various realizations of the non-AdS holography; see e.g. [5–9].

Similar to the extension of Riemannian geometry to Weyl geometry, the conformal extension of NCG has also been investigated from different perspectives. Following the experience of “deriving” Einstein gravity (more properly, Einstein–Cartan theory) and Riemann–Cartan geometry through gauging the relativistic Poincaré algebra, NCG

is obtained by gauging the Bargmann algebra, which is the centrally extended Galilean algebra [10]. Then the conformal generalization of NCG is also got by performing a similar gauging procedure to the Schrödinger algebra, the conformal version of Bargmann algebra [8, 11]. On the other hand, inspired by the celebrated Poincaré gauge theory [12–14] (see [15] for a detailed exposition) in which Riemann–Cartan geometry emerges naturally by localizing the global Poincaré symmetry of a field theory in Minkowski spacetime, it has been shown that NCG can be obtained by localizing the global Galilean symmetry of a field theory in 3d Galilean space with universal time [16]. Such a method also provides a systematic way for reformulating an originally Galilean-invariant theory in Euclidean space with universal time into a diffeomorphism-invariant theory in curved space [16–18]. Following this train of thought, the conformal extension of NCG is also obtained by localizing the global Galilean and scale symmetries [19].

The above two methods are not merely straightforward repetitions of the corresponding process in the relativistic case. There the space and time are on an equal footing and thus can be treated uniformly which makes the whole procedure relatively clear and concise. However, for non-relativistic cases, the space is relative while the time is not [20, 21]. To preserve the absoluteness of universal time and the relative character of space, special attention is needed throughout the whole procedure; see [10, 16–18] for details. Such different concepts of space and time are also reflected in the consistent form of the conformal transformation. In relativistic cases, conformal transformation must be isotropic due to the equal footing of space and time. Nevertheless, in non-relativistic case, the absolute nature of the universal time makes the concept of anisotropic conformal transformation emerge naturally; see [8, 11, 19] for further details.

On the other hand, by noting the existence of conformally related Riemannian geometries in scalar-tensor theory, Bekenstein considered the possibility of more general phys-

^a e-mail: huangp46@mail.sysu.edu.cn

^b e-mail: ffyuan@nankai.edu.cn

ical geometry which may have a Finslerian structure [22]. On account of the weak equivalence principle and causality, he found that the physical metric should be related to the gravitational metric by a generalization of the conventional conformal transformation, i.e. the disformal transformation

$$\bar{g}_{\mu\nu} = A(\phi, X)g_{\mu\nu} + B(\phi, X)\phi_\mu\phi_\nu, \quad (1)$$

with $\phi_\mu = \partial_\mu\phi$ and $X = \frac{1}{2}g^{\mu\nu}\phi_\mu\phi_\nu$. Note that the forms of the functionals A and B are not arbitrary which means they should ensure that the new metric is still a well-defined one, such as preserving the Lorentzian signature of the original spacetime, being invertible, etc. For detailed investigations of related issues, see [22–24].

The disformal transformation has attracted a lot of attention in recent years and found applications in e.g. varying speed of light models [25], inflation [26], dark energy models [27] and dark matter models [28]. In particular, it acts an important role in Horndeski theories and has led to many important results in modified gravity and cosmology [23, 24, 29–40]. It is interesting to note that there are very few discussions as regards disformal transformation in the non-relativistic realm. Even the related work on modified Newtonian dynamics paradigm has focused on the relativistic extensions [41]. Except for the obvious applications in gravity theories, a possible non-relativistic version of the disformal transformation may have implications for the non-relativistic holography given the power and utility of conformal transformations in this area.

With these observations in mind, together with all the important facts about NCG, some questions can be naturally proposed:

- Is there a consistent non-relativistic version of the disformal transformation?
- In view of the relation between conformal/disformal transformation and Weyl integrable geometry [42], one may wonder that can the NCG be disformally generalized?

It can be inferred that, in searching for the answer to the first question, the special role of the universal time in non-relativistic geometry may cause some novelty, and this is indeed the case as will be shown in Sect. 3. Once the form of the disformal transformation is given, then it seems that the answer to the second question can easily be achieved. One may seek to obtain the disformal extension of NCG by gauging the disformal extension of Bargmann algebra, or by localizing the global disformal symmetry of a disformally invariant field theory in 3d Galilean space with universal time.

However, the situation is more complicated than expected. On the one hand, what is a disformally invariant field theory in NCG is an open question. On the other hand, it is not clear

what exactly the disformal extension of Bargmann algebra is. To see the subtleties related to this unknown extension, it suffices to see the disformal transformation in Riemannian geometry for illustrating purpose. Then it is straightforward to check that two consecutive transformations do not lead to the conventional disformal transformation but a generalized one as follows:

$$\begin{aligned} \tilde{\tilde{g}}_{\mu\nu} &= C(\psi, Y)\bar{g}_{\mu\nu} + D(\psi, Y)\psi_\mu\psi_\nu \\ &= ACg_{\mu\nu} + BC\phi_\mu\phi_\nu + D\psi_\mu\psi_\nu. \end{aligned} \quad (2)$$

