

# Generalized teleparallel theory

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**Abstract** We construct a theory in which the gravitational interaction is described only by torsion, but that generalizes the teleparallel theory still keeping the invariance of local Lorentz transformations in one particular case. We show that our theory falls, in a certain limit of a real parameter, under  $f(\bar{R})$  gravity or, in another limit of the same real parameter, under modified  $f(T)$  gravity; on interpolating between these two theories it still can fall under several other theories. We explicitly show the equivalence with  $f(\bar{R})$  gravity for the cases of a Friedmann–Lemaître–Robertson–Walker flat metric for diagonal tetrads, and a metric with spherical symmetry for diagonal and non-diagonal tetrads. We study four applications, one in the reconstruction of the de Sitter universe cosmological model, for obtaining a static spherically symmetric solution of de Sitter type for a perfect fluid, for evolution of the state parameter  $\omega_{DE}$ , and for the thermodynamics of the apparent horizon.

## 1 Introduction

One of the most important findings in modern physics is that our universe has accelerated expansion [1–3]. However, a plausible common explanation for this is using the model of a very exotic fluid called dark energy, which has negative pressure. Another well-known possibility is to modify Einstein’s general relativity (GR) [4], making the action of the theory depend on a function of the curvature scalar  $R$ , but at a certain limit of parameters the theory falls under GR. This way to explain the accelerated expansion of our universe is known as modified or generalized gravity. Considering that the gravitational interaction is described only by the curvature of space-time, we can generalize the Einstein–

Hilbert action through an analytic function of scalars of the theory, as for example  $f(\bar{R})$  gravities [5–9], with  $\bar{R}$  being the Ricci scalar or curvature scalar,  $f(\bar{R}, \Theta)$  [10–13], with  $\Theta$  being the trace of energy-momentum tensor, or yet  $f(G)$  [14–18],  $f(\bar{R}, G)$  [19–24] and  $f(\bar{R}, \Theta, \bar{R}_{\mu\nu}\Theta^{\mu\nu})$  [25], with  $\Theta^{\mu\nu}$  being the energy-momentum tensor.

An alternative to consistently describe the gravitational interaction is one which only considers the torsion of space-time, thus canceling out any effect of the curvature. This approach is known as teleparallel theory (TT) [26–29], which is demonstrably equivalent to GR. In order to describe not only the gravitational interaction, but also the accelerated expansion of our universe, Ferraro and Fiorini [30] proposed a possible generalization of the TT, which became known as  $f(T)$  gravity [31–62], which up to now has provided good results in both cosmology and local phenomena of gravitation. A key problem in  $f(T)$  gravity is that it breaks the invariance under local Lorentz transformations complicating the interpretation of the relationship between all inertial frames of the tangent space to the differentiable manifold (space-time) [63,64]. This problem may lead to the emergence of spurious new degrees of freedom, which are responsible for the breakdown of the local Lorentz symmetry [65]. A consequence of the formulation of the theory using a scalar which is not invariant under local Lorentz transformations, the torsion scalar  $T$  in this case, is that instead of the theory presenting differential equations of motion of fourth order, as in the case of  $f(\bar{R})$  gravity, it has second-order differential equations. That seems like a benefit but it is a consequence of the fact of the local Lorentz symmetry. This generalization of the TT still is not equivalent to a generalization  $f(\bar{R})$  for RG.

This is the main reason why we will address the construction of a theory that generalizes the TT, but which still keeps the local Lorentz symmetry in a particular case. Therefore, it is clear that we must build the function of the action with

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dependence on a scalar that at some limit is invariant under local Lorentz transformations. This will be shown soon.

The paper is organized as follows. In Sect. 2 we present a review of  $f(T)$  gravity, introducing the functional variation method used in this work, obtaining the equations of motion of this theory, noting a poorly treated point at the limit to GR. In Sect. 3 we propose the action of generalized teleparallel theory, we obtain the equations of motion through functional variation of the same and compare with  $f(T)$  gravity. We show the equivalence of our theory with  $f(\bar{R})$  gravity, in the case of cosmology for the line element of flat FLRW metric in Sect. 4.1, and also in the case of a spherically symmetric line element in Sect. 4.2. We also show the equivalence of our theory with a particular case of  $f(T, B)$  gravity in Sect. 5. In Sect. 6 we make four applications, one where we reconstructed the action of our theory for the universe of the model of de Sitter, and another where we obtain a static de Sitter type solution; we analyze the evolution for the state parameter to dark energy and the thermodynamics for a cosmological model. We present our final considerations in Sect. 7.

