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Tetraquark state candidates: $Y(4260)$, $Y(4360)$, $Y(4660)$, and $Z_c(4020/4025)$

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Abstract In this article, we construct the axialvector-diquark–axialvector-antidiquark type tensor current to interpolate both the vector- and the axialvector-tetraquark states, then calculate the contributions of the vacuum condensates up to dimension 10 in the operator product expansion, and we obtain the QCD sum rules for both the vector- and the axialvector-tetraquark states. The numerical results support assigning the $Z_c(4020/4025)$ to be the $J^{PC} = 1^{+-}$ diquark–antidiquark type tetraquark state, and assigning the $Y(4660)$ to be the $J^{PC} = 1^{--}$ diquark–antidiquark type tetraquark state. Furthermore, we take the $Y(4260)$ and $Y(4360)$ as the mixed charmonium–tetraquark states, and we construct the two-quark–tetraquark type tensor currents to study the masses and pole residues. The numerical results support assigning the $Y(4260)$ and $Y(4360)$ to be the mixed charmonium–tetraquark states.

1 Introduction

In 2005, the BaBar Collaboration studied the initial-state radiation process $e^+e^- \rightarrow \gamma_{ISR}\pi^+\pi^- J/\psi$ and observed the $Y(4260)$ in the $\pi^+\pi^- J/\psi$ invariant-mass spectrum, the measured mass and width are $\left(4259 \pm 8^{+2}_{-6}\right)$ MeV and $\left(88 \pm 23^{+6}_{-4}\right)$ MeV, respectively [1]. In 2007, the Belle Collaboration studied the initial-state radiation process $e^+e^- \rightarrow \gamma_{ISR}\pi^+\pi^-\psi'$, and observed two structures $Y(4360)$ and $Y(4660)$ in the $\pi^+\pi^-\psi'$ invariant mass distributions at $(4361\pm9\pm9)$ MeV with a width of $(74\pm15\pm10)$ MeV and $(4664 \pm 11 \pm 5)$ MeV with a width of $(48 \pm 15 \pm 3)$ MeV, respectively [2,3]. In 2008, the Belle Collaboration studied the initial-state radiation process $e^+e^- \rightarrow \gamma_{ISR}\Lambda_c^+\Lambda_c^-$ and observed a clear peak $Y(4630)$ in the $\Lambda_c^+\Lambda_c^-$ invariant mass distribution just above the $\Lambda_c^+\Lambda_c^-$ threshold, and

determined the mass and width to be $\left(4634^{+8+5}_{-7-8}\right)$ MeV and $\left(92^{+40+10}_{-24-21}\right)$ MeV, respectively [4]. The $Y(4660)$ and $Y(4630)$ may be the same particle according to the uncertainties of the masses and widths.

In 2013, the BESIII Collaboration observed the $Z_c^{\pm}(4025)$ near the $(D^*\bar{D}^*)^{\pm}$ threshold in the $\pi^{\mp}$ recoil mass spectrum in the process $e^+e^- \rightarrow (D^*\bar{D}^*)^{\pm}\pi^{\mp}$, and determined the mass and width $M_{Z_c(4025)} = (4026.3 \pm 2.6 \pm 3.7)$ MeV and $\Gamma_{Z_c^{\pm}(4025)} = (24.8 \pm 5.6 \pm 7.7)$ MeV [5]. Furthermore, the BESIII Collaboration observed the $Z_c^{\pm}(4020)$ in the $\pi^{\pm}h_c$ mass spectrum in the process $e^+e^- \rightarrow \pi^+\pi^-h_c$, and determined the mass and width $M_{Z_c^{\pm}(4020)} = (4022.9 \pm 0.8 \pm 2.7)$ MeV and $\Gamma_{Z_c^{\pm}(4020)} = (7.9 \pm 2.7 \pm 2.6)$ MeV [6]. In 2014, the BESIII Collaboration observed the $Z_c^0(4020)$ in the π^0h_c mass spectrum in the process $e^+e^- \rightarrow \pi^0\pi^0h_c$ and determined the mass $M_{Z_c^0(4020)} = (4023.9 \pm 2.2 \pm 3.8)$ MeV [7]. In 2015, the BESIII Collaboration observed the $Z_c^0(4025)$ in the π^0 recoil mass spectrum in the process $e^+e^- \rightarrow (D^*\bar{D}^*)^0\pi^0$, and determined the mass and width $M_{Z_c^0(4025)} = (4025.5^{+2.0}_{-4.7}\pm3.1)$ MeV and $\Gamma_{Z_c^0(4025)} = (23.0\pm6.0\pm1.0)$ MeV [8]. It is natural to assign the $Z_c(4020)$ and $Z_c(4025)$ to be the same particle.

There have been several tentative assignments for the $Y(4260)$, $Y(4360)$, $Y(4660)$, and $Z_c(4020)$, such as tetraquark states, molecular states, re-scattering effects, etc., for more literature on the X, Y, Z mesons, one can consult the recent reviews [9,10]. In this article, we will focus on the scenario of tetraquark states based on the QCD sum rules.

The diquarks $q_j^T C\Gamma q_k'$ have five structures in Dirac spinor space, where $C\Gamma = C\gamma_5$, C, $C\gamma_\mu\gamma_5$, $C\gamma_\mu$, and $C\sigma_{\mu\nu}$ for the scalar, pseudoscalar, vector, axialvector, and tensor diquarks, respectively. The structures $C\gamma_\mu$ and $C\sigma_{\mu\nu}$ are symmetric, while the structures $C\gamma_5$, C, and $C\gamma_\mu\gamma_5$ are antisymmetric. The attractive interactions of one-gluon exchange favor formation of the diquarks in color antitriplet, flavor antitriplet and spin singlet [11,12], while the favored configurations are

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the scalar- ($C\gamma_5$) and axialvector- ($C\gamma_\mu$) diquark states [13–15]. The calculations based on the QCD sum rules indicate that the heavy-light scalar- and axialvector-diquark states have almost degenerate masses [13,14]. We can construct the diquark–antidiquark type hidden charm tetraquark states [16–18],

$$C\gamma_5 \otimes \gamma_5 C, \quad C\gamma_\mu \otimes \gamma^\mu C, \tag{1}$$

the $C\gamma_5 \otimes \gamma_5 C$ type and $C\gamma_\mu \otimes \gamma^\mu C$ type currents couple potentially to the lowest scalar tetraquark states with the masses about 3.82 GeV [19] and 3.85 GeV [20], respectively. If the contribution of an additional P-wave to the mass is about 0.5 GeV, we can construct the vector currents

$$C\gamma_\alpha \otimes \partial_\mu \gamma^\alpha C, \quad C\gamma_5 \otimes \partial_\mu \gamma_5 C, \tag{2}$$

to study the vector-tetraquark states, the estimated masses are about 4.35 GeV, which happens to be the value of the mass of the $Y(4360)$ [20]. In Refs. [21,22], Zhang and Huang take the $C\gamma_5 \otimes \partial_\mu \gamma_5 C$ type currents to study the $Y(4360)$ and $Y(4660)$ with the QCD sum rules, and obtain the values $M_{Y(4360)} = (4.32 \pm 0.20)$ GeV and $M_{Y(4660)} = (4.69 \pm 0.36)$ GeV, which are consistent with the rough estimation $M_{Y(4360)} = 4.35$ GeV.

We can also construct the

$$C \otimes \gamma_\mu C, C\gamma_5 \otimes \gamma_5 \gamma_\mu C, \tag{3}$$

type currents to study the vector-tetraquark states [23,24]. One can consult Ref. [25] for more interpolating currents for the vector-tetraquark states without introducing additional P-wave. In Refs. [23,24], we observe that the $C \otimes \gamma_\mu C$ type and $C\gamma_5 \otimes \gamma_5 \gamma_\mu C$ type tetraquark states have degenerate (or slightly different) masses based on the QCD sum rules, the ground state masses of the vector-tetraquark states with the symbolic quark constituent $\bar{c}c\bar{q}q$ are about 4.95 GeV, which is much larger than the mass of the $Y(4660)$. In Ref. [26], Albuquerque and Nielsen take the $C\gamma_5 \otimes \gamma_5 \gamma_\mu C$ type current to study the $Y(4660)$ with the QCD sum rules and obtain the value $M_{Y(4660)} = 4.65$ GeV, which is in excellent agreement with the mass of the $Y(4660)$. Although both in Refs. [23,24] and in Ref. [26], the standard values of the vacuum condensates are taken, in Refs. [23,24], the QCD spectral densities are calculated at the energy scale $\mu = 1$ GeV and the value $m_c(\mu = 1\text{GeV}) = 1.35$ GeV is taken; while in Ref. [26], the vacuum condensates are taken at the energy scale $\mu = 1$ GeV and the $\overline{MS}$ mass $m_c(m_c) = 1.23$ GeV is taken, the energy scales of the QCD spectral densities are not specified. In Ref. [27], we suggest the formula $\mu = \sqrt{M_{X/Y/Z}^2 - (2\mathbb{M}_c)^2}$ with the effective mass $\mathbb{M}_c$ to determine the energy scales of the QCD spectral densities of the hidden charmed tetraquark states, and we evolve the vacuum condensates and the $\overline{MS}$ mass to the energy scale μ using the $C \otimes \gamma_\mu C$ type current; we obtain the mass 4.66 or 4.70 GeV for the $Y(4660)$.