In addition, changing the order of the two different disformal transformations will not lead to the same results:

$$\begin{aligned} \bar{\bar{g}}_{\mu\nu} &= A(\phi, X)\tilde{g}_{\mu\nu} + B(\phi, X)\phi_\mu\phi_\nu \\ &= ACg_{\mu\nu} + AD\psi_\mu\psi_\nu + B\phi_\mu\phi_\nu. \end{aligned} \quad (3)$$

Furthermore, it can be shown that the inverse of $\tilde{\tilde{g}}_{\mu\nu}$ (or $\bar{\bar{g}}_{\mu\nu}$) are not in the form of $Eg^{\mu\nu} + F\psi^\mu\psi^\nu + G\phi^\mu\phi^\nu$ as expected, but in the form of $Eg^{\mu\nu} + F\psi^\mu\psi^\nu + G\phi^\mu\phi^\nu + H(\phi^\mu\psi^\nu + \psi^\mu\phi^\nu)$, otherwise there will be no solution to constraint equation $\tilde{\tilde{g}}^{\mu\nu}\tilde{\tilde{g}}_{\mu\rho} = \delta_\rho^\nu$ (or $\bar{\bar{g}}^{\mu\nu}\bar{\bar{g}}_{\mu\rho} = \delta_\rho^\nu$) [42]. Based on these observations, one can speculate that the possible disformal extension of Poincaré algebra may be very complicated or have a novel mathematical structure.

Simply speaking, the disformal extension of NCG cannot easily be achieved by the aforementioned two methods. This fact makes it an interesting problem from a mathematical point of view. What is more, a sound knowledge about the non-relativistic disformal transformation and disformal extension of NCG is also helpful for constructing disformally invariant field theories as well as gravity theories, which have the potential to enrich the investigations broaden the horizons of condensed matter theory and holography. All these considerations constitute the motivation for the present work. First of all, we will show that a consistent form of disformal transformation in NCG can indeed be given (see (30)). After that, we will use the newly obtained disformal transformation to induce a new geometry, not by gauging the disformal extension of Bargmann algebra or localizing the global symmetry of a disformally invariant field theory in NCG, but by using the method which has been used in [42] to get the disformal extension of Riemannian geometry.

This paper is organized as follows. In Sect. 2, a brief review of the general torsional NCG recently expatiated in [11] is given. We also demonstrate that the conformal extension of NCG can be induced through the anisotropic Weyl rescaling of original metrics. This serves as a complementary perspective to the standard gauging or localization procedure. In Sect. 3, we propose a method to find the most general anisotropic disformal transformation in NCG. Then, as an application of the newly obtained disformal transformation, we use it to induce the disformal extension of NCG in Sect. 4. The results consist of the basic disformal connection and

semi-metricity conditions. Our conclusion will be given in the last section.

2 Newton–Cartan geometry and its conformal extension

2.1 Newton–Cartan geometry

A first taste of NCG can be gained through a geometric reformulation of Newtonian gravity; see [43] for a clear exposition. Recently it has been shown that NCG can be obtained by gauging the centrally extended Galilean algebra, i.e., the Bargmann algebra [8, 10, 11]. A well-known fact is that general relativity can be obtained by gauging the Poincaré algebra. This method will naturally lead to the appearance of torsion and result in Riemann–Cartan geometry. Only after one imposes zero constraints on the curvature tensors of gauge fields related to generators of translational symmetries, then the torsion-free Riemannian geometry can be obtained. In non-relativistic cases, the situation is similar: gauging the Bargmann algebra will also lead to the appearance of torsion; however, after imposing several curvature constraints, the final NCG will have no torsion [10]. On the contrary, if no zero constraints were imposed, the geometry one obtained will be the torsional NCG [11].

Torsional NCG acts an important role in non-AdS holography. For example, the boundary geometry in Lifshitz holography is described by torsional NCG [7, 9]. For related studies on supergravity in NCG with or without torsion, see [44–48]. All in all, whether there is torsion or not can lead to very different results. While the dynamical NCG without torsion is related to projectable Hořava–Lifshitz gravity [49], the incorporation of twistless torsion leads to non-projectable Hořava–Lifshitz gravity [11]. Since the realization of disformal transformation in NCG and its potential applications are of our most concern, we would like to maintain as general as possible. In other words, no gauge fixing on the NCG will be imposed and a general torsion (with no twistless condition) is assumed. For our purpose, it is convenient to make use of the formalism of dynamical NCG developed in [11]. To begin with, let us list some relevant results which will be the starting point of our discussion.

The degenerated metrics and their inverses which transform covariantly under the local space and time translations, but are invariant under the remainder internal symmetries are τ_μ , $h^{\mu\nu}$, \hat{v}^μ and $H_{\mu\nu}$.¹ Then NCG is defined with these quantities as follows:

¹ For a detailed explanation of these quantities (as well as $\tilde{\Phi}$, which will appear soon) and their relevance in recent literature, see [11] and the references therein. See also [10, 50] for torsion-free cases while other aspects of NCG have been discussed in [50]. For present purposes, it suffices to content ourselves with the minimal introduction of NCG.

