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Dark energy and normalization of the cosmological wave function

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Abstract Dark energy is investigated from the perspective of quantum cosmology. It is found that, together with an appropriate normal ordering factor q, only when there is dark energy can the cosmological wave function be normalized. This interesting observation may require further attention.

The accelerating expansion of our universe has presented a very challenging problem in theoretical physics as well as in cosmology [1,2]. We believe that the character of dark energy is deeply associated with the nature of quantum gravity. In fact, one of the authors showed that dark energy can be well described with the help of the holographic principle, a characteristic feature of any viable theory of quantum gravity. This holographic dark energy model [3] has become one of the most competitive and popular dark energy candidates, which also implies that a deep relation between dark energy and quantum gravity is well appreciated in the community.

On the other hand, as an application of quantum physics to the dynamical systems describing closed universes, quantum cosmology is also tightly related to quantum gravity. In spite of some misgivings, interesting hints as regards fundamental mathematical and physical questions can be obtained from quantum cosmology. The studies on quantum cosmology as a generally covariant and highly interacting quantum theory may also provide answers to questions concerning the interplay of symmetries, discrete structures, and so forth.

Having noticed the common connection between dark energy, quantum cosmology, and quantum gravity, a natural and interesting question to ask is whether there is a relation between dark energy and quantum cosmology. An answer to this problem may deepen our understanding of the dark energy and shed light on problems related to it.

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For a theory which still retains its highly speculative and controversial nature, it is necessary to clarify the notation we will use in the following at first. For a general Wheeler–DeWitt equation for quantum cosmology we have

$$\left[-\hbar^2\frac{\partial^2}{\partial a^2}-\hbar^2\frac{q}{a}\frac{\partial}{\partial a}+\frac{9\pi^2}{4G^2}\left(a^2-\frac{\Lambda}{3}a^4\right)+\mathcal{H}_{\text{matter}}\right]\Psi(a)=0, \quad (1)$$

where we assume a positive spatial curvature as reflected in the term proportional to a^2. We define the inner product of the wave function as

$$\mathcal{P}=\int a^q|\Psi|^2\text{d}a \quad (2)$$

and interpret $\text{d}\mathcal{P}$ as the probability that the universe stays within scale factor between a and $a+\text{d}a$ when the wave function is normalizable. This quantity will be of our central interest. Of course, the probability interpretation is not simple, for instance to return to usual quantum mechanics one needs to introduce time, as proposed in [1], one can imagine introducing a scalar field to play the role of time.

To focus on the key problem without loss of generality, it is convenient to suppose that the universe is closed and filled only with a special dark energy which is just the cosmological constant. In this setup, the Wheeler–DeWitt equation has a simpler form:

$$\left[-\frac{\partial^2}{\partial a^2}-\frac{q}{a}\frac{\partial}{\partial a}+\frac{1}{l_p^4}\left(a^2-\frac{a^4}{l_\Lambda^2}\right)\right]\Psi(a)=0 \quad (3)$$

with $l_p=(\frac{4G^2\hbar^2}{9\pi^2})^{\frac{1}{4}}, l_\Lambda=(\frac{3}{\Lambda})^{\frac{1}{2}}$ corresponding to the Planck scale and a length scale introduced by the cosmological constant, respectively.

In the region $a\ll l_\Lambda$, the energy induced by spatial curvature dominates the universe, thus the term $\frac{a^4}{l_\Lambda^2}$ in Eq. (3) can be neglected. In this situation, the general solution of Eq. (3) has the form

$$\begin{cases} \Psi(a) = C_1(\frac{a}{l_p})^{\frac{1-q}{2}} I_{\frac{q-1}{4}}(\frac{a^2}{2l_p^2}) + C_2(\frac{a}{l_p})^{\frac{1-q}{2}} K_{\frac{q-1}{4}}(\frac{a^2}{2l_p^2}), & \text{for} \quad q \geq 1; \\ \Psi(a) = C_1(\frac{a}{l_p})^{\frac{1-q}{2}} I_{\frac{1-q}{4}}(\frac{a^2}{2l_p^2}) + C_2(\frac{a}{l_p})^{\frac{1-q}{2}} K_{\frac{1-q}{4}}(\frac{a^2}{2l_p^2}), & \text{for} \quad q < 1. \end{cases} \tag{4}$$