## 2 The equations of motion for $f(T)$ gravity

The geometry of a space-time can be characterized by the curvature and torsion. In the particular case in which we only consider the curvature and torsion being zero, we have defined, together with the metricity condition  $\nabla_\mu g_{\alpha\beta} \equiv 0$  where  $g_{\alpha\beta}$  are the components of the metric tensor, a Riemannian geometry where the connection  $\bar{\Gamma}^\mu_{\alpha\beta}$  is symmetric in the last two indices. Already in the particular case that we consider only torsion (Riemann tensor identically zero, the case without curvature) in the space-time, we can then work with objects that depend solely on the so-called tetrad matrices and their derivatives as dynamic fields.

In the space-time having only torsion, the line element can be represented through two standard forms

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} \theta^a \theta^b, \tag{1}$$

where we have the following relationships:  $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$ ,  $g^{\mu\nu} = \eta^{ab} e^a_\mu e^b_\nu$ ,  $\theta^a = e^a_\mu dx^\mu$ ,  $e^a_\mu e^b_\nu = \delta^a_b$ , with  $e^a_\mu$  being the tetrad matrices and  $e^\mu_a$  their inverse, and  $[\eta_{ab}] = \text{diag}[1, -1, -1, -1]$  the Minkowski metric. We adopt Latin indices for the tangent space and Greeks for space-time.

We will first establish the equations of motion for the theory  $f(T)$ , thus showing that the functional variation method adopted here is consistent.

We restrict the geometry to Weitzenböck geometry where we have the following connection:

$$\Gamma^\sigma_{\mu\nu} = e^\sigma_a \partial_\nu e^a_\mu = -e^a_\mu \partial_\nu e^\sigma_a. \tag{2}$$

All Riemann tensor components are identically zero for the connection (2). We can then define the components of the tensor of torsion and contortion as

$$T^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\mu\nu} = e^\sigma_a (\partial_\mu e^a_\nu - \partial_\nu e^a_\mu), \tag{3}$$

$$K^\mu_{\alpha}{}^{\nu} = -\frac{1}{2} (T^\mu_{\alpha}{}^{\nu} - T^\nu_{\alpha}{}^{\mu} - T^\mu_{\alpha}{}^{\nu}). \tag{4}$$

We can also define a new tensor, so we write the equations of motion in a more elegant way, through the components of the tensor torsion and contortion, as

$$S^{\mu\nu} = \frac{1}{2} (K^{\mu\nu} + \delta^\mu_\alpha T^{\beta\nu} - \delta^\nu_\alpha T^{\beta\mu}). \tag{5}$$

We define the torsion scalar as

$$T = T^\alpha_{\mu\nu} S^{\mu\nu} = \frac{1}{4} T^\alpha_{\mu\nu} T^\mu_{\alpha}{}^{\nu} + \frac{1}{2} T^\alpha_{\mu\nu} T^\nu_{\alpha}{}^{\mu} - T^\alpha_{\mu\alpha} T^{\beta\mu}. \tag{6}$$

Some observations are important here. The first is that there is a direct analogy to a space only with torsion and another considering only curvature in that the connections are related by

$$\bar{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - g_{\mu\lambda} K^\alpha_{\nu}{}^{\lambda}, \tag{7}$$

where  $\bar{\Gamma}^\alpha_{\mu\nu}$  is the Levi-Civita connection, which is symmetric in the last two indices. The second observation is that the torsion scalar  $T$  is not a Lorentz scalar (in the tangent space), being only a scalar in the tensorial indices (space-time) [66]. This is precisely the cause for that theory, being built starting from this scalar, to break down the invariance by local Lorentz transformations. We can in reality build the curvature scalar analog, through the torsion scalar, by the relation [66]

$$\bar{R} = -T - 2\bar{\nabla}^\mu T^\alpha_{\mu\alpha} = -T - 2e^{-1} \partial_\mu (e g^{\mu\lambda} T^\alpha_{\lambda\alpha}), \tag{8}$$

where  $e = \det[e^a_\mu] = \sqrt{-g}$ , with  $g = \det[g_{\mu\nu}]$ . The curvature scalar  $\bar{R}$  in (8) is a Lorentz scalar as well as a scalar on tensorial indices. That is why  $f(\bar{R})$  gravity is a theory that is invariant under local Lorentz transformations and general coordinates transformations (tensorial).