$$\begin{aligned} \hat{v}^\mu \tau_\mu &= -1, \quad \hat{v}^\mu H_{\mu\nu} = (2 - \alpha)\tilde{\Phi}\tau_\nu, \quad \tau_\mu h^{\mu\nu} = 0, \\ h^{\mu\nu} H_{\mu\rho} &= \delta_\rho^\nu + \hat{v}^\nu \tau_\rho, \end{aligned} \tag{4}$$

with α an arbitrary constant. The affine connection is defined as

$$\Gamma_{\mu\nu}^\lambda = -\hat{v}^\lambda \partial_\mu \tau_\nu + \frac{1}{2} h^{\lambda\sigma} (\partial_\mu H_{\sigma\nu} + \partial_\nu H_{\sigma\mu} - \partial_\sigma H_{\mu\nu}), \tag{5}$$

to obey the metricity postulates

$$\nabla_\mu \tau_\nu = 0, \quad \nabla_\mu h^{\nu\rho} = 0. \tag{6}$$

Here $H_{\mu\nu}$ is given by

$$H_{\mu\nu} = \hat{h}_{\mu\nu} + (\alpha - 2)\tau_\mu \tau_\nu \tilde{\Phi}. \tag{7}$$

One can observe from (5) that the first term on the right-hand side is not symmetric in μ and ν , thus the connection generally has torsion. This is different from the cases in e.g. [10, 50]. As explained at the beginning of this section, the appearance of torsion is a natural and expected result when no zero constraints on the curvature tensors of gauge fields have been imposed.

An intriguing and important point is that despite the difference in the explicit forms of $H_{\mu\nu}$ and $\hat{h}_{\mu\nu}$, they actually lead to the same results when constructing the effective actions for Hořava–Lifshitz gravity [11]. In the light of this observation, we will choose $\alpha = 2$ from now on. Then (4) and (5) change into

$$\begin{aligned} \hat{v}^\mu \tau_\mu &= -1, \quad \hat{v}^\mu \hat{h}_{\mu\nu} = 0, \quad \tau_\mu h^{\mu\nu} = 0, \quad h^{\mu\nu} \hat{h}_{\mu\rho} \\ &= \delta_\rho^\nu + \hat{v}^\nu \tau_\rho, \end{aligned} \tag{8}$$

$$\Gamma_{\mu\nu}^\lambda = -\hat{v}^\lambda \partial_\mu \tau_\nu + \frac{1}{2} h^{\lambda\sigma} (\partial_\mu \hat{h}_{\sigma\nu} + \partial_\nu \hat{h}_{\mu\sigma} - \partial_\sigma \hat{h}_{\mu\nu}). \tag{9}$$

2.2 Conformal extension of Newton–Cartan geometry

In principle, via conformal rescaling (which reduces to Weyl rescaling in the isotropic case), the conformal NCG could be induced from the original geometry. Our main task in this subsection is to demonstrate this point in detail.

In relativistic systems, the space and time have the same footing, and thus the conformal rescaling must be isotropic. Nevertheless, in non-relativistic systems, there is a preferred notion of time. So an anisotropic rescaling can be introduced which has the form

$$\tau_\mu \rightarrow \tilde{\tau}_\mu = e^{z\Omega} \tau_\mu, \quad h^{\mu\nu} \rightarrow \tilde{h}^{\mu\nu} = e^{-2\Omega} h^{\mu\nu}. \tag{10}$$

To preserve the defining relations of NCG in (8), the transformation rules for the inverses of τ_μ and $h^{\mu\nu}$ are constrained by

$$\hat{v}^\mu \rightarrow \tilde{v}^\mu = e^{-z\Omega} \hat{v}^\mu, \quad \hat{h}_{\mu\nu} \rightarrow \tilde{h}_{\mu\nu} = e^{2\Omega} \hat{h}_{\mu\nu}. \tag{11}$$

With these relations at hand, one can perform the following calculations:

$$\nabla_\mu \tau_\nu = 0 = -ze^{-z\Omega} \partial_\mu \Omega \cdot \tilde{\tau}_\nu + e^{-z\Omega} \nabla_\mu \tilde{\tau}_\nu, \tag{12}$$

$$\Rightarrow \nabla_\mu \tilde{\tau}_\nu = z \partial_\mu \Omega \cdot \tilde{\tau}_\nu;$$

$$\nabla_\mu h^{\alpha\beta} = 0 = 2e^{2\Omega} \partial_\mu \Omega \cdot \tilde{h}^{\alpha\beta} + e^{2\Omega} \nabla_\mu \tilde{h}^{\alpha\beta},$$

$$\Rightarrow \nabla_\mu \tilde{h}^{\alpha\beta} = -2 \partial_\mu \Omega \cdot \tilde{h}^{\alpha\beta}. \tag{13}$$

The original connection given in (9) can also be rewritten with the new variables as

$$\Gamma_{\mu\nu}^\lambda = -\tilde{v}^\lambda (\partial_\mu - z \partial_\mu \Omega) \tilde{\tau}_\nu$$

$$+ \frac{1}{2} \tilde{h}^{\lambda\sigma} [(\partial_\mu - 2 \partial_\mu \Omega) \tilde{h}_{\sigma\nu} + (\partial_\nu - 2 \partial_\nu \Omega) \tilde{h}_{\mu\sigma}$$

$$- (\partial_\sigma - 2 \partial_\sigma \Omega) \tilde{h}_{\mu\nu}]. \tag{14}$$

It is easy to indicate that the expressions in (12)–(14) are actually invariant under the transformation $\partial_\mu \Omega \rightarrow \partial_\mu \Omega + \partial_\mu \omega$.