It is well known that, for $0 <\mid z \mid \ll \sqrt{\alpha+1}$, the asymptotic behavior for modified Bessel functions is

$$I_\alpha(z) \sim \frac{1}{\Gamma(\alpha+1)}\left(\frac{z}{2}\right)^\alpha; \tag{5}$$

$$K_\alpha(z) \sim \begin{cases} -\ln(\frac{z}{2}) - \gamma, & \text{for} \quad \alpha = 0, \\ \frac{\Gamma(\alpha)}{2}(\frac{2}{z})^\alpha, & \text{for} \quad \alpha > 0. \end{cases}$$

These results imply the three following asymptotic behaviors for modified Bessel functions which are of our concern:

$$I_{\frac{q-1}{4}}\left(\frac{1}{2}\left(\frac{a}{l_p}\right)^2\right) \sim \left(\frac{a}{l_p}\right)^{\frac{q-1}{2}},$$
$$K_{\frac{q-1}{2}}\left(\frac{1}{4}\left(\frac{a}{l_p}\right)^2\right) \sim \left(\frac{a}{l_p}\right)^{\frac{1-q}{2}}, \quad \text{for} \quad q > 1; \tag{6}$$

$$I_0\left(\frac{1}{2}\left(\frac{a}{l_p}\right)^2\right) \sim A_1,$$
$$K_0\left(\frac{1}{2}\left(\frac{a}{l_p}\right)^2\right) \sim \ln a + A_2, \quad \text{for} \quad q = 1; \tag{7}$$

$$I_{\frac{1-q}{4}}\left(\frac{1}{2}\left(\frac{a}{l_p}\right)^2\right) \sim \left(\frac{a}{l_p}\right)^{\frac{1-q}{2}},$$
$$K_{\frac{1-q}{2}}\left(\frac{1}{4}\left(\frac{a}{l_p}\right)^2\right) \sim \left(\frac{a}{l_p}\right)^{\frac{q-1}{2}}, \quad \text{for} \quad q < 1. \tag{8}$$

After some tedious but straightforward calculations, one can find that the second term of the wave function (4) will always render the inner product $\mathcal{P} = \int_0^\epsilon a^q|\Psi|^2 \mathrm{d}a$ (ϵ is an arbitrary upper limit of the integral satisfying $\epsilon \ll l_\Lambda$) divergent unless the normal ordering ambiguity factor q is in the domain of $(-1, 3)$.

Another way to investigate the Wheeler–DeWitt equation in the region $a \ll l_\Lambda$ is to use the WKB method. Since the Wheeler–DeWitt equation is a second order differential function, boundary conditions are used in this method to select a particular solution. The two most famous wave functions are the Hartle–Hawking no-boundary wave function [4] and Vilenkin's tunneling wave function [5,6]. It will be interesting to find to which solutions their proposals correspond in the general solution space.

To proceed, one can write $\Psi(a)$ as $\Psi(a) = \mathrm{e}^{i\frac{S}{l_p^2}}$ with $S = S_0 + l_p^2\ S_1 + \mathcal{O}(l_p^4)$. Inserting this into Eq. (3) and arranging terms according to the order in l_p, one obtains the zeroth order and second order equations,

$$\partial_a S_0 = \pm i\sqrt{a^2 - \frac{a^4}{l_\Lambda^2}}, \quad \text{for} \quad a < l_\Lambda; \tag{9}$$

$$\partial_a S_0 \partial_a S_1 = \frac{i}{2}\left(\partial_a^2 S_0 + \frac{q}{a}\partial_a S_0\right). \tag{10}$$

The above two equations can be solved to obtain the WKB wave function

$$\Psi(a)_\pm = \mathrm{e}^{i(\frac{S_0}{l_p^2}+S_1)} = a^{-\frac{q}{2}}\frac{1}{\sqrt{p}}\mathrm{e}^{\pm\frac{1}{l_p^2}\int_a^{l_\Lambda}\mathrm{d}a'\,p(a')}, \quad \text{for} \quad a < l_\Lambda, \tag{11}$$