It is then possible to construct a generalization of the teleparallel theory (TT) using the following action of the  $f(T)$  gravity:

$$S_{f(T)} = \int d^4x \mathcal{L}_{f(T)} = \int d^4x \left[ \frac{e}{2\kappa^2} f(T) - \mathcal{L}_{\text{matter}} \right] \tag{9}$$

where  $\kappa^2 = 8\pi G_{\text{Newton}}$ ,  $f(T)$  is a function of the torsion scalar and  $\mathcal{L}_{\text{matter}}$  is the Lagrangian density of the matter content. We call attention to the true sign (-) in front of the matter term. This so far has not been explicitly addressed in the literature of this theory, because we still have few models that couple the matter contents that need to be obtained through functional variation in principle. This feature is essential if the theory is to be equivalent to GR at some limit. It will soon be made clear.

Addressing the functional variation of the action (9) we have

$$\begin{aligned} \delta S_{f(T)} &= \frac{1}{2\kappa^2} \int d^4x \left[ f \delta e + e \delta f - 2\kappa^2 \delta \mathcal{L}_{\text{matter}} \right], \\ &= \frac{1}{2\kappa^2} \int d^4x \left[ f \frac{\partial e}{\partial e_\sigma^a} \delta e_\sigma^a + e \frac{df}{dT} \delta T \right] \\ &\quad - \int d^4x \delta \mathcal{L}_{\text{matter}}, \\ &= \delta S_T - \delta S_{\text{matter}}, \end{aligned} \tag{10}$$

with  $\delta S_{\text{matter}} = \int d^4x \delta \mathcal{L}_{\text{matter}}$ . Now let us study first the functional variation of the matter term,

$$\begin{aligned} \delta S_{\text{matter}} &= \int d^4x \left[ \frac{\partial \mathcal{L}_{\text{matter}}}{\partial e_\sigma^a} \delta e_\sigma^a + \frac{\partial \mathcal{L}_{\text{matter}}}{\partial (\partial_\alpha e_\sigma^a)} \delta (\partial_\alpha e_\sigma^a) \right]; \\ &\text{performing the integration by parts of the latter term, considering } \delta e_\sigma^a|_{\text{surface}} \equiv 0, \text{ we have} \\ \delta S_{\text{matter}} &= \frac{1}{2\kappa^2} \int d^4x \left[ 2\kappa^2 \left[ \frac{\partial \mathcal{L}_{\text{matter}}}{\partial e_\sigma^a} \delta e_\sigma^a - \partial_\alpha \left( \frac{\partial \mathcal{L}_{\text{matter}}}{\partial (\partial_\alpha e_\sigma^a)} \right) \delta e_\sigma^a \right] \right. \\ &\quad \left. = \frac{1}{2\kappa^2} \int d^4x \left[ 2\kappa^2 e \Theta_\alpha^\sigma \delta e_\sigma^a \right] \right], \end{aligned} \tag{11}$$

where  $\Theta_\alpha^\sigma = e_a^\beta \Theta_\beta^\sigma$ , and we define  $\Theta_\nu^\sigma$  as the energy-momentum tensor.

We now have the functional variation of geometric part,

$$\begin{aligned} \delta S_T &= \frac{1}{2\kappa^2} \int d^4x \left\{ f \frac{\partial e}{\partial e_\sigma^a} \delta e_\sigma^a + e \frac{df}{dT} \left[ \frac{\partial T}{\partial e_\sigma^a} \delta e_\sigma^a + \frac{\partial T}{\partial (\partial_\alpha e_\sigma^a)} \delta (\partial_\alpha e_\sigma^a) \right] \right\}. \end{aligned}$$

for Performing integration by parts the last term, considering  $\delta e_\sigma^a|_{\text{surface}} \equiv 0$ , we obtain