One can also take a step further and promote the exact one-form (component form) $\partial_\mu \Omega$ into a general one-form b_μ . The resulting equations are still invariant under a similar transformation $b_\mu \rightarrow \partial_\mu \omega$. To exhibit this, we relabel $\tilde{\tau}_\mu, \tilde{v}^\mu, \tilde{h}_{\mu\nu}$ and $\tilde{h}^{\mu\nu}$ as $\tau_\mu, \hat{v}^\mu, \hat{h}_{\mu\nu}$ and $h^{\mu\nu}$, respectively. One then notices that these quantities satisfy the following conditions:

$$\hat{v}^\mu \tau_\mu = -1, \quad \hat{v}^\mu \hat{h}_{\mu\nu} = 0, \quad \tau_\mu h^{\mu\nu} = 0, \quad h^{\mu\nu} \hat{h}_{\mu\rho}$$

$$= \delta_\rho^\nu + \hat{v}^\nu \tau_\rho. \tag{15}$$

Furthermore, the semi-metricity conditions (the original metricity postulates are violated) can be rewritten as

$${}^W \nabla_\mu \tau_\nu = z b_\mu \tau_\nu, \quad {}^W \nabla_\mu h^{\alpha\beta} = -2 b_\mu h^{\alpha\beta}, \tag{16}$$

with the connection given by

$${}^W \Gamma_{\mu\nu}^\lambda = -\hat{v}^\lambda (\partial_\mu - z b_\mu) \tau_\nu$$

$$+ \frac{1}{2} h^{\lambda\sigma} [(\partial_\mu - 2 b_\mu) \hat{h}_{\sigma\nu} + (\partial_\nu - 2 b_\nu) \hat{h}_{\mu\sigma}$$

$$- (\partial_\sigma - 2 b_\sigma) \hat{h}_{\mu\nu}]. \tag{17}$$

It is obvious that (15) assumes the form of the defining relations (8) for metrics in NCG. Our notations are thus consistent with the fact that, before and after the conformal rescaling, these fundamental relations should be maintained. This also highlights the point that what one obtained through the procedure (10)–(14) is a new connection, *not* new metrics. From the view of basic geometric structures, the conformal rescaling of metrics for the original NCG may be deemed just an intermediary step.

We remark that the conformal extension of NCG, whose metric compatibility conditions and connection are given by (15), (16) and (17), respectively, can be seen as a non-relativistic version of the Weyl geometry (hence the superscripts “W”s in the above equations). Our results are also

consistent with those obtained through the gauging procedure [8, 11]. Furthermore, when b_μ (which is just the dilatation parameter in [8, 11]) can be expressed as the derivative of a scalar field $\partial_\mu b$, the corresponding geometry defined by (15)–(17) is just the Weyl integrable extension of NCG, whose relativistic cousin is the well-known Weyl integrable geometry.

To sum up, the defining relations (15)–(17) can be obtained through the rewriting and rearrangement of the original variables and expressions. It is just in this sense that we say the conformal extension of NCG can be *induced* from NCG by conformally rescaling the metrics. The potential advantage of this perspective is that one can use this approach to induce a new geometry even when the symmetries and algebraic structures of the underlying geometry are not well understood yet. This method has recently been utilized to induce a new geometry from Riemannian geometry through the disformal transformation [42].

3 Anisotropic disformal transformation in Newton–Cartan geometry

The disformal transformation is a generalization of the conventional conformal transformation of metrics which is defined as [22]

$$\bar{g}_{\mu\nu} = A(\phi, X) g_{\mu\nu} + B(\phi, X) \phi_\mu \phi_\nu, \tag{18}$$

where by construction the disformal functions A and B depend on the real scalar field ϕ and its kinetic term $X = \frac{1}{2} g^{\mu\nu} \phi_\mu \phi_\nu$ with $\phi_\mu \equiv \partial_\mu \phi$. In searching for the form of the disformal transformation in NCG, the first problem one encounters is how to construct the kinetic term (corresponding to X in relativistic case) with two degenerated metric at hand? For our purpose, it suffices to notice that both $\hat{v}^\mu \hat{v}^\nu \phi_\mu \phi_\nu$ and $\hat{h}^{\mu\nu} \phi_\mu \phi_\nu$ have potential capability to act the role as that of X in relativistic case. Nevertheless, due to the absolute distinguishability between space and time in NCG, it is quite possible that one of them are zero while the other is not. Therefore, instead of choosing only one of them to act as X in relativistic case, a better choice is to use a combination of them, i.e., $X = (\alpha \hat{v}^\mu \hat{v}^\nu + \beta \hat{h}^{\mu\nu}) \phi_\mu \phi_\nu$ with α and β two nonzero constants. In the present work, $\alpha = -1$ and $\beta = 1$ are chosen to agree with [9, 11]; furthermore, $(-\hat{v}^\mu \hat{v}^\nu + \hat{h}^{\mu\nu}) \phi_\mu \phi_\nu$ will be abbreviated as ϕ^μ .

