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The operator product expansion between the 16 lowest higher spin currents in the $\mathcal{N}=4$ superspace

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Abstract Some of the operator product expansions (OPEs) between the lowest 16 higher spin currents of spins $(1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 2, 2, 2, 2, 2, 2, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, 3)$ in an extension of the large $\mathcal{N}=4$ linear superconformal algebra were constructed in $\mathcal{N}=4$ superconformal coset $\frac{SU(5)}{SU(3)}$ theory previously. In this paper, by rewriting these OPEs in the $\mathcal{N}=4$ superspace developed by Schoutens (and other groups), the remaining undetermined OPEs in which the corresponding singular terms possess the composite fields with spins $s=\frac{7}{2}, 4, \frac{9}{2}, 5$ are completely determined. Furthermore, by introducing arbitrary coefficients in front of the composite fields on the right-hand sides of the above complete 136 OPEs, reexpressing them in the $\mathcal{N}=2$ superspace, and using the $\mathcal{N}=2$ OPEs Mathematica package by Krivonos and Thielemans, the complete structures of the above OPEs with fixed coefficient functions are obtained with the help of various Jacobi identities. We then obtain ten $\mathcal{N}=2$ super OPEs between the four $\mathcal{N}=2$ higher spin currents denoted by $(1, \frac{3}{2}, \frac{3}{2}, 2)$, $(\frac{3}{2}, 2, 2, \frac{5}{2})$, $(\frac{3}{2}, 2, 2, \frac{5}{2})$, and $(2, \frac{5}{2}, \frac{5}{2}, 3)$ (corresponding 136 OPEs in the component approach) in the $\mathcal{N}=4$ superconformal coset $\frac{SU(N+2)}{SU(N)}$ theory. Finally, we describe them as one single $\mathcal{N}=4$ super OPE between the above 16 higher spin currents in the $\mathcal{N}=4$ superspace. The fusion rule for this OPE contains the next 16 higher spin currents of spins of $(2, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, 3, 3, 3, 3, 3, 3, \frac{7}{2}, \frac{7}{2}, \frac{7}{2}, \frac{7}{2}, 4)$ in addition to the quadratic $\mathcal{N}=4$ lowest higher spin multiplet and the large $\mathcal{N}=4$ linear superconformal family of the identity operator. The various structure constants (fixed coefficient functions) appearing on the right-hand side of this OPE depend on N and the level k of the bosonic spin-1 affine Kac–Moody current. For convenience, the above 136 OPEs in the component approach for generic (N, k) with simplified notation are given.

Dedicated to the Department of Physics, Yonsei University, on the occasion of its 100th anniversary.

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1 Introduction

In the large $\mathcal{N}=4$ holography observed in [1], the duality between matrix extended higher spin theories on AdS_3 space with large $\mathcal{N}=4$ supersymmetry and large $\mathcal{N}=4$ coset theory in two-dimensional conformal field theory (CFT) was proposed.[1] One of the consistency checks for this duality is based on the matching of correlation functions. The simplest three-point functions consist of two scalar primaries and one higher spin current. Then the zero-mode eigenvalue equations of the higher spin Casimir current in the two-dimensional $\mathcal{N}=4$ coset model should coincide with the zero-mode eigenvalue equations of the higher spin field in an asymptotic (quantum) symmetry algebra of higher spin theory on the AdS_3 space. Recently, in [6], the zero-mode eigenvalue equations (and the corresponding three-point functions) of the bosonic (higher spin) currents of spins $s=1, 2, 3$ with two scalars for any finite N and k (and for large N 't Hooft limit) were obtained. (Recall that the group

[1] In the large-level limit of type IIB string theory on $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$, one of the two three-spheres becomes flat and decompactifies to R^3 (and to the three-torus $\mathbf{T}^3$) and then one obtains type IIB string theory on $AdS_3 \times \mathbf{S}^3 \times \mathbf{T}^4$. In [2], a better understanding of string theory in the tensionless limit was described from the viewpoint of Vasiliev higher spin theory. In particular, they found that perturbative Vasiliev theory is a subsector of tensionless string theory. In [3], unbroken stringy symmetry algebra is studied further. The corresponding $AdS_3 \times \mathbf{S}^3 \times \mathbf{K}^3$ theory on the tensionless point is described in [4]. Very recently, in [5], using conformal field perturbation theory that gives nonzero string tension, it is shown that the symmetry generators of the symmetric orbifold theory of $\mathbf{T}^4$ describe Regge trajectories. The leading Regge trajectory have lowest mass for a given spin and the higher spin generators are expressed in terms of quadratic free fields. The subleading Regge trajectories where the higher spin generators are expressed in terms of cubic- and higher-order free fields are also analyzed.

G of $\mathcal{N} = 4$ coset theory is given by $G = SU(N + 2)$ and the level of the spin-1 Kac–Moody current is given by the positive integer k.)

What happens for AdS_3 bulk theory? How can we observe the findings [6] on the eigenvalue equations and three-point functions in two-dimensional CFT computations in the context of higher spin theory on the AdS_3 space? It is well known, in the similar bosonic duality studied in [7,8], that the three-point functions for the spin-s ($s = 2, 3, 4, 5$) Casimir current with two scalars in two-dimensional CFT for any finite N and k (and for the large N 't Hooft limit) matched those for the spin-s field in the asymptotic symmetry algebra of higher spin theory on the AdS_3 space [9,10] (see also [11]). It was crucial for this consistency check to have asymptotic symmetry algebra for any N and k explicitly to calculate the eigenvalue equations of the higher spin-s field in an asymptotic quantum symmetry algebra of higher spin theory on the AdS_3 space. In the large $\mathcal{N} = 4$ holography, the corresponding asymptotic symmetry algebra of higher spin theory on the AdS_3 space is not yet known. Therefore, we should determine the asymptotic symmetry algebra to calculate the three-point functions of higher spin theory on AdS_3 space. Along the lines of [8–10], we would like to construct the operator product expansions (OPEs) between the lowest 16 higher spin currents for generic N and k.

One of the main results in [12] was that there exist complete OPEs, for $N = 3$ and arbitrary k, between the 16 higher spin currents (one spin-1 current, four spin-$\frac{3}{2}$ currents, six spin-2 currents, four spin-$\frac{5}{2}$ currents, and one spin-2 current) except for some singular terms possessing composite fields with spins $s = \frac{7}{2}, 4, \frac{9}{2}, 5$. We would like to construct the complete OPEs between the 16 higher spin currents for any N and k. The ultimate goal (which will appear in the near future) is to calculate the eigenvalue equations and corresponding three-point functions of the higher spin fields in an asymptotic symmetry algebra of higher spin theory on the AdS_3 space.

How can we obtain the general N behavior form the $N = 3$ results in [12]? One way to obtain the N behavior is to calculate the OPEs from the N-dependent higher spin currents (expressed in terms of various multiple products of WZW currents) manually as done in [9,10]. In principle, this is possible, but there exist too many (higher spin) currents: 16 higher spin currents as well as other 16 currents from the large $\mathcal{N} = 4$ superconformal algebra. We should calculate the $136(=16+15+\cdots+2+1)$ OPEs and rearrange them in terms of the composite fields from the above $(16 + 16)$ currents and their derivatives. Then we could use another approach to obtain the N dependence explicitly in the OPEs between the higher spin currents. We can also proceed from what has been done in [12] for different values of N. For example, we can increase the value of N as $N = 5, 7, 9, 11, 13, \ldots$. With the help of Thielemans' package [13], several N cases can give us the consistency check needed to ensure the particular singular term. From the explicit OPEs in [12], some of the OPEs have very complicated k dependence in the fractional structure constants, and the numerical values appearing in the power of k will be generalized to N-dependent terms (some power of N). For the fractional k-dependent coefficient functions, where the power of k is large, we need to consider more finite N values (for example, beyond $N = 13$). Therefore, we cannot determine the complete OPEs from these several N cases because there are still undetermined OPEs for $N = 3$ and we should repeat the above procedures without knowing the maximum value for N.

However, we can use the power of the $\mathcal{N} = 4$ supersymmetry. We expect that, by using this $\mathcal{N} = 4$ supersymmetry, some of the unknown structures in the (higher spin) currents or some OPEs between them can be fixed. In other words, the $\mathcal{N} = 4$ supersymmetry is the sought-after kind of constraint that preserves behind an extension of large $\mathcal{N} = 4$ linear superconformal algebra. We can also use the Jacobi identity between the (higher spin) currents. It is well known that the above WZW construction (that is, the construction of higher spin currents using the WZW currents) satisfies the Jacobi identities automatically [14]. We can easily see that the previous results for $N = 3$ [12] satisfy the Jacobi identities. In Thielemans' package [13], we can check the Jacobi identities for any given OPEs. It is natural to ask whether we can find the complete OPEs for generic N and k by taking the same OPEs for the $N = 3$ case and replacing the structure constants with undetermined coefficients. Of course, these coefficients reduce to the $N = 3$ results if we restrict the problem to the particular $N = 3$ case, as done in [12]. It will turn out that there exist consistent solutions for the above coefficients that can be obtained by solving the various Jacobi identities.

Thielemans' package [13] is based on the OPEs in the component approach. However, if we use the Jacobi identities inside of this package, too many unknown coefficients (which should be determined later) are introduced. Some experience in using this OPE package indicates that it is better to solve the Jacobi identities with a small number of unknown coefficients. Fortunately, there exists the other package by Krivonos and Thielemans [15], which is based on the OPEs in the $\mathcal{N} = 2$ superspace. Because any current in the $\mathcal{N} = 2$ superspace contains four component currents, we obtain a reduced number of unknown coefficients. Therefore, we use the Jacobi identities in the $\mathcal{N} = 2$ superspace while to check the consistency we can use the Jacobi identities in the component approach (since for fixed structure constants it does not take too much time to check the Jacobi identities). We can easily proceed from the component results to the $\mathcal{N} = 2$ superspace results of [15] and vice versa.

How can we determine the complete structures (that is, the possible composite fields up to spin-5 where the OPE between the higher spin-3 currents can have composite fields with spin-5 at the first-order pole) appearing on the right-hand side of the above 136 OPEs? As mentioned before, we can proceed to the $\mathcal{N}=4$ superspace (or to the $\mathcal{N}=2$ superspace) from the component results in [12]. The $\mathcal{N}=4$ single OPE between the $\mathcal{N}=4$ multiplet and itself can be expanded in terms of Grassmann (or fermionic) coordinates. Then the coordinate difference for the singular terms is given by the difference for the ordinary coordinate. To derive the OPE in the $\mathcal{N}=4$ superspace, we should reexpress this difference for the ordinary coordinate in terms of the difference for the $\mathcal{N}=4$ superspace coordinate by subtracting the product of Grassmann coordinates (and adding the same quantity) from the above difference for the ordinary coordinate. Then any fractional power of the difference for the ordinary coordinate can be expanded in terms of the sum of fractional power of the difference for the $\mathcal{N}=4$ superspace coordinate by using a Taylor expansion with the help of the property of Grassmann coordinates. Then any singular term in the single $\mathcal{N}=4$ OPE contains the product of Grassmann coordinates, the fractional power of the difference for the $\mathcal{N}=4$ superspace coordinate, and the ordinary coordinate-dependent pole terms from the OPEs between the various 16 higher spin currents. After simplifying these complicated singular terms in an $\mathcal{N}=4$ supersymmetric way, we can obtain the results in [12] in terms of a single $\mathcal{N}=4$ OPE. In other words, the undetermined parts of the 136 OPEs in [12] are completely fixed.

Now we can proceed to the $\mathcal{N}=2$ superspace from these component results and all the expressions are given for $N=3$. Let us replace the structure constants, which have k dependence explicitly with arbitrary coefficients. Then we have the complete OPEs in the $\mathcal{N}=2$ superspace with undetermined structure constants.[2] We use the Jacobi identities to fix the structure constants. In general, the new $\mathcal{N}=2$ primary current (which transforms as a primary current under the $\mathcal{N}=2$ stress energy tensor) can appear on the right-hand side of the OPEs as the spins of the currents increase. The above 16 higher spin currents can be represented by four $\mathcal{N}=2$ multiplets. Similarly, the 16 currents of the large $\mathcal{N}=4$ linear superconformal algebra can be combined into four $\mathcal{N}=2$ multiplets (where one of them is given by chiral and antichiral currents). Then we can use the Jacobi identities by choosing one $\mathcal{N}=2$ current and two $\mathcal{N}=2$ higher spin currents. We cannot use the Jacobi identities by taking three $\mathcal{N}=2$ higher spin currents because, if we consider the OPE between any $\mathcal{N}=2$ higher spin currents and another new $\mathcal{N}=2$ higher spin current, we do not know this OPE at this level. Therefore, the three quantities used for the Jacobi identities are given by one $\mathcal{N}=2$ current and two $\mathcal{N}=2$ higher spin currents. We can also consider the combination of one $\mathcal{N}=2$ higher spin current and two $\mathcal{N}=2$ currents or the case with three $\mathcal{N}=2$ currents, but these will do not produce any nontrivial equations for the unknown coefficients. They are satisfied trivially.

There is a concern about whether we can fix all the unknown coefficients completely even though we do not exhaust all the Jacobi identities. It will turn out that all the structure constants are determined except for one arbitrary unknown coefficient. In other words, all the known structure constants are expressed in terms of this unknown coefficient. How can we fix this unknown quantity? One way to determine this quantity is by considering the other four $\mathcal{N}=2$ higher spin currents in principle, but this is not so useful. This leads to another new problem because we should consider the OPE between the previous $\mathcal{N}=4$ higher spin multiplet of superspin 1 and the new $\mathcal{N}=4$ higher spin multiplet of superspin 2. We can also find the above unknown coefficient by looking at the particular OPE as values of N are varied. Eventually, we can determine the final unknown coefficient by using this analysis. In the component approach, the OPE between the higher spin-1 current and the higher spin-3 current for several N values provides the (N, k) dependence of the above unknown coefficient.

In Sect. 2, the large $\mathcal{N}=4$ linear superconformal algebra is reviewed. The 16 currents are expanded in an expansion of the Grassmann coordinates in the $\mathcal{N}=4$ superspace. The single OPE between the 16 currents in the $\mathcal{N}=4$ superspace is given. The various OPEs between the 16 currents in the component approach are also described. Another basis for the same large $\mathcal{N}=4$ linear superconformal algebra is also provided. The coset realization for the large $\mathcal{N}=4$ linear superconformal algebra is given by identifying the two levels of the large $\mathcal{N}=4$ linear superconformal algebra with two parameters appearing in $\mathcal{N}=4$ coset theory.

In Sect. 3, the 16 higher spin-s currents are introduced in the $\mathcal{N}=4$ superspace. The definition for the $\mathcal{N}=4$ primary current in the $\mathcal{N}=4$ superspace is described. The component results by projection into this definition can be obtained. We also present the above 16 higher spin-s currents in the primary basis. In other words, they are primary currents under the stress energy tensor. The other component results,

[2] So far, the field contents on the right-hand side of the OPEs are the same as the ones in [12] in the $\mathcal{N}=2$ superspace. For the lowest $\mathcal{N}=4$ (higher spin) multiplet, the exact relations between the 16 higher spin currents and the ones in this paper (or the ones in [24]) are known explicitly. For the next lowest $\mathcal{N}=4$ higher spin multiplet, they will depend on (N, k) in a complicated way. For example, the higher spin-2 current residing in the next lowest higher spin current denoted by $\mathbf{P}^{(2)}(z)$ in [12] contains $\Phi_0^{(2)}(z)$ (or, equivalently, $V_0^{(2)}(z)$), as well as many extra terms with (N, k)-dependent fractional coefficient functions. The merit of the Jacobi identity is that we do not have to determine the exact relations between the next lowest higher spin currents and the ones in this paper. We simply introduce the unknown coefficient functions in front of the possible composite fields. They will be fixed by using the Jacobi identity. However, it will turn out that, if we obtain the complete OPEs between the higher spin-1 current and the 16 higher spin current in the SO(4) manifest basis for generic (N, k), then all the remaining structure constants are automatically determined via $\mathcal{N}=4$ supersymmetry.

where SU(2) × SU(2) symmetry is manifest, are described. In this basis, the higher spin-$(s+2)$ current, which is the last component of this $\mathcal{N}=4$ multiplet (and its spin is maximum inside of this multiplet), is not a primary current under the stress energy tensor. The precise simple relations between the above lowest 16 higher spin currents with the ones in [12] are given explicitly.

Section 4 is similar to Sect. 3, but it is for the $\mathcal{N}=2$ superspace. The $\mathcal{N}=4$ primary current condition introduced in Sect. 3 is rewritten in the $\mathcal{N}=2$ superspace. The four $\mathcal{N}=2$ multiplets of spins s, $(s+\frac{1}{2})$, $(s+\frac{1}{2})$, and $(s+1)$ are located at the particular components of the above $\mathcal{N}=4$ multiplet in an expansion of two Grassmann coordinates. The precise relations between the four $\mathcal{N}=2$ multiplets and their 16 component currents are described.

In Sect. 5, based on the previous results for $N=3$ in [12], we proceed to the $\mathcal{N}=4$ superspace. By introducing the arbitrary coefficients in the ten $\mathcal{N}=2$ OPEs, we solve the various Jacobi identities to determine the above unknown coefficients, which depend on (N,k). The final ten OPEs in the $\mathcal{N}=2$ superspace are given, and the fusion rules are described.

In Sect. 6, we proceed to the component approach from the $\mathcal{N}=2$ results in Sect. 5. From the precise relations found in Sect. 5, we obtain the final $\mathcal{N}=4$ OPE for generic N and k. The fusion rule is also given.

In Sect. 7, a summary of this paper is given and possible future work is presented.

In Appendices A–J, which appear in the arXiv version only, the detailed computations in Sects. 2–7 are presented.

For a reader interested in the construction of the $\mathcal{N}=2$ superspace description, the main text and Appendix G are useful. For the component approach, the main text and Appendix H are crucial. For a description of the $\mathcal{N}=4$ superspace, refer to the main text and Appendix I.

The packages in [13,15] are used in this paper.[3]

2 Review of the OPEs between the 16 currents in $\mathcal{N}=4$ superspace

In this section, we describe the 16 currents of the large $\mathcal{N}=4$ linear superconformal algebra in the $\mathcal{N}=4$ superspace, where SO(4) symmetry is manifest. Then the corresponding large $\mathcal{N}=4$ linear superconformal algebra, which consists of 13 nontrivial OPEs in the component approach (in Appendix A), can be expressed in terms of a single $\mathcal{N}=4$ (super) OPE. A description in another basis, where SU(2) × SU(2) symmetry is manifest, is also reviewed.

Finally, the realization for the large $\mathcal{N}=4$ linear superconformal algebra is also reviewed.

2.1 SO(4)-singlet $\mathcal{N}=4$ stress energy tensor of superspin 0

The coordinates of the $\mathcal{N}=4$ (extended) superspace can be described as $(Z,\overline{Z})$, where $Z=(z,\theta^i)$, $\overline{Z}=(\bar{z},\bar{\theta}^i)$, and $i=1,2,3,4$ and the index i is the SO(4)-vector index. The left covariant spinor derivative is given by $D^i=\theta^i\frac{\partial}{\partial z}+\frac{\partial}{\partial\theta^i}$ and satisfies the following nontrivial anticommutators: $\{D^i,D^j\}=2\delta^{ij}\frac{\partial}{\partial z}$. Here the Kronecker delta δ^{ij} is the rank 2 SO(4) symmetric invariant tensor. Then the $\mathcal{N}=4$ stress energy tensor can be described as follows [18] (or see [19], where the $\mathcal{N}=1$ and $\mathcal{N}=2$ superspace descriptions are given):

$$
\begin{aligned}
\mathbf{J^{(4)}}(Z) &= -\Delta(z)+i\theta^j\Gamma^j(z)-i\theta^{4-jk}T^{jk}(z)\\
&\quad -\theta^{4-j}(G^j-2\alpha i\partial\Gamma^j)(z)\\
&\quad +\theta^{4-0}(2L-2\alpha\partial^2\Delta)(z)\\
&= -\Delta(z)+i\,\theta^1\,\Gamma^1(z)+i\,\theta^2\,\Gamma^2(z)\\
&\quad +i\,\theta^3\,\Gamma^3(z)+i\,\theta^4\,\Gamma^4(z)\\
&\quad -i\,\theta^1\,\theta^2\,T^{34}(z)+i\,\theta^1\theta^3\,T^{24}(z)-i\,\theta^1\,\theta^4\,T^{23}(z)\\
&\quad -i\,\theta^2\,\theta^3\,T^{14}(z)+i\,\theta^2\,\theta^4\,T^{13}(z)\\
&\quad -i\,\theta^3\,\theta^4\,T^{12}(z)-\theta^2\,\theta^4\,\theta^3\,(G^1-2\alpha\,i\,\partial\Gamma^1)(z)\\
&\quad -\theta^1\,\theta^3\,\theta^4\,(G^2-2\alpha\,i\,\partial\Gamma^2)(z)\\
&\quad -\theta^1\,\theta^4\,\theta^2\,(G^3-2\alpha\,i\,\partial\Gamma^3)(z)-\theta^1\,\theta^2\,\theta^3\\
&\quad \times(G^4-2\alpha\,i\,\partial\Gamma^4)(z)\\
&\quad +\theta^1\,\theta^2\,\theta^3\,\theta^4\,(2L-2\alpha\,\partial^2\Delta)(z).
\end{aligned}
\tag{2.1}
$$

In the first line of (2.1), the summation over repeated indices (which implies that the $\mathcal{N}=4$ stress energy tensor $\mathbf{J^{(4)}}(Z)$ is an SO(4) singlet) is taken.[4] The simplified notation (in the multiple product) θ^{4-0} is used for $\theta^1\,\theta^2\,\theta^3\,\theta^4$. The complement $4-i$ is defined such that $\theta^4=\theta^{4-i}\,\theta^i$. In the second line of (2.1), the complete 16 currents for the $\mathcal{N}=4$ stress energy tensor are described in an expansion of Grassmann coordinates completely. The quintic- and higher-order terms in θ^i vanish owing to the property of θ^i. The 16 currents are given by a single spin-0 current $\Delta(z)$, four spin-$\frac{1}{2}$ currents

[3] We describe the 16 currents of the large $\mathcal{N}=4$ linear superconformal algebra as "the 16 currents" simply and the 16 lowest higher spin currents of the $\mathcal{N}=4$ multiplet as "the 16 higher spin currents". Sometimes we ignore the "super" appearing in supercurrent, superfield, or super OPE because it occurs many times in this paper.

