

Time-dependent scalar fields in modified gravities in a stationary spacetime

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Received: 7 October 2015 / Accepted: 23 June 2016 / Published online: 6 July 2016
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Abstract Most no-hair theorems involve the assumption that the scalar field is independent of time. Recently in Graham and Jha (Phys. Rev. D90: 041501, 2014) the existence of time-dependent scalar hair outside a stationary black hole in general relativity was ruled out. We generalize this work to modified gravities and non-minimally coupled scalar field with the additional assumption that the spacetime is axisymmetric. It is shown that in higher-order gravity such as metric $f(R)$ gravity the time-dependent scalar hair does not exist. In Palatini $f(R)$ gravity and the non-minimally coupled case the time-dependent scalar hair may exist.

1 Introduction

It is well known that black holes have no hair except the parameters of mass, electric charge, and angular momentum. More precisely, the no-hair theorem claims that all black hole solutions of the Einstein–Maxwell equations of gravitation and electromagnetism in general relativity can be completely characterized by only the three parameters. Since the long-range field in the standard model of particle physics is electromagnetism, the matter field considered in the original no-hair theorem is the electromagnetic field only. Nevertheless, it is still worth thinking about what the result is if we take other matter fields such as scalar fields into account.

The issue of the scalar-vacuum was first considered in 1970 in Ref. [1]. Canonical scalar hair was ruled out for scalar fields with various kinds of potential [2–4]. Recently, this proof was extended to non-canonical scalar fields [5] and Galileons [6, 7]. Besides, black holes in Brans–Dicke

and scalar–tensor theories of gravity were studied in Refs. [8, 9], which showed that the isolated stationary black holes in scalar–tensor theories of gravity are not different from general relativity. In another word, non-minimally coupled scalar hair is also ruled out for stationary and conformally flat black holes. However, there is still the case that scalar hair does exist. Coexistence of black holes and a long-range scalar field in cosmology was presented in Refs. [10, 11]. Other scalar hair cases can be found in Refs. [7, 12–15]. These results are based on a same assumption: the scalar field is time-independent. In Ref. [16], the authors considered Einstein gravity minimally coupled to a dilaton scalar field and obtained an exact time-dependent spherically symmetric solution, which describes gravitational collapse to a static scalar-hairy black hole. If the scalar field is time-dependent, we should be more careful when defining the scalar hair in order to distinguish with some trivial situations. If a time-dependent scalar field is compatible with a stationary black hole metric (the back action of the scalar field to the spacetime is taken into account), it is called time-dependent scalar hair. Hence an in-falling flux of scalar waves is not time-independent scalar hair outside of a black hole, because the metric is no longer stationary if the back reaction to the spacetime is taken into account. It is important that the metric should be stationary because the no-hair theorem is about stationary black holes and the end state of the collapse of a star is stationary. It was shown that the system of a charged scalar field coupled to an electromagnetic field settles down to a stationary black hole with oscillating scalar hair [17]. It has been shown that scalar fields do not necessarily share the symmetries with the spacetime [18]. In Ref. [19] it was shown that the stationary spacetime does not ensure that the scalar field is time-independent and time-dependent real non-canonical scalar hair was ruled out in Einstein gravity. While for the complex scalar field these arguments do not apply. Indeed Kerr black holes were found to be having time-dependent massive complex scalar hair [20, 21].

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In this paper, we would like to generalize the work of Ref. [19] to some modified gravities. Since the proof only needs a small subset of the Einstein equations [19], this generalization is turned out to be possible for some cases. Among numerous modified gravities, $f(R)$ gravity, which is motivated by high-energy physics, cosmology, and astrophysics, has received increased attention. It is interesting to consider the generalization of Ref. [19] to $f(R)$ gravity. For metric $f(R)$ gravity the scalar curvature R in the action is constructed from the metric only. For Palatini $f(R)$ gravity the scalar curvature $R = g^{\mu\nu} R_{\mu\nu}$ where the Ricci curvature $R_{\mu\nu}$ is constructed from the independent connection. Besides, since the time-independent non-minimally coupled scalar hair is ruled out [8,9], we also investigate the case that the scalar field is time-dependent and we will find non-trivial results.

This paper is organized as follows. In Sect. 2 we first investigate the time-dependent scalar field in metric $f(R)$ gravity, then generalized it to other higher-order gravity and Eddington-inspired Born–Infeld (EiBI) gravity [22]. In Sect. 3 time-dependent scalar field in Palatini $f(R)$ gravity is investigated. In Sect. 4 we investigate the time-independent non-minimally coupled scalar. Finally the conclusion is given in Sect. 5.