with $p(a) = \sqrt{a^2 - \frac{a^4}{l_\Lambda^2}}$. We have denoted the two linearly independent solutions by "±" according to Vilenkin. It is then easy to see that Hartle–Hawking's choice was Ψ_- while Vilenkin's choice was $\Psi_+ - \frac{i}{2}\Psi_-$ [5]. However, Vilenkin argued that Ψ_- is negligible except in the vicinity of l_Λ so that he actually used Ψ_+ in the region $a < l_\Lambda$. Notice that the behavior of wave function as $a \to 0$ is in fact not derivable from the WKB wave function, because the WKB method is valid in the region $a \gg l_p$. Thus, to see the true behavior of wave functions according to their choices, one needs to figure out what wave functions they chose out of the general solution (4). The strategy is to match Hartle–Hawking's and Vilenkin's WKB wave functions with (4) in a region where both the WKB solution and the asymptotic solution are valid, i.e. the region $l_p \ll a \ll l_\Lambda$. In this region, the WKB wave functions can be approximated by

$$\Psi_\pm \approx a^{-\frac{q+1}{2}}\mathrm{e}^{\pm\frac{1}{3}(\frac{l_\Lambda}{l_p})^2}\mathrm{e}^{\mp\frac{1}{2}(\frac{a}{l_p})^2}. \tag{12}$$

For the asymptotic solution (4), one needs to use the asymptotic form of modified Bessel functions for large arguments,

$$I_\alpha \sim \frac{\mathrm{e}^z}{\sqrt{2\pi z}}\left(1 - \frac{4z^2-1}{8z} + \mathcal{O}(z^{-2})\right), \quad \text{for} \quad |argz| < \frac{\pi}{2}, \tag{13}$$

$$K_\alpha \sim \frac{\mathrm{e}^{-z}}{\sqrt{2\pi z}}\left(1 + \frac{4z^2-1}{8z} + \mathcal{O}(z^{-2})\right), \quad \text{for} \quad |argz| < \frac{3\pi}{2}. \tag{14}$$

So Eq. (4) approximates

$$\Psi \sim C_1 a^{-\frac{q+1}{2}}\mathrm{e}^{\frac{1}{2}(\frac{a}{l_p})^2} + C_2 a^{-\frac{q+1}{2}}\mathrm{e}^{-\frac{1}{2}(\frac{a}{l_p})^2}. \tag{15}$$

Now it can be seen that Hartle–Hawking's choice corresponds to the modified Bessel function of the second kind, I, while Vilenkin's choice corresponds to the modified Bessel function of the first kind, K. It could be seen that Hartle–Hawking's wave function leads to a convergent probability whatever q is, while the inner product of Vilenkin's wave function is convergent only when $-1 < q < 3$.

In the region $a \gg l_\Lambda$, the cosmological-constant term prevails over the spatial curvature, and the term a^2 in Eq. (3) can be neglected. The general solution of Eq. (3) then is

$$\Psi(a) = C_1\left(\frac{a^3}{l_p^2 l_\Lambda}\right)^{\frac{1-q}{6}} J_{\frac{q-1}{6}}\left(\frac{1}{3}\frac{a^3}{l_p^2 l_\Lambda}\right) + C_2\left(\frac{a^3}{l_p^2 l_\Lambda}\right)^{\frac{1-q}{6}} Y_{\frac{q-1}{6}}\left(\frac{1}{3}\frac{a^3}{l_p^2 l_\Lambda}\right). \tag{16}$$

Using the asymptotic expansion of Bessel functions at a large argument,

$$J_\alpha(z) = \sqrt{\frac{2}{\pi z}}\left[\cos\left(z - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + e^{|Im(z)|}\mathcal{O}(|z|^{-1})\right], \quad \text{for} \quad |argz| < \pi, \tag{17}$$

$$Y_\alpha(z) = \sqrt{\frac{2}{\pi z}}\left[\sin\left(z - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + e^{|Im(z)|}\mathcal{O}(|z|^{-1})\right], \quad \text{for} \quad |argz| < \pi, \tag{18}$$

one can get

$$J_{\frac{q-1}{6}}\left(\frac{1}{3}\frac{a^3}{l_p^2 l_\Lambda}\right) \sim a^{-\frac{3}{2}}\cos\left(\frac{1}{3}\frac{a^3}{l_p^2 l_\Lambda} + \text{phase}\right); \tag{19}$$

$$Y_{\frac{q-1}{6}}\left(\frac{1}{3}\frac{a^3}{l_p^2 l_\Lambda}\right) \sim a^{-\frac{3}{2}}\sin\left(\frac{1}{3}\frac{a^3}{l_p^2 l_\Lambda} + \text{phase}\right). \tag{20}$$

Then it is easy to see that the inner product $\mathcal{P} = \int_\lambda^{+\infty} a^q|\Psi|^2 \mathrm{d}a$ (λ is an arbitrary lower limit of the integral satisfying $\lambda \gg l_\Lambda$) always converges regardless of the normal ordering factor ambiguity q.