[4] In this notation, the superscript (4) has nothing to do with the spin. It stands for the number of the supersymmetry. Later we introduce the $\mathcal{N}=4$ (higher spin) multiplet $\Phi^{(\mathbf{s})}(Z)$ of superspin s. Then it is more appropriate to write the $\mathcal{N}=4$ stress energy tensor $\mathbf{J^{(4)}}(Z)$ as $\Phi^{(\mathbf{0})}(Z)$, but we follow the original notation in [18]. We use boldface notation for the $\mathcal{N}=4$ or $\mathcal{N}=2$ multiplet to emphasize the fact that the corresponding multiplet has many component currents. For the $\mathcal{N}=4$ multiplet, 16 independent component currents arise while, for the $\mathcal{N}=2$ multiplet, 4 independent components arise (if they are unconstrained currents). Sometimes the right-hand side of the given OPE contains too many multiplets and then we ignore the boldface notation for simplicity. See also Sect. 6.

$\Gamma^i(z)$ transforming as a vector representation under SO(4), six spin-1 currents $T^{ij}(z)$ transforming as an adjoint representation under SO(4), four spin-$\frac{3}{2}$ currents $G^i(z)$ transforming as a vector representation under SO(4), and the spin-2 current $L(z)$. In particular, the spin-0 and spin-2 currents are SO(4) singlets. The spin of θ^i is given by $-\frac{1}{2}$ (and the covariant spinor derivative D^i has spin $\frac{1}{2}$) and therefore the $\mathcal{N}=4$ (super)spin of the stress energy tensor $\mathbf{J^{(4)}}(Z)$ is equal to zero. Each term in (2.1) has a spin-0 value.

In the cubic and quartic terms in the θ^i, there exist α-dependent terms, where the parameter is introduced in [18] as follows:

$$\alpha = \frac{1}{2}\frac{(k^+ - k^-)}{(k^+ + k^-)}. \tag{2.2}$$

The self-dual and anti-self-dual combinations of the spin-1 current $T^{ij}(z)$ have levels k^+ and k^-, respectively. That is, two commuting SU(2) Kac–Moody algebras have their levels $k^{\pm}$. The parameter α in (2.2) reflects the asymmetry between the occurrences of these two SU(2) Kac–Moody subalgebras in the (anti)commutators involving odd currents. More precisely, in Appendix A, the α dependence appears in the OPEs between the spin-$\frac{3}{2}$ currents and in the OPEs between the spin-$\frac{3}{2}$ currents and the spin-1 currents. For $\alpha = \pm\frac{1}{2}$ (or $k^{\pm} \to \infty$), one of the SU(2) subalgebras with the currents $\Gamma^i(z)$ and $\Delta(z)$ decouple and the above large $\mathcal{N}=4$ linear superconformal algebra is reduced to the SU(2)-extended $\mathcal{N}=4$ superconformal algebra in [20].

Furthermore, the central charge appearing in the OPE between the bosonic stress energy tensor $L(z)$ is given by [18]

$$c = \frac{6k^+k^-}{(k^+ + k^-)}.$$

When $k^+ = k^-$ (that is, $\alpha = 0$), this central charge is a positive multiple of 3 and can be realized by one real scalar and four Majorana fermions [21]. Note that there is no twisted anomaly, where the spin-0 current $\Delta(w)$ transforms as a primary field under the stress energy tensor $L(z)$. Later we will see the realization of the $\mathcal{N}=4$ stress energy tensor in $\mathcal{N}=4$ coset theory and the above two parameters will be given by two parameters $(k+1)$ and $(N+1)$. Accordingly, the central charge can be characterized by k and N.

2.2 OPE of the $\mathcal{N}=4$ stress energy tensor

The $\mathcal{N}=4$ super OPE between the $\mathcal{N}=4$ stress energy tensor and itself can be summarized by [18]

$$\mathbf{J^{(4)}}(Z_1)\,\mathbf{J^{(4)}}(Z_2) = \frac{\theta_{12}^4}{z_{12}^2}\,\frac{1}{2}(k^+ - k^-) + \frac{\theta_{12}^{4-i}}{z_{12}}\,D^i\mathbf{J^{(4)}}(Z_2) + \frac{\theta_{12}^4}{z_{12}}\,2\,\partial\mathbf{J^{(4)}}(Z_2) - \frac{1}{2}(k^+ + k^-)\,\log(z_{12}) + \cdots, \tag{2.3}$$

where summation over the repeated indices is assumed (the OPE between the SO(4)-singlet current and itself), the fermionic coordinate difference for given index i is defined as $\theta_{12}^i = \theta_1^i - \theta_2^i$, and the bosonic coordinate difference is given by $z_{12} = z_1 - z_2 - \theta_1^i\theta_2^i$. By introducing the spin-$\frac{1}{2}$ field in $J^i(Z) \equiv D^i\mathbf{J^{(4)}}(Z)$, the OPE $J^i(Z_1)\,J^j(Z_2)$ from (2.3) does not contain the $\log(z_{12})$ term and then the nonlocal operator associated with the central term disappears [18,22,23].[5] Note that there is no $\mathbf{J^{(4)}}(Z_2)$ term on the right-hand side of (2.3). However, for a less supersymmetric theory with $\mathcal{N} \le 3$, the corresponding stress energy tensor term appears on the right-hand side of the OPE from $\frac{\theta_{12}^{\mathcal{N}}}{z_{12}^2}\,(4-\mathcal{N})\mathbf{J}^{(\mathcal{N})}(Z_2)$. Also the explicit component results (which will be described in next subsection) will be given in Appendix A.

2.3 OPEs of the $\mathcal{N}=4$ stress energy tensor in the component approach

With the $\mathcal{N}=4$ stress energy tensor given by (2.1) and its $\mathcal{N}=4$ OPE given by (2.3), it is straightforward to reexpress (2.3) in the component approach by substituting (2.1) into (2.3). In Appendix A, the SO(4)-extended linear superconformal algebra [18] is presented. Let us read off the OPE $\Gamma^1(z_1)\,T^{12}(z_2)$ related to the 11th equation of Appendix A.1 from (2.3). Multiplying the differential operator $D_1^1\,D_2^3\,D_2^4$ on the left-hand side of (2.3), we obtain the OPE $D_1^1\,\mathbf{J^{(4)}}(Z_1)\,D_2^3\,D_2^4\,\mathbf{J^{(4)}}(Z_2)$, where the (bosonic) operator $D_2^3\,D_2^4$ can pass the (bosonic) current $\mathbf{J^{(4)}}(Z_1)$ because the differential operator with respect to the supercoordinate Z_2 commutes with the current $\mathbf{J^{(4)}}(Z_1)$, which depends on the supercoordinate Z_1 only. Now we take $\theta_1^i = \theta_2^i = 0$ and have the component OPE $-\Gamma^1(z_1)\,T^{12}(z_2)$ from the left-hand side.[6] We then identify the above result with the 11th equation of Appendix A.1 by using the property in the OPE [14]. In this way, we can check that the $\mathcal{N}=4$ OPE in (2.3) is equivalent to the component results in Appendix A.1 by acting the differential operators D_1^i and D_2^j on (2.3) and setting $\theta_1^i = \theta_2^i = 0$.

[5] Recall that the superspin of the $\mathcal{N}=4$ stress energy tensor is zero and the OPE between the spin-0 current, $\Delta(z_1)\,\Delta(z_2)$, has a term $\log(z_1 - z_2)$. One can see this feature from (2.3) by taking $\theta^i = 0$ on both sides. Then the left-hand side is given by the OPE $\Delta(z_1)\,\Delta(z_2)$ while the right-hand side is given by $-\frac{1}{2}(k^+ + k^-)\,\log(z_1 - z_2)$. The associated OPE is in the last equation of Appendix A.1, where we introduce $U(z) \equiv -\partial\Delta(z)$.

[6] Note that cubic terms in θ^i can arise from the second term of (2.3). The nontrivial terms occur from the particular term $\theta_{12}^3\,\theta_{12}^4\,\theta_{12}^1$. It is easy to see that the above differential operator $D_1^1\,D_2^3\,D_2^4$ acting on this leads to -1. Finally, we obtain the OPE $\Gamma^1(z_1)\,T^{12}(z_2) = \frac{1}{(z_1-z_2)}\,i\,\Gamma^2(w) + \cdots$, where z_{12} is reduced to $(z_1 - z_2)$ after setting the condition $\theta_1^i = \theta_2^i = 0$. Note that the $D_2^2\,\mathbf{J^{(4)}}(Z_2)$ term reduces to the expression $i\Gamma^2(z_2)$.

We can check the $\mathcal{N}=4$ OPE from its component results by using the following identity:

$$\frac{1}{(z_1-z_2)^n} = \frac{1}{z_{12}^n} - n\,\frac{\theta_1^i\theta_2^i}{z_{12}^{n+1}} + \frac{1}{2!}n(n+1)\,\frac{\theta_1^i\theta_2^i\theta_1^j\theta_2^j}{z_{12}^{n+2}} - \frac{1}{3!}n(n+1)(n+2)\,\frac{\theta_1^i\theta_2^i\theta_1^j\theta_2^j\theta_1^k\theta_2^k}{z_{12}^{n+3}} + \frac{1}{4!}n(n+1)(n+2)(n+3) \times \frac{\theta_1^i\theta_2^i\theta_1^j\theta_2^j\theta_1^k\theta_2^k\theta_1^l\theta_2^l}{z_{12}^{n+4}},$$
$$z_{12} \equiv z_1 - z_2 - \theta_1^i\theta_2^i. \quad (2.4)$$

This connects the structure of the nth-order pole in the bosonic coordinate with those (plus other poles) in the $\mathcal{N}=4$ superspace coordinate.

The next higher-order terms in (2.4) contain the expression $(\theta_1^i\theta_2^i)^5$ and this is identically zero owing to the property of the fermionic coordinates. The positive integer n can be $n=1,2,3,$ or 4 for the 16 currents of the large $\mathcal{N}=4$ linear superconformal algebra because the highest spin among them is given by 2 and the highest singular term is the fourth-order pole in the OPE.[7]

2.4 $\mathcal{N}=4$ stress energy tensor in another basis

It is useful to express (2.1) in the basis of [24], where the spin-1 currents are represented in an SU(2) × SU(2) symmetric way, rather than in an SO(4) symmetric way. Then we would expect that the previous SO(4) adjoint six spin-1 currents $T^{ij}(z)$ can be expressed in terms of two SU(2) adjoints. The final result is

$$\begin{aligned}\mathbf{J}^{(4)}_{\mathbf{bcg}}(Z) &= -\Delta(z) - \theta^1\,Q^1(z) + \theta^2\,Q^2(z) + \theta^3\,Q^3(z) \\ &+ \theta^4\,Q^4(z) + \theta^{12}\,(-A^{+3}+A^{-3})(z) \\ &+ \theta^{13}(A^{+2}-A^{-2})(z) - \theta^{14}(A^{+1}+A^{-1})(z) \\ &+ \theta^{23}(A^{+1}-A^{-1})(z) + \theta^{24}(A^{+2}+A^{-2})(z) \\ &+ \theta^{34}\,(A^{+3}+A^{-3})(z) - \theta^{243} \\ &\times (G^1_{bcg} + 2\alpha\,\partial Q^1)(z) \\ &+ \theta^{134}\,(G^2_{bcg} + 2\alpha\,\partial Q^2)(z) \\ &+ \theta^{142}\,(G^3_{bcg} + 2\alpha\,\partial Q^3)(z) \\ &+ \theta^{123}\,(G^4_{bcg} + 2\alpha\,\partial Q^4)(z) \\ &+ \theta^{1234}\,(2L + 2\alpha\,\partial U)(z), \end{aligned} \quad (2.5)$$

where $U(z) = -\partial\Delta(z)$. Let us consider the spin-$\frac{1}{2}$ currents. In Appendix A.1, the OPE $\Gamma^i(z)\,\Gamma^j(w)$ has a positive sign on the right-hand side. In Appendix B.1, where the SU(2) × SU(2) subalgebra is manifest, the corresponding OPE appearing in the second line from the bottom has a negative sign. This implies that the spin-$\frac{1}{2}$ current $Q^a(z)$ in [25] is given by $\pm i$ times the spin-$\frac{1}{2}$ current $\Gamma^i(z)$ in the previous subsection. Furthermore, the OPE $U(z)\,G^i(w)$ in Appendix A.1 has a second-order pole with the term $-i\,\Gamma^i(w)$, and in the corresponding OPE of Appendix B there is no $-i$ factor. If the spin-$\frac{3}{2}$ current remains the same (the spin-1 current $U(z)$ being common), then the corresponding spin-$\frac{1}{2}$ currents have an extra $-i$, but this is not the case. The reason is as follows: In the OPE of $G^i(z)\,\Gamma^j(w)$ with $j=i$ of Appendix A, there exists a term $i\,U(w)$ on the right-hand side. In contrast, the corresponding OPE of Appendix B does not have the complex number i.

Therefore, the product of $\Gamma^i(z)$ and $G^i(z)$ should contain the extra $-i$. This implies that some of the spin-$\frac{3}{2}$ current changes by a minus sign. Once the spin-$\frac{1}{2}$ currents are fixed, then the spin-$\frac{3}{2}$ currents are also determined completely from this property. There are still other constraints in Appendix A. So we can proceed by assuming that the spin-$\frac{1}{2}$ currents $Q^a(z)$ can be written as $\pm i$ times the spin-$\frac{1}{2}$ currents $\Gamma^i(z)$ and then use other nontrivial relations. It turns out that, as in (2.5), the first component of the spin-$\frac{1}{2}$ current has an extra $-i$ ($Q^1(z) = -i\Gamma^1(z)$) and the remaining ones have the extra i ($Q^j(z) = i\Gamma^j(z)$, where $j=2,3,4$). Furthermore, the first component of the spin-$\frac{3}{2}$ current remains unchanged ($G^1_{bcg}(z) = G^1(z)$) whereas the others change by a minus sign ($G^i_{bcg}(z) = -G^i(z)$, where $i=2,3,4$) as in (2.5).

What about the spin-1 currents? For example, let us focus on the OPE $G^1(z)\,\Gamma^4(w)$ in Appendix A, which has a first-order pole, $-T^{23}(w)$. The corresponding OPE from Appendix B is given by the expression $G^1(z)\,(-i)\,Q^4(w)$, which contains the first-order pole $(-i)\times 2(-\frac{1}{2}A^{+1} - \frac{1}{2}A^{-1})(w)$ from Appendix B by substituting the values of the α tensor explicitly. Then we conclude that the spin-1 current $T^{23}(z)$ is equal to the expression $-i(A^{+1}+A^{-1})(z)$. In this way we can determine the precise relations, as in (2.5), between the spin-1 currents with SO(4) indices and those with SU(2) × SU(2) indices as follows:

$$\begin{aligned} &T^{12}(z) = i(A^{+3}+A^{-3})(z), \quad T^{13}(z) = -i(A^{+2}+A^{-2})(z), \\ &T^{14}(z) = i(A^{+1}-A^{-1})(z), \quad T^{23}(z) = -i(A^{+1}+A^{-1})(z), \\ &T^{24}(z) = -i(A^{+2}-A^{-2})(z), \; T^{34}(z) = -i(A^{+3}-A^{-3})(z). \end{aligned} \quad (2.6)$$

[7] Let us consider the OPE $\Gamma^1(z_1)\,T^{12}(z_2)$. Then the corresponding term on the left-hand side of (2.3) is given by $i\,\theta_1^1\,\Gamma^1(z_1)\,(-i)\,\theta_2^3\,\theta_2^4\,T^{12}(z_2)$. This becomes $\theta_1^1\,\theta_2^3\,\theta_2^4\,\Gamma^1(z_1)\,T^{12}(z_2)$ (where there is no sign change). Now we can use the component result between the spin-$\frac{1}{2}$ current and spin-1 current in Appendix A. Then the above expression leads to the singular term $\theta_1^1\,\theta_2^3\,\theta_2^4\,\frac{1}{(z_1-z_2)}\,i\,\Gamma^2(w) + \cdots$. By using Eq. (2.4), we obtain the nontrivial singular term $\frac{\theta_1^1\,\theta_2^3\,\theta_2^4}{z_{12}}\,i\,\Gamma^2(z_2)$, which can be written as $\frac{\theta_{12}^3\,\theta_{12}^4\,\theta_{12}^1}{z_{12}}\,D^2\,\mathbf{J}^{(4)}(Z_2)$ by considering other nontrivial terms also. In this way, we can check that all the component results in Appendix A can be rewritten in terms of a single OPE (2.3) in the $\mathcal{N}=4$ superspace.

Of course, with (2.6), we can also express the spin-1 currents $A^{\pm i}(z)$ in terms of the spin-1 currents $T^{ij}(z)$. We also identify the spin-2 current $L(z)$ on both sides. Therefore, the two Appendices A and B are equivalent to each other via the explicit current identifications described above.

2.5 Realization of the $\mathcal{N}=4$ stress energy tensor in $\mathcal{N}=4$ coset theory

It is well known that the above 16 currents of the large $\mathcal{N}=4$ linear superconformal algebra can be realized by the spin-$\frac{1}{2}$ and the spin-1 currents of $\mathcal{N}=4$ coset $\frac{SU(N+2)}{SU(N)}$ theory [6,26]. The two parameters are related to the two quantities in $\mathcal{N}=4$ coset theory as follows [27]:

$$k^+ = (k+1), \quad k^- = (N+1). \tag{2.7}$$

Here the level k appears in the central term $k\,g^{ab}$ of the OPE between the spin-1 currents $V^a(z_1)\,V^b(z_2)$, where the indices $a, b, \ldots$ are the adjoint indices of the group $G = SU(N+2)$. The metric, structure constants, and almost complex structures in the 16 currents occur in their coefficient functions in the multiple product of spin-$\frac{1}{2}$ and spin-1 currents. Furthermore, it is natural to ask whether there exist the $\mathcal{N}=4$ affine Kac–Moody current $Q^a(Z)$ of superspin $\frac{1}{2}$ and the corresponding Sugawara construction for the $\mathcal{N}=4$ stress energy tensor $\mathbf{J^{(4)}}(Z)$. It would be interesting to study this aspect in detail. The point here is how we can generalize the known $\mathcal{N}=2$ superspace description with constraints in the $\mathcal{N}=4$ superspace.

3 OPEs between the 16 currents and the 16 higher spin currents in the $\mathcal{N}=4$ superspace

In this section, we describe the $\mathcal{N}=4$ (super) primary current, in an SO(4) symmetric way, under the $\mathcal{N}=4$ stress energy tensor explained in the previous section. Although these 16 higher spin currents, in general, transform as a nontrivial representation under group SO(4), only the SO(4)-singlet 16 higher spin currents are described. Furthermore, the superspin is, in general, given by the positive integer s, but its lowest value $s=1$ will be considered later when the OPEs between them are calculated for generic N and k. The OPEs between the 16 currents and the 16 higher spin currents in the component approach are also given. Another basis, where the $SU(2)\times SU(2)$ symmetry is manifest, is compared (the higher spin-3 current not being a primary current under the stress energy tensor) and we also give the precise relations with the ones in [12].