$$\begin{aligned} \delta S_T &= \frac{1}{2\kappa^2} \int d^4x \left\{ f \frac{\partial e}{\partial e_\sigma^a} + e f_T \frac{\partial T}{\partial e_\sigma^a} - \partial_\alpha \left[ e f_T \frac{\partial T}{\partial (\partial_\alpha e_\sigma^a)} \right] \right\} \delta e_\sigma^a, \end{aligned} \tag{12}$$

where  $f_T = df/dT$ . Taking (11) and (12) and replacing in (10), and imposing the principle of least action  $\delta S_{f(T)} \equiv 0$  and multiplying by  $e^{-1} e_\nu^a/2$ , we have the following equation of motion:

$$\begin{aligned} \frac{1}{2} f \left( e^{-1} e_\nu^a \frac{\partial e}{\partial e_\sigma^a} \right) + \frac{1}{2} f_T e_\nu^a \frac{\partial T}{\partial e_\sigma^a} - \frac{1}{2} e^{-1} e_\nu^a \partial_\alpha \left[ e f_T \frac{\partial T}{\partial (\partial_\alpha e_\sigma^a)} \right] - \kappa^2 \Theta_\nu^\sigma = 0. \end{aligned} \tag{13}$$

Substituting the derivatives [66]

$$\frac{\partial e}{\partial e_\sigma^a} = e e_a^\sigma, \quad \frac{\partial T}{\partial e_\sigma^a} = -4e_a^\lambda T_{\nu\lambda}^\alpha S_\alpha^{\nu\sigma}, \quad \frac{\partial T}{\partial (\partial_\alpha e_\sigma^a)} = 4e_a^\lambda S_\lambda^{\alpha\sigma}, \tag{14}$$

in (13) we finally have the equations of motion of  $f(T)$  gravity

$$\frac{1}{2} f \delta_\nu^\sigma - 2 f_T T_{\beta\nu}^\alpha S_\alpha^{\beta\sigma} - 2e^{-1} e_\nu^a \partial_\alpha \left[ e f_T e_a^\beta S_\beta^{\alpha\sigma} \right] - \kappa^2 \Theta_\nu^\sigma = 0. \tag{15}$$

Now we make use of the identity [66]

$$\left[ e^{-1} e_\nu^a \partial_\alpha \left( e e_a^\beta S_\beta^{\alpha\sigma} \right) + T_{\beta\nu}^\alpha S_\alpha^{\beta\sigma} \right] = -\frac{1}{2} \left[ G_\nu^\sigma - \frac{1}{2} \delta_\nu^\sigma T \right], \tag{16}$$

with  $G_\nu^\sigma$  being the mixed components of the Einstein tensor, for rewriting (15) as

$$-2S_\nu^{\alpha\sigma} \partial_\alpha f_T + f_T G_\nu^\mu + \frac{1}{2} \delta_\nu^\sigma [f - f_T T] = \kappa^2 \Theta_\nu^\sigma. \tag{17}$$

This theory falls under Einstein’s general relativity with a cosmological constant, when we make  $f(T) = T - 2\Lambda$ . Here it becomes clear that if we do not consider the sign (–) in front of the matter term in the action (9) in the theory, we do not return to GR for a linear  $f(T)$  function, reaching the case opposite to Einstein’s equation. This fact will be crucial in showing later that an invariant theory by local Lorentz transformations, as  $f(\bar{R})$  gravity, cannot fall under  $f(T)$  gravity, since these have opposite coupling signs to the matter term.

Sotiriou et al. [63,64] have shown that  $f(T)$  gravity does not preserve its equations of motion invariant by local Lorentz transformations. It is in relation to this problem that we then construct a generalization of the teleparallel theory that preserves the invariance of the equations of motion for a local Lorentz transformation. This will be addressed in the next section.

### 3 Equations of motion on generalized teleparallel theory

An important identity is given by  $\bar{R} = -T - 2\bar{\nabla}^\mu T_{\mu\beta}^\beta$ , where  $\bar{R}$  is the curvature scalar associated with a Riemann tensor defined solely by the Levi-Civita connection  $\bar{\Gamma}_{\mu\nu}^\alpha$ , where the indices  $(\mu\nu)$  are symmetric, and the covariant derivative  $\bar{\nabla}$  is defined by this connection. The curvature scalar is by definition invariant under a local Lorentz transformation, but it is also invariant under a general coordinate transformation. So it would be interesting to develop a theory that generalizes the TT but for which the functional action depends on an invariant under local Lorentz transformations. This is not the case of  $f(T)$  gravity.