2 Time-dependent scalar field in $f(R)$ gravity

The action of $f(R)$ gravity is

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_M(g_{\mu\nu}, \varphi), \tag{1}$$

where φ denotes the matter field. The variation of the action (1) with respect to the metric $g_{\mu\nu}$ leads to the equation of motion (EoM) in $f(R)$ gravity:

$$f_R R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + [g_{\mu\nu} \square - \nabla_\mu \nabla_\nu] f_R = \kappa T_{\mu\nu}, \tag{2}$$

where $f_R \equiv \frac{\partial f(R)}{\partial R}$, $\square = \nabla^\mu \nabla_\mu$, and the energy-momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{3}$$

In general relativity, if the null energy condition holds, the rigidity theorem ensures that stationary spacetime must be axisymmetric [23,24]. In $f(R)$ gravity, the null energy condition does not lead to $R_{\mu\nu} l^\mu l^\nu \geq 0$ for all timelike vector l^μ . Therefore, the null energy condition of the matter fields does not lead to the conclusion that stationary spacetime must be axisymmetric. We have to assume that the spacetime is axisymmetric and we choose coordinates (t, r, θ, ϕ) so that the metric takes the form [25]

$$ds^2 = -e^{u(r,\theta)} dt^2 + 2\rho(r, \theta) dt d\phi + e^{v(r,\theta)} d\phi^2$$

$$+ e^{A(r,\theta)} dr^2 + e^{B(r,\theta)} d\theta^2. \tag{4}$$

One can easily verify that the following components of the Ricci tensor and Christoffel symbol vanish:

$$R_{tr} = R_{t\theta} = R_{r\phi} = R_{\theta\phi} = 0, \tag{5}$$

$$\Gamma_{tr}^r = \Gamma_{tr}^\theta = \Gamma_{t\theta}^r = \Gamma_{t\theta}^\theta = 0. \tag{6}$$

The action of the K-essence is [26–29]

$$S_M = \int d^4x \sqrt{-g} P(\varphi, X), \tag{7}$$

where the kinetic term is $X = -\frac{1}{2} \nabla_\mu \varphi \nabla^\mu \varphi$. When $P = X - V(\varphi)$, Eq. (7) reduces to the action of a canonical scalar field. Varying the action (7) with respect to the scalar field φ we obtain the EoM of the non-canonical scalar field,

$$P_\varphi + P_X \square \varphi + (\nabla^\mu \varphi) \nabla_\mu P_X = 0. \tag{8}$$

The energy-momentum of the scalar field can be obtained by varying the action (7) with respect to the metric:

$$T_{\mu\nu} = P_X \partial_\mu \varphi \partial_\nu \varphi + P g_{\mu\nu}. \tag{9}$$

the tr and $t\theta$ components of Eq. (2) imply that

$$T_{tr} = P_X \partial_t \varphi \partial_r \varphi = 0, \tag{10}$$

$$T_{t\theta} = P_X \partial_t \varphi \partial_\theta \varphi = 0. \tag{11}$$

Note that $P_X \neq 0$, as otherwise the action (7) would depend on φ only and the EoM of the scalar field would be an algebraic equation. Moreover, the scalar field is time-dependent, i.e. $\partial_t \varphi \neq 0$. Thus, Eqs. (10) and (11) yield

$$\partial_r \varphi = 0, \quad \partial_\theta \varphi = 0, \tag{12}$$

or, equivalently,

$$\varphi = \varphi(t, \phi). \tag{13}$$

On the other hand, considering the rr component of Eq. (2) and noting that the metric is independent of time, we have

$$\begin{aligned} \partial_t T_{rr} &= \partial_t (P_X \partial_r \varphi \partial_r \varphi) + g_{rr} \partial_t P \\ &= g_{rr} \partial_t P = 0. \end{aligned} \tag{14}$$

Thus, $\partial_t P = 0$. Similarly, the tt component of Eq. (2) shows that $P_X \dot{\varphi}^2$ is independent of time. For general actions these yield $P_\varphi = 0$ and φ depends at most linearly upon ϕ , as otherwise there will be two equations for one unknown φ , the system will be overdetermined [19].