3.1 SO(4)-singlet $\mathcal{N}=4$ multiplet of superspin s

For general superspin s, we can express the $\mathcal{N}=4$ (higher spin) multiplet as follows:[8]

$$\begin{aligned}
\Phi^{(\mathbf{s})}(Z) &= \Phi_0^{(s)}(z) + \theta^i\,\Phi_{\frac{1}{2}}^{(s),i}(z) + \theta^{4-ij}\,\Phi_1^{(s),ij}(z) \\
&\quad + \theta^{4-i}\,\Phi_{\frac{3}{2}}^{(s),i}(z) + \theta^{4-0}\,\Phi_2^{(s)}(z) \\
&= \Phi_0^{(s)}(z) + \theta^1\,\Phi_{\frac{1}{2}}^{(s),1}(z) + \theta^2\,\Phi_{\frac{1}{2}}^{(s),2}(z) \\
&\quad + \theta^3\,\Phi_{\frac{1}{2}}^{(s),3}(z) + \theta^4\,\Phi_{\frac{1}{2}}^{(s),4}(z) + \theta^{12}\Phi_1^{(s),34}(z) \\
&\quad + \theta^{13}\Phi_1^{(s),42}(z) + \theta^{14}\Phi_1^{(s),23}(z) + \theta^{23}\Phi_1^{(s),14}(z) \\
&\quad + \theta^{24}\Phi_1^{(s),31}(z) + \theta^{34}\Phi_1^{(s),12}(z) \\
&\quad + \theta^{324}\Phi_{\frac{3}{2}}^{(s),1}(z) + \theta^{134}\,\Phi_{\frac{3}{2}}^{(s),2}(z) + \theta^{142}\,\Phi_{\frac{3}{2}}^{(s),3}(z) \\
&\quad + \theta^{123}\,\Phi_{\frac{3}{2}}^{(s),4}(z) + \theta^{1234}\,\Phi_2^{(s)}(z),
\end{aligned} \tag{3.1}$$

where we introduce $\theta^{ij} \equiv \theta^i\,\theta^j$, $\theta^{ijk} \equiv \theta^i\,\theta^j\,\theta^k$, and $\theta^{1234} \equiv \theta^1\,\theta^2\,\theta^3\,\theta^4$. In components, there are a single higher spin-s current $\Phi_0^{(s)}(z)$, four higher spin-$(s+\frac{1}{2})$ currents $\Phi_{\frac{1}{2}}^{(s),i}(z)$ transforming as a vector representation under SO(4), six higher spin-$(s+1)$ currents $\Phi_1^{(s),ij}(z)$ transforming as an adjoint representation under SO(4), four higher spin-$(s+\frac{3}{2})$ currents $\Phi_{\frac{3}{2}}^{(s),i}(z)$, and the higher spin-$(s+2)$ current $\Phi_2^{(s)}(z)$. The higher spin-s and the higher spin-$(s+2)$ currents are SO(4) singlets. We can easily see that the subscript $(0, \frac{1}{2}, 1, \frac{3}{2}, 2)$ in the component currents stands for the number of Grassmann coordinates in the $\mathcal{N}=4$ superspace description. Depending on the superspin s, the above $\mathcal{N}=4$ (higher spin) multiplet is a bosonic higher spin current for integer spin s or a fermionic higher spin current for half-integer spin s. In general, the $\mathcal{N}=4$ (higher spin) multiplet has a nontrivial SO(4) representation [18].

3.2 $\mathcal{N}=4$ primary condition

Because the superspin of $\mathbf{J^{(4)}}(Z_1)$ is zero, the right-hand side of the OPE $\mathbf{J^{(4)}}(Z_1)\,\Phi^{(\mathbf{s})}(Z_2)$ has a superspin s. The pole structure of the linear term in $\Phi^{(\mathbf{s})}(Z_2)$ on the right-hand side should have spin 0 without any SO(4) indices. This implies that the structure should be $\frac{\theta_{12}^4}{z_{12}^2}$, where the spin of $\frac{1}{z_{12}}$ is equal to 1 and the spin of θ_{12} is equal to $-\frac{1}{2}$. The ordinary derivative term can occur at the singular term $\frac{\theta_{12}^4}{z_{12}}$. Furthermore, the spinor derivative terms (descendant terms) with a triple product of θ_{12} arise. Finally, we obtain the following

[8] In the notation of [5], this corresponds to $R^{(s)}(\mathbf{1},\mathbf{1})(Z)$, where the representation $(\mathbf{1},\mathbf{1})$ stands for the singlet under $SU(2)\times SU(2)$. In our case, there is no SO(4) index in the $\mathcal{N}=4$ (higher spin) multiplet $\Phi^{(\mathbf{s})}(Z)$.

$\mathcal{N}=4$ primary condition for the 16 higher spin currents in the $\mathcal{N}=4$ superspace [18]:

$$\mathbf{J^{(4)}}(Z_1)\,\Phi^{(\mathbf{s})}(Z_2) = \frac{\theta_{12}^4}{z_{12}^2}\,2s\,\Phi^{(\mathbf{s})}(Z_2) + \frac{\theta_{12}^{4-i}}{z_{12}}\,D^i\,\Phi^{(\mathbf{s})}(Z_2) + \frac{\theta_{12}^4}{z_{12}}\,2\,\partial\Phi^{(\mathbf{s})}(Z_2) + \cdots, \quad (3.2)$$

where $\theta_{12}^4 = \theta_{12}^1\,\theta_{12}^2\,\theta_{12}^3\,\theta_{12}^4$. We can understand the numerical factor of 2 in the first and the last terms of (3.2) by taking $D_1^1\,D_1^2\,D_1^3\,D_1^4$ on both sides and setting $\theta_1^i = 0 = \theta_2^i$. Then the left-hand side is given by the OPE $2L(z_1)\,\Phi_0^{(s)}(z_2)$ and the right-hand side is given by the expression $\frac{1}{(z_1-z_2)^2}\,2s\,\Phi_0^{(s)}(z_2) + \frac{1}{(z_1-z_2)}\,2\,\partial\Phi_0^{(s)}(z_2)+\cdots$. Therefore, by canceling out the factor of 2 in this OPE, we obtain the usual primary condition for the higher spin current $\Phi_0^{(s)}(z_2)$.[9]

How is the coefficient of the second term in (3.2) affected? In this case, we can apply the differential operator $D_1^1\,D_1^2\,D_1^3\,D_1^4\,D_2^1$ with the condition $\theta_1^i = 0 = \theta_2^i$ to both sides of (3.2). After this action, the left-hand side is given by the OPE $2L(z_1)\,\Phi_{\frac{1}{2}}^{(s),1}(z_2)$, where we also use the fact that the OPE $U(z_1)\,\Phi_{\frac{1}{2}}^{(s),1}(z_2)$ does not have any singular terms. Also we assume that the superspin s is an integer.[10]

The other cases for the remaining higher spin-$(s+\frac{1}{2})$ currents can be analyzed similarly. Let us emphasize that, for the nonsinglet SO(4) representation for the $\mathcal{N}=4$ higher spin multiplet, there exist nontrivial extra terms in the above OPE [18]. In particular, the $\frac{\theta_{12}^{4-ij}}{z_{12}}$ term contracted with the SO(4) representation T^{ij} (which is not a spin-1 current) appears. Furthermore, one of the extra indices in T^{ij} is contracted with the index of $\mathcal{N}=4$ (higher spin) multiplet. (See also [22,23].)

9 We used the fact that there are no singular terms in the OPE of $U(z_1)\,\Phi_0^{(s)}(z_2)$ by taking $\theta_1^i = 0 = \theta_2^i$ on both sides of (3.2). For $s=1$, this is true because the lowest higher spin-1 current commutes with the spin-1 current $U(z_1)$ (and four spin-$\frac{1}{2}$ currents); this is the so-called Goddard–Schwimmer mechanism [28]. For arbitrary s, the description of [24] implies that this is also true. In this calculation, we can easily see that there is no contribution from the second term of (3.2) because the action of $D_1^1\,D_1^2\,D_1^3\,D_1^4$ on this second term, with the condition $\theta_1^i = 0 = \theta_2^i$, vanishes.

10 The right-hand side from the first term in (3.2) is given by $\frac{1}{(z_1-z_2)^2}\,2s\,\Phi_{\frac{1}{2}}^{(s),1}(z_2)$. The second term gives the singular term $\frac{1}{(z_1-z_2)^2}\,\Phi_{\frac{1}{2}}^{(s),1}(z_2)$, where the identity $D_2^1\,\frac{1}{z_{12}} = -\frac{\theta_{12}^1}{z_{12}^2}$ is used. The third term gives $\frac{1}{(z_1-z_2)}\,2\,\partial\Phi_{\frac{1}{2}}^{(s),1}(z_2)$. Combining all these contributions, we obtain the following expression: $L(z_1)\,\Phi_{\frac{1}{2}}^{(s),1}(z_2) = \frac{1}{(z_1-z_2)^2}\,(s+\frac{1}{2})\,\Phi_{\frac{1}{2}}^{(s),1}(z_2) + \frac{1}{(z_1-z_2)}\,\partial\Phi_{\frac{1}{2}}^{(s),1}(z_2)+\cdots$, which will appear in Appendix C. The coefficient of the second term of (3.2) is fixed by the $\frac{1}{2}$ of the second-order pole for the primary condition.

3.3 $\mathcal{N}=4$ primary condition in the component approach

From (3.2), we can read off its component expressions. For example, let us consider the OPE $G^1(z_1)\,\Phi_1^{(s),12}(z_2)$. Then we should multiply both sides of (3.2) by the differential operator $D_1^2\,D_1^3\,D_1^4\,D_2^3\,D_2^4$. At the final stage, we set the condition $\theta_1^i = \theta_2^i = 0$. Then the left-hand side of the OPE is given by $(G^1 - 2\,\alpha\,i\,\partial\Gamma^1)(z_1)\,\Phi_1^{(s),12}(z_2)$ while the right-hand side is given by the contribution from the second term of (3.2). That is, the action of $D_1^2\,D_1^3\,D_1^4$ into $\frac{\theta_{12}^{234}}{z_{12}}$ gives the singular term $\frac{1}{z_{12}}$ and the factor $D_2^3\,D_2^4\,D_2^1\,\Phi^{(\mathbf{s})}(Z_2)$ provides the expression $-\Phi_{\frac{3}{2}}^{(s),2}(z_2)$ after the projection of $\theta_1^i = 0 = \theta_2^i$. Combining all these factors leads to the final expression $-\frac{1}{(z_1-z_2)}\,\Phi_{\frac{3}{2}}^{(s),2}(z_2)+\cdots$. To see the contribution from the left-hand side in the above, we should consider the OPE $\Gamma^1(z_1)\,\Phi_1^{(s),12}(z_2)$.[11] Therefore, we obtain the following OPE $G^1(z_1)\,\Phi_1^{(s),12}(z_2)$ as $-\frac{1}{(z_1-z_2)^2}\,2\,\alpha\,\Phi_{\frac{1}{2}}^{(s),2}(z_2) - \frac{1}{(z_1-z_2)}\,\Phi_{\frac{3}{2}}^{(s),2}(z_2)+\cdots$, which can be seen from Appendix C. In this way, we can obtain all the component results in Appendix C.

3.4 $\mathcal{N}=4$ multiplet where all the component currents are primary under the bosonic stress energy tensor

According to the OPEs in Appendix C, the higher spin currents $\Phi_0^{(s)}(w)$, $\Phi_{\frac{1}{2}}^{(s),i}(w)$, and $\Phi_1^{(s),ij}(w)$ of spins s, $(s+\frac{1}{2})$, and $(s+1)$, respectively, are primary fields under the stress energy tensor $L(z)$. However, the higher spin currents $\Phi_{\frac{3}{2}}^{(s),i}(w)$ and $\Phi_2^{(s)}(w)$ are not primary fields. We can make them primary by introducing other composite or derivative terms as follows:

$$\Phi^{(\mathbf{s})}(Z) = \Phi_0^{(s)}(z) + \theta^i\,\Phi_{\frac{1}{2}}^{(s),i}(z) + \theta^{4-ij}\,\Phi_1^{(s),ij}(z) + \theta^{4-i}\left[\tilde{\Phi}_{\frac{3}{2}}^{(s),i}(z) + \frac{2}{(2s+1)}\alpha\,\partial\Phi_{\frac{1}{2}}^{(s),i}(z)\right] + \theta^{4-0}\left[\tilde{\Phi}_2^{(s)}(z) + p_1\,\partial^2\Phi_0^{(s)}(z) + p_2 L\,\Phi_0^{(s)}(z)\right], \quad (3.3)$$

where the coefficients depend on the spin s, N, and k as

11 Again, let us multiply both sides of (3.2) by $D_1^1\,D_2^3\,D_2^4$. Then we obtain that the left-hand side is given by the OPE $-i\,\Gamma^1(z_1)\,\Phi_1^{(s),12}(z_2)$ after projection. The nontrivial contribution from the right-hand side can be written as $-\frac{1}{(z_1-z_2)}\,\Phi_{\frac{1}{2}}^{(s),2}(z_2)+\cdots$. We conclude that the corresponding OPE can be written as $\Gamma^1(z_1)\,\Phi_1^{(s),12}(z_2) = -\frac{1}{(z_1-z_2)}\,i\,\Phi_{\frac{1}{2}}^{(s),2}(z_2)+\cdots$, which can be seen from Appendix C. Furthermore, we have $\partial\Gamma^1(z_1)\,\Phi_1^{(s),12}(z_2) = \frac{1}{(z_1-z_2)^2}\,i\,\Phi_{\frac{1}{2}}^{(s),2}(z_2)+\cdots$.

$$p_1 = \frac{2(k-N)(3+3k+3N+3kN+26s+13ks+13Ns)}{(2+k+N)(3+3k+3N+3kN-4s+ks+Ns+6kNs+16s^2+8ks^2+8Ns^2)},$$
$$p_2 = -\frac{12\,(k-N)\,s\,(1+s)}{(3+3k+3N+3kN-4s+ks+Ns+6kNs+16s^2+8ks^2+8Ns^2)}, \qquad (3.4)$$

For $N = k$, these coefficients vanish. It is straightforward to obtain these two coefficients explicitly by requiring that $\widetilde{\Phi}_2^{(s)}(w) \equiv \Phi_2^{(s)}(w) - p_1\,\partial^2\Phi_0^{(s)}(w) - p_2\,L\,\Phi_0^{(s)}(w)$ with (3.4) transform as a primary current under the stress energy tensor $L(z)$ via Appendix C. For $s = 1$, this observation was made in [12].

3.5 $\mathcal{N} = 4$ multiplet in another basis

As in previous sections, we can also compare expression (3.1) with the corresponding quantity in [24] by following the previous procedure for the 16 currents. Let us express the answer as follows:

$$\begin{aligned}
\mathbf{V^{(s)}}(Z) &= i\,V_0^{(s)}(z) - i\,\theta^1\,V_{\frac{1}{2}}^{(s),1}(z) + i\,\theta^2\,V_{\frac{1}{2}}^{(s),2}(z)\\
&\quad + i\,\theta^3\,V_{\frac{1}{2}}^{(s),3}(z) + i\,\theta^4\,V_{\frac{1}{2}}^{(s),4}(z)\\
&\quad + \theta^{12}\,\frac{i}{2}\left(V_1^{(s),+3} + V_1^{(s),-3}\right)(z)\\
&\quad - \theta^{13}\,\frac{i}{2}\left(V_1^{(s),+2} + V_1^{(s),-2}\right)(z)\\
&\quad + \theta^{14}\,\frac{i}{2}\left(V_1^{(s),+1} - V_1^{(s),-1}\right)(z)\\
&\quad + \theta^{23}\,\frac{i}{2}\left(-V_1^{(s),+1} - V_1^{(s),-1}\right)(z)\\
&\quad - \theta^{24}\,\frac{i}{2}\left(V_1^{(s),+2} - V_1^{(s),-2}\right)(z)\\
&\quad + \theta^{34}\,\frac{i}{2}\left(-V_1^{(s),+3} + V_1^{(s)-3}\right)(z)\\
&\quad + \theta^{243}\,\frac{i}{2}\left[V_{\frac{3}{2}}^{(s),1} - \frac{4\alpha}{(2s+1)}\partial V_{\frac{1}{2}}^{(s),1}\right](z)\\
&\quad - \theta^{134}\,\frac{i}{2}\left[V_{\frac{3}{2}}^{(s),2} - \frac{4\alpha}{(2s+1)}\partial V_{\frac{1}{2}}^{(s),2}\right](z)\\
&\quad - \theta^{142}\,\frac{i}{2}\left[V_{\frac{3}{2}}^{(s),3} - \frac{4\alpha}{(2s+1)}\partial V_{\frac{1}{2}}^{(s),3}\right]\\
&\quad - \theta^{123}\,\frac{i}{2}\left[V_{\frac{3}{2}}^{(s),4} - \frac{4\alpha}{(2s+1)}\partial V_{\frac{1}{2}}^{(s),4}\right](z)\\
&\quad - \theta^{1234}\,\frac{i}{2}\left[V_2^{(s)} - \frac{4\alpha}{(2s+1)}\partial^2 V_0^{(s)}\right](z). \qquad (3.5)
\end{aligned}$$

Let us consider the lowest higher spin-1 current. In [24], the normalization for the OPE $V_0^{(s)}(z)\,V_0^{(s)}(w)$ for $s = 1$ has an extra minus sign whereas in our case there is a positive sign. Therefore, we introduce the complex number i as in (3.5). Now we consider the next higher spin-$(s+\frac{1}{2})$ current. Let us consider the OPE $G^i(z)\,\Phi_0^{(s)}(w)$ in Appendix C. For the index $i = 1$, the spin-$\frac{3}{2}$ current $G^1(z)$ is the same as the one in [24]. In other words, we obtained $G^1_{bcg}(z) = G^1(z)$ before. By substituting $\Phi_0^{(s)}(w)$ for $i\,V_0^{(s)}(w)$, the left-hand side is given by i times the OPE $G^1_{bcg}(z)\,V_0^{(s)}(w)$, which is equal to the expression $-\Phi_{\frac{1}{2}}^{(s),1}(w)$ from Appendix C. This implies that the expression $i\,\Phi_{\frac{1}{2}}^{(s),1}(w)$ is equal to the expression $V_{\frac{1}{2}}^{(s),1}(w)$ from Appendix D. Therefore, we conclude that the higher spin current $\Phi_{\frac{1}{2}}^{(s),1}(w) = -i\,V_{\frac{1}{2}}^{(s),1}(w)$, as in (3.5). For the indices $i = 2, 3, 4$, the corresponding spin-$\frac{3}{2}$ currents have an extra minus sign (that is, $G^i_{bcg}(z) = -G^i(z)$) and this reflects the signs for the higher spin-$(s+\frac{1}{2})$ currents with these indices as in (3.5).

Let us consider the next higher spin-$(s+1)$ currents. Let us look at the OPE $G^i(z)\,\Phi_{\frac{1}{2}}^{(s),j}(w)$ with indices $(i, j) = (1, 2)$, which contains the first-order pole $\Phi_1^{(s),34}(w)$. Then the left-hand side corresponds to the OPE $G^1_{bcg}(z)\,(i)\,V_{\frac{1}{2}}^{(s),2}(w)$. According to Appendix D, the corresponding OPE is given by the third OPE. The first-order pole is given by the expression $i\,(\alpha_{12}^{+,i}\,V_1^{(s),+i} + \alpha_{12}^{-,i}\,V_1^{(s),-i})(w) = \frac{i}{2}(V_1^{(s),+3} + V_1^{(s),-3})(w)$. This is exactly the one given in (3.5). Then we arrive at the following results for the higher spin-$(s+1)$ currents:

$$\begin{aligned}
\Phi_1^{(s),14}(z) &= -\frac{i}{2}\left(V_1^{(s),+1} + V_1^{(s),-1}\right)(z),\\
\Phi_1^{(s),23}(z) &= \frac{i}{2}\left(V_1^{(s),+1} - V_1^{(s),-1}\right)(z),\\
\Phi_1^{(s),42}(z) &= -\frac{i}{2}\left(V_1^{(s),+2} + V_1^{(s),-2}\right)(z),\\
\Phi_1^{(s),31}(z) &= -\frac{i}{2}\left(V_1^{(s),+2} - V_1^{(s),-2}\right)(z),\\
\Phi_1^{(s),34}(z) &= \frac{i}{2}\left(V_1^{(s),+3} + V_1^{(s),-3}\right)(z),\\
\Phi_1^{(s),12}(z) &= -\frac{i}{2}\left(V_1^{(s),+3} - V_1^{(s),-3}\right)(z). \qquad (3.6)
\end{aligned}$$

Let us continue to describe the next higher spin-$(s+\frac{3}{2})$ currents by starting from the OPE $G^1(z)\,\Phi_1^{(s),12}(w)$ in Appendix C. The first-order pole gives $-\Phi_{\frac{3}{2}}^{(s),2}(w)$. From Eq. (3.6), we can calculate the OPE $(-\frac{i}{2})\,G^1_{bcg}(z)\,(V_1^{(s),+3} - V_1^{(s),-3})(w)$ using the results of Appendix D. It turns out that the first-order pole is given by the expression $\frac{i}{2}\,(V_{\frac{3}{2}}^{(s),2} - \frac{4\alpha}{(2s+1)}\partial V_{\frac{1}{2}}^{(s),2})$, where the $\alpha_{ab}^{\pm,i}$ are substituted. By considering the appropriate coefficients, we arrive at the final expression located at the θ^{134} term in (3.5). A similar analysis for other types of higher spin-$(s+\frac{3}{2})$ currents can be performed.[12]

3.6 Explicit relations between the higher spin currents in different bases

Starting from the higher spin-1 current

$$V_0^{(1)}(z) = -i\,T^{(1)}(z) = -i\,\Phi_0^{(1)}(z), \tag{3.7}$$

where relations (3.1) and (3.5) are used, we can express the higher spin-$\frac{3}{2}$ currents as follows:

$$\begin{aligned}
&V_{\frac{1}{2}}^{(1),1}(z)\\
&= i\left(-i\,G^2+\sqrt{2}\,T_+^{(\frac{3}{2})}+\sqrt{2}\,T_-^{(\frac{3}{2})}\right)(z) = i\,\Phi_{\frac{1}{2}}^{(1),1}(z),\\
&V_{\frac{1}{2}}^{(1),2}(z)\\
&= -\left(G^1+\sqrt{2}\,T_+^{(\frac{3}{2})}-\sqrt{2}\,T_-^{(\frac{3}{2})}\right)(z) = -i\,\Phi_{\frac{1}{2}}^{(1),2}(z),\\
&V_{\frac{1}{2}}^{(1),3}(z)\\
&= i\left(i\,G^4+\sqrt{2}\,U^{(\frac{3}{2})}+\sqrt{2}\,V^{(\frac{3}{2})}\right)(z) = -i\,\Phi_{\frac{1}{2}}^{(1),3}(z),\\
&V_{\frac{1}{2}}^{(1),4}(z)\\
&= \left(G^3+\sqrt{2}\,U^{(\frac{3}{2})}-\sqrt{2}\,V^{(\frac{3}{2})}\right)(z) = -i\,\Phi_{\frac{1}{2}}^{(1),4}(z).
\end{aligned} \tag{3.8}$$

The first relation of (3.8) was obtained in [12] and the second can be obtained from (3.1) and (3.5). Obviously, the higher spin-$\frac{3}{2}$ currents $T_{\pm}^{(\frac{3}{2})}(z)$, $U^{(\frac{3}{2})}(z)$, and $V^{(\frac{3}{2})}(z)$ can be expressed in terms of linear combinations of the higher spin current $\Phi_{\frac{1}{2}}^{(1),i}(z)$.