In Refs. [28,29], the molecule currents,

$$J_\mu(x) = \bar{c}(x)\gamma_\mu c(x) \bar{q}(x)q(x), \tag{4}$$

are chosen to study the $Y(4260)$ and $Y(4660)$ in the QCD sum rules, and it is observed that the $Y(4660)$ can be assigned to be the $\psi' f_0(980)$ molecular state [28], and the $Y(4260)$ cannot be assigned to be the $J/\psi f_0(980)$ molecular state [29]. Again the parameters are taken as in Refs. [23,24] and in Ref. [26], respectively.

In Ref. [30], Dias et al. take the $Y(4260)$ as a mixed charmonium–tetraquark state and choose the current $J_\mu(x)$,

$$J_\mu(x) = J_\mu^2(x)\cos\theta + J_\mu^4(x)\sin\theta, \tag{5}$$

where

$$
\begin{aligned}
J_\mu^4(x) &= \frac{\epsilon^{ijk}\epsilon^{imn}}{\sqrt{2}}\left\{ q_j^T(x)C\gamma_5 c_k(x)\bar{q}_m(x)\gamma_\mu\gamma_5 C\bar{c}_n^T(x) \right. \\
&\quad \left. + q_j^T(x)C\gamma_5\gamma_\mu c_k(x)\bar{q}_m(x)\gamma_5 C\bar{c}_n^T(x) \right\}, \\
J_\mu^2(x) &= \frac{1}{\sqrt{2}}\langle\bar{q}q\rangle\,\bar{c}(x)\gamma_\mu c(x),
\end{aligned}
\tag{6}
$$

to study its mass and decay width with the QCD sum rules, and observe that at the mixing angle around $\theta \approx (53.0 \pm 0.5)°$, the mass of the $Y(4260)$ can be reproduced but the decay width is far below the experimental value.

In this article, we take the axialvector- ($C\gamma_\mu$) diquark states as the basic constituents [13–15], construct the

$$C\gamma_\mu \otimes \gamma_\nu C - C\gamma_\nu \otimes \gamma_\mu C, \tag{7}$$

type tensor current without introducing the additional P-wave to interpolate both the vector- and the axialvector-tetraquark states, and study the $Y(4260)$, $Y(4360)$, $Y(4660/4630)$, and $Z_c(4020/4025)$ with the QCD sum rules by calculating the operator product expansion up to the vacuum condensates of dimension 10. The tensor current is expected to couple to the vector-tetraquark state with smaller mass compared to the $C\gamma_\alpha \otimes \partial_\mu \gamma^\alpha C$, $C\gamma_5 \otimes \partial_\mu \gamma_5 C$, $C \otimes \gamma_\mu C$, $C\gamma_5 \otimes \gamma_5\gamma_\mu C$ type axialvector currents, so as to reproduce the mass of the $Y(4260)$ as the vector-tetraquark state. Furthermore, we study the $Z_c^0(4020/4025)$ as the axialvector-tetraquark state consists of an axialvector-diquark pair, which is expected to have slight larger mass than the $C\gamma_5 \otimes \gamma_\mu C$ type tetraquark state [13–15]. In Ref. [31], we choose the $C\gamma_5 \otimes \gamma_\mu C$ type current to study the axialvector-tetraquark states, and we obtain the mass $M_{Z_c(3900)} = 3.91^{+0.11}_{-0.09}$ GeV for the $Z_c(3900)$ with the assignment $J^{PC} = 1^{+-}$.

The article is arranged as follows: we derive the QCD sum rules for the masses and pole residues of the $Y(4260)$, $Y(4360)$, $Y(4660)$, and $Z_c(4020)$ as pure tetraquark states in Sect. 2; in Sect. 3, we derive the QCD sum rules for the masses and pole residues of the $Y(4260)$ and $Y(4360)$ as mixed charmonium–tetraquark states; Sect. 4 is reserved for our conclusion.

2 QCD sum rules for the $Y(4260)$, $Y(4360)$, $Y(4660)$, and $Z_c(4020)$ as pure tetraquark states

In the following, we write down the two-point correlation function $\Pi_{\mu\nu\alpha\beta}(p)$ in the QCD sum rules,

$$\Pi_{\mu\nu\alpha\beta}(p) = i \int d^4x e^{ip\cdot x} \langle 0|T\left\{\eta_{\mu\nu}(x)\eta_{\alpha\beta}^\dagger(0)\right\}|0\rangle, \qquad (8)$$

$$\eta_{\mu\nu}(x) = \frac{\epsilon^{ijk}\epsilon^{imn}}{2}\left\{u_j^T(x)C\gamma_\mu c_k(x)\bar{u}_m(x)\gamma_\nu C\bar{c}_n^T(x)\right.$$
$$+d_j^T(x)C\gamma_\mu c_k(x)\bar{d}_m(x)\gamma_\nu C\bar{c}_n^T(x)$$
$$-u_j^T(x)C\gamma_\nu c_k(x)\bar{u}_m(x)\gamma_\mu C\bar{c}_n^T(x)$$
$$\left.-d_j^T(x)C\gamma_\nu c_k(x)\bar{d}_m(x)\gamma_\mu C\bar{c}_n^T(x)\right\}, \qquad (9)$$

where the i, j, k, m, n are color indexes, the C is the charge conjugation matrix. The charged partner $\tilde{\eta}_{\mu\nu}(x)$,

$$\tilde{\eta}_{\mu\nu}(x) = \frac{\epsilon^{ijk}\epsilon^{imn}}{\sqrt{2}}\left\{u_j^T(x)C\gamma_\mu c_k(x)\bar{d}_m(x)\gamma_\nu C\bar{c}_n^T(x)\right.$$
$$\left.-u_j^T(x)C\gamma_\nu c_k(x)\bar{d}_m(x)\gamma_\mu C\bar{c}_n^T(x)\right\}, \qquad (10)$$

couples to the $Z_c^+(4020/4025)$ potentially. In the isospin limit, the currents $\eta_{\mu\nu}(x)$ and $\tilde{\eta}_{\mu\nu}(x)$ couple to the tetraquark states with degenerate masses.

At the hadronic side, we can insert a complete set of intermediate hadronic states with the same quantum numbers as the current operator $\eta_{\mu\nu}(x)$ into the correlation function $\Pi_{\mu\nu\alpha\beta}(p)$ to obtain the hadronic representation [32–34]. After isolating the ground state contributions of the axialvector- and vector-tetraquark states, we get the following results:

$$\Pi_{\mu\nu\alpha\beta}(p) = \frac{\lambda_Z^2}{M_Z^2 - p^2}\left(p^2 g_{\mu\alpha}g_{\nu\beta} - p^2 g_{\mu\beta}g_{\nu\alpha}\right.$$
$$- g_{\mu\alpha}p_\nu p_\beta - g_{\nu\beta}p_\mu p_\alpha + g_{\mu\beta}p_\nu p_\alpha + g_{\nu\alpha}p_\mu p_\beta)$$
$$+\frac{\lambda_Y^2}{M_Y^2 - p^2}\left(-g_{\mu\alpha}p_\nu p_\beta - g_{\nu\beta}p_\mu p_\alpha\right.$$
$$\left.+g_{\mu\beta}p_\nu p_\alpha + g_{\nu\alpha}p_\mu p_\beta\right) + \cdots, \qquad (11)$$

where the Z denotes the axialvector-tetraquark state $Z_c(4020)$, the Y denotes the vector-tetraquark state $Y(4260)$, $Y(4360)$ or $Y(4660)$, the pole residues λ_Z and λ_Y are defined by

$$\langle 0|\eta_{\mu\nu}(0)|Z_c(p)\rangle = \lambda_Z \epsilon_{\mu\nu\alpha\beta}\varepsilon^\alpha p^\beta,$$
$$\langle 0|\eta_{\mu\nu}(0)|Y(p)\rangle = \lambda_Y\left(\varepsilon_\mu p_\nu - \varepsilon_\nu p_\mu\right), \qquad (12)$$

the ε_μ are the polarization vectors of the vector- and axialvector-tetraquark states with the following property:

$$\sum_\lambda \varepsilon_\mu^*(\lambda, p)\varepsilon_\nu(\lambda, p) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}. \qquad (13)$$

We can rewrite the correlation function $\Pi_{\mu\nu\alpha\beta}(p)$ into the following form according to Lorentz covariance:

$$\Pi_{\mu\nu\alpha\beta}(p) = \Pi_Z(p^2)\left(p^2 g_{\mu\alpha}g_{\nu\beta} - p^2 g_{\mu\beta}g_{\nu\alpha}\right.$$
$$- g_{\mu\alpha}p_\nu p_\beta - g_{\nu\beta}p_\mu p_\alpha + g_{\mu\beta}p_\nu p_\alpha + g_{\nu\alpha}p_\mu p_\beta)$$
$$+\Pi_Y(p^2)\left(-g_{\mu\alpha}p_\nu p_\beta - g_{\nu\beta}p_\mu p_\alpha\right.$$
$$\left.+g_{\mu\beta}p_\nu p_\alpha + g_{\nu\alpha}p_\mu p_\beta\right). \qquad (14)$$