We propose the following action:

$$S_{GTT} = \int d^4x \left[ \frac{e}{2\kappa^2} f(T) + \mathcal{L}_{\text{matter}} \right], \tag{18}$$

where we define

$$\mathcal{T} = -T - 2a_1 \bar{\nabla}^\mu T_{\mu\beta}^\beta = -T - 2a_1 e^{-1} \partial_\mu (e g^{\mu\lambda} T_{\lambda\alpha}^\alpha). \tag{19}$$

This action generalizes TT and falls under a modified  $f(T)$  gravity as well as  $f(\bar{R})$  gravity. We can show this by making the limit  $a_1 \rightarrow 0$ , where we have  $\mathcal{T} \rightarrow -T$ , therefore  $f(\mathcal{T}) \rightarrow f(-T)$ ,  $f_{\mathcal{T}} \rightarrow -f_T$  and the theory must be equivalent to a modified  $f(T)$  (we shall see this later). Moreover, we can regain  $f(\bar{R})$  gravity, making the limit  $a_1 \rightarrow 1$ , where we have  $\mathcal{T} \rightarrow \bar{R}$ ; then the theory must be equivalent to  $f(\bar{R})$ . We show this explicitly through the equations of motion later on.

By performing the functional variation of the action (18) we obtain

$$\delta S_{GTT} = \frac{1}{2\kappa^2} \int d^4x \left[ f \delta e + e \delta f + 2\kappa^2 \delta \mathcal{L}_{\text{matter}} \right]. \tag{20}$$

As  $S_{GTT} \equiv S_{GTT} [e_\sigma^a, \partial_\alpha e_\sigma^a, \Phi^A]$ , in which  $\Phi^A$  are the matter fields, we have

$$\delta S_{GTT} = \delta S_{\mathcal{T}} + \delta S_{\text{matter}}, \tag{21}$$

with  $\delta S_{\text{matter}} = \int d^4x \delta \mathcal{L}_{\text{matter}}$  in the same manner as in  $f(T)$  gravity. The functional variation of the matter term (21) is exactly the same as given in (11).

The geometric part is

$$\begin{aligned} \delta S_{\mathcal{T}} &= \frac{1}{2\kappa^2} \int d^4x [f \delta e + e \delta f] \\ &= \frac{1}{2\kappa^2} \int d^4x \left[ f \frac{\partial e}{\partial e_\sigma^a} \delta e_\sigma^a + e f_{\mathcal{T}} \delta \mathcal{T} \right], \end{aligned} \tag{22}$$

where we use  $f_{\mathcal{T}} = df/d\mathcal{T}$ . The first term in (22) is already known, and we will pay attention to the second term.

Performing the functional variation to  $\mathcal{T}$  in (19) we obtain

$$\begin{aligned} \delta \mathcal{T} &= -\delta T - 2a_1 \delta \left[ e^{-1} \partial_\mu (e g^{\mu\beta} T_{\beta\alpha}^\alpha) \right] \\ &= -\delta T - 2a_1 \left[ -e^{-2} \partial_\mu (e g^{\mu\beta} T_{\beta\alpha}^\alpha) \delta e \right. \\ &\quad \left. + e^{-1} \delta \partial_\mu (e g^{\mu\beta} T_{\beta\alpha}^\alpha) \right], \end{aligned} \tag{23}$$

in (22) taking into account the functional variation of  $T$  and  $e$  we have

$$\begin{aligned} \delta S_{\mathcal{T}} &= \frac{1}{2\kappa^2} \int d^4x \left\{ f \frac{\partial e}{\partial e_\sigma^a} \delta e_\sigma^a \right. \\ &\quad - e f_{\mathcal{T}} \left[ \frac{\partial T}{\partial e_\sigma^a} \delta e_\sigma^a + \frac{\partial T}{\partial (\partial_\alpha e_\sigma^a)} \delta (\partial_\alpha e_\sigma^a) \right] \\ &\quad + 2a_1 \left[ e^{-1} f_{\mathcal{T}} \partial_\mu (e g^{\mu\beta} T_{\beta\alpha}^\alpha) \frac{\partial e}{\partial e_\sigma^a} \delta e_\sigma^a \right. \\ &\quad \left. - f_{\mathcal{T}} \delta \partial_\mu (e g^{\mu\beta} T_{\beta\alpha}^\alpha) \right] \Big\}. \end{aligned} \tag{24}$$