The result that $P_\varphi = 0$ was deduced from a highbrow point of view in Ref. [18]. Here we briefly introduce the derivation. We start from a K-essence minimally coupled to gravity (not necessarily $f(R)$ gravity) in a stationary spacetime. It is easy to verify that

$$T = g^{\mu\nu} T_{\mu\nu} = -2X P_X + 4P \tag{15}$$

and

$$P = \frac{1}{4}T \pm \frac{1}{4}\sqrt{\frac{3T_{\mu\nu}T^{\mu\nu} - T^2}{3}}. \tag{16}$$

Since the spacetime is stationary, from Eqs. (15) and (16) we have

$$0 = \xi_\xi P = P_{,X}\xi_\xi X + P_{,\phi}\xi_\xi \phi \tag{17}$$

and

$$0 = \xi_\xi T = -2(\xi_\xi X)P_{,X} - 2X\xi_\xi(P_{,X}) + 4\xi_\xi P, \tag{18}$$

where ξ is the timelike Killing vector. Thus we have $\xi_\xi(P_{,X}) = (\xi_\xi P)_{,X} = 0$, which together with Eqs. (17) and (18) yields $P_{,\phi}\xi_\xi \phi = 0$. So, the condition for the scalar field not to inherit the symmetry of the spacetime is $P_{,\phi} = 0$.

Similarly one can deduce that the scalar field φ depends at most linearly upon ϕ . Moreover, since φ should depend periodically upon ϕ , it is incompatible if φ depends linearly upon ϕ . Hence, we finally deduce that the only possible configuration of the scalar field is

$$\varphi = at + b, \tag{19}$$

where a and b are constants. So far we have proved that in $f(R)$ gravity, the time-dependent non-canonical scalar field in a stationary spacetime is only a linear function of t . This conclusion is the same as that of Ref. [19]. Following the procedure of Ref. [19], for asymptotically flat and (anti-) de Sitter stationary black holes, there is no time-dependent scalar hair. Here we give a brief demonstration.

2.1 Boundary conditions

Let us first consider the asymptotic flat condition, i.e. $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ as the radial coordinate $r \rightarrow \infty$, for which $X \rightarrow a^2/2$, and the tt and rr components of the energy-momentum tensor tend to

$$T_{tt} \rightarrow a^2 P_X \left(\frac{a^2}{2}\right) - P \left(\frac{a^2}{2}\right), \tag{20}$$

$$T_{rr} \rightarrow P \left(\frac{a^2}{2}\right). \tag{21}$$

The EoMs demand $T_{tt} = 0$ and $T_{rr} = 0$, thus either $a = 0$ or $P_X(\frac{a^2}{2})$. However, $P_X(\frac{a^2}{2})$ together with $P(\frac{a^2}{2})$ compose two equations for one unknown, which is overdetermined. Hence we have $a = 0$, and the scalar field is a constant.

For the case of an asymptotically anti-de Sitter spacetime, $g^{tt} \rightarrow 0$ as $r \rightarrow \infty$, which yields $X \rightarrow 0$. In the static spherically symmetric coordinates, the anti-de Sitter metric reads

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right)dt^2 + \left(1 - \frac{\Lambda}{3}r^2\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{22}$$

The tt and rr components of the energy-momentum tensor at infinity tend to

$$T_{tt} \rightarrow P_X P(0) - \left(1 + \frac{|\Lambda|r^2}{3}\right)P(0), \tag{23}$$

$$T_{tt} \rightarrow a^2 P_X(0) - \left(1 + \frac{|\Lambda|r^2}{3}\right)P(0), \tag{24}$$

$$T_{rr} \rightarrow 0. \tag{25}$$

It is clear that $T_{tt} = 0$ and $T_{rr} = 0$ yield $P(0) = 0$ and $a = 0$. So there is no time-dependent scalar hair.

For the case of an asymptotically de Sitter spacetime, in the static coordinates the metric is the same as Eq. (22). As $\Lambda > 0$, there is an event horizon at $r = \sqrt{3/\Lambda}$. Thus, $g_{rr}(r \rightarrow \sqrt{3/\Lambda}) \rightarrow \infty$ leads to $T_{rr}(r \rightarrow \sqrt{3/\Lambda}) \rightarrow \infty$, which is incompatible with the geometry. Hence there is no time-dependent scalar hair.