Now we project out the components $\Pi_Z(p^2)$ and $\Pi_Y(p^2)$ by introducing the operators $P_Z^{\mu\nu\alpha\beta}$ and $P_Y^{\mu\nu\alpha\beta}$,

$$\tilde{\Pi}_Z(p^2) = p^2\Pi_Z(p^2) = P_Z^{\mu\nu\alpha\beta}\Pi_{\mu\nu\alpha\beta}(p),$$
$$\tilde{\Pi}_Y(p^2) = p^2\Pi_Y(p^2) = P_Y^{\mu\nu\alpha\beta}\Pi_{\mu\nu\alpha\beta}(p), \qquad (15)$$

where

$$P_Z^{\mu\nu\alpha\beta} = \frac{1}{6}\left(g^{\mu\alpha} - \frac{p^\mu p^\alpha}{p^2}\right)\left(g^{\nu\beta} - \frac{p^\nu p^\beta}{p^2}\right),$$
$$P_Y^{\mu\nu\alpha\beta} = \frac{1}{6}\left(g^{\mu\alpha} - \frac{p^\mu p^\alpha}{p^2}\right)\left(g^{\nu\beta} - \frac{p^\nu p^\beta}{p^2}\right) - \frac{1}{6}g^{\mu\alpha}g^{\nu\beta}. \qquad (16)$$

In the following, we carry out the operator product expansion for the correlation function $\Pi_{\mu\nu\alpha\beta}(p)$ up to the vacuum condensates of dimension 10, and project out the components

$$\tilde{\Pi}_Z(p^2) = P_Z^{\mu\nu\alpha\beta}\Pi_{\mu\nu\alpha\beta}(p),$$
$$\tilde{\Pi}_Y(p^2) = P_Y^{\mu\nu\alpha\beta}\Pi_{\mu\nu\alpha\beta}(p), \qquad (17)$$

at the QCD side, and we obtain the QCD spectral densities through dispersion relation,

$$\rho_Z(s) = \frac{\mathrm{Im}\tilde{\Pi}_Z(s)}{\pi},$$
$$\rho_Y(s) = \frac{\mathrm{Im}\tilde{\Pi}_Y(s)}{\pi}, \qquad (18)$$

where we take into account the contributions of the terms D_0, D_3, D_4, D_5, D_6, D_7, D_8, and D_{10},

$$D_0 = \text{perturbative terms},$$
$$D_3 \propto \langle\bar{q}q\rangle,$$
$$D_4 \propto \left\langle\frac{\alpha_s GG}{\pi}\right\rangle,$$
$$D_5 \propto \langle\bar{q}g_s\sigma Gq\rangle,$$
$$D_6 \propto \langle\bar{q}q\rangle^2, \ g_s^2\langle\bar{q}q\rangle^2,$$
$$D_7 \propto \langle\bar{q}q\rangle\left\langle\frac{\alpha_s GG}{\pi}\right\rangle,$$
$$D_8 \propto \langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle,$$
$$D_{10} \propto \langle\bar{q}g_s\sigma Gq\rangle^2, \ \langle\bar{q}q\rangle^2\left\langle\frac{\alpha_s GG}{\pi}\right\rangle. \qquad (19)$$

The explicit expressions of the QCD spectral densities $\rho_Z(s)$ and $\rho_Y(s)$ are given in the appendix. The four-quark condensate $g_s^2\langle\bar{q}q\rangle^2$ comes from the terms $\langle\bar{q}\gamma_\mu t^a q g_s D_\eta G_{\lambda\tau}^a\rangle$, $\langle\bar{q}_j D_\mu^\dagger D_\nu^\dagger D_\alpha^\dagger q_i\rangle$ and $\langle\bar{q}_j D_\mu D_\nu D_\alpha q_i\rangle$, rather than comes

from the perturbative corrections of $\langle \bar{q}q \rangle^2$ (see Ref. [31] for the technical details). The condensates $\langle g_s^3 GGG \rangle$, $\langle \frac{\alpha_s GG}{\pi} \rangle^2$, $\langle \frac{\alpha_s GG}{\pi} \rangle \langle \bar{q} g_s \sigma Gq \rangle$ have the dimensions 6, 8, 9, respectively, but they are the vacuum expectations of the operators of the order $\mathcal{O}(\alpha_s^{3/2})$, $\mathcal{O}(\alpha_s^2)$, $\mathcal{O}(\alpha_s^{3/2})$, respectively, and neglected. We take the truncations $n \leq 10$ and $k \leq 1$ in a consistent way, the operators of the orders $\mathcal{O}(\alpha_s^k)$ with $k > 1$ are discarded. In Tables 1 and 2, we show the contributions of the vacuum condensates of dimension 4 and 10 explicitly, $|D_4| = 1\%$, $(1-2)\%$, $(2-3)\%$, 2% in the Borel windows for the $Z_c(4020)$, $Y(4660)$, $Y(4260)$, $Y(4360)$, respectively; and $D_{10} \ll 1\%$, $\ll 1\%$, $1 \leq \%$, $<1\%$ in the Borel windows for the $Z_c(4020)$, $Y(4660)$, $Y(4260)$, $Y(4360)$, respectively. Although the vacuum condensates are vacuum expectations of the operators of the order $\mathcal{O}(\alpha_s)$ both in the terms D_4 and D_{10}, $|D_4| \gg D_{10}$, as there are additional factors $\frac{1}{T^2}$, $\frac{1}{T^4}$ and $\frac{1}{T^6}$ in the D_{10}, which suppress the contributions greatly. The operators in the condensates $\langle g_s^3 GGG \rangle$, $\langle \frac{\alpha_s GG}{\pi} \rangle^2$, $\langle \frac{\alpha_s GG}{\pi} \rangle \langle \bar{q} g_s \sigma Gq \rangle$ are suppressed by additional factors $\mathcal{O}(\alpha_s^{1/2})$, $\mathcal{O}(\alpha_s)$, $\mathcal{O}(\alpha_s^{1/2})$, respectively and additional factor $\frac{1}{T^2}$ compared with the operator in the D_4 or $\langle \frac{\alpha_s GG}{\pi} \rangle$, their contributions are expected to be of the same order as the D_{10} and negligible. In Ref. [35], Zhang calculates the contributions of the $\langle g_s^3 GGG \rangle$, $\langle \frac{\alpha_s GG}{\pi} \rangle^2$, $\langle \frac{\alpha_s GG}{\pi} \rangle \langle \bar{q} g_s \sigma Gq \rangle$ explicitly in the QCD sum rules for the $Z_c(3900)$ as a $\bar{D}D^*$ molecular state, their contributions are tiny in the Borel window.

Once the analytical expressions of the QCD spectral densities $\rho_Z(s)$ and $\rho_Y(s)$ are obtained, we can take the quark–hadron duality below the continuum thresholds s_0 and perform a Borel transform with respect to the variable $P^2 = -p^2$ to obtain the QCD sum rules:

$$\lambda_Z^2 M_Z^2 \exp\left(-\frac{M_Z^2}{T^2}\right) = \int_{4m_c^2}^{s_0} ds\, \rho_Z(s) \exp\left(-\frac{s}{T^2}\right), \quad (20)$$

$$\lambda_Y^2 M_Y^2 \exp\left(-\frac{M_Y^2}{T^2}\right) = \int_{4m_c^2}^{s_0} ds\, \rho_Y(s) \exp\left(-\frac{s}{T^2}\right). \quad (21)$$

We differentiate Eqs. (20) and (21) with respect to $\frac{1}{T^2}$, eliminate the pole residues λ_Z and λ_Y, and we obtain the QCD sum rules for the masses of the axialvector- and vector-tetraquark states,

$$M_Z^2 = \frac{\int_{4m_c^2}^{s_0} ds\, \frac{d}{d(-1/T^2)} \rho_Z(s) \exp\left(-\frac{s}{T^2}\right)}{\int_{4m_c^2}^{s_0} ds\, \rho_Z(s) \exp\left(-\frac{s}{T^2}\right)}, \quad (22)$$

$$M_Y^2 = \frac{\int_{4m_c^2}^{s_0} ds\, \frac{d}{d(-1/T^2)} \rho_Y(s) \exp\left(-\frac{s}{T^2}\right)}{\int_{4m_c^2}^{s_0} ds\, \rho_Y(s) \exp\left(-\frac{s}{T^2}\right)}. \quad (23)$$

We take the standard values of the vacuum condensates, $\langle \bar{q}q \rangle = -(0.24 \pm 0.01\,\text{GeV})^3$, $\langle \bar{q} g_s \sigma Gq \rangle = m_0^2 \langle \bar{q}q \rangle$,

$m_0^2 = (0.8 \pm 0.1)\,\text{GeV}^2$, $\langle \frac{\alpha_s GG}{\pi} \rangle = (0.33\,\text{GeV})^4$ at the energy scale $\mu = 1\,\text{GeV}$ [32–34]. The quark condensate and mixed quark condensate evolve with the renormalization group equation, $\langle \bar{q}q \rangle(\mu) = \langle \bar{q}q \rangle(Q) \left[\frac{\alpha_s(Q)}{\alpha_s(\mu)}\right]^{\frac{4}{9}}$ and $\langle \bar{q} g_s \sigma Gq \rangle(\mu) = \langle \bar{q} g_s \sigma Gq \rangle(Q) \left[\frac{\alpha_s(Q)}{\alpha_s(\mu)}\right]^{\frac{2}{27}}$.