Similarly, the higher spin-2 currents can be expressed in terms of linear combinations of the higher spin current $\Phi_1^{(1),ij}(z)$ as follows:

$$\begin{aligned}
V_1^{(1),\pm 1}(z) &= \pm 2\left(U_{\mp}^{(2)} \mp V_{\pm}^{(2)}\right)(z)\\
&= i\left(\Phi_1^{(1),14} \mp \Phi_1^{(1),23}\right)(z),\\
V_1^{(1),\pm 2}(z) &= -2i\left(U_{\mp}^{(2)} + V_{\pm}^{(2)}\right)(z)\\
&= i\left(\Phi_1^{(1),42} \pm \Phi_1^{(1),31}\right)(z),\\
V_1^{(1),\pm 3}(z) &= 2\left(T^{(2)} \mp W^{(2)}\right)(z)\\
&= i\left(-\Phi_1^{(1),34} \pm \Phi_1^{(1),12}\right)(z).
\end{aligned} \tag{3.9}$$

In (3.9), the last relation is obtained from (3.5) and (3.1).

Furthermore, we have the following relations for the higher spin-$\frac{5}{2}$ currents:

$$\begin{aligned}
V_{\frac{3}{2}}^{(1),1}(z) &= -2i\sqrt{2}\left(W_+^{(\frac{5}{2})} - W_-^{(\frac{5}{2})}\right)(z)\\
&= -2i\left(\Phi_{\frac{3}{2}}^{(1),1} + \frac{2\alpha}{3}\partial\Phi_{\frac{1}{2}}^{(1),1}\right)(z),\\
V_{\frac{3}{2}}^{(1),2}(z) &= 2\sqrt{2}\left(W_+^{(\frac{5}{2})} + W_-^{(\frac{5}{2})}\right)(z)\\
&= 2i\left(\Phi_{\frac{3}{2}}^{(1),2} - \frac{2\alpha}{3}\partial\Phi_{\frac{1}{2}}^{(1),2}\right)(z),\\
V_{\frac{3}{2}}^{(1),3}(z) &= -2i\sqrt{2}\left(U^{(\frac{5}{2})} - V^{(\frac{5}{2})}\right)(z)\\
&= 2i\left(\Phi_{\frac{3}{2}}^{(1),3} - \frac{2\alpha}{3}\partial\Phi_{\frac{1}{2}}^{(1),3}\right)(z),\\
V_{\frac{3}{2}}^{(1),4}(z) &= -2\sqrt{2}\left(U^{(\frac{5}{2})} + V^{(\frac{5}{2})}\right)(z)\\
&= 2i\left(\Phi_{\frac{3}{2}}^{(1),4} - \frac{2\alpha}{3}\partial\Phi_{\frac{1}{2}}^{(1),4}\right)(z).
\end{aligned} \tag{3.10}$$

The first relation of (3.10) was obtained from [12] and the last relation was obtained from the previous results in (3.1) and (3.5).

Finally, the higher spin-3 current obeys the following relations [6]:

$$\begin{aligned}
V_2^{(1)}(z) &= 4i\Bigg[W^{(3)} + \frac{4(k-N)}{((4N+5)+(3N+4)k)}\\
&\quad \times\left(T^{(1)}\,L - \frac{1}{2}\,\partial^2 T^{(1)}\right)\Bigg](z)\\
&= 2i\left(\Phi_2^{(1)} - \frac{2\alpha}{3}\,\partial^2\Phi_0^{(1)}\right)(z).
\end{aligned} \tag{3.11}$$

In (3.11), the primary higher spin-3 current $W^{(3)}(z)$ under the stress energy tensor $L(z)$ can be expressed in terms of $\Phi_2^{(1)}(z)$, $\partial^2\Phi_0^{(1)}$, and $\Phi_0^{(1)}\,L(z)$ with (3.7).

For the next higher spin $\mathcal{N}=4$ multiplet of superspin 2, the exact relations between the ones in [12] and the ones in (3.1) with $s=2$, for general N and k, are rather complicated.

[12] Let us describe the final higher spin-$(s+2)$ current. As done before, we consider the OPE $G^1(z)\,\Phi_{\frac{3}{2}}^{(s),1(w)}$ with the help of Appendix C. The first-order pole is given by $-\Phi_2^{(s)}(w)$. However, the OPE $G_{bcg}^1(z)\,V_{\frac{3}{2}}^{(s),1(w)}$ leads to the first-order pole $V_2^{(s)}(w)$. The OPE $G_{bcg}^1(z)\,\partial V_{\frac{1}{2}}^{(s),1(w)}(w)$ has a first-order pole of $\partial^2 V_0^{(s)}(w)$. Combining all the coefficients gives the final expression in (3.5).

3.7 Realization of the lowest $\mathcal{N}=4$ multiplet in $\mathcal{N}=4$ coset theory

As done in the realization of the large $\mathcal{N}=4$ linear superconformal algebra in $\mathcal{N}=4$ coset theory, we can construct the 16 higher spin currents in $\mathcal{N}=4$ coset theory, where there are two fundamental currents (the spin-1 current and the spin-$\frac{1}{2}$ current) [6]. (See Sect. 2.5.) It would be interesting to see whether we can construct the 16 higher spin currents using the $\mathcal{N}=4$ affine Kac–Moody current $Q^a(Z)$ of superspin $\frac{1}{2}$ via the generalized Sugawara construction.

4 OPEs between the 16 currents and the 16 higher spin currents in the $\mathcal{N}=2$ superspace

From the component results in Appendix C, we would like to construct them in the $\mathcal{N}=2$ superspace (in an SO(2) symmetric way) explicitly. The exact relations between the component higher spin currents and their $\mathcal{N}=2$ higher spin currents are given. Furthermore, we also express the $\mathcal{N}=4$ higher spin current multiplet in terms of four $\mathcal{N}=2$ higher spin currents by expansion of Grassmann coordinates.

4.1 The $\mathcal{N}=4$ primary current condition in the $\mathcal{N}=2$ superspace

The 16 higher spin currents can be represented by $\mathcal{N}=2$ higher spin-s current $\mathbf{T^{(s)}}(Z)$, two higher spin-$(s+\frac{1}{2})$ currents $\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}(Z)$, and higher spin-$(s+1)$ current $\mathbf{W^{(s+1)}}(Z)$. Each $\mathcal{N}=2$ higher spin current has four component currents and therefore the number of component currents is given by 16. The $U(1)$ charges of the $\mathcal{N}=2$ superconformal algebra are given by zero for both $\mathbf{T^{(s)}}(Z)$ and $\mathbf{W^{(s+1)}}(Z)$ and 2α for $\mathbf{U^{(s+\frac{1}{2})}}(Z)$ and -2α for $\mathbf{V^{(s+\frac{1}{2})}}(Z)$.

It turns out that there exists the following $\mathcal{N}=2$ version corresponding to (3.2):

$$
\begin{aligned}
&\mathbf{T}(Z_1)\,\mathbf{T^{(s)}}(Z_2)=\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2}s\mathbf{T^{(s)}}(Z_2)-\frac{\theta_{12}}{z_{12}}D\mathbf{T^{(s)}}(Z_2)\\
&\quad+\frac{\bar{\theta}_{12}}{z_{12}}\overline{D}\mathbf{T^{(s)}}(Z_2)+\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}}\partial\mathbf{T^{(s)}}(Z_2)+\cdots,\\
&\mathbf{T}(Z_1)\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}(Z_2)=\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2}\,(\frac{1}{2}+s)\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}(Z_2)\\
&\quad\pm\frac{1}{z_{12}}\,2\,\alpha\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}(Z_2)\\
&\quad-\frac{\theta_{12}}{z_{12}}\,D\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}(Z_2)+\frac{\bar{\theta}_{12}}{z_{12}}\,\overline{D}\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}(Z_2)\\
&\quad+\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}}\,\partial\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}(Z_2)+\cdots,\\
&\mathbf{T}(Z_1)\,\mathbf{W^{(s+1)}}(Z_2)=\frac{1}{z_{12}^2}\,\frac{64\alpha}{3}\,\mathbf{T^{(s)}}(Z_2)\\
&\quad+\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2}\,(1+s)\,\mathbf{W^{(s+1)}}(Z_2)-\frac{\theta_{12}}{z_{12}}\,D\mathbf{W^{(s+1)}}(Z_2)\\
&\quad+\frac{\bar{\theta}_{12}}{z_{12}}\,\overline{D}\mathbf{W^{(s+1)}}(Z_2)+\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}}\,\partial\mathbf{W^{(s+1)}}(Z_2)+\cdots,\\
&\mathbf{H}(Z_1)\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}(Z_2)=\mp\frac{\bar{\theta}_{12}}{z_{12}}\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}(Z_2)+\cdots,\\
&\overline{\mathbf{H}}(Z_1)\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}(Z_2)=\pm\frac{\theta_{12}}{z_{12}}\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}(Z_2)+\cdots,\\
&\begin{pmatrix}\mathbf{H}\\ \overline{\mathbf{H}}\end{pmatrix}(Z_1)\,\mathbf{W^{(s+1)}}(Z_2)=-\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2}\,\frac{4(1+2s)}{(1+2s^2)}\begin{pmatrix}D\mathbf{T^{(s)}}\\ \overline{D}\mathbf{T^{(s)}}\end{pmatrix}(Z_2)\\
&\quad\mp\frac{1}{z_{12}^2}\begin{pmatrix}\bar{\theta}_{12}\\ \theta_{12}\end{pmatrix}\frac{8s(1+2s)}{(1+2s^2)}\,\mathbf{T^{(s)}}(Z_2)\\
&\quad\pm\frac{1}{z_{12}}\,\frac{8(1+2s)}{(1+2s^2)}\begin{pmatrix}D\mathbf{T^{(s)}}\\ \overline{D}\mathbf{T^{(s)}}\end{pmatrix}(Z_2)+\cdots,\\
&\begin{pmatrix}\mathbf{G}\\ \overline{\mathbf{G}}\end{pmatrix}(Z_1)\,\mathbf{T^{(s)}}(Z_2)=\pm\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}}\,\frac{1}{2}\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}(Z_2)+\cdots,\\
&\begin{pmatrix}\mathbf{G}\\ \overline{\mathbf{G}}\end{pmatrix}(Z_1)\begin{pmatrix}\mathbf{V^{(s+\frac{1}{2})}}\\ \mathbf{U^{(s+\frac{1}{2})}}\end{pmatrix}(Z_2)=\pm\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2}\,2s\,\mathbf{T^{(s)}}(Z_2)\\
&\quad\mp\frac{\theta_{12}}{z_{12}}\,2\,D\mathbf{T^{(s)}}(Z_2)\pm\frac{\bar{\theta}_{12}}{z_{12}}\,2\,\overline{D}\mathbf{T^{(s)}}(Z_2)\\
&\quad+\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}}\Bigg[\frac{2}{(1+2s)}\,\alpha\,[D,\overline{D}]\,\mathbf{T^{(s)}}\\
&\quad+\frac{(1+2s^2)}{4(1+2s)}\,\mathbf{W^{(s+1)}}\pm\partial\mathbf{T^{(s)}}\Bigg](Z_2)+\cdots,\\
&\begin{pmatrix}\mathbf{G}\\ \overline{\mathbf{G}}\end{pmatrix}(Z_1)\,\mathbf{W^{(s+1)}}(Z_2)\\
&\quad=-\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2}\,\frac{2(1+2s)^2}{(1+2s^2)}\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}(Z_2)\\
&\quad\pm\frac{1}{z_{12}}\,\frac{8}{(1+2s^2)}\alpha\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}(Z_2)\\
&\quad+\frac{\theta_{12}}{z_{12}}\,\frac{8((1+s)k^{\pm}+sk^{\mp})}{(k^++k^-)(1+2s^2)}\,D\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}(Z_2)\\
&\quad-\frac{\bar{\theta}_{12}}{z_{12}}\,\frac{8((1+s)k^{\mp}+sk^{\pm})}{(k^++k^-)(1+2s^2)}\,\overline{D}\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}(Z_2)\\
&\quad+\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}}\Bigg[\mp\frac{4}{(1+2s^2)}\,\alpha\,[D,\overline{D}]\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}\\
&\quad-\frac{2(1+2s)}{(1+2s^2)}\,\partial\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}\Bigg](Z_2)+\cdots
\end{aligned}
\tag{4.1}
$$

As described before, the $\mathcal{N}=2$ spin of four $\mathcal{N}=2$ higher spin currents can be read off from the $\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2}$ term in the OPEs with the $\mathcal{N}=2$ stress energy tensor $\mathbf{T}(Z_1)$. The $\mathcal{N}=2$ higher spin current $\mathbf{W^{(s+1)}}(Z_2)$ is not an $\mathcal{N}=2$ primary under the $\mathcal{N}=2$ stress energy tensor $\mathbf{T}(Z_1)$ because there exists a $\frac{1}{z_{12}^2}$ term in (4.1). As in the standard $\mathcal{N}=2$ superspace formalism, by acting the $\mathcal{N}=2$ currents $\mathbf{G}(Z_1)$ and $\overline{\mathbf{G}}(Z_1)$ on the $\mathcal{N}=2$ higher spin current $\mathbf{T^{(s)}}(Z_2)$, the other $\mathcal{N}=2$ higher spin currents $\mathbf{U^{(s+\frac{1}{2})}}(Z_2)$ and $\mathbf{V^{(s+\frac{1}{2})}}(Z_2)$ can be generated. Furthermore, by acting the $\mathcal{N}=2$ currents $\mathbf{G}(Z_1)$ (or $\overline{\mathbf{G}}(Z_1)$) on the $\mathcal{N}=2$ higher spin currents $\mathbf{V^{(s+\frac{1}{2})}}(Z_2)$ (or $\mathbf{U^{(s+\frac{1}{2})}}(Z_2)$), the $\mathcal{N}=2$ higher spin current $\mathbf{W^{(s+1)}}(Z_2)$ is obtained.

How can we obtain the component results from the above OPEs (4.1)? Let us consider the OPE $G^1(z)\,\Phi_{\frac{1}{2}}^{(1),2}(w)$. From Appendix C, we expect that the first-order pole of this OPE is given by $\Phi_1^{(1),34}(w)$. We would like to see this from the OPEs (4.1). From Appendix E, we should take $(D_1+\overline{D}_1)\,\mathbf{T}(Z_1)$. Furthermore, for the other superspace coordinate we should take $(-i)(D_2+\overline{D}_2)\,\mathbf{T^{(1)}}(Z_2)$ because of the decompositions, which will appear in the next subsection. In other words, let us multiply both sides of the first equation of (4.1) with the conditions $\theta=\overline{\theta}=0$ by $(-i)(D_1+\overline{D}_1)(D_2+\overline{D}_2)$. Then the left-hand side of the OPE becomes the OPE $G^1(z_1)\,\Phi_{\frac{1}{2}}^{(1),2}(z_2)$.[13] We can also easily see that other contributions from the above differential operator acting on the other parts on the right-hand side will vanish.

In this way, we can check all the component results in Appendix C from its $\mathcal{N}=2$ version in (4.1) and vice versa.

4.2 Component currents in the four higher spin-s, $(s+\frac{1}{2})$, $(s+\frac{1}{2})$, $(s+1)$ currents in the $\mathcal{N}=2$ superspace

We also have the following relations between the $\mathcal{N}=2$ higher spin currents and their components:

$$
\begin{aligned}
&\mathbf{T}^{(s)}\,|_{\theta=\bar{\theta}=0}\,(z)=\Phi_0^{(s)}(z),\\
&D\mathbf{T}^{(s)}\,|_{\theta=\bar{\theta}=0}\,(z)=\frac{1}{2}\Big(\Phi_{\frac{1}{2}}^{(s),1}+i\,\Phi_{\frac{1}{2}}^{(s),2}\Big)(z),\\
&\overline{D}\mathbf{T}^{(s)}\,|_{\theta=\bar{\theta}=0}\,(z)=\frac{1}{2}\Big(-\Phi_{\frac{1}{2}}^{(s),1}+i\,\Phi_{\frac{1}{2}}^{(s),2}\Big)(z),\\
&-\frac{1}{2}[D,\overline{D}]\mathbf{T}^{(s)}\,|_{\theta=\bar{\theta}=0}\,(z)=\frac{1}{2}\,i\,\Phi_1^{(s),34}(z),\\
&\mathbf{U}^{(s+\frac{1}{2})}\,|_{\theta=\bar{\theta}=0}\,(z)=\Big(\Phi_{\frac{1}{2}}^{(s),3}+i\,\Phi_{\frac{1}{2}}^{(s),4}\Big)(z),\\
&D\mathbf{U}^{(s+\frac{1}{2})}\,|_{\theta=\bar{\theta}=0}\,(z)=\frac{1}{2}\Big(-\Phi_1^{(s),13}-i\,\Phi_1^{(s),14}\\
&-i\,\Phi_1^{(s),23}+\Phi_1^{(s),24}\Big)(z),\\
&\overline{D}\mathbf{U}^{(s+\frac{1}{2})}\,|_{\theta=\bar{\theta}=0}\,(z)=\frac{1}{2}\Big(-\Phi_1^{(s),13}-i\,\Phi_1^{(s),14}\\
&+i\,\Phi_1^{(s),23}-\Phi_1^{(s),24}\Big)(z),\\
&-\frac{1}{2}[D,\overline{D}]\mathbf{U}^{(s+\frac{1}{2})}\,|_{\theta=\bar{\theta}=0}\,(z)=\frac{1}{2}\Big(\Phi_{\frac{3}{2}}^{(s),3}+i\,\Phi_{\frac{3}{2}}^{(s),4}\Big)(z),\\
&\mathbf{V}^{(s+\frac{1}{2})}\,|_{\theta=\bar{\theta}=0}\,(z)=\Big(\Phi_{\frac{1}{2}}^{(s),3}-i\,\Phi_{\frac{1}{2}}^{(s),4}\Big)(z),\\
&D\mathbf{V}^{(s+\frac{1}{2})}\,|_{\theta=\bar{\theta}=0}\,(z)=\frac{1}{2}\Big(\Phi_1^{(s),13}-i\,\Phi_1^{(s),14}\\
&+i\,\Phi_1^{(s),23}+\Phi_1^{(s),24}\Big)(z),\\
&\overline{D}\mathbf{V}^{(s+\frac{1}{2})}\,|_{\theta=\bar{\theta}=0}\,(z)=\frac{1}{2}\Big(\Phi_1^{(s),13}-i\,\Phi_1^{(s),14}\\
&-i\,\Phi_1^{(s),23}-\Phi_1^{(s),24}\Big)(z),\\
&-\frac{1}{2}[D,\overline{D}]\mathbf{V}^{(s+\frac{1}{2})}\,|_{\theta=\bar{\theta}=0}\,(z)=\frac{1}{2}\Big(-\Phi_{\frac{3}{2}}^{(s),3}+i\,\Phi_{\frac{3}{2}}^{(s),4}\Big)(z),\\
&\mathbf{W}^{(s+1)}\,|_{\theta=\bar{\theta}=0}\,(z)=\frac{4i\,(1+2s)}{(1+2s^2)}\Phi_1^{(s),12}(z)\\
&+\frac{8i\,\alpha}{(1+2s^2)}\Phi_1^{(s),34}(z),\\
&D\mathbf{W}^{(s+1)}\,|_{\theta=\bar{\theta}=0}\,(z)=\frac{2\,(1+2s)}{(1+2s^2)}\Big(\Phi_{\frac{3}{2}}^{(s),1}+i\,\Phi_{\frac{3}{2}}^{(s),2}\Big)(z)\\
&-\frac{4\alpha}{(1+2s^2)}\,\partial\Big(\Phi_{\frac{1}{2}}^{(s),1}+i\,\Phi_{\frac{1}{2}}^{(s),2}\Big)(z),\\
&\overline{D}\mathbf{W}^{(s+1)}\,|_{\theta=\bar{\theta}=0}\,(z)=\frac{2\,(1+2s)}{(1+2s^2)}\Big(\Phi_{\frac{3}{2}}^{(s),1}-i\,\Phi_{\frac{3}{2}}^{(s),2}\Big)(z)\\
&-\frac{4\alpha}{(1+2s^2)}\,\partial\Big(\Phi_{\frac{1}{2}}^{(s),1}-i\,\Phi_{\frac{1}{2}}^{(s),2}\Big)(z),\\
&-\frac{1}{2}[D,\overline{D}]\mathbf{W}^{(s+1)}\,|_{\theta=\bar{\theta}=0}\,(z)=-\frac{2\,(1+2s)}{(1+2s^2)}\Phi_2^{(s)}(z)\\
&+\frac{4\alpha}{(1+2s^2)}\partial^2\Phi_0^{(s)}(z).
\end{aligned}
\tag{4.2}
$$

For $s=1$, we can also express the right-hand sides of (4.2) using (3.8), (3.9), (3.10), and (3.11).[14] The second

[13] The nonzero contributions from the right-hand side arise from the second and third terms of the first equation of (4.1). The former gives $(-\frac{1}{2}[D,\overline{D}]-\frac{1}{2}\partial)\mathbf{T^{(1)}}(Z_2)|_{\theta=\bar{\theta}=0}$ with the singular term $\frac{(-i)}{z_{12}}$ (where we use the fact that $D_1\,\frac{\theta_{12}}{z_{12}}=\frac{1}{(z_1-z_2)}$ after a projection) while the latter gives $(-\frac{1}{2}[D,\overline{D}]+\frac{1}{2}\partial)\mathbf{T^{(1)}}(Z_2)|_{\theta=\bar{\theta}=0}$ with the singular term $\frac{(-i)}{z_{12}}$, where the property that $\overline{D}_1\,\frac{\bar{\theta}_{12}}{z_{12}}$ goes to $\frac{1}{(z_1-z_2)}$ is used. This leads to the expression $-(-i)[D,\overline{D}]\mathbf{T^{(1)}}(Z_2)|_{\theta=\bar{\theta}=0}$ in the first-order pole, which is equal to the above $\Phi_1^{(1),34}(z_2)$, where the decompositions (which will appear in the next subsection) are needed.