The derivation of the time-dependent scalar field in $f(R)$ gravity can be generalized to a large class of alternative theories of gravity under the metric form (4). As an example, consider a higher-order gravity with the action

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g}(R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}) + S_\varphi. \tag{26}$$

The field equations read

$$\begin{aligned} G_{\mu\nu} + 2\alpha R \left(R_{\mu\nu} - \frac{1}{4}R g_{\mu\nu}\right) \\ + (2\alpha + \beta)(g_{\mu\nu}\square - \nabla_\mu \nabla_\nu)R \\ + 2\beta R^{\rho\sigma} \left(R_{\mu\rho\nu\sigma} - \frac{1}{4}R_{\rho\sigma} g_{\mu\nu}\right) \\ + \beta \square \left(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}\right) = \kappa T_{\mu\nu}^\varphi. \end{aligned} \tag{27}$$

Since the metric is stationary and axisymmetric, the tr and $t\theta$ components of Eq. (27) vanish and we still have Eqs. (10) and (11). The rest of the derivation is the same as that in $f(R)$ gravity. It is obvious that these arguments can apply to some other alternative gravities like EiBI gravity theory.

2.2 Double scalar fields

We consider the case that the matter fields are consisted of two coupled non-canonical scalar fields φ_1 and φ_2 , of which the generalized action is

$$S_M = \int d^4x \sqrt{-g} P(\varphi_1, \varphi_2, X_1, X_2). \tag{28}$$

This action contains the case of a complex scalar field as a special case. The energy-momentum tensor is

$$T_{\mu\nu} = P_{X_1} \partial_\mu \varphi_1 \partial_\nu \varphi_1 + P_{X_2} \partial_\mu \varphi_2 \partial_\nu \varphi_2 + P g_{\mu\nu}. \tag{29}$$

Therefore $T_{0i} = 0$ does not necessarily lead to $\partial_0 \varphi_1 \partial_i \varphi_1 = 0$ or $\partial_0 \varphi_2 \partial_i \varphi_2 = 0$, and the argument given above does not work for the double scalar field case any more.

3 Time-dependent scalar field in Palatini $f(R)$ gravity

Now we turn to the time-dependent scalar field in Palatini $f(R)$ gravity. The action of Palatini $f(R)$ gravity is [30–32]

$$S_{\text{Pal}} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(\mathcal{R}) + S_M(g_{\mu\nu}, \varphi), \tag{30}$$

where the Ricci tensor $\mathcal{R}_{\mu\nu}$ is constructed with the independent connection $\Gamma^\lambda_{\mu\nu}$ and the corresponding Ricci scalar is $\mathcal{R} = g^{\mu\nu} \mathcal{R}_{\mu\nu}$. Here we still assume that the spacetime is stationary and axisymmetric. Then the metric has the same form of (4) and \mathcal{R} is independent on t and ϕ .

Varying the action (30) independently with respect to the metric and connection, one can obtain the EoMs of Palatini $f(R)$ gravity,

$$f_{\mathcal{R}} \mathcal{R}_{(\mu\nu)} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{31}$$

$$\tilde{\nabla}_\lambda (\sqrt{-g} f_{\mathcal{R}} g^{\mu\nu}) = 0, \tag{32}$$

where $\tilde{\nabla}_\lambda$ ined with the independent connection $\Gamma^\lambda_{\mu\nu}$. Let us define a conformal metric $q_{\mu\nu}$,

$$q_{\mu\nu} \equiv f_{\mathcal{R}} g_{\mu\nu}. \tag{33}$$

Then Eq. (32) implies that the independent connection $\Gamma^\lambda_{\mu\nu}$ is the Levi-Civita connection of the conformal metric $q_{\mu\nu}$. Under conformal transformations, the Ricci tensor $R_{\mu\nu}$ transforms as

$$\begin{aligned} \mathcal{R}_{\mu\nu} = R_{\mu\nu} + \frac{3}{2} \frac{1}{f_{\mathcal{R}}^2} (\nabla_\mu f_{\mathcal{R}}) (\nabla_\nu f_{\mathcal{R}}) \\ - \frac{1}{f_{\mathcal{R}}} \left(\nabla_\mu \nabla_\nu + \frac{1}{2} g_{\mu\nu} \square \right) f_{\mathcal{R}}, \end{aligned} \tag{34}$$

where the Ricci tensor $R_{\mu\nu}$ and ∇_μ are constructed by the spacetime metric $g_{\mu\nu}$. Contraction with $g^{\mu\nu}$ yields

$$\begin{aligned} \mathcal{R} = R + \frac{3}{2} \frac{1}{f_{\mathcal{R}}^2} (\nabla_\mu f_{\mathcal{R}}) (\nabla^\mu f_{\mathcal{R}}) \\ - \frac{3}{f_{\mathcal{R}}} \square f_{\mathcal{R}}. \end{aligned} \tag{35}$$