It should be noticed that the quantity $\alpha \hat{v}^\mu \hat{v}^\nu + \beta \hat{h}^{\mu\nu}$, with special nonzero parameters, is sometimes referred to as the *extended metric* and has arisen frequently in the literature [3, 9, 11, 51, 55]. Such an extended metric is a nondegenerate symmetric rank 2 tensor with Lorentzian signature. However, this metric should not be treated as a fundamental dynamical variable in NCG. Indeed, if a single nondegenerate metric

could be defined in NCG, there would be no need for introducing two degenerate metrics. It is precisely the lack of such a nondegenerate metric that led Cartan to formulate NCG in this way.

After setting up the notation and clarifying potential confusions, a direct generalization of anisotropic disformal transformation for NCG can be made by inspection of (10), (11), and (18):

$$\begin{aligned} \tau_\mu &\rightarrow \bar{\tau}_\mu = A^z \tau_\mu + B \phi_\mu, & h^{\mu\nu} &\rightarrow \bar{h}^{\mu\nu} = A^{-2} h^{\mu\nu} \\ &+ C \phi^\mu \phi^\nu, \\ \hat{v}^\mu &\rightarrow \bar{v}^\mu = A^{-z} \hat{v}^\mu + E \phi^\mu, & \hat{h}_{\mu\nu} &\rightarrow \bar{h}_{\mu\nu} = A^2 \hat{h}_{\mu\nu} \\ &+ D \phi_\mu \phi_\nu, \end{aligned} \tag{19}$$

where $A, B, C, D,$ and E are all functions of the real scalar field ϕ and its kinetic term X .

To ensure the defining relations given in (8), one has the following constraint equations:

$$A^z E \phi^\mu \tau_\mu + A^{-z} B \hat{v}^\mu \phi_\mu + 2BEX = 0, \tag{20}$$

$$A^z C \tau_\mu \phi^\mu \phi^\nu + A^{-2} B \phi_\mu h^{\mu\nu} + 2BCX \phi^\nu = 0, \tag{21}$$

$$\begin{aligned} A^{-2} D h^{\mu\nu} \phi_\mu \phi_\nu + A^2 C \phi^\mu \phi^\nu \hat{h}_{\mu\rho} + 2DCX \phi^\nu \phi_\rho \\ - A^{-z} B \hat{v}^\nu \phi_\rho - A^z E \phi^\nu \tau_\rho - BE \phi^\nu \phi_\rho = 0, \end{aligned} \tag{22}$$

$$A^{-z} D \hat{v}^\mu \phi_\mu \phi_\nu + A^2 E \phi^\mu \hat{h}_{\mu\nu} + 2DEX \phi_\nu = 0. \tag{23}$$

Then it is easy to check that generally only when $B = C = D = E = 0$ can (20)–(23) be satisfied, which just corresponds to the conventional anisotropic conformal transformation.

Alternatively, if we consider the following novel possibility (with F a function):

$$\phi_\mu = F \tau_\mu, \quad \phi^\mu = F \hat{v}^\mu, \tag{24}$$

then one is left with a particular form of disformal transformation (after the constraint equations for coefficients being used)

$$\begin{aligned} \bar{\tau}_\mu &= (A^z + BF) \tau_\mu, & \bar{h}^{\mu\nu} &= A^{-2} h^{\mu\nu}, \\ \bar{v}^\mu &= (A^z + BF)^{-1} \hat{v}^\mu, & \bar{h}_{\mu\nu} &= A^2 \hat{h}_{\mu\nu}. \end{aligned} \tag{25}$$

One can absorb F into B by redefining it for convenience. Notice that $A^z + B$ generally cannot be rewritten as $A^{z'}$ with z' a constant as the new dynamical exponent, thus (25) is the desired form of disformal transformation with the assumption of (19). Furthermore, one can argue that the conventional anisotropic conformal rescaling can be enlarged to general cases where the conformal factors for temporal and spatial components cannot be related through the dynamical component z [19]. This is in fact equivalent to the case given in (25), which is naturally required in seeking a direct and consistent form of the disformal transformation in NCG.

Although such result is consistent from beginning to end, the disturbing aspect of (25) is that the metric on spatial slices

is not involved. Apparently with this method we could not find the most general form of disformal transformation. Thus, we have to tackle this problem from another perspective. Recall that there is an absolute distinction between space and time in NCG, while (19) did not reflect this character (especially for ϕ_μ and ϕ^μ). To retain this character of NCG, one can introduce the projectors onto the temporal and spatial directions, respectively, as

$$\begin{aligned} \text{temporal projector: } & -\hat{v}^\mu \tau_\nu, & \text{spatial projector:} \\ h_\nu^\mu &= h^{\rho\mu} \hat{h}_{\rho\nu} = \delta_\nu^\mu + \hat{v}^\mu \tau_\nu. \end{aligned} \tag{26}$$