[14] In [15], the command `N2OPEToComponents[T_]` provides the components of the $\mathcal{N}=2$ current $\mathbf{T}(Z)$. We can then write

(third) component of $\mathbf{T^{(1)}}(Z)$ contains the higher spin currents $T_{\pm}^{(\frac{3}{2})}(z)$, in addition to the other $\frac{3}{2}$ currents from (3.8), and the last component is proportional to the higher spin current $T^{(2)}(z)$ from (3.9). The first component of $\mathbf{U^{(\frac{3}{2})}}(Z)$ contains the higher spin current $U^{(\frac{3}{2})}(z)$. The second and third components are given by the linear combinations between the higher spin currents $U_{\pm}^{(2)}(z)$ and $V_{\pm}^{(2)}(z)$ from (3.9). The fourth component has the higher spin currents $U^{(\frac{5}{2})}(z)$ and $\partial U^{(\frac{3}{2})}(z)$. Similarly, the first component of $\mathbf{V^{(\frac{3}{2})}}(Z)$ contains the higher spin current $V^{(\frac{3}{2})}(z)$, the second and third components are given by the linear combinations between the higher spin currents $U_{\pm}^{(2)}(z)$ and $V_{\pm}^{(2)}(z)$, and the fourth component has the higher spin currents $V^{(\frac{5}{2})}(z)$ and $\partial V^{(\frac{3}{2})}(z)$. The first component of $\mathbf{W^{(2)}}(Z)$ contains the higher spin current $W^{(2)}(z)$ as well as the higher spin current $T^{(2)}(z)$. The second (third) component contains the higher spin currents $W_{\pm}^{(\frac{5}{2})}(z)$, $\partial T_{\pm}^{(\frac{3}{2})}(z)$, and other spin-$\frac{3}{2}$ currents. The fourth component contains the higher spin current $W^{(3)}(z)$ (and other nonlinear terms) and is a quasiprimary field.

We can also rewrite the $\mathcal{N}=4$ higher spin-s current (3.3) in the expansion of the Grassmann coordinates θ^3 and θ^4 as follows:

$$\begin{aligned}
\Phi^{(\mathbf{s})}(Z) &= \Phi_0^{(s)}(z) + \theta^1 \Phi_{\frac{1}{2}}^{(s),1}(z) + \theta^2 \Phi_{\frac{1}{2}}^{(s),2}(z) \\
&\quad + \theta^1\theta^2 \Phi_1^{(s),34}(z) \\
&\quad + \theta^3\Big[\Phi_{\frac{1}{2}}^{(s),3}(z) - \theta^1\,\Phi_1^{(s),24}(z) \\
&\quad - \theta^2\,\Phi_1^{(s),14}(z) + \theta^1\,\theta^2\,\Phi_{\frac{3}{2}}^{(s),4}(z)\Big] \\
&\quad + \theta^4\Big[\Phi_{\frac{1}{2}}^{(s),4}(z) - \theta^1\,\Phi_1^{(s),23}(z) \\
&\quad - \theta^2\,\Phi_1^{(s),31}(z) - \theta^1\,\theta^2\,\Phi_{\frac{3}{2}}^{(s),3}(z)\Big] \\
&\quad + \theta^3\,\theta^4\Big[\Phi_1^{(s),12}(z) + \theta^1\,\Phi_{\frac{3}{2}}^{(s),2}(z) \\
&\quad + \theta^2\,\Phi_{\frac{3}{2}}^{(s),1}(z) + \theta^1\,\theta^2\,\Phi_2^{(s)}(z)\Big]. \qquad (4.3)
\end{aligned}$$

As done with the $\mathbf{J^{(4)}}(Z)$ in terms of the $\mathcal{N}=2$ currents, this expression can be rewritten without any difficulty by using (4.2) in the next subsection. We can obtain the higher spin currents in terms of the $\mathcal{N}=3$ multiplets or $\mathcal{N}=1$ multiplets, as done for the $\mathcal{N}=4$ stress energy tensor. Furthermore, as done before, the corresponding OPEs between the $\mathcal{N}=3$ currents can be obtained from (3.2) in a different basis. A similar analysis for the $\mathcal{N}=1$ superspace description in the OPE between the stress energy tensor and the higher spin currents can be made to the one above.

4.3 $\mathcal{N}=2$ superspace description of the $\mathcal{N}=4$ higher spin-s current multiplet

We can rewrite (4.3) in terms of its $\mathcal{N}=2$ version. From the explicit results in (4.2), we obtain

$$\begin{aligned}
\Phi^{(\mathbf{s})}(Z) &= \mathbf{T^{(s)}}(z,\theta,\overline{\theta}) + \theta^3\,\frac{1}{2}\,\left(\mathbf{U^{(s+\frac{1}{2})}} + \mathbf{V^{(s+\frac{1}{2})}}\right)(z,\theta,\overline{\theta}) \\
&\quad + \theta^4\,\frac{i}{2}\,\left(-\mathbf{U^{(s+\frac{1}{2})}} + \mathbf{V^{(s+\frac{1}{2})}}\right)(z,\theta,\overline{\theta}) \\
&\quad - \theta^3\,\theta^4\,\frac{i}{4(1+2s)}\,\Big[(1+2s^2)\mathbf{W^{(s+1)}} \\
&\quad + 8\,\alpha\,[D,\overline{D}]\,\mathbf{T^{(s)}}\Big](z,\theta,\overline{\theta}). \qquad (4.4)
\end{aligned}$$

Let us try to understand the $\mathcal{N}=4$ primary condition (3.2) in the $\mathcal{N}=2$ superspace described in (4.1). Let us consider the action of D_1^3 on Eq. (3.2) and set the condition $\theta^3=0=\theta^4$. Then the left-hand side of this OPE is given by the expression $\frac{1}{2}(\mathbf{G}+\overline{\mathbf{G}})(Z_1)\,\mathbf{T^{(s)}}(Z_2)$.[15] Note that care must be taken about the sign when operating the differential operator into θ_{12}^{4-i}. The convention in [18] is such that $\theta^4=\theta^{4-i}\,\theta^i$.

5 OPEs between the 16 higher spin currents in the $\mathcal{N}=2$ superspace

To obtain the complete OPEs between the 16 higher spin currents in the $\mathcal{N}=2$ superspace, we need to know the complete composite fields appearing in the OPEs. It is well known that some of the OPEs between the 16 higher spin currents in the component approach are found explicitly for $N=3$ [12]. One way to obtain the possible candidates for the composite fields in the OPEs (in the $\mathcal{N}=2$ superspace) is to find the corresponding OPEs in the component approach.

Footnote 14 continued
$\mathbf{T}(Z) = \mathbf{T}|_{\theta=\overline{\theta}=0} + \theta\,D\,\mathbf{T}|_{\theta=\overline{\theta}=0} + \overline{D}\,\mathbf{T}|_{\theta=\overline{\theta}=0} - \theta\,\overline{\theta}\,\frac{1}{2}\,[D,\overline{D}]\mathbf{T}|_{\theta=\overline{\theta}=0}$. These components are presented in the first four equations of Appendix E. We can also analyze the other $\mathcal{N}=2$ (higher spin) currents similarly.

15 In contrast, the right-hand side of this OPE (coming from the second term) is given by the expression $\frac{\theta_{12}^1\,\theta_{12}^2}{z_{12}}\,\frac{i}{2}\,(-\mathbf{U^{(s+\frac{1}{2})}}+\mathbf{V^{(s+\frac{1}{2})}})(Z_2)$. Similarly, the action of D_1^4 on Eq. (3.2) with the condition $\theta^3=0=\theta^4$ leads to a left-hand side given by $\frac{i}{2}(-\mathbf{G}+\overline{\mathbf{G}})(Z_1)\,\mathbf{T^{(s)}}(Z_2)$. Moreover, the right-hand side of this OPE (coming from the second term) becomes the following result: $-\frac{\theta_{12}^1\,\theta_{12}^2}{z_{12}}\,\frac{1}{2}\,(\mathbf{U^{(s+\frac{1}{2})}}+\mathbf{V^{(s+\frac{1}{2})}})(Z_2)$. By combining these equations, we obtain the OPE $\begin{pmatrix}\mathbf{G}\\ \overline{\mathbf{G}}\end{pmatrix}(Z_1)\,\mathbf{T^{(s)}}(Z_2) = \pm\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}}\,\frac{1}{2}\,\begin{pmatrix}\mathbf{U^{(s+\frac{1}{2})}}\\ \mathbf{V^{(s+\frac{1}{2})}}\end{pmatrix}(Z_2)+\cdots$, which is exactly what is expected. We also use the fact that $i\theta_{12}^1\,\theta_{12}^2 = -\frac{1}{2}\,\theta_{12}\overline{\theta}_{12}$.

We can proceed to follow the method in [12] and obtain the unknown OPEs, but in this paper we use the power of $\mathcal{N}=4$ supersymmetry, as emphasized in the introduction. We can determine the undetermined OPEs in [12] by moving to the $\mathcal{N}=4$ superspace and arranging the known OPEs in appropriate places in a single OPE in the $\mathcal{N}=4$ superspace. Then we can move to the $\mathcal{N}=2$ superspace by collecting those OPEs in the component approach and rearranging them in an $\mathcal{N}=2$ supersymmetric way. So far, all the coefficients in the OPEs are given with fixed $N=3$ and arbitrary k. Now we set these coefficients as functions of N and k and use Jacobi identities between the $\mathcal{N}=2$ currents or higher spin currents. Eventually, we obtain the complete structure constants with arbitrary N and k (2.7) appearing in the complete OPEs in the $\mathcal{N}=2$ superspace.

5.1 Ansatz from the 136 OPEs in the component approach

Because the spin of $\Phi^{(\mathbf{1})}(Z)$ is given by $s=1$, the OPE between this $\mathcal{N}=4$ multiplet and itself has a spin $s=2$. Then the right-hand side of this OPE should preserve the total spin with $s=2$. We can then consider the singular terms $\frac{\theta_{12}^{4}}{z_{12}^{n}}$, $\frac{\theta_{12}^{4-i}}{z_{12}^{n}}$, $\frac{\theta_{12}^{4-ij}}{z_{12}^{n}}$, $\frac{\theta_{12}^{4-ijk}}{z_{12}^{n}}$, and $\frac{1}{z_{12}^{n}}$ in the $\mathcal{N}=4$ superspace. Recall that the spin of θ_{12}^{i} is given by $-\frac{1}{2}$ and the spin of $\frac{1}{z_{12}}$ is given by 1. The first singular term has a spin of $(n-2)$ and then the possible n values are given by 4, 3, 2, and 1. The right-hand sides should contain the composite fields with spin $s=0$, 1, 2, or 3 at each singular term, respectively. For the second singular term, the right-hand sides should contain the composite fields with spin $s=\frac{1}{2},\frac{3}{2}$, or $\frac{5}{2}$ at the singular term, respectively (in this case, the spin is given by $(n-\frac{3}{2})$ and the possible n values are given by $n=3$, 2, or 1). Note that the derivative D^{i} has a spin of $\frac{1}{2}$. For the third singular term, the right-hand sides should contain the composite fields with spin $s=0$, 1, or 2 at the singular term, respectively (in this case the spin is given by $(n-1)$ and the possible n values are given by $n=3$, 2, or 1). $n=3$ is not allowed because the spin $s=0$ composite field does not contain any SO(4) indices, which should be contracted with two SO(4) indices ij. For the fourth singular term, the possible n values are given by 2 or 1. Then the right-hand sides should contain the composite fields with spin $s=\frac{1}{2}$ or $\frac{3}{2}$. The only $s=\frac{3}{2}$ case can have three SO(4) indices that are contracted with the SO(4) indices ijk. For the last singular term, the n values can be 2 or 1, the former having a central term and the latter having composite fields with spin $s=1$.

We can then rearrange the 136 OPEs in the component approach with the help of (3.1) and Sect. 3.6 to rewrite the corresponding single $\mathcal{N}=4$ OPE in the $\mathcal{N}=4$ superspace. (See also Appendix F.) In other words, it turns out that all the structures (the possible composite terms) appearing on the right-hand side of the OPEs in the component approach are determined completely. In the next subsection, we proceed to the $\mathcal{N}=2$ superspace by taking the ansatz from the component approach with arbitrary coefficients.

5.2 Jacobi identities

The general graded Jacobi identity (that is, (3.24) of [14]) reads

$$(-1)^{\mathbf{AC}}\,[\,[\mathbf{A},\mathbf{B}\},\mathbf{C}\}(Z)+\text{cycl.}=0, \tag{5.1}$$

for general currents $\mathbf{A}(Z)$, $\mathbf{B}(Z)$, and $\mathbf{C}(Z)$. The command `OPEJacobi[op1_, op2_, op3_]` in [15] calculates the Jacobi identities (5.1) for the singular part of the OPEs of the three arguments. In general, all different orderings have been done. `OPEJacobi` returns a list in which all should be zero up to null fields to be associative. The $\mathcal{N}=1$ example of (5.1) appears in the $\mathcal{N}=1$ extension of W_3 algebra studied in [29] (which was done manually). Of course, we can check the Jacobi identities using the definition of a normal ordered product between any two currents in the $\mathcal{N}=4$ superspace (for example [22,23]) manually.[16]

5.3 Determination of the structure constant

In this subsection, the three OPEs in the $\mathcal{N}=2$ superspace are given explicitly with Appendices G.1–G.3, where the structure constants are presented explicitly, and the remaining OPEs will be given in Appendices G.4–G.7 explicitly.

Because the OPEs $\mathbf{T}^{(\mathbf{1})}(Z_1)\,\mathbf{T}^{(\mathbf{1})}(Z_2)$, $\mathbf{U}^{(\frac{\mathbf{3}}{\mathbf{2}})}(Z_1)\,\mathbf{U}^{(\frac{\mathbf{3}}{\mathbf{2}})}(Z_2)$, and $\mathbf{V}^{(\frac{\mathbf{3}}{\mathbf{2}})}(Z_1)\,\mathbf{V}^{(\frac{\mathbf{3}}{\mathbf{2}})}(Z_2)$ have no higher spin currents, the structure constants of these OPEs are completely determined by the Jacobi identities without knowledge of the OPEs between the higher spin currents and the next higher spin currents. However, the remaining seven OPEs contain the next higher spin currents, as we will see later, and we do not have any information as regards the OPEs between the higher spin currents and the next higher spin current at the moment. Consequently, the insufficient Jacobi identities lead to the one unknown structure constant.

We can fix the unknown structure constant via the coefficient of the energy momentum tensor $T(w)$ in the second-order pole of the OPE $T^{(1)}(z)\,\mathbf{W}^{(\mathbf{3})}(w)$ [12]. However, since the result in [12] is for $N=3$, we need to obtain the coefficient for arbitrary N. Let us introduce the coefficient c_1 in front of $T(w)$ in the second-order pole of the OPE

[16] Inside of [15], by using the command `N2OPEToComponents[ope_, J1_, J2_]` one can calculate the 16 OPEs of the components of $\mathbf{J_1}(Z_1)$ and $\mathbf{J_2}(Z_2)$, which are any two $\mathcal{N}=2$ (higher spin) currents. It gives a double list, where the (m,n)th element is the component OPE between the mth component of $\mathbf{J_1}(Z_1)$ and the nth component of $\mathbf{J_2}(Z_2)$.

$T^{(1)}(z)\,\mathbf{W^{(3)}}(w)$. More explicitly, we have $\{T^{(1)}\,\mathbf{W^{(3)}}\}_{-2} = -\mathbf{P^{(2)}}(w) + c_1\,T(w) + \cdots$. Let us introduce the coefficient c_2 of $T(z)$, where in the relation between $\mathbf{P^{(2)}}$ and $V_0^{(2)}$, $\mathbf{P^{(2)}}(z) = c_2\,T(z) + V_0^{(2)}(z) + \cdots$. We then have $\{T^{(1)}\,\mathbf{W^{(3)}}\}_{-2} = (c_1 - c_2)\,T(w) + \cdots$. We can obtain their N generalization by considering for low N values ($N = 3, 5, 7, \ldots$) the following:[17]

$$c_1(N,k) = \frac{8(k-N)(4N+4k+5)}{3(N+k+2)(3Nk+4N+4k+5)},$$
$$c_2(N,k) = \frac{2(2k+N+3)(5k+8N+10)}{3(N+k+2)^2}. \qquad (5.2)$$

Of course, we obtain the 16 OPEs between the higher spin-1 current and the 16 higher spin currents for several N values and their complete 16 OPEs can be expressed for generic N and k. In particular, there is no new primary current in the first-order pole of the OPE $T^{(1)}(z)\,\mathbf{W^{(3)}}(w)$. This implies that the 16 higher spin currents from the WZW currents in $\mathcal{N}=4$ coset theory in the final $\mathcal{N}=4$ OPE $\Phi^{(\mathbf{1})}(Z_1)\,\Phi^{(\mathbf{1})}(Z_2)$ do not generate the third $\mathcal{N}=4$ multiplet $\Phi^{(\mathbf{3})}(Z_2)$ on the right-hand side.

We also obtain the same quantity from the Jacobi identity only and solve the following equation with (5.2):

$$\begin{aligned} c_1 - c_2 = {} & \frac{1}{3(k-N)(1+N)(5+4k+4N+3kN)}\,4(40 \\ & +62k+24k^2+102N+114kN \\ & +36k^2N+86N^2+64kN^2+12k^2N^2+24N^3 \\ & +12kN^3+160X+448kX+496k^2X \\ & +272k^3X+74k^4X+8k^5X+448NX \\ & +1128kNX+1100k^2NX+518k^3NX \\ & -117k^4NX+10k^5NX+496N^2X \\ & +1100kN^2X+912k^2N^2X+347k^3N^2X \\ & -58k^4N^2X+3k^5N^2X+272N^3X+518kN^3X \\ & +347k^2N^3X+96k^3N^3X+9k^4N^3X \\ & -74N^4X+117kN^4X+58k^2N^4X+9k^3N^4X \\ & +8N^5X+10kN^5X+3k^2N^5X). \end{aligned} \qquad (5.3)$$

The unknown quantity (appearing as the c_{51} coefficient in Sect. 5.3.2 below) from (5.3) is determined by

$$X = -\frac{(1+N)(32+55k+18k^2+41N+35kN+11N^2)}{2(2+N)(2+k+N)^5}. \qquad (5.4)$$

Therefore, all the structure constants are determined completely. In the following, we use the following currents without boldface notation for simplicity:

$$\mathbf{H}(Z) \equiv H(Z), \quad \overline{\mathbf{H}}(Z) \equiv \overline{H}(Z), \quad \mathbf{G}(Z) \equiv G(Z),$$
$$\overline{\mathbf{G}}(Z) \equiv \overline{G}(Z), \quad \mathbf{T}(Z) \equiv T(Z). \qquad (5.5)$$

[17] We thank H. Kim for this calculation and other related computations.