With Eqs. (34) and (35), Eq. (31) is reduced to

$$\begin{aligned} G_{\mu\nu} = \frac{\kappa}{f_{\mathcal{R}}} T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(\mathcal{R} - \frac{f}{f_{\mathcal{R}}} \right) \\ + \frac{1}{f_{\mathcal{R}}} (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) f_{\mathcal{R}} \\ - \frac{3}{2} \frac{1}{f_{\mathcal{R}}^2} \left[(\nabla_\mu f_{\mathcal{R}}) (\nabla_\nu f_{\mathcal{R}}) - \frac{1}{2} g_{\mu\nu} (\nabla f_{\mathcal{R}})^2 \right], \end{aligned} \tag{36}$$

from which we can see that we still have Eqs. (10), (11), and Eqs. (14)–(18) for Palatini $f(R)$ gravity. Thus, the only possible configuration of the scalar field is (19).

Now we consider whether the configuration of the scalar field can be compatible to the boundary conditions. First we consider the asymptotic flat boundary condition. Note that

the asymptotic flat boundary condition implies that the metric $g_{\mu\nu}$ approaches the Minkowski metric $\eta_{\mu\nu}$, while $q_{\mu\nu}$ approaches conformally Minkowski. Thus Eqs. (20) and (21) no longer hold. On the other hand, $\partial_t P = 0$ and $\partial_t \varphi = 0$ yield $\partial_\varphi P = 0$, and $P_X = P_X(a^2/2)$ is a constant. Thus $\nabla_\mu P_X = 0$. It is easy to verify $\square \varphi = 0$. Therefore, the configuration of the scalar field of Eq. (19) is compatible with Eq. (8), the EoM of the scalar field. The asymptotic flat boundary condition no longer yields $a = 0$. Similar argument can be made in the asymptotic AdS/dS cases. Hence for all the three kinds of boundary conditions the time-dependent scalar hair may exist in Palatini $f(R)$ gravity.

Though the discussion above failed to exclude the time-dependent scalar hair for an arbitrary Palatini $f(R)$ gravity, it works if

$$f(\mathcal{R}) = \mathcal{R} + \sum_{n=2}^N a_n \mathcal{R}^n. \tag{37}$$

Taking the trace of Eq. (31), we have

$$f_{\mathcal{R}} \mathcal{R} - 2f(\mathcal{R}) = \kappa T. \tag{38}$$

Using Eq. (37) and noting that $T_{\mu\nu}$ approaches a constant at infinity, Eq. (38) yields $\mathcal{R} = \text{constant}$. Considering the asymptotic flat boundary condition, Eq. (35) yields $\mathcal{R} = 0$, $f(0) = 0$ and $f_{\mathcal{R}}(0) = 1$. Finally Eq. (36) results in $T_{\mu\nu} = 0$ at infinity and the time-dependent scalar hair is ruled out. For the asymptotic AdS/dS we have the same conclusion.

We can also investigate the time-dependent scalar field φ in scalar–tensor gravity with the action

$$\begin{aligned} S_{\text{st}} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[U(\psi) R - \frac{1}{2} h(\psi) \nabla_\mu \psi \nabla^\mu \psi - V(\psi) \right] \\ + S_M(g_{\mu\nu}, \varphi), \end{aligned} \tag{39}$$

where the action of the scalar field φ is still given by Eq. (7). Here we also assume that $g_{\mu\nu}$ is in the form of Eq. (4) and $\psi = \psi(r, \theta)$. This action is equivalent to the action (1) of metric $f(R)$ gravity if $U(\psi) = \psi$ and $h(\psi) = 0$ [31–34], and equivalent to the action (30) of Palatini $f(R)$ gravity if $U(\psi) = \psi$ and $h(\psi) = -\frac{3}{2\psi}$ [31,32,34,35]. This case will be the same as that of the time-dependent scalar field in Palatini $f(\mathcal{R})$ gravity: we can educe the conclusion that the scalar field φ only depend linearly on t , but the boundary conditions do not exclude the scalar hair, thus the scalar hair may exist. While for some specific $V(\psi)$ corresponding to the Lagrangian of Palatini $f(\mathcal{R})$ gravity given by Eq. (37) we can rule out the scalar hair.