In the article, we take the $\overline{MS}$ mass $m_c(m_c) = (1.275 \pm 0.025)\,\text{GeV}$ from the Particle Data Group [36], and take into account the energy-scale dependence of the $\overline{MS}$ mass from the renormalization group equation,

$$m_c(\mu) = m_c(m_c) \left[\frac{\alpha_s(\mu)}{\alpha_s(m_c)}\right]^{\frac{12}{25}},$$

$$\alpha_s(\mu) = \frac{1}{b_0 t} \left[1 - \frac{b_1}{b_0^2} \frac{\log t}{t} + \frac{b_1^2(\log^2 t - \log t - 1) + b_0 b_2}{b_0^4 t^2}\right], \quad (24)$$

where $t = \log \frac{\mu^2}{\Lambda^2}$, $b_0 = \frac{33 - 2n_f}{12\pi}$, $b_1 = \frac{153 - 19n_f}{24\pi^2}$, $b_2 = \frac{2857 - \frac{5033}{9}n_f + \frac{325}{27}n_f^2}{128\pi^3}$, $\Lambda = 213$, 296 and 339 MeV for the flavors $n_f = 5$, 4, and 3, respectively [36].

In previous work, we described the hidden charm (or bottom) four-quark systems $q\bar{q}'Q\bar{Q}$ by a double-well potential [20,27,37–42]. In the four-quark system $q\bar{q}'Q\bar{Q}$, the Q-quark serves as a static well potential and combines with the light quark q to form a heavy diquark $\mathcal{D}_{qQ}^i$ in color antitriplet $q + Q \to \mathcal{D}_{qQ}^i$ [20,27,37–39], or combines with the light antiquark $\bar{q}'$ to form a heavy meson in color singlet (meson-like state in color octet) $\bar{q}' + Q \to \bar{q}'Q$ $(\bar{q}'\lambda^a Q)$ [40–42]; the $\bar{Q}$-quark serves as another static well potential and combines with the light antiquark $\bar{q}'$ to form a heavy antidiquark $\mathcal{D}_{\bar{q}'\bar{Q}}^i$ in color triplet $\bar{q}' + \bar{Q} \to \mathcal{D}_{\bar{q}'\bar{Q}}^i$ [20,27,37–39], or combines with the light quark q to form a heavy meson in color singlet (meson-like state in color octet) $q + \bar{Q} \to \bar{Q}q$ $(\bar{Q}\lambda^a q)$ [40–42], where the i is color index, the λ^a is Gell-Mann matrix. Then

$$\mathcal{D}_{qQ}^i + \mathcal{D}_{\bar{q}'\bar{Q}}^i \to \text{compact tetraquark states},$$
$$\bar{q}'Q + \bar{Q}q \to \text{loose molecular states},$$
$$\bar{q}'\lambda^a Q + \bar{Q}\lambda^a q \to \text{molecule-like states}, \quad (25)$$

the two heavy quarks Q and $\bar{Q}$ stabilize the four-quark systems $q\bar{q}'Q\bar{Q}$, just as in the case of the $(\mu^- e^+)(\mu^+ e^-)$ molecule in QED [43].

In Refs. [20,27,31,37–42], we study the acceptable energy scales of the QCD spectral densities for the hidden charm (bottom) four-quark systems $q\bar{q}'Q\bar{Q}$ with the QCD sum rules in detail for the first time, and suggest the formula
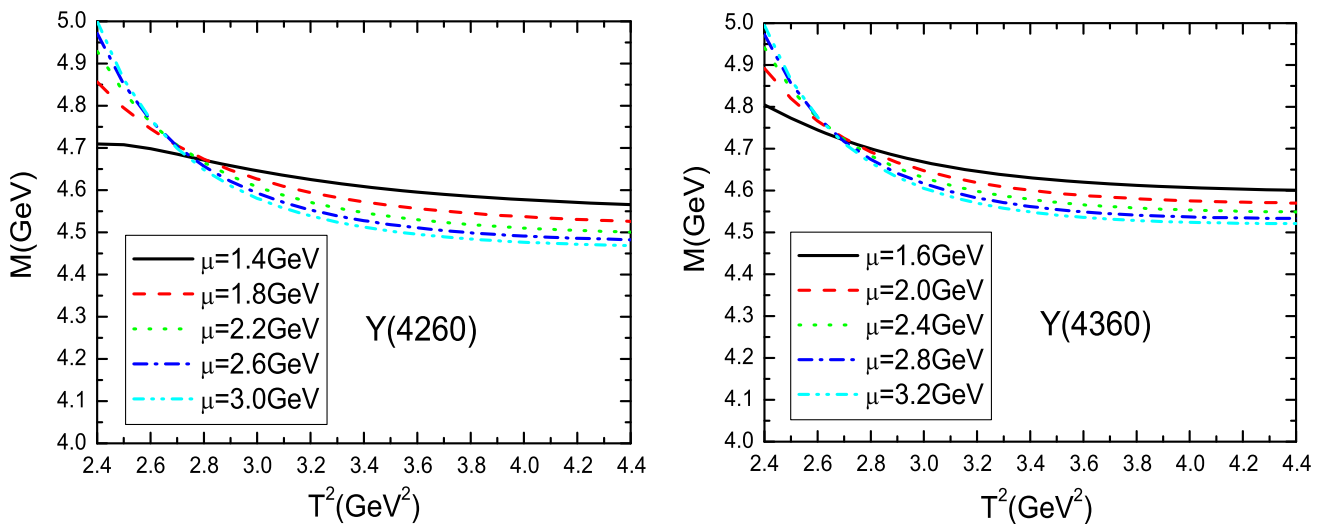
$$\mu = \sqrt{M_{X/Y/Z}^2 - (2\mathbb{M}_Q)^2}, \quad (26)$$

Fig. 1 The predicted masses with variations of the Borel parameters T^2 and the energy scales μ

to determine the energy scales, where the X, Y, Z denote the four-quark systems, and the $\mathbb{M}_Q$ denotes the effective heavy quark masses. In Refs. [31,37–39], we obtain the optimal value of the effective mass for the diquark–antidiquark type tetraquark states, $\mathbb{M}_c = 1.8$ GeV. Recently, we re-checked the numerical calculations and found that there exists a small error involving the mixed condensates. The Borel windows are modified slightly and the numerical results are also improved slightly after the small error is corrected, the conclusions survive, the optimal value of the effective mass is $\mathbb{M}_c = 1.82$ GeV for the diquark–antidiquark type tetraquark states. In this article, we choose the value $\mathbb{M}_c = 1.82$ GeV.

First of all, we assume that the $Y(4260)$ and $Y(4360)$ are the ground state vector-tetraquark states, the energy gap between the ground states and the first radial excited states is about $(0.4$–$0.6)$ GeV, just like that of the conventional mesons. In case I, the $Y(4260)$ is the ground state vector-tetraquark state; in case II, the $Y(4360)$ is the ground state vector-tetraquark state.

In Fig. 1, we plot the masses of the vector-tetraquark states with variations of the Borel parameters T^2 and energy scales μ for the continuum threshold parameters $s^0_{Y(4260)} = 23$ GeV^2 and $s^0_{Y(4360)} = 24$ GeV^2, respectively. According to the formula in Eq. (26), the energy scales $\mu_{Y(4260)} = 2.2$ GeV and $\mu_{Y(4360)} = 2.4$ GeV are the optimal energy scales. From Fig. 1, we can see that the masses decrease monotonously with increase of the energy scales at the value $T^2 > 2.7$ GeV^2. However, it is impossible to reproduce the experimental values even if much larger energy scales are taken, the QCD sum rules do not support assigning the $Y(4260)$ and $Y(4360)$ to be the vector-tetraquark states.

In the conventional QCD sum rules [32–34], there are two criteria (pole dominance at the phenomenological side

and convergence of the operator product expansion) for choosing the Borel parameters T^2 and continuum threshold parameters s_0. Now we assume the tensor current couples potentially to the vector-tetraquark state $Y(4660)$ and the axialvector-tetraquark state $Z_c(4020)$, and search for the Borel parameters T^2 and continuum threshold parameters s_0. The resulting Borel parameters, continuum threshold parameters, energy scales, pole contributions, and contributions of the vacuum condensates of dimension 10 are shown in Table 1.