Now we do the integration by parts in the terms containing  $\delta(\partial_\alpha e_\sigma^a)$  and  $\delta \partial_\mu (e g^{\mu\beta} T_{\beta\nu}^\nu)$ . The first integration by parts is given by

$$\begin{aligned} & -\frac{1}{2\kappa^2} \int d^4x e f_{\mathcal{T}} \frac{\partial T}{\partial (\partial_\alpha e_\sigma^a)} \delta (\partial_\alpha e_\sigma^a) \\ &= -\frac{1}{2\kappa^2} \int d^4x \partial_\alpha \left[ e f_{\mathcal{T}} \frac{\partial T}{\partial (\partial_\alpha e_\sigma^a)} \delta e_\sigma^a \right] \\ &\quad + \frac{1}{2\kappa^2} \int d^4x \partial_\alpha \left[ e f_{\mathcal{T}} \frac{\partial T}{\partial (\partial_\alpha e_\sigma^a)} \right] \delta e_\sigma^a, \end{aligned} \tag{25}$$

where the first term is zero because it is a surface term, which we consider  $\delta e_\sigma^a|_{\text{surface}} \equiv 0$ . The second integration by parts is given by

$$\begin{aligned} & -\frac{2a_1}{2\kappa^2} \int d^4x f_{\mathcal{T}} \delta \partial_\mu (e g^{\mu\beta} T_{\beta\nu}^\nu) \\ &= -\frac{2a_1}{2\kappa^2} \int d^4x \partial_\mu \left[ f_{\mathcal{T}} \delta (e g^{\mu\beta} T_{\beta\nu}^\nu) \right] \\ &\quad + \frac{2a_1}{2\kappa^2} \int d^4x (\partial_\mu f_{\mathcal{T}}) \delta (e g^{\mu\beta} T_{\beta\nu}^\nu), \end{aligned} \tag{26}$$

with the first term is null for being a surface term. Then we have

$$\begin{aligned} & -\frac{2a_1}{2\kappa^2} \int d^4x f_{\mathcal{T}} \delta \partial_\mu (e g^{\mu\beta} T_{\beta\nu}^\nu) \\ &= \frac{2a_1}{2\kappa^2} \int d^4x (\partial_\mu f_{\mathcal{T}}) \left[ g^{\mu\beta} T_{\beta\nu}^\nu \frac{\partial e}{\partial e_\sigma^a} \delta e_\sigma^a \right. \\ &\quad \left. + e T_{\beta\nu}^\nu \delta g^{\mu\beta} + e g^{\mu\beta} \delta T_{\beta\nu}^\nu \right]. \end{aligned} \tag{27}$$

Making use of the following relationship:

$$\begin{aligned} \delta g^{\mu\beta} &= \delta (\eta^{ab} e_a^\mu e_b^\beta) \\ &= -(g^{\beta\sigma} e_a^\mu \delta e_\sigma^a + g^{\mu\sigma} e_a^\beta \delta e_\sigma^a) \end{aligned} \tag{28}$$

and replacing (25) and (27) in (24), developing the terms of  $\delta T_{\beta\nu}^\nu$  we have