These notations are used in Appendix G also.

5.3.1 OPE between the $\mathcal{N}=2$ higher spin-1 currents

The OPE between the $\mathcal{N}=2$ higher spin-1 currents, where the four components of the $\mathcal{N}=2$ higher spin-1 current are given by the first four equations of (4.2) with $s=1$, is summarized by

$$\begin{aligned} \mathbf{T^{(1)}}(Z_1)\,\mathbf{T^{(1)}}(Z_2) = {} & \frac{1}{z_{12}^2}\,c_1 + \frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2}\Big[c_2\,\overline{D}H + c_3\,D\overline{H} \\ & + c_4\,T + c_5\,H\overline{H} + c_6\,G\overline{G}\Big](Z_2) \\ & + \frac{\theta_{12}}{z_{12}}\Big[c_7\,DT + c_8\,\partial H + c_9\,DG\overline{G} \\ & + c_{10}\,GD\overline{G} + c_{11}\,D\overline{H}H + c_{12}\,HG\overline{G}\Big](Z_2) \\ & + \frac{\bar{\theta}_{12}}{z_{12}}\Big[c_{13}\,\overline{D}T + c_{14}\,\partial\overline{H} + c_{15}\,\overline{D}G\overline{G} + c_{16}\,G\overline{D}\overline{G} \\ & + c_{17}\,\overline{D}H\overline{H} + c_{18}\,\overline{H}G\overline{G}\Big](Z_2) \\ & + \frac{\theta_{12}\bar{\theta}_{12}}{z_{12}}\Big[c_{19}\,\partial T + c_{20}\,\partial\overline{D}H + c_{21}\,\partial D\overline{H} \\ & + c_{22}\,[D,\overline{D}]G\overline{G} + c_{23}\,G[D,\overline{D}]\overline{G} \\ & + c_{24}\,\partial H\overline{H} + c_{25}\,H\partial\overline{H} + c_{26}\,\partial G\overline{G} + c_{27}\,G\partial\overline{G} \\ & + c_{28}\,\overline{H}DG\overline{G} + c_{29}\,H\overline{D}G\overline{G} \\ & + c_{30}\,\overline{H}GD\overline{G} + c_{31}\,HG\overline{D}\overline{G} \\ & + c_{32}\,\overline{D}HG\overline{G} + c_{33}\,D\overline{H}G\overline{G}\Big](Z_2) + \cdots, \end{aligned} \qquad (5.6)$$

where the coefficients are given in Appendix G.1 and we use the notation in (5.5).[18]

The fusion rule between the $\mathcal{N}=2$ higher spin-1 currents is given by

$$[\mathbf{T^{(1)}}] \cdot [\mathbf{T^{(1)}}] = [\mathbf{I}], \qquad (5.7)$$

where $[\mathbf{I}]$ denotes the large $\mathcal{N}=4$ linear superconformal family of the identity operator. Note that there are also

[18] Let us multiply both sides of (5.6) with the condition $\theta = 0 = \bar{\theta}$ by $(D_2 - \overline{D}_2)$. Then the left-hand side is given by $\Phi_0^{(1)}(z_1)\,\Phi_{\frac{1}{2}}^{(1),1}(z_2)$. The nontrivial contributions come from the singular terms $\frac{\theta_{12}}{z_{12}}$ and $\frac{\bar{\theta}_{12}}{z_{12}}$ because the D_2 action on the former gives the singular term $-\frac{1}{(z_1-z_2)}$ and the $\overline{D}_2$ action on the latter gives the singular term $-\frac{1}{(z_1-z_2)}$. Then the first-order pole of $\Phi_0^{(1)}(z_1)\,\Phi_{\frac{1}{2}}^{(1),1}(z_2)$ contains the expression $(-c_7 DT + c_{13}\overline{D}T)$ with the condition $\theta = 0 = \bar{\theta}$. By substituting the coefficients, this becomes the result $(D_2 + \overline{D}_2)T(Z_2)$, which reduces to $G^1(z_2)$. This term is what we expect from the component result in Appendix H.1. The other terms can be identified similarly.

$H(Z_2)$-, $\overline{H}(Z_2)$-, $G(Z_2)$-, and $\overline{G}(Z_2)$-dependent terms as well as $T(Z_2)$-dependent terms in (5.6).

5.3.2 OPE between the $\mathcal{N}=2$ higher spin-1 current and the $\mathcal{N}=2$ higher spin-$\frac{3}{2}$ currents

The OPE between the $\mathcal{N}=2$ higher spin-1 current and the $\mathcal{N}=2$ higher spin-$\frac{3}{2}$ currents, where four components of the $\mathcal{N}=2$ higher spin-1 and $\frac{3}{2}$ currents are given by the first four equations of (4.2) and the next eight equations with $s=1$, is described by

$$
\begin{aligned}
&\mathbf{T^{(1)}}(Z_1)\begin{pmatrix}\mathbf{U}^{(\frac{3}{2})}\\ \mathbf{V}^{(\frac{3}{2})}\end{pmatrix}(Z_2) = \frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^3}\, c_{\pm 1}\, G_{\pm}(Z_2)\\
&+\frac{\theta_{12}}{z_{12}^2}\Big[c_{\pm 2}\, DG_{\pm}+c_{\pm 3}\, HG_{\pm}\Big](Z_2)\\
&+\frac{\bar{\theta}_{12}}{z_{12}^2}\Big[c_{\pm 4}\, \overline{D}G_{\pm}+c_{\pm 5}\, \overline{H}G_{\pm}\Big](Z_2)\\
&+\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2}\Big[c_{\pm 6}\,[D,\overline{D}]G_{\pm}+c_{\pm 7}\, H\overline{D}G_{\pm}+c_{\pm 8}\, \overline{H}DG_{\pm}\\
&+c_{\pm 9}\, TG_{\pm}+c_{\pm 10}\, \overline{D}HG_{\pm}\\
&+c_{\pm 11}\, D\overline{H}G_{\pm}+c_{\pm 12}\, \partial G_{\pm}\Big](Z_2)\\
&+\frac{1}{z_{12}}\Big[c_{\pm 13}\,[D,\overline{D}]G_{\pm}+c_{\pm 14}\, H\overline{D}G_{\pm}\\
&+c_{15}\, H\overline{H}\, G_{\pm}+c_{\pm 16}\, \overline{H}DG_{\pm}\\
&+c_{\pm 17}\, TG_{\pm}+c_{18}\, \overline{D}HG_{\pm}+c_{\pm 19}\, D\overline{H}G_{\pm}\\
&+c_{\pm 20}\, \partial G_{\pm}\Big](Z_2)+\frac{\theta_{12}}{z_{12}}\Big[c_{\pm 21}\, \partial DG_{\pm}\\
&+c_{\pm 22}\, G_{+}DG_{\pm}G_{-}+c_{\pm 23}\, H\overline{H}DG_{\pm}\\
&+c_{\pm 24}\, H\overline{D}HG_{\pm}+c_{\pm 25}\, HD\overline{H}G_{\pm}\\
&+c_{\pm 26}\, H\partial G_{\pm}+c_{\pm 27}\, TDG_{\pm}+c_{\pm 28}\, THG_{\pm}\\
&+c_{\pm 29}\, \overline{D}HDG_{\pm}+c_{\pm 30}\, D\overline{H}DG_{\pm}+c_{\pm 31}\, \partial HG_{\pm}\Big]\\
&+\frac{\bar{\theta}_{12}}{z_{12}}\Big[c_{\pm 32}\, \partial\overline{D}G_{\pm}+c_{\pm 33}\, G_{+}\overline{D}G_{\pm}G_{-}\\
&+c_{\pm 34}\, H\overline{H}\overline{D}G_{\pm}+\, c_{\pm 35}\, \overline{H}D\overline{H}G_{\pm}+c_{\pm 36}\, \overline{H}\partial G_{\pm}\\
&+c_{\pm 37}\, T\overline{D}G_{\pm}+c_{\pm 38}\, T\overline{H}G_{\pm}+c_{\pm 39}\, \overline{D}H\overline{D}G_{\pm}\\
&+c_{\pm 40}\, \overline{D}H\overline{H}G_{\pm}+c_{\pm 41}\, D\overline{H}\overline{D}G_{\pm}+c_{\pm 42}\, \partial\overline{H}G_{\pm}\Big]\\
&+\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}}\Big[c_{\pm 43}\begin{pmatrix}\mathbf{U}^{(\frac{5}{2})}\\ \mathbf{V}^{(\frac{5}{2})}\end{pmatrix}+c_{\pm 44}\,\mathbf{T^{(1)}}\begin{pmatrix}\mathbf{U}^{(\frac{3}{2})}\\ \mathbf{V}^{(\frac{3}{2})}\end{pmatrix}\\
&+c_{\pm 45}\, G_{+}\overline{D}G_{\pm}DG_{-}+c_{\pm 46}\, G_{+}[D,\overline{D}]G_{\pm}G_{-}\\
&+c_{\pm 47}\, DG_{+}G_{\pm}\overline{D}G_{-}+c_{\pm 48}\, H\partial\overline{D}G_{\pm}\\
&+c_{\pm 49}\, HG_{+}\overline{D}G_{\pm}G_{-}+c_{\pm 50}\, H\overline{H}[D,\overline{D}]G_{\pm}\\
&+c_{\pm 51}\, H\overline{H}D\overline{H}G_{\pm}+c_{\pm 52}\, H\overline{H}\partial G_{\pm}\\
&+c_{\pm 53}\, H\overline{D}H\overline{D}G_{\pm}+c_{\pm 54}\, H\overline{D}H\overline{H}G_{\pm}\\
&+c_{\pm 55}\, HD\overline{H}\overline{D}G_{\pm}+c_{\pm 56}\, H\partial\overline{H}G_{\pm}+c_{\pm 57}\, \overline{H}\partial DG_{\pm}\\
&+c_{\pm 58}\, \overline{H}G_{+}DG_{\pm}G_{-}+c_{\pm 59}\, \overline{H}D\overline{H}DG_{\pm}\\
&+c_{\pm 60}\, T[D,\overline{D}]G_{\pm}+c_{\pm 61}\, TH\overline{D}G_{\pm}+c_{\pm 62}\, TH\overline{H}G_{\pm}\\
&+c_{\pm 63}\, T\overline{H}DG_{\pm}+c_{\pm 64}\, TTG_{\pm}\\
&+c_{\pm 65}\, T\overline{D}HG_{\pm}+c_{\pm 66}\, TD\overline{H}G_{\pm}+c_{\pm 67}\, T\partial G_{\pm}\\
&+c_{\pm 68}\, \partial[D,\overline{D}]G_{\pm}+c_{\pm 69}\, \overline{D}G_{+}DG_{\pm}G_{-}\\
&+c_{\pm 70}\, \overline{D}H[D,\overline{D}]G_{\pm}+c_{\pm 71}\, \overline{D}H\overline{H}DG_{\pm}\\
&+c_{\pm 72}\, \overline{D}H\overline{D}HG_{\pm}+c_{\pm 73}\, \overline{D}HD\overline{H}G_{\pm}\\
&+c_{\pm 74}\, \overline{D}H\partial G_{\pm}+c_{\pm 75}\, \overline{D}TDG_{\pm}+c_{\pm 76}\, \overline{D}THG_{\pm}\\
&+c_{\pm 77}\, \partial\overline{D}HG_{\pm}+c_{\pm 78}\,[D,\overline{D}]TG_{\pm}\\
&+c_{\pm 79}\, D\overline{H}[D,\overline{D}]G_{\pm}+c_{\pm 80}\, D\overline{H}D\overline{H}G_{\pm}\\
&+c_{\pm 81}\, D\overline{H}\partial G_{\pm}+c_{\pm 82}\, DT\overline{D}G_{\pm}+c_{\pm 83}\, DT\overline{H}G_{\pm}\\
&+c_{\pm 84}\, \partial D\overline{H}G_{\pm}+c_{\pm 85}\, \partial G_{\pm}G_{+}G_{-}+c_{\pm 86}\, \partial H\overline{D}G_{\pm}\\
&+c_{\pm 87}\, \partial H\overline{H}G_{\pm}+c_{\pm 88}\, \partial\overline{H}DG_{\pm}\\
&+c_{\pm 89}\, \partial TG_{\pm}+c_{\pm 90}\, \partial^2 G_{\pm}\Big](Z_2)+\cdots, \qquad (5.8)
\end{aligned}
$$

where the coefficients are given in Appendix G.2 after substituting (5.4) explicitly. We use the notation in (5.5) and introduce

$$G\equiv G_{+},\quad \overline{G}\equiv G_{-},\quad H\equiv H_{+},\quad \overline{H}\equiv H_{-}$$

to be able to express the two similar equations together.[19]

The fusion rule between the $\mathcal{N}=2$ higher spin-1 current and the $\mathcal{N}=2$ higher spin-$\frac{3}{2}$ currents is given by

$$[\mathbf{T^{(1)}}]\cdot\begin{bmatrix}\mathbf{U}^{(\frac{3}{2})}\\ \mathbf{V}^{(\frac{3}{2})}\end{bmatrix}=[\mathbf{I}]+\begin{bmatrix}\mathbf{T^{(1)}}\,\mathbf{U}^{(\frac{3}{2})}\\ \mathbf{T^{(1)}}\,\mathbf{V}^{(\frac{3}{2})}\end{bmatrix}+\begin{bmatrix}\mathbf{U}^{(\frac{5}{2})}\\ \mathbf{V}^{(\frac{5}{2})}\end{bmatrix}, \qquad (5.9)$$

where the last term, which resides in the next 16 higher spin currents, has its component expressions in the fifth to 12th equations of (4.2) with $s=2$.[20]

[19] Let us set $\theta=0=\bar{\theta}$ in (5.8). Then the left-hand side is given by $\Phi_0^{(1)}(z_1)\, 2\,\Phi_{\frac{1}{2}}^{(1),3}(z_2)$. The right-hand side contains $(c_{+13}[D,\overline{D}]G_{+}+c_{-13}[D,\overline{D}]G_{-})$, which is equal to $([D,\overline{D}]G_{+}-[D,\overline{D}]G_{-})$. This will further reduce to $2\,G^3(z_2)$ with other terms. Therefore, the first-order term of the OPE $\Phi_0^{(1)}(z_1)\,\Phi_{\frac{1}{2}}^{(1),3}(z_2)$ contains the $G^3(z_2)$ term, as we expect in the component approach.

[20] We use nonstandard notation in (5.9). If we use the $\pm$ notation in the higher spin currents $\mathbf{U}^{(\frac{3}{2})}$, $\mathbf{V}^{(\frac{3}{2})}$, $\mathbf{U}^{(\frac{5}{2})}$, and $\mathbf{V}^{(\frac{5}{2})}$ (that is, $\mathbf{U}_{\pm}^{(\frac{3}{2})}$ for the first two and $\mathbf{U}_{\pm}^{(\frac{5}{2})}$ for the last two), then we can write them in one line rather than as 2×1 matrices as in (5.9). In our notation, the currents $\mathbf{U}^{(\frac{3}{2})}(Z)$ and $\mathbf{U}^{(\frac{5}{2})}(Z)$ reside in the different $\mathcal{N}=2$ multiplets. The former has $s=1$ (the component of the lowest $\mathcal{N}=4$ multiplet) and the latter has $s=2$ (the component of the next $\mathcal{N}=4$ multiplet).

5.3.3 OPE between the $\mathcal{N}=2$ higher spin-1 current and the $\mathcal{N}=2$ higher spin-2 current

The OPE between the $\mathcal{N}=2$ higher spin-1 current and the $\mathcal{N}=2$ higher spin-2 current, where four components of the $\mathcal{N}=2$ higher spin-1 and spin-2 currents are given by the first four equations of (4.2) and the last four equations with $s=1$, is summarized by