Then we take into account all uncertainties of the input parameters, and obtain the values of the masses (and pole residues) of the axialvector- and vector-tetraquark states, which are shown in Fig. 2,

$$M_{Z_c(4020)} = (4.01 \pm 0.08)\ \text{GeV},$$
$$\lambda_{Z_c(4020)} = (7.31 \pm 0.99) \times 10^{-3}\ \text{GeV}^4, \tag{27}$$
$$M_{Y(4660)} = (4.66 \pm 0.09)\ \text{GeV},$$
$$\lambda_{Y(4660)} = (1.33 \pm 0.15) \times 10^{-2}\ \text{GeV}^4. \tag{28}$$

The present prediction $M_{Z_c(4020)} = (4.01 \pm 0.08)$ GeV is consistent with the experimental values $M_{Z_c^\pm(4025)} = (4026.3 \pm 2.6 \pm 3.7)$ MeV, $M_{Z_c^\pm(4020)} = (4022.9 \pm 0.8 \pm 2.7)$ MeV, $M_{Z_c^0(4020)} = (4023.9 \pm 2.2 \pm 3.8)$ MeV, $M_{Z_c^0(4025)} = (4025.5^{+2.0}_{-4.7} \pm 3.1)$ MeV from the BESIII Collaboration [5–8], which favors assigning the $Z_c(4020/4025)$ to be the $J^{PC} = 1^{+-}$ diquark–antidiquark type tetraquark state. In Ref. [20], the contributions of the vector- and axialvector-tetraquark states are not separated explicitly, the prediction $M_{Z_c(4020/4025)} = \left(4.02^{+0.07}_{-0.08}\right)$ GeV is consistent with the present value $M_{Z_c(4020)} = (4.01 \pm 0.08)$ GeV, which indicates the contamination from the vector-tetraquark state $Y(4660)$ is small, as the energy gap $M_{Y(4660)} - M_{Z_c(4020)} \approx 0.65$ GeV. The present prediction $M_{Y(4660)} =$

Table 1 The Borel parameters, continuum threshold parameters, energy scales, pole contributions, and contributions of the vacuum condensates of dimension 4 and 10 for the $Z_c(4020)$ and $Y(4660)$

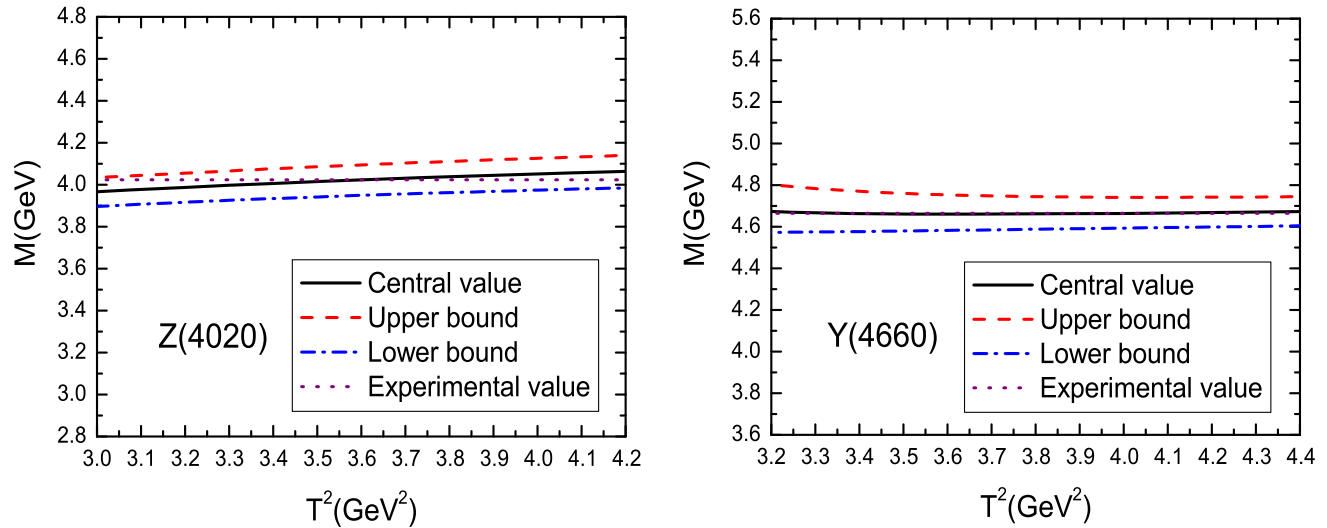|	$T^2\,(\mathrm{GeV}^2)$	$s_0\,(\mathrm{GeV}^2)$	$\mu\,(\mathrm{GeV})$	Pole	$|D_4|$	D_{10}
$Z_c(4020)$	3.2–3.6	21.0 ± 1.0	1.7	(40–61) %	1 %	≪1 %
$Y(4660)$	3.5–3.9	26.5 ± 1.0	2.9	(46–64) %	(1–2) %	≪1 %

Fig. 2 The masses with variations of the Borel parameters T^2 for the tetraquark states $Z_c(4020)$ and $Y(4660)$

(4.66 ± 0.09) GeV is consistent with the experimental value $M_{Y(4660)} = (4665 \pm 10)$ MeV within uncertainty [36], which favors assigning the $Y(4660)$ to be the vector-diquark–antidiquark type tetraquark state.

Now we can see that all the three diquark–antidiquark type currents $C \otimes \gamma_\mu C$, $C\gamma_5 \otimes \gamma_5\gamma_\mu C$ [23,24,26,27], $C\gamma_\mu \otimes \gamma_\nu C - C\gamma_\nu \otimes \gamma_\mu C$, couple potentially to the vector-tetraquark state $Y(4660)$. In Ref. [25], Chen and Zhu observe that the $C\gamma^\nu \otimes \sigma_{\mu\nu} C$ type current also couples potentially to the $Y(4660)$. The interpolating currents of the types

$$C \otimes \gamma_\mu C, \quad C\gamma_5 \otimes \gamma_5\gamma_\mu C, \quad C\gamma^\nu \otimes \sigma_{\mu\nu} C, \tag{29}$$

have unstable diquarks, such as the pseudoscalar C, vector $C\gamma_\mu\gamma_5$, tensor $C\sigma_{\mu\nu}$ diquarks, and couple potentially to the tetraquark states with the additional P-wave [44,45]. In this article, we observe that the $C\gamma_\mu \otimes \gamma_\nu C - C\gamma_\nu \otimes \gamma_\mu C$ type current without unstable diquarks also couples potentially to the vector-tetraquark state with the additional P-wave, however, the large mass (4.66 ± 0.09) GeV disfavors assigning the $Y(4260)$ and $Y(4360)$ to be the vector-tetraquark states. In Refs. [44,45], the $Y(4260)$ is identified as the $C\gamma_5 \otimes \gamma_5 C$ type vector-tetraquark state with an additional P-wave. On the other hand, we can also construct the $C\gamma_\alpha \otimes \partial_\mu \gamma^\alpha C$ type and $C\gamma_5 \otimes \partial_\mu \gamma_5 C$ type diquark–antidiquark currents to interpolate the vector-tetraquark states [21,22].

3 QCD sum rules for the $Y(4260)$ and $Y(4360)$ as mixed charmonium–tetraquark states

Now we take the $Y(4260)$ and $Y(4360)$ to be the mixed charmonium–tetraquark states, and we study the masses and pole residues with the QCD sum rules. First, let us write down the interpolating current,

$$J_{\mu\nu}(x) = \eta_{\mu\nu}(x) \cos\theta + \frac{i}{3} \langle \bar{q}q \rangle \bar{c}(x)\sigma_{\mu\nu}c(x) \sin\theta, \tag{30}$$

where the θ is the mixing angle, the $\frac{i}{3}\langle \bar{q}q \rangle$ is normalization factor [46]. The calculations can be carried out straightforwardly with the simple replacement

$$\eta_{\mu\nu}(x) \rightarrow J_{\mu\nu}(x) \tag{31}$$

in the correlation function $\Pi_{\mu\nu\alpha\beta}(p)$ in Eq. (1). The resulting QCD sum rules are

$$\lambda_Y^2 M_Y^2 \exp\left(-\frac{M_Y^2}{T^2}\right)$$
$$= \int_{4m_c^2}^{s_0} \mathrm{d}s \left[\cos^2\theta\, \rho_Y(s) + 2\sin\theta\cos\theta\, \rho_m(s)\right.$$
$$\left. + \sin^2\theta\, \rho_2(s)\right] \exp\left(-\frac{s}{T^2}\right), \tag{32}$$

where the $\rho_Y(s)$ is the QCD spectral density of the tetraquark component shown in Eq. (18), and

Table 2 The mixing angles, Borel parameters, continuum threshold parameters, energy scales, pole contributions, and contributions of the vacuum condensates of dimension 4 and 10 for the $Y(4260)$ and $Y(4360)$