$$\begin{aligned} \delta S_{\mathcal{T}} &= \frac{1}{2\kappa^2} \int d^4x \left\{ f \frac{\partial e}{\partial e_\sigma^a} \delta e_\sigma^a - e f_{\mathcal{T}} \frac{\partial T}{\partial e_\sigma^a} \delta e_\sigma^a \right. \\ &\quad + \partial_\alpha \left[ e f_{\mathcal{T}} \frac{\partial T}{\partial (\partial_\alpha e_\sigma^a)} \right] \delta e_\sigma^a \\ &\quad + 2a_1 \left[ e^{-1} f_{\mathcal{T}} \partial_\mu (e g^{\mu\beta} T_{\beta\nu}^\nu) \frac{\partial e}{\partial e_\sigma^a} \delta e_\sigma^a \right. \\ &\quad + (\partial_\mu f_{\mathcal{T}}) \left[ g^{\mu\beta} T_{\beta\nu}^\nu \frac{\partial e}{\partial e_\sigma^a} \delta e_\sigma^a \right. \\ &\quad \left. - e T_{\beta\nu}^\nu (g^{\beta\sigma} e_a^\mu \delta e_\sigma^a + g^{\mu\sigma} e_a^\beta \delta e_\sigma^a) + e g^{\mu\beta} \frac{\partial T_{\beta\nu}^\nu}{\partial e_\sigma^a} \delta e_\sigma^a \right. \\ &\quad \left. \left. + e g^{\mu\beta} \frac{\partial T_{\beta\nu}^\nu}{\partial (\partial_\alpha e_\sigma^a)} \delta (\partial_\alpha e_\sigma^a) \right] \right\}. \end{aligned} \tag{29}$$

At this point we see that we still have to do an integration by parts in the last term, i.e.,

$$\begin{aligned} & \frac{2a_1}{2\kappa^2} \int d^4x (\partial_\mu f_T) e g^{\mu\beta} \frac{\partial T_{\beta\nu}^v}{\partial(\partial_\alpha e_\sigma^a)} \delta(\partial_\alpha e_\sigma^a) \\ &= \frac{2a_1}{2\kappa^2} \int d^4x \partial_\alpha \left[ (\partial_\mu f_T) e g^{\mu\beta} \frac{\partial T_{\beta\nu}^v}{\partial(\partial_\alpha e_\sigma^a)} \delta e_\sigma^a \right] \\ & \quad - \frac{2a_1}{2\kappa^2} \int d^4x \partial_\alpha \left[ (\partial_\mu f_T) e g^{\mu\beta} \frac{\partial T_{\beta\nu}^v}{\partial(\partial_\alpha e_\sigma^a)} \right] \delta e_\sigma^a, \end{aligned} \tag{30}$$

where once again the first term vanishes due to being a surface term. Substituting this result in (29) we obtain

$$\begin{aligned} \delta S_T &= \frac{1}{2\kappa^2} \int d^4x \left\{ f \frac{\partial e}{\partial e_\sigma^a} - e f_T \frac{\partial T}{\partial e_\sigma^a} \right. \\ & \quad + \partial_\alpha \left[ e f_T \frac{\partial T}{\partial(\partial_\alpha e_\sigma^a)} \right] + 2a_1 \left[ e^{-1} f_T \partial_\mu (e g^{\mu\beta} T_{\beta\nu}^v) \frac{\partial e}{\partial e_\sigma^a} \right. \\ & \quad + (\partial_\mu f_T) \left[ g^{\mu\beta} T_{\beta\nu}^v \frac{\partial e}{\partial e_\sigma^a} - e T_{\beta\nu}^v (g^{\beta\sigma} e_a^\mu + g^{\mu\sigma} e_a^\beta) \right. \\ & \quad \left. \left. \left. + e g^{\mu\beta} \frac{\partial T_{\beta\nu}^v}{\partial e_\sigma^a} \right] - \partial_\alpha \left[ (\partial_\mu f_T) e g^{\mu\beta} \frac{\partial T_{\beta\nu}^v}{\partial(\partial_\alpha e_\sigma^a)} \right] \right] \right\} \delta e_\sigma^a. \end{aligned} \tag{31}$$

Now we must replace the results derived from  $T$ ,  $e$  and  $T_{\beta\nu}^v$  in the relation as regards the tetrads and the derivatives. Taking into account the results of  $f(T)$ , we have the following derivative:

$$\begin{aligned} \frac{\partial e}{\partial e_\sigma^a} &= e e_\sigma^a, \quad \frac{\partial T}{\partial e_\sigma^a} = -4e^\lambda T_{\nu\lambda}^\alpha S_\alpha^{\nu\sigma}, \\ \frac{\partial T}{\partial(\partial_\alpha e_\sigma^a)} &= 4e_a^\lambda S_\lambda^{\alpha\sigma}, \quad \frac{\partial T_{\beta\nu}^v}{\partial e_\sigma^a} = -e_a^\nu T_{\beta\nu}^\sigma, \end{aligned} \tag{32}$$

$$\frac{\partial T_{\beta\nu}^v}{\partial(\partial_\alpha e_\sigma^a)} = e_b^\nu \delta_a^b \left( \delta_\beta^\alpha \delta_\nu^\sigma - \delta_\nu^\alpha \delta_\beta^\sigma \right). \tag{33}$$