$$
\begin{aligned}
\mathbf{T^{(1)}}(Z_1)\,\mathbf{W^{(2)}}(Z_2) = {} & \frac{\theta_{12}}{z_{12}^3}\,c_1\,H(Z_2)+\frac{\overline{\theta}_{12}}{z_{12}^3}\,c_2\,\overline{H}(Z_2)\\
& +\frac{1}{z_{12}^2}\Big[c_3\,T+c_4\,\overline{D}H+c_5\,D\overline{H}+c_6\,G\overline{G}+c_7\,H\overline{H}\Big](Z_2)\\
& +\frac{\theta_{12}}{z_{12}^2}\Big[c_8\,DT+c_9\,GD\overline{G}+c_{10}\,H\overline{D}H+c_{11}\,HD\overline{H}\\
& +c_{12}\,HG\overline{G}+c_{13}\,TH+c_{14}\,DG\overline{G}+c_{15}\,\partial H\Big]\\
& +\frac{\overline{\theta}_{12}}{z_{12}^2}\Big[c_{16}\,\overline{D}T+c_{17}\,G\overline{DG}+c_{18}\,\overline{H}D\overline{H}+c_{19}\,\overline{H}G\overline{G}\\
& +c_{20}\,T\overline{H}+c_{21}\,\overline{D}G\overline{G}+c_{22}\,\overline{D}H\overline{H}+c_{23}\,\partial\overline{H}\Big]\\
& +\frac{\theta_{12}\overline{\theta}_{12}}{z_{12}^2}\Big[c_{24}\,\mathbf{T^{(2)}}+c_{25}\,\mathbf{T^{(1)}}\mathbf{T^{(1)}}+c_{26}\,[D,\overline{D}]T\\
& +c_{27}\,\partial D\overline{H}+c_{28}\,G[D,\overline{D}]\overline{G}+c_{29}\,G\partial\overline{G}\\
& +c_{30}\,HG\overline{DG}+c_{31}\,H\overline{H}D\overline{H}+c_{32}\,H\overline{H}G\overline{G}\\
& +c_{33}\,H\overline{D}G\overline{G}+c_{34}\,H\overline{D}H\overline{H}+c_{35}\,H\partial\overline{H}\\
& +c_{36}\,\overline{H}GD\overline{G}+c_{37}\,\overline{H}DG\,\overline{G}+c_{38}\,TT+c_{39}\,T\overline{D}H\\
& +c_{40}\,TD\overline{H}+c_{41}\,TG\overline{G}+c_{42}\,TH\overline{H}\\
& +c_{43}\,\partial\overline{D}H+c_{44}\,\overline{D}GD\overline{G}+c_{45}\,\overline{D}H\overline{D}H+c_{46}\,\overline{D}HD\overline{H}\\
& +c_{47}\,\overline{D}HG\overline{G}+c_{48}\,\overline{D}TH+c_{49}\,[D,\overline{D}]G\overline{G}\\
& +c_{50}\,DG\overline{DG}+c_{51}\,D\overline{H}D\overline{H}+c_{52}\,D\overline{H}G\overline{G}\\
& +c_{53}\,DT\overline{H}+c_{54}\,\partial G\overline{G}+c_{55}\,\partial H\overline{H}+c_{56}\,\partial T\Big](Z_2)\\
& +\frac{1}{z_{12}}\Big[c_{57}\,\partial\overline{D}H+c_{58}\,\partial D\overline{H}+c_{59}\,G[D,\overline{D}]\overline{G}\\
& +c_{60}\,G\partial\overline{G}+c_{61}\,HG\overline{DG}+c_{62}\,H\overline{H}D\overline{H}\\
& +c_{63}\,H\overline{D}G\overline{G}+c_{64}\,H\,\overline{D}H\overline{H}+c_{65}\,H\partial\overline{H}\\
& +c_{66}\,\overline{H}GD\overline{G}+c_{67}\,\overline{H}DG\overline{G}+c_{68}\,\overline{D}HG\overline{G}\\
& +c_{69}\,\overline{D}TH+c_{70}\,[D,\overline{D}]G\overline{G}+c_{71}\,D\overline{H}G\overline{G}\\
& +c_{72}\,DT\overline{H}+c_{73}\,\partial G\overline{G}+c_{74}\,\partial H\overline{H}+c_{75}\,\partial T\Big](Z_2)\\
& +\frac{\theta_{12}}{z_{12}}\Big[c_{76}\,D\mathbf{T^{(2)}}+c_{77}\,\mathbf{T^{(1)}}D\mathbf{T^{(1)}}+c_{78}\,G\partial D\overline{G}\\
& +c_{79}\,H\partial D\overline{H}+c_{80}\,HG[D,\overline{D}]\overline{G}\\
& +c_{81}\,HG\partial\overline{G}+c_{82}\,H\overline{H}GD\overline{G}+c_{83}\,H\overline{H}DG\overline{G}\\
& +c_{84}\,H\overline{D}GD\overline{G}+c_{85}\,H\overline{D}HD\overline{H}\\
& +c_{86}\,H[D,\overline{D}]G\overline{G}+c_{87}\,HDG\overline{DG}\\
& +c_{88}\,HD\overline{H}D\overline{H}+c_{89}\,HD\overline{H}G\overline{G}+c_{90}\,H\partial G\overline{G}\\
& +c_{91}\,\overline{H}DGD\overline{G}+c_{92}\,TDT+c_{93}\,TGD\overline{G}\\
& +c_{94}\,THD\overline{H}+c_{95}\,TDG\overline{G}+c_{96}\,T\partial H\\
& +c_{97}\,\partial DT+c_{98}\,\overline{D}HGD\overline{G}+c_{99}\,\overline{D}HDG\overline{G}\\
& +c_{100}\,\partial\overline{D}HH+c_{101}\,[D,\overline{D}]G\,D\overline{G}\\
& +c_{102}\,[D,\overline{D}]TH+c_{103}\,DG[D,\overline{D}]\overline{G}+c_{104}\,DG\partial\overline{G}\\
& +c_{105}\,D\overline{H}GD\overline{G}+c_{106}\,D\overline{H}DG\overline{G}\\
& +c_{107}\,DT\overline{D}H+c_{108}\,DTD\overline{H}+c_{109}\,DTG\overline{G}\\
& +c_{110}\,DTH\overline{H}+c_{111}\,\partial DG\overline{G}+c_{112}\,\partial GD\overline{G}\\
& +c_{113}\,\partial H\,\overline{D}H+c_{114}\,\partial HD\overline{H}+c_{115}\,\partial HG\overline{G}\\
& +c_{116}\,\partial HH\overline{H}+c_{117}\,\partial TH+c_{118}\,\partial^2 H\Big](Z_2)\\
& +\frac{\overline{\theta}_{12}}{z_{12}^2}\Big[c_{119}\,\overline{D}\mathbf{T^{(2)}}+c_{120}\,\mathbf{T^{(1)}}\overline{D}\mathbf{T^{(1)}}+c_{121}\,G\partial\overline{DG}\\
& +c_{122}\,H\overline{H}G\overline{DG}+c_{123}\,H\overline{H}DG\overline{G}\\
& +c_{124}\,H\overline{D}G\overline{DG}+c_{125}\,H\partial\overline{HH}+c_{126}\,\overline{H}G[D,\overline{D}]\overline{G}\\
& +c_{127}\,\overline{H}G\partial\overline{G}+c_{128}\,\overline{HDG}D\overline{G}\\
& +c_{129}\,\overline{H}[D,\overline{D}]G\,\overline{G}+c_{130}\,\overline{H}DG\overline{DG}+c_{131}\,\overline{H}\partial G\overline{G}\\
& +c_{132}\,T\overline{D}T+c_{133}\,TG\overline{DG}\\
& +c_{134}\,T\overline{D}G\overline{G}+c_{135}\,T\overline{D}H\overline{H}+c_{136}\,T\partial\overline{H}\\
& +c_{137}\,\partial\overline{D}T+c_{138}\,\overline{D}G[D,\overline{D}]\overline{G}+c_{139}\,\overline{D}G\partial\overline{G}\\
& +c_{140}\,\overline{D}TG\overline{G}+c_{141}\,\overline{D}H\overline{H}D\overline{H}+c_{142}\,\overline{D}H\overline{H}G\overline{G}\\
& +c_{143}\,\overline{D}H\overline{D}G\overline{G}+c_{144}\,\overline{D}H\overline{D}H\overline{H}\\
& +c_{145}\,\overline{D}H\partial\overline{H}+c_{146}\,\overline{D}T\overline{D}H+c_{147}\,\overline{D}TD\overline{H}\\
& +c_{148}\,\overline{D}HG\overline{DG}+c_{149}\,\overline{D}TH\overline{H}\\
& +c_{150}\,\partial\overline{D}G\overline{G}+c_{151}\partial\overline{D}H\overline{H}+c_{152}\,[D,\overline{D}]G\overline{DG}\\
& +c_{153}\,[D,\overline{D}]T\overline{H}+c_{154}\,D\overline{H}G\overline{DG}\\
& +c_{155}\,D\overline{H}\overline{D}G\overline{G}+c_{156}\,\partial D\overline{HH}\\
& +c_{157}\,\partial G\overline{DG}+c_{158}\,\partial\overline{H}D\overline{H}\\
& +c_{159}\,\partial\overline{H}G\overline{G}+c_{160}\,\partial T\overline{H}+c_{161}\,\partial^2\overline{H}\Big](Z_2)\\
& +\frac{\theta_{12}\overline{\theta}_{12}}{z_{12}}\Big[c_{162}\,\partial\mathbf{T^{(2)}}+c_{163}\,\partial\mathbf{T^{(1)}}\mathbf{T^{(1)}}+c_{164}\,\partial^2 D\overline{H}\\
& +c_{165}\,G\partial[D,\overline{D}]\overline{G}+c_{166}\,G\partial^2\overline{G}\\
& +c_{167}\,HG\partial\overline{DG}+c_{168}\,H\overline{H}G\partial\overline{G}+c_{169}\,H\overline{H}\partial G\overline{G}\\
& +c_{170}\,H\overline{D}G[D,\overline{D}]\overline{G}+c_{171}\,H\overline{D}G\partial\overline{G}\\
& +c_{172}\,H\overline{D}H\partial\overline{H}+c_{173}\,H\partial\overline{D}G\overline{G}+c_{174}\,H[D,\overline{D}]G\overline{DG}\\
& +c_{175}\,H\partial D\overline{HH}+c_{176}\,H\partial G\overline{DG}\\
& +c_{177}\,H\partial\overline{H}D\overline{H}+c_{178}\,H\partial\overline{H}G\overline{G}+c_{179}\,H\partial^2\overline{H}\\
& +c_{180}\,\overline{H}G\partial D\overline{G}+c_{181}\,\overline{H}[D,\overline{D}]GD\overline{G}\\
& +c_{182}\,\overline{H}DG[D,\overline{D}]\overline{G}+c_{183}\,\overline{H}DG\partial\overline{G}+c_{184}\,\overline{H}\partial DG\overline{G}\\
& +c_{185}\,\overline{H}\partial GD\bar{G}+c_{186}\,T\partial\overline{D}H
\end{aligned}
$$

$$
\begin{aligned}
&+c_{187}\, T\partial D\overline{H}+c_{188}\, TG\partial\overline{G}+c_{189}\, TH\partial\overline{H}+c_{190}\, T\partial G\overline{G}\\
&+c_{191}\, T\partial H\overline{H}+c_{192}\, \overline{D}G\partial D\overline{G}\\
&+c_{193}\, \overline{D}H\partial D\overline{H}+c_{194}\, \overline{D}HG\partial\overline{G}+c_{195}\, \overline{D}H\partial G\overline{G}\\
&+c_{196}\, \overline{D}TGD\overline{G}+c_{197}\, \overline{D}TDG\overline{G}\\
&+c_{198}\, \overline{D}T\partial H+c_{199}\, \partial\overline{D}GD\overline{G}+c_{200}\, \partial\overline{D}H\overline{D}H\\
&+c_{201}\, \partial\overline{D}HD\overline{H}+c_{202}\, \partial\overline{D}HG\overline{G}\\
&+c_{203}\, \partial\overline{D}HH\overline{H}+c_{204}\, \partial\overline{D}TH+c_{205}\, [D,\overline{D}]G\partial\overline{G}\\
&+c_{206}\, \partial[D,\overline{D}]G\overline{G}+c_{207}\, DG\partial\overline{DG}\\
&+c_{208}\, D\overline{H}G\partial\overline{G}+c_{209}\, D\overline{H}\partial G\overline{G}+c_{210}\, DTG\overline{DG}\\
&+c_{211}\, DT\overline{D}G\overline{G}+c_{212}\, DT\partial\overline{H}\\
&+c_{213}\, \partial DG\overline{DG}+c_{214}\, \partial D\overline{H}D\overline{H}+c_{215}\, \partial D\overline{H}G\overline{G}\\
&+c_{216}\, \partial DT\overline{H}+c_{217}\, \partial G[D,\overline{D}]\overline{G}\\
&+c_{218}\, \partial HG\overline{DG}+c_{219}\, \partial H\overline{H}\overline{D}H+c_{220}\, \partial H\overline{H}G\overline{G}\\
&+c_{221}\, \partial H\overline{D}G\overline{G}+c_{222}\, \partial H\overline{D}H\overline{H}\\
&+c_{223}\, \partial H\partial\overline{H}+c_{224}\, \partial\overline{H}GD\overline{G}+c_{225}\, \partial\overline{H}DG\overline{G}\\
&+c_{226}\, \partial TT+c_{227}\, \partial T\overline{D}H+c_{228}\, \partial TD\overline{H}\\
&+c_{229}\, \partial TG\overline{G}+c_{230}\, \partial TH\overline{H}+c_{231}\, \partial[D,\overline{D}]T\\
&+c_{232}\, \partial^2 G\overline{G}+c_{233}\, \partial^2 H\overline{H}\\
&+c_{234}\, \partial^2\overline{D}H+c_{235}\, \partial^2 T\Big](Z_2)+\cdots, \qquad (5.10)
\end{aligned}
$$

where the coefficients are given in Appendix G.3 and we use the notation in (5.5).[21]

The fusion rule between the $\mathcal{N}=2$ higher spin-1 current and the $\mathcal{N}=2$ higher spin-2 current is given by

$$[\mathbf{T^{(1)}}]\cdot[\mathbf{W^{(2)}}]=[\mathbf{I}]+[\mathbf{T^{(1)}}\,\mathbf{T^{(1)}}]+[\mathbf{T^{(2)}}], \qquad (5.11)$$

where the last term, which resides in the next 16 higher spin currents, has its component expressions in the first four equations of (4.2) with $s=2$. Recall that the next higher $\mathcal{N}=4$ multiplet has four $\mathcal{N}=2$ higher spin currents denoted by $\mathbf{T^{(2)}}(Z)$, $\mathbf{U}^{(\frac{5}{2})}(Z)$, $\mathbf{V}^{(\frac{5}{2})}(Z)$, and $\mathbf{W^{(3)}}(Z)$. Some of the terms on the right-hand side of (5.10) occur in (5.6) but most of the terms in (5.10) are new. For example, the c_{10} and c_{13} terms do not appear in (5.6).

5.3.4 *Other remaining OPEs in the $\mathcal{N}=2$ superspace*

From Appendix G.4, the fusion rule between the $\mathcal{N}=2$ higher spin-$\frac{3}{2}$ currents, where four components of the $\mathcal{N}=2$ higher spin-$\frac{3}{2}$ currents are given by the fifth to 12th equations of (4.2) with $s=1$, is summarized by

$$\begin{bmatrix}\mathbf{U}^{(\frac{3}{2})}\\ \mathbf{V}^{(\frac{3}{2})}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{U}^{(\frac{3}{2})}\\ \mathbf{V}^{(\frac{3}{2})}\end{bmatrix}=[\mathbf{I}]. \qquad (5.12)$$

From Appendix G.5, the other fusion rule between the $\mathcal{N}=2$ higher spin-$\frac{3}{2}$ currents, where four components of the $\mathcal{N}=2$ higher spin-$\frac{3}{2}$ currents are given by the fifth to 12th equations of (4.2) with $s=1$, is summarized by

$$[\mathbf{U}^{(\frac{3}{2})}]\cdot[\mathbf{V}^{(\frac{3}{2})}]=[\mathbf{I}]+[\mathbf{T^{(1)}}\,\mathbf{T^{(1)}}]+[\mathbf{T^{(1)}}\,\mathbf{W^{(2)}}]+[\mathbf{U}^{(\frac{3}{2})}\,\mathbf{V}^{(\frac{3}{2})}]+[\mathbf{T^{(2)}}]+[\mathbf{W^{(3)}}], \qquad (5.13)$$

where the last two terms, which reside in the next 16 higher spin currents, have their component expressions in the first four and the last four equations of (4.2) with $s=2$. Therefore, we see that the right-hand sides of (5.9), (5.11), and (5.13) contain the next 16 higher spin currents in terms of its $\mathcal{N}=2$ version.

From Appendix G.6, the fusion rule between the $\mathcal{N}=2$ higher spin-$\frac{3}{2}$ current and the $\mathcal{N}=2$ higher spin-2 current, where four components of the $\mathcal{N}=2$ higher spin-$\frac{3}{2}$ currents are given by the fifth to 12th equations of (4.2) and four components of the $\mathcal{N}=2$ higher spin-2 currents are given by the last four equations of (4.2) with $s=1$, is summarized by

$$\begin{bmatrix}\mathbf{U}^{(\frac{3}{2})}\\ \mathbf{V}^{(\frac{3}{2})}\end{bmatrix}\cdot[\mathbf{W^{(2)}}]=[\mathbf{I}]+\begin{bmatrix}\mathbf{T^{(1)}}\,\mathbf{U}^{(\frac{3}{2})}\\ \mathbf{T^{(1)}}\,\mathbf{V}^{(\frac{3}{2})}\end{bmatrix}+\begin{bmatrix}\mathbf{U}^{(\frac{5}{2})}\\ \mathbf{V}^{(\frac{5}{2})}\end{bmatrix}, \qquad (5.14)$$

where the last term, which resides in the next 16 higher spin currents, has its component expressions in the fifth to 12th equations of (4.2) with $s=2$.

Finally, from Appendix G.7, the fusion rule between the $\mathcal{N}=2$ higher spin-2 currents, where four components of the $\mathcal{N}=2$ higher spin-2 current are given by the last four equations of (4.2) with $s=1$, is described by

$$[\mathbf{W^{(2)}}]\cdot[\mathbf{W^{(2)}}]=[\mathbf{I}]+[\mathbf{T^{(1)}}\,\mathbf{T^{(1)}}]+[\mathbf{T^{(1)}}\,\mathbf{W^{(2)}}]+[\mathbf{U}^{(\frac{3}{2})}\,\mathbf{V}^{(\frac{3}{2})}]+[\mathbf{T^{(2)}}]+[\mathbf{W^{(3)}}], \qquad (5.15)$$

where the last two terms, which reside in the next 16 higher spin currents, have their component expressions in the first four and the last four equations of (4.2) with $s=2$. The presence of $[\mathbf{W^{(3)}}]$ in (5.15) appears also in the $\mathcal{N}=2$ $\mathcal{W}_{N+1}$ algebra studied in [30,31].

By adding (5.7), (5.9), (5.11), (5.12), (5.13), (5.14), and (5.15) together, we obtain the fusion rule for the lowest 16 higher spin currents in the $\mathcal{N}=2$ superspace:

$$[\mathbf{I}]+\Big([\mathbf{T^{(1)}}\,\mathbf{T^{(1)}}]+[\mathbf{T^{(1)}}\,\mathbf{U}^{(\frac{3}{2})}]+[\mathbf{T^{(1)}}\,\mathbf{V}^{(\frac{3}{2})}]+[\mathbf{T^{(1)}}\,\mathbf{W^{(2)}}]+[\mathbf{U}^{(\frac{3}{2})}\,\mathbf{V}^{(\frac{3}{2})}]\Big)+\Big([\mathbf{T^{(2)}}]+[\mathbf{U}^{(\frac{5}{2})}]+[\mathbf{V}^{(\frac{5}{2})}]+[\mathbf{W^{(3)}}]\Big). \qquad (5.16)$$

[21] Let us take $\theta=0=\overline{\theta}$ in (5.10). Then the left-hand side is given by $\Phi_0^{(1)}(z_1)\left(4\,i\,\Phi_1^{(1),12}+\frac{8\,i\,\alpha}{3}\Phi_1^{(1),34}\right)(z_2)$. The right-hand side contains $c_6(G\,\overline{G})(Z_2)$, which reduces to $2\,i\,c_6\,(\Gamma^3\,\Gamma^4)(z_2)$ in the second-order pole. This is what we expect from the component approach from Appendix H.1.

In the next section, this fusion rule will be expressed in the $\mathcal{N}=4$ superspace. At the moment, it is not clear why the terms $[\mathbf{U}^{(\frac{3}{2})}\,\mathbf{U}^{(\frac{3}{2})}]$, $[\mathbf{U}^{(\frac{3}{2})}\,\mathbf{W}^{(2)}]$, $[\mathbf{V}^{(\frac{3}{2})}\,\mathbf{V}^{(\frac{3}{2})}]$, $[\mathbf{V}^{(\frac{3}{2})}\,\mathbf{W}^{(2)}]$, and $[\mathbf{W}^{(2)}\,\mathbf{W}^{(2)}]$ do not appear in the above fusion rule, but if we proceed to the $\mathcal{N}=4$ superspace, we can easily understand the reason as follows. In the fusion rule of $[\Phi^{(1)}]\cdot[\Phi^{(1)}]$, Eq. (4.4) has an expansion of θ^3 and θ^4 and the normal ordered product $\Phi^{(1)}\,\Phi^{(1)}$ at the same point of the $\mathcal{N}=4$ superspace coordinate contains the above normal ordered product in (5.16) because those terms are linear in θ^3- or θ^4- or $\theta^{3,4}$-independent terms. However, the above five normal ordered products, which are not allowed in (5.16), have quadratic terms in $\theta^{3,4}$ and therefore they vanish from the viewpoint of the $\mathcal{N}=4$ superspace.[22]

6 OPEs between the 16 higher spin currents in the $\mathcal{N}=4$ superspace

In this final section, we summarize what has been obtained in previous sections in the $\mathcal{N}=4$ superspace.

6.1 The 136 OPEs from the $\mathcal{N}=2$ superspace results

In the previous section, the complete $\mathcal{N}=2$ OPEs with complete structure constants are determined. Again, using the package in [15], we can proceed to the component approach where the 136 OPEs are determined completely. They are presented in Appendix H with simplified notation. We might ask whether or not there exists the possibility of having new primary currents in these 136 OPEs. Because we do not check for them from the 16 higher spin currents in $\mathcal{N}=4$ coset theory for generic N manually, we should be careful about the occurrence of new primary currents in the OPEs. For example, in the bosonic coset theory described in [9,10], the higher spin-4 current appears as long as $N\geq 4$; for $N=3$, there is no higher spin-4 current because the structure constant appearing in the higher spin-4 current contains the factor $(N-3)$. Furthermore, the higher spin-5 current appears as long as $N\geq 5$; for $N=4$, there is no higher spin-5 current because the structure constant appearing in the higher spin-5 current contains the factor $(N-4)$. Therefore, as the number N increases, extra new primary currents appear.

However, in the present $\mathcal{N}=4$ coset theory, such a feature does not arise. As explained with respect to (5.2) before, we have confirmed that there are no extra primary currents in the basic 16 OPEs between the higher spin-1 current and 16 higher spin currents using the WZW currents for several N values. We believe that these basic 16 OPEs are satisfied even if we try to calculate them manually. This behavior is rather different from that from purely bosonic coset theory.