	θ	$T^2(\text{GeV}^2)$	$s_0(\text{GeV}^2)$	$\mu(\text{GeV})$	Pole	$\lvert D_4\rvert$	D_{10}
$Y(4260)$	$5.84°$	2.9–3.3	23.0 ± 1.0	2.2	$(40\text{–}63)\,\%$	$(2\text{–}3)\,\%$	$\leq 1\,\%$
$Y(4360)$	$5.61°$	3.1–3.5	24.0 ± 1.0	2.4	$(42\text{–}64)\,\%$	$2\,\%$	$< 1\,\%$

$$\rho_m(s) = \rho_2(s) + \frac{\langle \bar{q}q \rangle \langle \bar{q}g_s \sigma Gq \rangle}{144\pi^2} \int_{y_i}^{y_f} \mathrm{d}y$$

$$\times \left[1 + \frac{\widetilde{m}_c^2}{2} \delta(s - \widetilde{m}_c^2) \right], \tag{33}$$

$$\rho_2(s) = \frac{\langle \bar{q}q \rangle^2}{12\pi^2} \int_{y_i}^{y_f} \mathrm{d}y \left[y(1-y)s + m_c^2 \right]$$

$$+ \frac{\langle \bar{q}q \rangle^2}{72} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int_0^1 \mathrm{d}y \left(\frac{1}{3} - \frac{\widetilde{m}_c^2}{T^2} \right) \delta(s - \widetilde{m}_c^2)$$

$$+ \frac{m_c^2 \langle \bar{q}q \rangle^2}{72T^2} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle$$

$$\times \int_0^1 \mathrm{d}y \left[\frac{1}{y^2} + \frac{1}{(1-y)^2} \right] \delta(s - \widetilde{m}_c^2)$$

$$+ \frac{m_c^2 \langle \bar{q}q \rangle^2}{108T^2} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int_0^1 \mathrm{d}y \left[\frac{1-y}{y^2} + \frac{y}{(1-y)^2} \right]$$

$$\times \left(1 - \frac{\widetilde{m}_c^2}{T^2} \right) \delta(s - \widetilde{m}_c^2), \tag{34}$$

$y_f = \frac{1 + \sqrt{1 - 4m_c^2/s}}{2}$, $y_i = \frac{1 - \sqrt{1 - 4m_c^2/s}}{2}$, $\widetilde{m}_c^2 = \frac{m_c^2}{y(1-y)}$, $\int_{y_i}^{y_f} \mathrm{d}y \to \int_0^1 \mathrm{d}y$, when the δ function $\delta\left(s - \widetilde{m}_c^2\right)$ appears.

We differentiate Eq. (32) with respect to $\frac{1}{T^2}$, then eliminate the pole residues λ_Y, and obtain the QCD sum rules for the masses of the mixed charmonium–tetraquark states,

We take into account all uncertainties of the input parameters, and obtain the values of the masses (and pole residues) of the $Y(4260)$ and $Y(4360)$ as mixed charmonium–tetraquark states, which are shown explicitly in Fig. 3,

$$M_{Y(4260)} = (4.26 \pm 0.11)\,\text{GeV},$$

$$\lambda_{Y(4260)} = (6.72 \pm 1.33) \times 10^{-3}\,\text{GeV}^4, \tag{36}$$

$$M_{Y(4360)} = (4.36 \pm 0.10)\,\text{GeV},$$

$$\lambda_{Y(4360)} = (8.32 \pm 1.36) \times 10^{-3}\,\text{GeV}^4. \tag{37}$$

The prediction $M_{Y(4260)} = (4.26 \pm 0.11)\,\text{GeV}$ is consistent with the experimental value $M_{Y(4260)} = \left(4259 \pm 8^{+2}_{-6}\right)\,\text{MeV}$ [1], which favors assigning the $Y(4260)$ to be the mixed charmonium–tetraquark state. On the other hand, the prediction $M_{Y(4360)} = (4.36 \pm 0.10)\,\text{GeV}$ is consistent with the experimental value $M_{Y(4360)} = (4361 \pm 9 \pm 9)\,\text{MeV}$ [2,3], which also favors assigning the $Y(4360)$ to be the mixed charmonium–tetraquark state. In the two cases, $\cos^2 \theta \approx 0.99$, the dominant components are the tetraquark states, $2 \sin \theta \cos \theta \approx 0.20$ or 0.19, the mixing effects are also considerable. In Ref. [26], the tetraquark component of the $Y(4260)$ is about $\sin^2 \theta \approx 0.64$, the conclusion is quite different from the present work. The difference maybe originate from the interpolating currents and the truncation of the operator product expansion.

$$M_Y^2 = \frac{\int_{4m_c^2}^{s_0} \mathrm{d}s \, \frac{\mathrm{d}}{\mathrm{d}(-1/T^2)} \left[\cos^2 \theta \, \rho_Y(s) + 2 \sin \theta \cos \theta \, \rho_m(s) + \sin^2 \theta \, \rho_2(s) \right] \exp\left(-\frac{s}{T^2} \right)}{\int_{4m_c^2}^{s_0} \mathrm{d}s \left[\cos^2 \theta \, \rho_Y(s) + 2 \sin \theta \cos \theta \, \rho_m(s) + \sin^2 \theta \, \rho_2(s) \right] \exp\left(-\frac{s}{T^2} \right)}. \tag{35}$$

In case I, we take the $Y(4260)$ as the ground state mixed charmonium–tetraquark state, and choose the optimal energy scale $\mu = 2.2\,\text{GeV}$. In case II, we take the $Y(4360)$ as the ground state mixed charmonium–tetraquark state, and choose the optimal energy scale $\mu = 2.4\,\text{GeV}$. Then we impose the two criteria (pole dominance at the phenomenological side and convergence of the operator product expansion) of the QCD sum rules on the $Y(4260)$ and $Y(4360)$, and search for the mixing angles θ, Borel parameters T^2, and continuum threshold parameters s_0. The resulting mixing angles, Borel parameters, continuum threshold parameters, energy scales, pole contributions, and contributions of the vacuum condensates of dimension 10 are shown in Table 2. From the table, we can see that the two criteria of the conventional QCD sum rules can be satisfied, so we expect to make reasonable predictions.

4 Conclusion

In this article, we construct the axialvector-diquark–axial-vector-antidiquark type tensor current to interpolate both the vector- and the axialvector-tetraquark states, then we calculate the contributions of the vacuum condensates up to dimension 10 in the operator product expansion, and we obtain the QCD sum rules for both the vector- and the axialvector-tetraquark states. In calculations, we use the formula $\mu = \sqrt{M_{X/Y/Z}^2 - (2\mathbb{M}_c)^2}$ suggested in our previous work to determine the energy scales of the QCD spectral densities, which works well. The numerical results support assigning the $Z_c(4020/4025)$ to be the $J^{PC} = 1^{+-}$ diquark–antidiquark type tetraquark state, and assigning the $Y(4660)$ to be the $J^{PC} = 1^{--}$ diquark–antidiquark type tetraquark state. Furthermore, we take the $Y(4260)$ and $Y(4360)$ as the
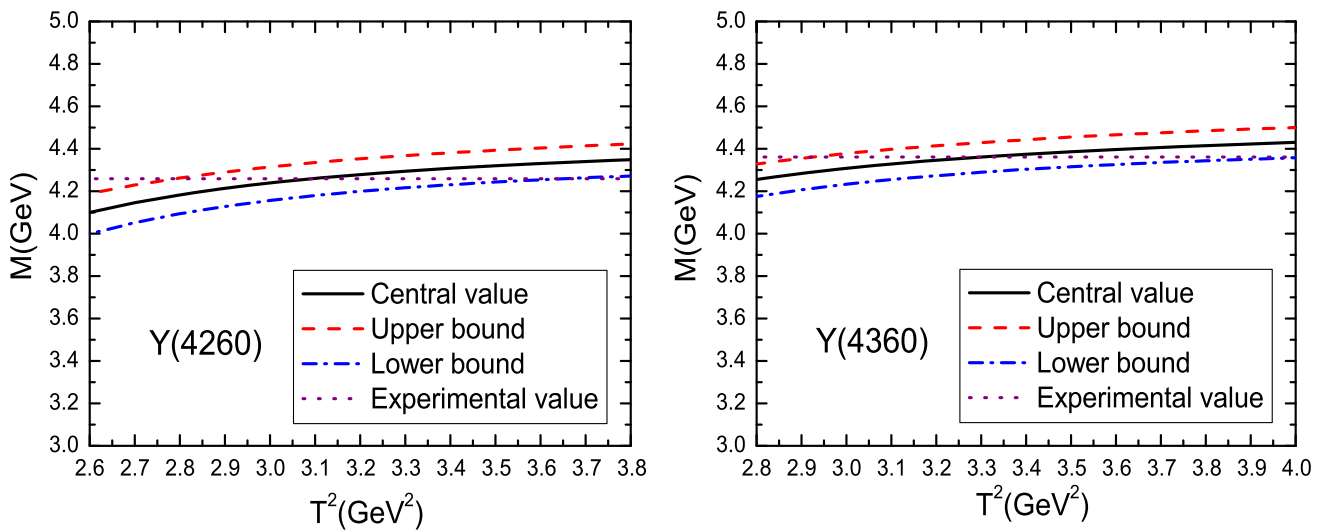
Fig. 3 The masses with variations of the Borel parameters T^2 for the $Y(4260)$ and $Y(4360)$ as mixed charmonium–tetraquark states

mixed charmonium–tetraquark states, introduce the mixing angle and construct the two-quark–tetraquark type tensor currents to study the masses and pole residues. The experimental values of the masses can be reproduced with suitable mixing angles, the QCD sum rules support assigning the $Y(4260)$ and $Y(4360)$ to be the mixed charmonium–tetraquark states.