4 Non-minimally coupled scalar field

We now give the argument for a time-dependent non-minimally coupled scalar field in a stationary spacetime. It

should be note that this is different from the case of the time-dependent scalar field in scalar–tensor. The action for the non-minimally coupled scalar field is

$$S = \int d^4x \sqrt{-g} \left[\varphi R - \frac{\omega(\varphi)}{\varphi} \nabla^\mu \varphi \nabla_\mu \varphi - V(\varphi) \right]. \quad (40)$$

By varying with respect to $g_{\mu\nu}$ and φ , one obtains the field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{\omega(\varphi)}{\varphi^2} \left(\nabla_\mu \varphi \nabla_\nu \varphi - \frac{1}{2} g_{\mu\nu} \nabla^\lambda \varphi \nabla_\lambda \varphi \right) + \frac{1}{\varphi} \left(\nabla_\mu \nabla_\nu \varphi - g_{\mu\nu} \square \varphi \right) - \frac{V(\varphi)}{2\varphi} g_{\mu\nu},$$

$$(2\omega + 3)\square \varphi = -\omega_\varphi \nabla^\lambda \varphi \nabla_\lambda \varphi + \varphi V_\varphi - 2V. \quad (41)$$

Here we still assume that the spacetime is stationary and axisymmetric. Thus we have the metric (4) and Eqs. (5) and (6). The tr and $t\theta$ components of Eq. (2) imply that

$$\frac{\omega(\varphi)}{\varphi^2} \partial_t \varphi \partial_r \varphi + \frac{1}{\varphi} \nabla_t \nabla_r \varphi = 0, \quad (42)$$

$$\frac{\omega(\varphi)}{\varphi^2} \partial_r \varphi \partial_\theta \varphi + \frac{1}{\varphi} \nabla_r \nabla_\theta \varphi = 0. \quad (43)$$

Now it is clear that $\partial_t \varphi \neq 0$ no longer educes $\partial_r \varphi = 0$ or $\partial_\theta \varphi = 0$, thus the arguments in Sect. 2 no longer apply and the time-dependent non-minimally coupled scalar hair may exist.

5 Conclusion

In this paper, we investigated the non-canonical time-dependent scalar field in a stationary and axisymmetric spacetime in modified gravities. For a single real scalar field in metric $f(R)$ gravity, we proved that the time-dependent scalar hair does not exist for the three kinds of boundary conditions (asymptotically flat, anti-de Sitter, and de Sitter). It was shown that the demonstration can be generalized to a large class of alternative theories of gravity like the higher-order gravity described by the action (26) and EiBI gravity. While for two coupled scalar fields, these arguments do not apply. These conclusions are the same as for the time-dependent scalar field in general relativity in Ref. [19].

Though the demonstrations for a single scalar hair in general relativity and metric $f(R)$ only use a small subset of the field equations [19], the generalization to other alternative gravities may not be correct. For Palatini $f(R)$ gravity coupled with a scalar field, as the boundary conditions no longer rule out the non-trivial configuration of the scalar field, the time-dependent scalar hair may exist outside a stationary and axisymmetric black hole.

However, for the case that $f(\mathcal{R})$ is given by Eq. (37), the time-dependent scalar hair is ruled out. This conclusion

can be widely generalized. The keypoint is to ensure that $f(\mathcal{R} = 0) = 0$ and $f_{\mathcal{R}}(\mathcal{R} = 0) \neq 0$.

Since Palatini $f(R)$ gravity is equivalent to scalar–tensor gravity, similar argument can be applied to time-dependent scalar field in scalar–tensor gravity. For some specific $V(\psi)$ we can rule out the scalar hair. For the time-independent non-minimally coupled scalar field, since the effective energy-momentum of the scalar field contains the second derivative of the scalar field, the derivations of Sect. 2 do not apply any more and non-minimally coupled scalar hair may exist outside a stationary and axisymmetric black hole.

Acknowledgments We would like to thank the referee for helping us to clarify the concept of the black hole scalar hair in the time-dependent case. This work was supported by the National Natural Science Foundation of China (Grants No. 11205074, No. 11375075, and No. 11522541) and the Fundamental Research Funds for the Central Universities (Grants No. lzujbky-2015-jl11, No. lzujbky-2015-207 and No. lzujbky-2016-k04). Y.Z. was also supported by the scholarship granted by the Chinese Scholarship Council (CSC).

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