The interesting thing is that, together with an appropriate normal ordering factor q, the cosmological-constant dark energy causes a convergent and thus normalizable cosmological wave function. It is natural to wonder whether the relation between cosmological-constant dark energy and the normalization of the cosmological wave function is only a coincidence or a profound observation which deserves more attention. To investigate this, we consider following the Wheeler–DeWitt equation:

$$\left[-\frac{\partial^2}{\partial a^2} - \frac{q}{a}\frac{\partial}{\partial a} + \frac{1}{l_p^4}\left(a^2 - \frac{1}{f(h)}\frac{a^h}{l_p^{h-2}}\right)\right]\Psi(a) = 0 \tag{21}$$

with $f(h)$ a general function of the parameter h and $f(4) = (\frac{l_\Lambda}{l_p})^2$ is required. Apparently, the term $\frac{1}{f(h)}\frac{a^h}{l_p^{h-2}}$ corresponds to a general energy component whose index of equation of state has the value of $w = \frac{1-h}{3}$. Thus, $h = 0, 1, 4$ correspond to radiation, matter, and the cosmological constant, respectively. When $h > 2$, in the region $a \to 0$, it is easy to see that the energy induced by spatial curvature again dominates the universe, thus the results got here will be the same as that for Eq. (3), i.e., the inner product for the wave function corresponding to the universe described by Eq. (21) with a small scale factor is convergent only when the normal ordering ambiguity factor q takes its value on the domain of $(-1, 3)$. For large a, Eq. (21) becomes

$$\left[-\frac{\partial^2}{\partial a^2} - \frac{q}{a}\frac{\partial}{\partial a} - \left(\frac{1}{f(h)}\frac{a^h}{l_p^{h+2}}\right)\right]\Psi(a) = 0. \tag{22}$$

Its general solution is

$$\Psi(a) = C_1 a^{\frac{1-q}{2}} J_{\frac{q-1}{h+2}}\left(\frac{2}{h+2}\frac{a^{\frac{h+2}{2}}}{l_p^{\frac{h+2}{2}}\sqrt{f}}\right) + C_2 a^{\frac{1-q}{2}} Y_{\frac{q-1}{h+2}}\left(\frac{2}{h+2}\frac{a^{\frac{h+2}{2}}}{l_p^{\frac{h+2}{2}}\sqrt{f}}\right). \tag{23}$$

Using the asymptotic forms of Bessel functions given in Eqs. (19) and (20), one can find that the asymptotic form of the general solution is

$$\Psi(a) \sim C_1 a^{-\frac{h+2}{4}}\cos\left[\frac{2}{h+2}\frac{a^{\frac{h+2}{2}}}{l_p^{\frac{h+2}{2}}\sqrt{f}} + \text{phase}\right] + C_2 a^{-\frac{h+2}{4}}\sin\left[\frac{2}{h+2}\frac{a^{\frac{h+2}{2}}}{l_p^{\frac{h+2}{2}}\sqrt{f}} + \text{phase}\right]. \tag{24}$$

Then it can be shown that the integral of probability still converges on the large a side for $h > 2$. It can also easily be seen that, after a similar process, the wave function is divergent when $h = 2$. When $h < 2$, the asymptotic wave function on the large a side becomes the same as Eq. (4). Notice that for large argument, the modified Bessel function of the first kind has an exponentially divergent part, while the modified Bessel function of the second kind has an exponentially convergent part. So the integral of the generic probability for $h < 2$ is divergent.