Appendix

The QCD spectral densities $\rho_Y(s)$ and $\rho_Z(s)$,

$$
\rho_Y(s) = \frac{1}{6144\pi^6} \int \mathrm{d}y\mathrm{d}z \, yz(1-y-z)^3 \left(s-\overline{m}_c^2\right)^2
$$
$$
\times \left(35s^2 - 18s\overline{m}_c^2 - \overline{m}_c^4\right)
$$
$$
+ \frac{1}{6144\pi^6} \int \mathrm{d}y\mathrm{d}z \, yz(1-y-z)^2
$$
$$
\times \left(s-\overline{m}_c^2\right)^3 \left(9s-\overline{m}_c^2\right)
$$
$$
+ \frac{m_c\langle\bar{q}q\rangle}{32\pi^4} \int \mathrm{d}y\mathrm{d}z \, (y+z)(1-y-z)\left(s-\overline{m}_c^2\right)^2
$$
$$
- \frac{m_c^2}{4608\pi^4} \left\langle\frac{\alpha_s GG}{\pi}\right\rangle \int \mathrm{d}y\mathrm{d}z \left(\frac{z}{y^2} + \frac{y}{z^2}\right)
$$
$$
\times (1-y-z)^3 \left\{4s + \overline{m}_c^2 + \frac{4}{3}s^2 \delta\left(s-\overline{m}_c^2\right)\right\}
$$

$$
- \frac{m_c^2}{4608\pi^4} \left\langle\frac{\alpha_s GG}{\pi}\right\rangle \int \mathrm{d}y\mathrm{d}z \left(\frac{z}{y^2} + \frac{y}{z^2}\right)
$$
$$
\times (1-y-z)^2 \left(3s-\overline{m}_c^2\right)
$$
$$
+ \frac{1}{18432\pi^4} \left\langle\frac{\alpha_s GG}{\pi}\right\rangle \int \mathrm{d}y\mathrm{d}z \, (y+z)(1-y-z)^2
$$
$$
\times \left(95s^2 - 120s\overline{m}_c^2 + 33\overline{m}_c^4\right)
$$
$$
+ \frac{1}{9216\pi^4} \left\langle\frac{\alpha_s GG}{\pi}\right\rangle \int \mathrm{d}y\mathrm{d}z \, (y+z)(1-y-z)
$$
$$
\times \left(s-\overline{m}_c^2\right)\left(5s-\overline{m}_c^2\right)
$$
$$
- \frac{1}{4608\pi^4} \left\langle\frac{\alpha_s GG}{\pi}\right\rangle \int \mathrm{d}y\mathrm{d}z \, (y+z)(1-y-z)^2
$$
$$
\times \left(s-\overline{m}_c^2\right)\left(5s-3\overline{m}_c^2\right)
$$
$$
+ \frac{1}{82944\pi^4} \left\langle\frac{\alpha_s GG}{\pi}\right\rangle \int \mathrm{d}y\mathrm{d}z \, (1-y-z)^3
$$
$$
\times \left(35s^2 - 24s\overline{m}_c^2 - 3\overline{m}_c^4\right)
$$
$$
- \frac{1}{4608\pi^4} \left\langle\frac{\alpha_s GG}{\pi}\right\rangle \int \mathrm{d}y\mathrm{d}z \, yz(1-y-z)
$$
$$
\times \left(s-\overline{m}_c^2\right)\left(5s-3\overline{m}_c^2\right)
$$
$$
- \frac{1}{27648\pi^4} \left\langle\frac{\alpha_s GG}{\pi}\right\rangle \int \mathrm{d}y\mathrm{d}z \, (1-y-z)^2
$$
$$
\times \left(s-\overline{m}_c^2\right)\left(13s-5\overline{m}_c^2\right)
$$
$$
- \frac{1}{4608\pi^4} \left\langle\frac{\alpha_s GG}{\pi}\right\rangle \int \mathrm{d}y\mathrm{d}z \, yz\left(s-\overline{m}_c^2\right)^2
$$
$$
- \frac{m_c\langle\bar{q}g_s\sigma Gq\rangle}{64\pi^4} \int \mathrm{d}y\mathrm{d}z \, (y+z)\left(s-\overline{m}_c^2\right)
$$
$$
+ \frac{m_c\langle\bar{q}g_s\sigma Gq\rangle}{288\pi^4} \int \mathrm{d}y\mathrm{d}z \, (1-y-z)\left(s-\overline{m}_c^2\right)
$$
$$
- \frac{m_c^2\langle\bar{q}q\rangle^2}{12\pi^2} \int \mathrm{d}y
$$

$$+\frac{g_s^2\langle\bar{q}q\rangle^2}{1296\pi^4}\int dydz\,yz\left\{(4+\overline{m}_c^2)+\frac{4}{3}s^2\,\delta\left(s-\overline{m}_c^2\right)\right\}$$

$$+\frac{g_s^2\langle\bar{q}q\rangle^2}{3888\pi^4}\int dy\,y(1-y)\left(3s-\overline{m}_c^2\right)$$

$$+\frac{g_s^2\langle\bar{q}q\rangle^2}{3888\pi^4}\int dydz\,(1-y-z)\left\{\left(\frac{z}{y}+\frac{y}{z}\right)9\left(s-\overline{m}_c^2\right)\right.$$

$$+\left(\frac{z}{y^2}+\frac{y}{z^2}\right)m_c^2\left[5+s\,\delta\left(s-\overline{m}_c^2\right)\right]-(y+z)$$

$$\times\left[(24s-6\overline{m}_c^2)+4s^2\,\delta\left(s-\overline{m}_c^2\right)\right]\right\}$$

$$-\frac{g_s^2\langle\bar{q}q\rangle^2}{11664\pi^4}\int dydz\,(1-y-z)$$

$$\times\left\{\left(\frac{z}{y}+\frac{y}{z}\right)9\left(3s-\overline{m}_c^2\right)+\left(\frac{z}{y^2}+\frac{y}{z^2}\right)\right.$$

$$\times m_c^2\left[4+8s\,\delta\left(s-\overline{m}_c^2\right)\right]+(y+z)$$

$$\times\left[12s+3\overline{m}_c^2+4s^2\,\delta\left(s-\overline{m}_c^2\right)\right]\right\}$$

$$-\frac{m_c^3\langle\bar{q}q\rangle}{288\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int dydz\left(\frac{y}{z^3}+\frac{z}{y^3}+\frac{1}{y^2}+\frac{1}{z^2}\right)$$

$$\times(1-y-z)\,\delta\left(s-\overline{m}_c^2\right)$$

$$+\frac{m_c\langle\bar{q}q\rangle}{96\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle$$

$$\times\int dydz\left(\frac{y}{z^2}+\frac{z}{y^2}\right)(1-y-z)$$

$$-\frac{m_c\langle\bar{q}q\rangle}{96\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int dydz\left\{1+\frac{2}{3}s\,\delta\left(s-\overline{m}_c^2\right)\right\}$$

$$+\frac{m_c\langle\bar{q}q\rangle}{864\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle$$

$$\int dydz\left\{\frac{y}{z}+\frac{z}{y}+\left(\frac{1}{y}+\frac{1}{z}\right)(1-y-z)\right\}$$

$$+\frac{m_c\langle\bar{q}q\rangle}{576\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int dy$$

$$+\frac{m_c^2\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle}{24\pi^2}\int_0^1 dy\left(1+\frac{s}{T^2}\right)\delta\left(s-\widetilde{m}_c^2\right)$$

$$-\frac{\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle}{432\pi^2}\int_0^1 dy\,s\,\delta\left(s-\widetilde{m}_c^2\right)$$

$$-\frac{m_c^2\langle\bar{q}g_s\sigma Gq\rangle^2}{192\pi^2 T^6}\int_0^1 dy\,s^2\,\delta\left(s-\widetilde{m}_c^2\right)$$

$$+\frac{m_c^4\langle\bar{q}q\rangle^2}{216 T^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_0^1 dy\left\{\frac{1}{y^3}+\frac{1}{(1-y)^3}\right\}\delta\left(s-\widetilde{m}_c^2\right)$$

$$-\frac{m_c^2\langle\bar{q}q\rangle^2}{72 T^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_0^1 dy\left\{\frac{1}{y^2}+\frac{1}{(1-y)^2}\right\}\delta\left(s-\widetilde{m}_c^2\right)$$