6.2 $\mathcal{N}=4$ superspace description

The final single $\mathcal{N}=4$ OPE between the $\mathcal{N}=4$ (higher spin current) multiplet of superspin 1 can be described as

$$
\begin{aligned}
\Phi^{(1)}(Z_1)\,\Phi^{(1)}(Z_2) &= \frac{\theta_{12}^{4-0}}{z_{12}^4}\,c_1+\frac{\theta_{12}^{4-i}}{z_{12}^3}\,c_2\,J^i(Z_2)\\
&+\frac{\theta_{12}^{4-0}}{z_{12}^3}\,c_3\,\partial J(Z_2)+\frac{1}{z_{12}^2}\,c_4\\
&+\frac{\theta_{12}^{4-ij}}{z_{12}^2}\Big[c_5\,J^iJ^j+c_6\,J^{ij}+c_7\,\varepsilon^{ijkl}J^kJ^l+c_8\,J^{4-ij}\Big](Z_2)\\
&+\frac{\theta_{12}^{4-i}}{z_{12}^2}\Big[c_9\,J^{4-i}+c_{10}\,\partial J^i+c_{11}\,\varepsilon^{ijkl}\,J^{jk}J^l\\
&+c_{12}\,J^{ij}J^j+c_{13}\,\partial J\,J^i\Big](Z_2)\\
&+\frac{\theta_{12}^{4-0}}{z_{12}^2}\Big[c_{14}\,\Phi^{(2)}+c_{15}\,\Phi^{(1)}\Phi^{(1)}+c_{16}\,J^{4-0}\\
&+c_{17}\,J^{4-i}J^i+\varepsilon^{ijkl}\Big\{c_{18}\,J^iJ^jJ^kJ^l+c_{19}\,J^{ij}J^{kl}\Big\}\\
&+c_{20}\,J^{4-ij}J^{4-ij}+c_{21}\,J^{ij}J^iJ^j+c_{22}\,J^{4-ij}J^iJ^j\\
&+c_{23}\,\partial^2J+c_{24}\,\partial J^iJ^i+c_{25}\,\partial J\,\partial J\Big](Z_2)\\
&+\frac{\theta_{12}^{4-jkl}}{z_{12}}\Big[c_{26}\,J^{jkl}+\varepsilon^{ijkl}(c_{27}\partial J^i+c_{28}\,J^{ij}J^j)+c_{29}J^jJ^kJ^l\Big](Z_2)\\
&+\frac{\theta_{12}^{4-ij}}{z_{12}}\Big[c_{30}\,(J^{4-i}J^j-J^{4-j}J^i)+c_{31}\,\partial J^{4-ij}\\
&+c_{32}\,\partial J^{ij}+c_{33}\,\partial(J^iJ^j)\\
&+c_{34}\,\varepsilon^{ijkl}\,\partial(J^k\,J^l)+c_{35}\,(J^{ik}J^kJ^j-J^{jk}J^kJ^i)\Big](Z_2)\\
&+\frac{\theta_{12}^{4-i}}{z_{12}}\Big[c_{36}\,D^i\Phi^{(2)}+c_{37}\,\Phi^{(1)}D^i\Phi^{(1)}\\
&+c_{38}(J^{ij}J^{ij}J^i-J^{4-ij}J^{ik}J^{4-ijk})+c_{39}\,\partial^2J\,J^i\\
&+c_{40}\,\partial^2J^i+c_{41}\,J^{ij}\partial J^j+c_{42}\,J^i\partial J^jJ^j\\
&+c_{43}\,J^{ij}J^{4-j}+c_{44}\,\partial J^i\,\partial J^i+c_{45}\,\partial J^{4-i}\\
&+c_{46}\,\partial J^{ij}J^j+c_{47}\,J^{4-ij}J^{4-j}+c_{48}\,J^{ij}\partial J\,J^j+c_{49}\,\partial J\,J^{4-i}\\
&+\varepsilon^{ijkl}\Big\{c_{50}\,J^jJ^kJ^{ijk}+c_{51}\,(J^{ij}J^{kl}J^i-\partial J\,J^{jk}J^l-J^iJ^jJ^{ikl})\\
&+c_{52}\,J^{ij}J^iJ^kJ^l+c_{53}\,J^{jk}\partial J^l\\
&\times c_{54}\,\partial J\,J^jJ^kJ^l+c_{55}\,\partial(J^jJ^kJ^l)+c_{56}\,\partial J^{jk}J^l\Big\}\Big](Z_2)\\
&+\frac{\theta_{12}^{4-0}}{z_{12}}\Big[c_{57}\,\partial\Phi^{(2)}+c_{58}\,\partial\Phi^{(1)}\Phi^{(1)}+c_{59}\,J^{4-i}\partial J^i+c_{60}\,\partial J^{4-0}\\
&+c_{61}\,\partial(J^{4-ij}J^{4-ij})+c_{62}\,\partial^2J\,\partial J+c_{63}\,\partial^3J+c_{64}\,\partial^2J^iJ^i\\
&+c_{65}\,\partial(J^{ij}J^iJ^j)+c_{66}\,J^iJ^j\partial J^{ij}+c_{67}\,\partial J\,\partial J^iJ^i\\
&+\varepsilon^{ijkl}\Big\{c_{68}\,\partial(J^iJ^jJ^kJ^l)+c_{69}\,\partial(J^{ij}J^{kl})\\
&+c_{70}\,\partial(J^{ij}J^kJ^l)\Big\}\Big](Z_2)+\cdots,
\end{aligned}
\tag{6.1}
$$

[22] Let us emphasize, as noticed in [12], that, if we take the basis in [24] where the higher spin-3 current is not a primary current, then the above nonlinear terms in the $\mathcal{N}=2$ superspace vanish with the redefinitions of the next 16 higher spin currents.

where the following simplified notations are introduced:

$$\mathbf{J^{(4)}}(Z) \equiv J(Z), \quad D^i \mathbf{J^{(4)}}(Z) \equiv J^i(Z),$$
$$D^i D^j \mathbf{J^{(4)}}(Z) \equiv J^{ij}(Z), \quad D^i D^j D^k \mathbf{J^{(4)}}(Z) \equiv J^{ijk}(Z),$$
$$D^1 D^2 D^3 D^4 \mathbf{J^{(4)}}(Z) \equiv J^{1234}(Z) \equiv J^{4-0}(Z).$$

Note that $\theta^{4-ijk} = \varepsilon^{lijk}\theta^l$ and we use unusual notation for the summation (e.g., see the c_{39} term, where the dummy index i arises four times). We can easily see that there are consistent SO(4) index contractions with SO(4)-invariant tensors ε^{ijkl} and δ^{ij} on the right-hand side of the OPE (6.1).[23] The structure constants appearing in (6.1) can be summarized by

$$c_1 = \frac{4k(k-N)N}{(2+k+N)^2}, \quad c_2 = \frac{8kN}{(2+k+N)^2},$$
$$c_3 = \frac{16kN}{(2+k+N)^2}, \quad c_4 = \frac{2kN}{(2+k+N)},$$
$$c_5 = -\frac{2(k-N)}{(2+k+N)^2}, \quad c_6 = \frac{(k+N)}{(2+k+N)},$$
$$c_7 = -\frac{(k+N)}{(2+k+N)^2},$$
$$c_8 = \frac{(k-N)}{(2+k+N)}, \quad c_9 = \frac{3(k-N)}{(2+k+N)},$$
$$c_{10} = \frac{(12k+3k^2+12N+10kN+3N^2)}{(2+k+N)^2},$$
$$c_{11} = \frac{2(k+N)}{(2+k+N)^2}, \quad c_{12} = \frac{2(-k+N)}{(2+k+N)^2},$$
$$c_{13} = \frac{4(-k+N)}{(2+k+N)^2}, \quad c_{14} = 2,$$

$$c_{15} = -\frac{(60+77k+22k^2+121N+115kN+20k^2N+79N^2+42kN^2+16N^3)}{(2+N)(2+k+N)^2},$$

$$c_{16} = \frac{2(10+9k+2k^2+14N+7kN+4N^2)}{(2+k+N)^2},$$
$$c_{17} = -\frac{2}{(2+k+N)},$$
$$c_{18} = \frac{(32+55k+18k^2+41N+35kN+11N^2)}{3(2+N)(2+k+N)^4},$$
$$c_{19} = \frac{(13k+6k^2+3N+9kN+N^2)}{4(2+N)(2+k+N)^2},$$
$$c_{20} = \frac{(20+21k+6k^2+25N+13kN+7N^2)}{2(2+N)(2+k+N)^2},$$
$$c_{21} = -\frac{2(8+17k+6k^2+11N+11kN+3N^2)}{(2+N)(2+k+N)^3},$$
$$c_{22} = -\frac{2(20+21k+6k^2+25N+13kN+7N^2)}{(2+N)(2+k+N)^3},$$
$$c_{23} = -\frac{2(16+26k+13k^2+2k^3+6N-11kN-7k^2N-10N^2-15kN^2-4N^3)}{(2+k+N)^3},$$
$$c_{24} = \frac{2(100+95k+24k^2+155N+83kN+3k^2N+69N^2+14kN^2+9N^3)}{(2+N)(2+k+N)^3},$$
$$c_{25} = \frac{4(10+9k+2k^2+14N+7kN+4N^2)}{(2+k+N)^3},$$
$$c_{26} = \frac{1}{6}, \quad c_{27} = \frac{(k-N)}{6(2+k+N)},$$
$$c_{28} = -\frac{1}{(2+k+N)}, \quad c_{29} = -\frac{4}{3(2+k+N)^2},$$
$$c_{30} = \frac{1}{(2+k+N)},$$
$$c_{31} = \frac{(k-N)}{(2+k+N)}, \quad c_{32} = 1,$$
$$c_{33} = -\frac{(k-N)}{(2+k+N)^2}, \quad c_{34} = -\frac{1}{(2+k+N)},$$
$$c_{35} = -\frac{2}{(2+k+N)^2}, \quad c_{36} = \frac{1}{2},$$

[23] Let us take a simple example to examine how the $\mathcal{N}=4$ OPE can be reduced to the corresponding OPE in the component approach. Nonlinear terms appear from the c_5 term to the c_8 term in (6.1). Let us multiply both sides of (6.1) by the operator $D_2^1 D_2^2$ with the condition of $\theta_1^i = 0 = \theta_2^i$. Then the left-hand side is given by $\Phi_0^{(1)}(z_1)\,(-1)\,\Phi_1^{(1),34}(z_2)$. In contrast, the right-hand side contains $D_2^1 D_2^2 \frac{\theta_{12}^1\theta_{12}^2}{z_{12}^2}$ with current-dependent terms. This leads to the singular term $-\frac{1}{(z_1-z_2)^2}$. Therefore, the second-order terms of the OPE $\Phi_0^{(1)}(z_1)\,\Phi_1^{(1),34}(z_2)$ are given by $2(c_5 J^3 J^4 + c_6 J^{34} + 2c_7 J^1 J^2 + c_8 J^{12})(z_2)$ at vanishing θ^i. Then we obtain $2(-c_5\Gamma^3\Gamma^4 + ic_6 T^{12} - 2c_7\Gamma^1\Gamma^2 + ic_8 T^{34})(z_2)$, where the coefficients are given in Appendix I. We can then check that this is equal to the particular singular terms in the corresponding OPE in Appendix H.1.

$$c_{37} = -\frac{(60 + 77k + 22k^2 + 121N + 115k\,N + 20k^2N + 79N^2 + 42k\,N^2 + 16N^3)}{2(2+N)(2+k+N)^2},$$
$$c_{38} = \frac{(16 + 21k + 6k^2 + 19N + 13k\,N + 5N^2)}{(2+N)(2+k+N)^3},$$
$$c_{39} = -\frac{(20 + 37k + 14k^2 + 9N + 21k\,N + 4k^2N - 9N^2 - 4N^3)}{(2+N)(2+k+N)^3},$$
$$c_{40} = \frac{(34k + 29k^2 + 6k^3 + 14N + 37k\,N + 13k^2N + 6N^2 + 5k\,N^2)}{(2+k+N)^3},$$
$$c_{41} = -\frac{(80 + 88k + 11k^2 - 6k^3 + 136N + 110k\,N + 9k^2N + 75N^2 + 36k\,N^2 + 13N^3)}{2(2+N)(2+k+N)^3},$$
$$c_{42} = -\frac{(-72k - 45k^2 - 6k^3 + 8N - 40k\,N - 7k^2N + 21N^2 + 6k\,N^2 + 7N^3)}{(2+N)(2+k+N)^4},$$
$$c_{43} = \frac{(16 + 21k + 6k^2 + 19N + 13k\,N + 5N^2)}{2(2+N)(2+k+N)^2},$$
$$c_{44} = -\frac{(k-N)(32 + 29k + 6k^2 + 35N + 17k\,N + 9N^2)}{2(2+N)(2+k+N)^3},$$
$$c_{45} = \frac{(10 + 17k + 6k^2 + 6N + 7k\,N)}{(2+k+N)^2},$$
$$c_{46} = -\frac{(20 + 31k + 10k^2 + 35N + 41k\,N + 8k^2N + 21N^2 + 14k\,N^2 + 4N^3)}{(2+N)(2+k+N)^3},$$
$$c_{47} = \frac{(20 + 21k + 6k^2 + 25N + 13k\,N + 7N^2)}{2(2+N)(2+k+N)^2},$$
$$c_{48} = -\frac{(32 + 29k + 6k^2 + 35N + 17k\,N + 9N^2)}{(2+N)(2+k+N)^3},$$
$$c_{49} = -\frac{(13k + 6k^2 + 3N + 9k\,N + N^2)}{2(2+N)(2+k+N)^2},$$
$$c_{50} = -\frac{(13k + 6k^2 + 3N + 9k\,N + N^2)}{2(2+N)(2+k+N)^3},$$
$$c_{51} = \frac{(20 + 21k + 6k^2 + 25N + 13k\,N + 7N^2)}{2(2+N)(2+k+N)^3},$$
$$c_{52} = -\frac{(32 + 55k + 18k^2 + 41N + 35k\,N + 11N^2)}{(2+N)(2+k+N)^4},$$
$$c_{53} = \frac{(4+k+N)(13k + 6k^2 + 3N + 9k\,N + N^2)}{4(2+N)(2+k+N)^3},$$
$$c_{54} = -\frac{(32 + 55k + 18k^2 + 41N + 35k\,N + 11N^2)}{3(2+N)(2+k+N)^4},$$
$$c_{55} = -\frac{(80 + 92k + 11k^2 - 6k^3 + 172N + 158k\,N + 21k^2N + 119N^2 + 64k\,N^2 + 25N^3)}{6(2+N)(2+k+N)^4},$$
$$c_{56} = \frac{(16 + 29k + 10k^2 + 27N + 25k\,N + 2k^2N + 13N^2 + 4k\,N^2 + 2N^3)}{(2+N)(2+k+N)^3}, \quad c_{57} = 2,$$
$$c_{58} = -\frac{2(60 + 77k + 22k^2 + 121N + 115k\,N + 20k^2N + 79N^2 + 42k\,N^2 + 16N^3)}{(2+N)(2+k+N)^2},$$
$$c_{59} = -\frac{8}{(2+k+N)},$$

$$c_{60}=\frac{2(10+9k+2k^2+14N+7kN+4N^2)}{(2+k+N)^2},$$
$$c_{61}=\frac{(20+21k+6k^2+25N+13kN+7N^2)}{2(2+N)(2+k+N)^2},$$
$$c_{62}=\frac{8(10+9k+2k^2+14N+7kN+4N^2)}{(2+k+N)^3},$$
$$c_{63}=-\frac{2(32+42k+17k^2+2k^3+22N+5kN-3k^2N-6N^2-11kN^2-4N^3)}{(2+k+N)^3},$$
$$c_{64}=\frac{2(100+99k+26k^2+151N+85kN+4k^2N+65N^2+14kN^2+8N^3)}{(2+N)(2+k+N)^3},$$
$$c_{65}=-\frac{2(16+21k+6k^2+19N+13kN+5N^2)}{(2+N)(2+k+N)^3},$$
$$c_{66}=\frac{8}{(2+k+N)^2},$$
$$c_{67}=-\frac{16}{(2+k+N)^2},$$
$$c_{68}=\frac{(32+55k+18k^2+41N+35kN+11N^2)}{3(2+N)(2+k+N)^4},$$
$$c_{69}=\frac{(13k+6k^2+3N+9kN+N^2)}{4(2+N)(2+k+N)^2},$$
$$c_{70}=-\frac{(20+21k+6k^2+25N+13kN+7N^2)}{(2+N)(2+k+N)^3}.$$

Of course, if one uses the description in the basis of [24] or of [12], then the corresponding OPE, which is equivalent to (6.1), can be obtained similarly.

Therefore, the fusion rule for the lowest 16 higher spin currents in the $\mathcal{N}=4$ superspace can be written as

$$[\Phi^{(\mathbf{1})}]\cdot[\Phi^{(\mathbf{1})}]=[\mathbf{I}]+[\Phi^{(\mathbf{1})}\,\Phi^{(\mathbf{1})}]+[\Phi^{(\mathbf{2})}], \tag{6.2}$$

where $[\mathbf{I}]$ denotes the large $\mathcal{N}=4$ linear superconformal family of the identity operator. This is equivalent to the previous result (5.16) in the $\mathcal{N}=2$ superspace. Note that the $\mathcal{N}=4$ multiplet $\Phi^{(\mathbf{1})}(Z)$ stands for the four $\mathcal{N}=2$ higher spin currents $\mathbf{T}^{(\mathbf{1})}(Z)$, $\mathbf{U}^{(\frac{3}{2})}(Z)$, $\mathbf{V}^{(\frac{3}{2})}(Z)$, and $\mathbf{W}^{(\mathbf{2})}(Z)$ while the next $\mathcal{N}=4$ multiplet $\Phi^{(\mathbf{2})}(Z)$ stands for the four $\mathcal{N}=2$ next higher spin currents $\mathbf{T}^{(\mathbf{2})}(Z)$, $\mathbf{U}^{(\frac{5}{2})}(Z)$, $\mathbf{V}^{(\frac{5}{2})}(Z)$, and $\mathbf{W}^{(\mathbf{3})}(Z)$. The explicit component expansions for the two $\mathcal{N}=4$ higher spin currents with $s=1,2$ were presented in (3.1) previously.[24]

[24] It is useful to see how we can obtain the $\mathcal{N}=2$ superspace description starting from its $\mathcal{N}=4$ version in (6.1). Let us focus on the simplest OPE given by (5.6). Setting $\theta^3=0=\theta^4$ in (6.1) gives rise to $\mathbf{T}^{(\mathbf{1})}(Z_1)\,\mathbf{T}^{(\mathbf{1})}(Z_2)$ on the left-hand side. How then can we obtain the nontrivial contributions from the right-hand side with the above conditions? If we have θ_{12}^4, then this does not produce nontrivial contributions. For the terms θ^{4-i}, we do not have any nonzero contributions because it contains either θ^3 or θ^4. For the terms θ^{4-ij}, we can have nonzero contributions for the indices i, j being 3, 4. For the terms θ^{4-ijk}, we

7 Conclusions and outlook

The single OPE between the $\mathcal{N}=4$ higher spin-1 multiplet is given by (6.1) with the structure constants presented in (6.2). From the fusion rule in (6.2), the $\mathcal{N}=4$ higher spin-2 multiplet occurs on the right-hand side of this OPE.

Let us list some possible future research directions:

- Orthogonal coset theory

We can also apply the present analysis of this paper to $\mathcal{N}=4$ orthogonal coset theory. So far the higher spin extension of the large $\mathcal{N}=4$ nonlinear superconformal algebra in the realization of large $\mathcal{N}=4$ orthogonal coset theory was obtained in [32]. It is straightforward to construct the

Footnote 24 continued
can have nonzero contributions. Finally, for the term $\frac{1}{z_{12}^2}$, the nonzero contribution appears. We can easily see that the above coefficient c_4 in (6.1) is equal to the coefficient c_1 in (5.6). Furthermore, the expression $(c_5\,J^3\,J^4+c_6\,J^{34}+c_7\,\varepsilon^{34kl}\,J^k\,J^l+c_8\,J^{12})$ should reproduce the corresponding quantity in (5.6). For example, the c_5 term here can be obtained and gives the expression $\frac{i}{2}G\,\overline{G}(Z_2)$. Also we have the relation $\frac{\theta_{12}^1\theta_{12}^2}{z_{12}^2}=\frac{i}{2}\,\frac{\theta_{12}\overline{\theta}_{12}}{z_{12}^2}$ and we see that the coefficient of $\frac{\theta_{12}\overline{\theta}_{12}}{z_{12}^2}$ is given by $-\frac{1}{4}\,c_5\,G\,\overline{G}(Z_2)$, where the coefficient c_5 is given in Appendix I. This becomes $c_6\,G\,\overline{G}(Z_2)$ in (5.6) with Appendix G.1, as we expect. It is straightforward to check the other remaining terms explicitly.

extension of the large $\mathcal{N}=4$ linear superconformal algebra, where the group $G = SO(N+4)$ appears in the large $\mathcal{N}=4$ orthogonal coset theory for $N=4$. It would be interesting to obtain the final $\mathcal{N}=4$ single OPE $\Phi^{(2)}(Z_1)\,\Phi^{(2)}(Z_2)$, where the $\Phi^{(2)}(Z)$ is the lowest $\mathcal{N}=4$ higher spin multiplet of superspin 2, corresponding to Eq. (6.1). Note that the lowest 16 higher spin currents have the lowest higher spin-2 current, whose OPE is nontrivial, compared to the corresponding higher spin-1 current for the unitary case, because the second-order pole contains the nontrivial composite spin-2 fields. In this direction, the work of [33–37] will be useful.

- Extension of the large $\mathcal{N}=4$ nonlinear superconformal algebra in the $\mathcal{N}=4$ superspace

In this paper, the extension of the large $\mathcal{N}=4$ linear superconformal algebra was described. What about the extension of the large $\mathcal{N}=4$ nonlinear superconformal algebra by decoupling the spin-1 current $U(z)$ and four spin-$\frac{1}{2}$ currents $\Gamma^i(z)$ [38–40]? Is it possible to express the 11 currents in the $\mathcal{N}=4$ superspace (or the $\mathcal{N}=2$ superspace)?

- $\mathcal{N}=3$ holography

In the work of [41–43], $\mathcal{N}=3$ holography was found in the context of the Kazama–Suzuki model for the particular level (see also [44,45]). It would be interesting to construct the higher spin currents. We can easily see the $\mathcal{N}=3$ primary current condition for any $\mathcal{N}=3$ multiplet, which consists of one higher spin-s current, three higher spin-$(s+\frac{1}{2})$ currents, three higher spin-$(s+1)$ currents, and one higher spin-$(s+\frac{3}{2})$ current in the component approach. It is nontrivial to obtain the $\mathcal{N}=3$ OPE between the lowest $\mathcal{N}=3$ higher spin multiplet and itself. Several (N, M) cases are needed to understand the structure of this OPE. Therefore, the $\mathcal{N}=3$ description in this paper can provide some hints for the index structure of the SO(3) group.

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