These covariant temporal and spatial projectors have been introduced in [52, 53] in the context of Lifshitz holography. Their applications can be found in e.g. [54, 55]. For our purposes, ϕ_μ and ϕ^μ now can be divided into temporal and spatial parts as

$$\begin{aligned} \phi_\mu &= -2Y \tau_\mu + (\phi_\mu + 2Y \tau_\mu), & \phi^\mu &= -2Y \hat{v}^\mu \\ &+ (\phi^\mu + 2Y \hat{v}^\mu), \end{aligned} \tag{27}$$

where $Y \equiv \frac{1}{2} \hat{v}^\mu \phi_\mu = \frac{1}{2} \tau_\mu \phi^\mu$. The disformal transformation then takes the following form:

$$\begin{aligned} \tau_\mu &\rightarrow \bar{\tau}_\mu = A^z \tau_\mu + B(-2Y \tau_\mu), \\ h^{\mu\nu} &\rightarrow \bar{h}^{\mu\nu} = A^{-2} h^{\mu\nu} + C(\phi^\mu + 2Y \hat{v}^\mu)(\phi^\nu + 2Y \hat{v}^\nu), \\ \hat{v}^\mu &\rightarrow \bar{v}^\mu = A^{-z} \hat{v}^\mu + E(-2Y \hat{v}^\mu), \\ \hat{h}_{\mu\nu} &\rightarrow \bar{h}_{\mu\nu} = A^2 \hat{h}_{\mu\nu} + D(\phi_\mu + 2Y \tau_\mu)(\phi_\nu + 2Y \tau_\nu). \end{aligned} \tag{28}$$

Through a similar procedure as above, one gets two constraint equations

$$\begin{aligned} (A^z - 2BY)(A^{-z} - 2EY) &= 1, \\ A^{-2}D + A^2C + DC(2X + 4Y^2) &= 0. \end{aligned} \tag{29}$$

Together with (28), the final form of the disformal transformation in NCG is obtained:

$$\begin{aligned} \bar{\tau}_\mu &= (A^z - 2BY) \tau_\mu, \\ \bar{h}^{\mu\nu} &= A^{-2} h^{\mu\nu} + C \Phi^\mu \Phi^\nu, \\ \bar{v}^\mu &= (A^z - 2BY)^{-1} \hat{v}^\mu, \\ \bar{h}_{\mu\nu} &= A^2 \hat{h}_{\mu\nu} - \frac{A^4 C}{1 + 2A^2 C Z} \Phi_\mu \Phi_\nu. \end{aligned} \tag{30}$$

Here we have defined $\Phi_\mu \equiv \phi_\mu + 2Y \tau_\mu$, $\Phi^\mu = \Phi_\nu g^{\mu\nu} = \phi^\mu + 2Y \hat{v}^\mu$ and $Z \equiv X + 2Y^2 = \frac{1}{2} \Phi^\mu \Phi_\mu$. Recall that the independent functions A, B, C depend on the real scalar field ϕ and its kinetic term $X = \frac{1}{2} g^{\mu\nu} \phi_\mu \phi_\nu$. In contrast with the original forms in (19), here (see (28)) we have eliminated two functions D, E using the constraint equations. On the other hand, $Y \equiv \frac{1}{2} \tau_\mu \phi^\mu$ comes from the temporal part of scalar field (see (27)), and it is uniquely determined by the latter. Obviously the case of Z is similar and should also not be regarded as an independent function.

Before closing this section, several comments are in order:

- The first two expressions in (30) suffice to give the definition of the corresponding disformal transformation. This is because their inverses can be derived from them and thus are not independent.
- The previous result in (25) simply corresponds to the special case when $Y = -\frac{1}{2}F$. Obviously in this case ϕ_μ has only a temporal component.
- With respect to the disformal transformation (18) in relativistic gravity theories, it has been shown that $A(\phi, X)$ must be positive definite, and $B(\phi, X)$ should satisfy some constraints to ensure a healthy definition [22–24]. The similar situation is also expected to happen in non-relativistic cases. Nevertheless, this is not of the present concern and is left for future study.

4 Disformal extension of Newton–Cartan geometry

As an application of the newly obtained disformal transformation (30), we would like to use it to induce a new geometry which is a disformal extension of NCG. Noticing that the relativistic disformal transformation in the form $\bar{g}_{\alpha\beta} = A(\phi)g_{\alpha\beta} + B(\phi)\phi_\alpha\phi_\beta$ is of particular interest in recent literature [23, 24, 34–40], we will restrict our attention to the special case where the disformal parameters A, B , and C in (30) are only functions of the scalar field ϕ throughout this section.

A key procedure to induce a new geometry is to rewrite the original geometrical variables with new variables obtained through disformal transformation. To implement this, one first notices that $\bar{Y} = \frac{1}{2}\bar{v}^\mu\phi_\mu = (A^z - 2BY)^{-1}Y$, which leads to

$$Y = \frac{A^z\bar{Y}}{1 + 2B\bar{Y}}. \tag{31}$$

With this relation, it is easy to get

$$\tau_\mu = A^{-z}(1 + 2B\bar{Y})\bar{\tau}_\mu, \tag{32}$$

$$\hat{v}^\mu = A^z(1 + 2B\bar{Y})^{-1}\bar{v}^\mu, \tag{33}$$

$$\begin{aligned} \phi_\mu &= -2Y\tau_\mu + (\phi_\mu + 2Y\tau_\mu) \\ &= -2\bar{Y}\bar{\tau}_\mu + (\phi_\mu + 2\bar{Y}\bar{\tau}_\mu). \end{aligned} \tag{34}$$