$$-\frac{\langle\bar{q}q\rangle^2}{648 T^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_0^1 dy\,s\,\delta\left(s-\widetilde{m}_c^2\right)$$

$$+\frac{\langle\bar{q}g_s\sigma Gq\rangle^2}{384\pi^2 T^2}\int_0^1 dy\,s\left(1+\frac{2s}{9T^2}\right)\delta\left(s-\widetilde{m}_c^2\right)$$

$$-\frac{m_c^2\langle\bar{q}q\rangle^2}{216 T^6}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_0^1 dy\,s^2\,\delta\left(s-\widetilde{m}_c^2\right), \tag{38}$$

$$\rho_Z(s)=\frac{1}{2048\pi^6}\int dydz\,yz(1-y-z)^3\left(s-\overline{m}_c^2\right)^2$$

$$\times\left(21s^2-14s\overline{m}_c^2+\overline{m}_c^4\right)$$

$$-\frac{1}{2048\pi^6}\int dydz\,yz(1-y-z)^2\left(s-\overline{m}_c^2\right)^4$$

$$-\frac{m_c\langle\bar{q}q\rangle}{32\pi^4}\int dydz\,(y+z)(1-y-z)\left(s-\overline{m}_c^2\right)$$

$$\times\left(3s-\overline{m}_c^2\right)$$

$$-\frac{m_c^2}{1536\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int dydz\left(\frac{z}{y^2}+\frac{y}{z^2}\right)(1-y-z)^3$$

$$\times\left\{4s-\overline{m}_c^2+\frac{2}{3}s^2\,\delta\left(s-\overline{m}_c^2\right)\right\}$$

$$+\frac{m_c^2}{1536\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int dydz\left(\frac{z}{y^2}+\frac{y}{z^2}\right)(1-y-z)^2$$

$$\times\left(s-\overline{m}_c^2\right)$$

$$-\frac{1}{6144\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int dydz\,(y+z)(1-y-z)^2$$

$$\times\left(35s^2-40s\overline{m}_c^2+9\overline{m}_c^4\right)$$

$$-\frac{1}{3072\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int dydz\,(y+z)(1-y-z)$$

$$\times\left(s-\overline{m}_c^2\right)^2$$

$$+\frac{1}{4608\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int dydz\,(y+z)(1-y-z)^2$$

$$\times\left(15s^2-16s\overline{m}_c^2+3\overline{m}_c^4\right)$$

$$+\frac{1}{27648\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int dydz\,(1-y-z)^3$$

$$\times\left(25s^2-24s\overline{m}_c^2+3\overline{m}_c^4\right)$$

$$+\frac{1}{13824\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int dydz\,yz(1-y-z)$$

$$\times\left(25s^2-24s\overline{m}_c^2+3\overline{m}_c^4\right)$$

$$+\frac{1}{9216\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int dydz\,(1-y-z)^2$$

$$\times\left(s-\overline{m}_c^2\right)^2$$

$$+\frac{1}{13824\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int dydz\,yz\left(s-\overline{m}_c^2\right)\left(13s-5\overline{m}_c^2\right)$$

$$+\frac{m_c\langle\bar{q}g_s\sigma Gq\rangle}{64\pi^4}\int dydz\,(y+z)\left(2s-\overline{m}_c^2\right)$$

$$-\frac{m_c\langle\bar{q}g_s\sigma Gq\rangle}{288\pi^4}\int dydz\,(1-y-z)\left(2s-\overline{m}_c^2\right)$$

$$+\frac{m_c^2\langle\bar{q}q\rangle^2}{12\pi^2}\int dy$$

$$+\frac{g_s^2\langle\bar{q}q\rangle^2}{432\pi^4}\int dydz\,yz\left\{(4-\overline{m}_c^2)+\frac{2}{3}s^2\,\delta\left(s-\overline{m}_c^2\right)\right\}$$

$$-\frac{g_s^2\langle\bar{q}q\rangle^2}{1296\pi^4}\int dy\,y(1-y)\left(s-\overline{m}_c^2\right)$$

$$-\frac{g_s^2\langle\bar{q}q\rangle^2}{3888\pi^4}\int dydz\,(1-y-z)\left\{\left(\frac{z}{y}+\frac{y}{z}\right)9\left(2s-\overline{m}_c^2\right)\right.$$

$$+\left(\frac{z}{y^2}+\frac{y}{z^2}\right)$$

$$\times m_c^2\left[5+4s\,\delta\left(s-\overline{m}_c^2\right)\right]+(y+z)$$

$$\times\left[6\overline{m}_c^2+2s^2\,\delta\left(s-\overline{m}_c^2\right)\right]\right\}$$

$$+\frac{g_s^2\langle\bar{q}q\rangle^2}{3888\pi^4}\int dydz\,(1-y-z)$$

$$\times\left\{\left(\frac{z}{y}+\frac{y}{z}\right)\left(-3\overline{m}_c^2\right)+\left(\frac{z}{y^2}+\frac{y}{z^2}\right)\right.$$

$$\times m_c^2\left[2-s\,\delta\left(s-\overline{m}_c^2\right)\right]-(y+z)$$

$$\times\left[12s-3\overline{m}_c^2+2s^2\,\delta\left(s-\overline{m}_c^2\right)\right]\right\}$$

$$+\frac{m_c^3\langle\bar{q}q\rangle}{288\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int dydz\left(\frac{y}{z^3}+\frac{z}{y^3}+\frac{1}{y^2}+\frac{1}{z^2}\right)$$

$$\times(1-y-z)\,\delta\left(s-\overline{m}_c^2\right)$$

$$-\frac{m_c\langle\bar{q}q\rangle}{96\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int dydz\left(\frac{y}{z^2}+\frac{z}{y^2}\right)(1-y-z)$$

$$\times\left\{1+s\,\delta\left(s-\overline{m}_c^2\right)\right\}$$

$$+\frac{m_c\langle\bar{q}q\rangle}{96\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int dydz\left\{1+\frac{1}{3}s\,\delta\left(s-\overline{m}_c^2\right)\right\}$$

$$-\frac{m_c\langle\bar{q}q\rangle}{864\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int dydz\left\{\frac{y}{z}+\frac{z}{y}+\left(\frac{1}{y}+\frac{1}{z}\right)\right.$$

$$\times\,(1-y-z)\}\left\{1+s\,\delta\left(s-\overline{m}_c^2\right)\right\}$$

$$-\frac{m_c\langle\bar{q}q\rangle}{576\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int dy\left\{1+s\,\delta\left(s-\widetilde{m}_c^2\right)\right\}$$

$$-\frac{m_c^2\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle}{24\pi^2}\int_0^1 dy\left(1+\frac{s}{T^2}\right)\delta\left(s-\widetilde{m}_c^2\right)$$

$$+\frac{\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle}{432\pi^2}\int_0^1 dy\,s\,\delta\left(s-\widetilde{m}_c^2\right)$$

$$-\frac{m_c^4\langle\bar{q}q\rangle^2}{216T^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_0^1 dy\left\{\frac{1}{y^3}+\frac{1}{(1-y)^3}\right\}\delta\left(s-\widetilde{m}_c^2\right)$$

$$+\frac{m_c^2\langle\bar{q}q\rangle^2}{72T^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_0^1 dy\left\{\frac{1}{y^2}+\frac{1}{(1-y)^2}\right\}\delta\left(s-\widetilde{m}_c^2\right)$$

$$+\frac{\langle\bar{q}q\rangle^2}{648T^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_0^1 dy\,s\,\delta\left(s-\widetilde{m}_c^2\right)$$

$$-\frac{\langle\bar{q}g_s\sigma Gq\rangle^2}{384\pi^2 T^2}\int_0^1 dy\,s\left(1+\frac{2s}{9T^2}\right)\delta\left(s-\widetilde{m}_c^2\right)$$

$$+\frac{m_c^2\langle\bar{q}g_s\sigma Gq\rangle^2}{192\pi^2 T^6}\int_0^1 dy\,s^2\,\delta\left(s-\widetilde{m}_c^2\right)$$

$$+\frac{m_c^2\langle\bar{q}q\rangle^2}{216T^6}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_0^1 dy\,s^2\,\delta\left(s-\widetilde{m}_c^2\right),\tag{39}$$

where $\int dydz=\int_{y_i}^{y_f}dy\int_{z_i}^{1-y}dz$, $y_f=\frac{1+\sqrt{1-4m_c^2/s}}{2}$, $y_i=\frac{1-\sqrt{1-4m_c^2/s}}{2}$, $z_i=\frac{ym_c^2}{ys-m_c^2}$, $\overline{m}_c^2=\frac{(y+z)m_c^2}{yz}$, $\widetilde{m}_c^2=\frac{m_c^2}{y(1-y)}$, $\int_{y_i}^{y_f}dy\to\int_0^1 dy$, $\int_{z_i}^{1-y}dz\to\int_0^{1-y}dz$, when the δ functions $\delta\left(s-\overline{m}_c^2\right)$ and $\delta\left(s-\widetilde{m}_c^2\right)$ appear.

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