What has been shown is that, for an energy component whose index of equation of state is $w = \frac{1-h}{3}$, the cosmological wave function is normalizable only if the energy component is dark energy (with $h > 2$). It needs to be stressed that the above claim holds well only when the requirement $w < -\frac{1}{3}$ is always satisfied when a is large enough. To see this, let us consider a general energy component which leads to a Wheeler–DeWitt equation as follows:

$$\left[-\frac{\partial^2}{\partial a^2}-\frac{q}{a}\frac{\partial}{\partial a}+\frac{1}{l_p^4}\left(a^2-l_p^2 g\left(\frac{a}{l_p}\right)\right)\right]\Psi(a)=0 \quad (25)$$

with $g(\frac{a}{l_p})$ a function of $\frac{a}{l_p}$, which will prevail over a^2 when a is large enough. Denoting $\Psi(a)=\mathrm{e}^{i(\frac{S_0}{l_p^2}+S_1)}$, and supposing that the condition $|S'|^2 \gg l_p^2|S''+\frac{q}{x}S'|$, which takes the following form for our case:

$$g-x^2 \gg \frac{1}{2}\frac{g'-2x}{g-x^2}+\frac{q}{x}, \quad (26)$$

is satisfied by $g(x)$, then through the general procedure of the WKB method, one gets the wave function and the inner product corresponding to it as follows:

$$\Psi(a)=(S_0'x^2)^{-\frac{1}{2}}\mathrm{e}^{i\int\sqrt{g-x^2}\mathrm{d}a}, \quad (27)$$

$$\mathcal{P}\propto\int\frac{\mathrm{d}x}{\sqrt{g-x^2}}, \quad (28)$$

with $x=\frac{a}{l_p}$ and the prime denoting differentiation with respect to x. Since $g(\frac{a}{l_p})$ prevails over a^2 when a is large enough, Eq. (28) can be simplified further to

$$\Psi(a)=(S_0'x^2)^{-\frac{1}{2}}\mathrm{e}^{i\int\sqrt{g}\mathrm{d}a}, \quad (29)$$

$$\mathcal{P}\propto\int\frac{\mathrm{d}x}{\sqrt{g}}. \quad (30)$$

Now, considering a special energy component whose appearance in Wheeler–DeWitt equation is $g(x)=x^2\ln x$, the special character of this kind of energy component is that its index of the equation of state is $w=-\frac{1}{3}-\frac{1}{3\ln x}$, which will always be less than $-\frac{1}{3}$ (for a large) except at the point where the scale factor a is infinite. One can check that the term $g(x)=x^2\ln x$ indeed prevails over the term a^2 in Eq. (25) and it also satisfies condition (26); thus, one can safely insert it into (30) to find that the inner product is divergent. Now, consider another energy component for which the condition $g(x)>x^{2+\epsilon}$ for a large scale factor (ϵ is an arbitrary given positive constant) is satisfied. This function corresponds to an energy component with index of equation of state $w=-\frac{1+\epsilon}{3}$, which is less than $-\frac{1}{3}$ when a is large enough. Then, after a similar process, one can find

$$\mathcal{P}\propto\int_\lambda^{+\infty}\frac{\mathrm{d}x}{\sqrt{g}}<\int_\lambda^{+\infty}\frac{\mathrm{d}x}{x^{\frac{\epsilon}{2}+1}}=\frac{2}{\epsilon}\lambda^{-\frac{\epsilon}{2}}, \quad (31)$$

which is convergent.

The difference between the first and the second energy component considered above is that, while the second energy component always satisfies the condition $w<-\frac{1}{3}$ for large scale factor, the first energy component violates it at the point $a=\infty$. Then, despite the first energy component behaving precisely like ordinary dark energy most of the time in the sense that it can accelerate the universe, it still leads to a divergent inner product of the wave function because it deviates from dark energy at the point with $a=\infty$. Thus, as has been stressed, the claim that dark energy causes a normalizable cosmological wave function only holds in the circumstances that $w<-\frac{1}{3}$ is strictly satisfied when a is large enough.

To conclude, we study dark energy from the perspective of quantum cosmology in this work. It is found that, together with an appropriate normal ordering factor q (taking its value on the domain $(-1,3)$, dark energy leads to a normalizable cosmological wave function while the other energy components cannot. We think that the interesting relation between dark energy and normalization of the cosmological wave function may imply some deep features of dark energy which deserves more attention. For further investigation, it is an interesting question whether or not quantum cosmology gives us some hints on choosing the dark energy models. There might not be a good answer based on the cases considered in this paper as the functions representing dark energy are rather arbitrary. However, we believe that there exists an interesting answer to this question when the dynamics of the dark energy is known (in our case, when the dynamics of h is known). Furthermore, considering the fact that the exact meaning of the wave function of the universe and the problem of measurement is still under careful investigation in quantum cosmology, it may be interesting to consider this work in the light of a more precise version of quantum theory, like the many worlds interpretation or the de Broglie–Bohm theory.

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