A subtle point is that, with the definition $\bar{h}^{\mu\nu} = A^{-2}h^{\mu\nu} + C\Phi^\mu\Phi^\nu$ at hand, rewriting $h^{\mu\nu}$ with $\bar{h}^{\mu\nu}$ and Φ^μ will inevitably lead to an infinite iteration due to the hidden $h^{\mu\nu}$ in Φ^μ . A way to circumvent this problem is to notice that the disformal transformation of $h^{\mu\nu}$ and $\hat{h}_{\mu\nu}$ can be defined in an equivalent form as

$$\begin{aligned} \bar{h}^{\mu\nu} &= A^{-2}h^{\mu\nu} - \frac{D}{A^4 + 2A^2ZD}\Phi^\mu\Phi^\nu, \\ \bar{h}_{\mu\nu} &= A^2\hat{h}_{\mu\nu} + D\Phi_\mu\Phi_\nu, \end{aligned} \tag{35}$$

where we have introduced new coefficients $D(\phi)$. Then one immediately gets

$$\hat{h}_{\mu\nu} = A^{-2}\bar{h}_{\mu\nu} - \frac{D}{A^2}\Phi_\mu\Phi_\nu. \tag{36}$$

The inverse of $\hat{h}_{\mu\nu}$ is supposed to have the form $h^{\mu\nu} = A^2\bar{h}^{\mu\nu} + G\bar{\Phi}^\mu\bar{\Phi}^\nu$, where $\bar{\Phi}^\mu \equiv \Phi_\nu\bar{g}^{\mu\nu} = \Phi_\nu\bar{h}^{\mu\nu}$, and G is an unknown function to be determined. It is straightforward to find that

$$h^{\mu\nu} = A^2\bar{h}^{\mu\nu} + \frac{A^2D}{1 - 2D\bar{Z}}\bar{\Phi}^\mu\bar{\Phi}^\nu, \tag{37}$$

with $\bar{Z} = \frac{1}{2}\bar{\Phi}^\mu\Phi_\mu$.

Based on the above results, through some straightforward calculations, we find that the original connection (9) can be expressed in terms of the new variables as

$$\begin{aligned} \Gamma_{\mu\nu}^\lambda &= -\bar{v}^\lambda\left[\partial_\mu - \partial_\mu\left(\ln\frac{A^z}{1 + 2B\bar{Y}}\right)\right]\bar{\tau}_\nu \\ &+ \frac{1}{2}\left(\bar{h}^{\lambda\sigma} + \frac{D}{1 - 2D\bar{Z}}\bar{\Phi}^\lambda\bar{\Phi}^\sigma\right)\left[(\partial_\mu - \partial_\mu\ln A^2)(\bar{h}_{\sigma\nu} \right. \\ &- D\Phi_\sigma\Phi_\nu) + (\partial_\nu - \partial_\nu\ln A^2)(\bar{h}_{\mu\sigma} - D\Phi_\mu\Phi_\sigma) \\ &\left. - (\partial_\sigma - \partial_\sigma\ln A^2)(\bar{h}_{\mu\nu} - D\Phi_\mu\Phi_\nu)\right]. \end{aligned} \tag{38}$$

Furthermore, the semi-metricity conditions for the new geometry can also be induced. For $\bar{\tau}_\mu$, the result is

$$\begin{aligned} \nabla_\mu\tau_\nu = 0 &= \nabla_\mu\left(\frac{1 + 2B_2\bar{Y}}{A^z}\bar{\tau}_\nu\right), \\ \Rightarrow \nabla_\mu\bar{\tau}_\nu &= \partial_\mu\left(\ln\frac{A^z}{1 + 2B\bar{Y}}\right)\cdot\bar{\tau}_\nu. \end{aligned} \tag{39}$$

The semi-metricity condition for $\bar{h}^{\alpha\beta}$, on the other hand, takes a more complicated form. To induce it, the first step is to rearrange the original metric compatibility condition as

$$\begin{aligned} \nabla_\mu h^{\alpha\beta} = 0 &= \nabla_\mu\left[\left(A^2\bar{h}^{\alpha\beta} + \frac{A^2D}{1 - 2D\bar{Z}}\right)\bar{\Phi}^\alpha\bar{\Phi}^\beta\right], \\ \Rightarrow \nabla_\mu\left(\bar{h}^{\alpha\beta} + \frac{D}{1 - 2D\bar{Z}}\bar{\Phi}^\alpha\bar{\Phi}^\beta\right) \\ &= -\partial_\mu\ln A^2\cdot\left(\bar{h}^{\alpha\beta} + \frac{D}{1 - 2D\bar{Z}}\bar{\Phi}^\alpha\bar{\Phi}^\beta\right). \end{aligned} \tag{40}$$

As clearly demonstrated in the case of conformal extension of NCG, even though we have induced a new connection by implementing the disformal transformation of the original metrics as an intermediary step, the metrics themselves in fact do not change. Thus for clarity it is more proper to rewrite $\bar{\tau}_\mu, \bar{v}^\mu, \bar{h}_{\mu\nu}$ and $\bar{h}^{\mu\nu}$ as $\tau_\mu, \hat{v}^\mu, \hat{h}_{\mu\nu}$ and $h^{\mu\nu}$, respectively. Then it can be seen that the new induced geometry has two degenerate metrics, τ_μ and $h^{\mu\nu}$, which satisfy (as in (8))

$$\begin{aligned} \hat{v}^\mu\tau_\mu &= -1, \quad \hat{v}^\mu\hat{h}_{\mu\nu} = 0, \quad \tau_\mu h^{\mu\nu} = 0, \quad h^{\mu\nu}\hat{h}_{\mu\rho} \\ &= \delta_\rho^\nu + \hat{v}^\nu\tau_\rho. \end{aligned} \tag{41}$$

The corresponding connection and semi-metricity conditions are given as follows:

- Disformal connection:

$$\begin{aligned}
 {}^D\Gamma_{\mu\nu}^\lambda &= -\hat{v}^\lambda \left[\partial_\mu - \partial_\mu \left(\ln \frac{A^z}{1+2BY} \right) \right] \tau_\nu \\
 &+ \frac{1}{2} \left(h^{\lambda\sigma} + \frac{D}{1-2DZ} \Phi^\lambda \Phi^\sigma \right) \left[(\partial_\mu - \partial_\mu \ln A^2)(\hat{h}_{\sigma\nu} - D\Phi_\sigma \Phi_\nu) \right. \\
 &+ (\partial_\nu - \partial_\nu \ln A^2)(\hat{h}_{\mu\sigma} - D\Phi_\mu \Phi_\sigma) \\
 &\left. - (\partial_\sigma - \partial_\sigma \ln A^2)(\hat{h}_{\mu\nu} - D\Phi_\mu \Phi_\nu) \right]. \tag{42}
 \end{aligned}$$

Here $\partial_\mu - \partial_\mu \left(\ln \frac{A^z}{1+2BY} \right)$ can be interpreted as the Weyl covariant derivative along the time direction, and $\partial_\mu - \partial_\mu \ln A^2$ is the Weyl covariant derivative on the spatial slices. For the special case with $B = D = 0$, (42) reduces to (17), which is the result obtained in the conformal extension of NCG.

- Semi-metricity conditions:

$${}^D\nabla_\mu \tau_\nu = \partial_\mu \left(\ln \frac{A^z}{1+2BY} \right) \cdot \tau_\nu, \tag{43}$$

$$\begin{aligned}
 {}^D\nabla_\mu \left(h^{\alpha\beta} + \frac{D}{1-2DZ} \Phi^\alpha \Phi^\beta \right) \\
 = -\partial_\mu \ln A^2 \cdot \left(h^{\alpha\beta} + \frac{D}{1-2DZ} \Phi^\alpha \Phi^\beta \right). \tag{44}
 \end{aligned}$$

These two semi-metricity conditions together with the disformal connection (42) define the disformal extension of NCG. Note again that the conformal extension of NCG (16) is naturally included here as a special case. Furthermore, the geometry defined by (42)–(44) is obviously a non-relativistic and a disformally extended version of the conventional Weyl integrable geometry.

5 Conclusion and discussion

In this work, we have found the general anisotropic disformal transformation (30) in NCG. It is shown that a naive assumption of the form of the disformal transformation can only lead to a very special form which is consistent; see (25). To obtain the most general form, one needs to project the vector ϕ_μ in the relativistic disformal transformation (18) into temporal and spatial parts, and then impose the constraints from the defining relations in NCG.

As an application, this newly obtained disformal transformation has been used to induce a new geometry whose disformal connection and semi-metricity conditions are given in (42) and (43, 44), respectively. The key step to do this is to

rewrite the original geometrical variables with new variables obtained through disformal transformation. The corresponding subtle point is an infinite iteration in rewriting $h^{\mu\nu}$ with $h^{\mu\nu}$ and Φ^μ which is due to the hidden $h^{\mu\nu}$ in Φ^μ . This problem has been circumvented by using another equivalent (but more convenient) form of the disformal transformation; see (35).

Taking a more look at the semi-metricity conditions for the induced geometry, one can find that (43, 44) could be rewritten in a more illuminating form

$$\left({}^D\nabla_\mu - \partial_\mu \left(\ln \frac{A^z}{1+2BY} \right) \right) \tau_\nu = 0, \tag{45}$$

$$\left({}^D\nabla_\mu + \partial_\mu \ln A^2 \right) \left(h^{\alpha\beta} + \frac{D}{1-2DZ} \Phi^\alpha \Phi^\beta \right) = 0. \tag{46}$$

In this form, all the derivatives on the left-hand side are adjusted into Weyl covariant type, which is to say that these two equations in fact serve as a non-relativistic and disformal generalization of the metricity conditions for Weyl gauge theory [15].

For further investigation, it would be interesting to see whether the non-relativistic and anisotropic disformal transformation (see (30)) can be reproduced through holography. Concretely speaking, if the holography relates a bulk relativistic geometry to a boundary non-relativistic one and the disformal transformation for the bulk relativistic gravity is known, then one could in principle use this holography to obtain the corresponding anisotropic disformal transformation induced on the boundary. Second, it may also be interesting to figure out the disformal extension of Poincaré and Bargmann algebra exactly, then to obtain the disformal extension of Riemann–Cartan geometry, as well as NCG, via a gauging procedure. Third, as hinted at in the introduction, our results may be related to modified Newtonian dynamics paradigm [41,56] and other modified gravity theories. Finally, in view of the role of conformal transformation and Schrödinger algebra in non-relativistic holography as well as the recent discussions as regards disformal transformation in cosmology, one may speculate the possible utility of disformal transformation in Lifshitz holography [57] and holographic cosmology [58,59].

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