Substituting the above derivatives in (31), making  $\delta S_{GTT} \equiv 0$  in (21) and multiplying by  $\frac{1}{2}e^{-1}e_\omega^a$  we have the following equation of motion for the generalized teleparallel theory:

$$\begin{aligned} & \frac{1}{2} \delta_\omega^\sigma f + 2f_T T_{\nu\omega}^\beta S_\beta^{\nu\sigma} + 2e^{-1} e_\omega^a \partial_\alpha \left[ e f_T e_a^\beta S_\beta^{\alpha\sigma} \right] \\ & \quad + a_1 \left\{ e^{-1} f_T \delta_\omega^\sigma \partial_\mu (e g^{\mu\beta} T_{\beta\nu}^v) + (\partial_\mu f_T) \left[ \delta_\omega^\sigma g^{\mu\beta} T_{\beta\nu}^v \right. \right. \\ & \quad \left. \left. - \left( \delta_\omega^\mu g^{\beta\sigma} T_{\beta\nu}^v + g^{\mu\sigma} T_{\omega\nu}^v \right) - g^{\mu\beta} T_{\beta\omega}^\sigma \right] \right. \\ & \quad \left. - e^{-1} e_\omega^a \partial_\alpha \left[ e (\partial_\mu f_T) (g^{\mu\alpha} e_a^\sigma - g^{\mu\sigma} e_a^\alpha) \right] \right\} + \kappa^2 \Theta_\omega^\sigma = 0. \end{aligned} \tag{34}$$

Taking the limit in which  $a_1 \rightarrow 0$  ( $T \rightarrow -T$ ,  $f \equiv f(-T)$ ,  $f_T \rightarrow -f_T$ ), making  $T \rightarrow -T$  the equation of motion (34) does not fall exactly under the equation of motion of the  $f(T)$  gravity in (15). This happens due to the fact that the relationship between the curvature scalar and the torsion

scalar has a minus sign, which prevents a theory to be  $f(\bar{R})$  gravity, in which the coupling with matter is positive, which would fall under a theory like  $f(T)$  gravity, in which the coupling signal with the matter should be negative so that it falls under GR. In the next section we shall show the equivalence between GTT and  $f(\bar{R})$  gravity.

### 4 Equivalence between GTT and $f(\bar{R})$ gravity

Let us start this section showing the equivalence of GTT with  $f(\bar{R})$  gravity in the limit  $a_1 \rightarrow 1$ , to general tetrads.

Let us first establish some necessary identities, as arising from the condition of the metricity,

$$\begin{aligned} \bar{\nabla}_\alpha g_{\mu\nu} &= \bar{\nabla}_\alpha g^{\mu\nu} \equiv 0, \quad \partial_\alpha g_{\mu\nu} = \bar{\Gamma}_{\alpha\mu}^\lambda g_{\lambda\nu} \\ & \quad + \bar{\Gamma}_{\alpha\nu}^\lambda g_{\lambda\mu}, \quad \partial_\alpha g^{\mu\nu} = -\bar{\Gamma}_{\lambda\alpha}^\mu g^{\lambda\nu} - \bar{\Gamma}_{\lambda\alpha}^\nu g^{\lambda\mu}. \end{aligned} \tag{35}$$

With it the identity  $\partial_\alpha e = e g^{\mu\nu} \partial_\alpha g_{\mu\nu}$  becomes

$$\partial_\alpha e = 2e \bar{\Gamma}_{\alpha\nu}^\nu. \tag{36}$$

Now we can divide the equation of motion (34) in terms such as

$$T^{(1)} + T^{(2)} + T^{(3)} + \kappa^2 \Theta_\omega^\sigma = 0, \tag{37}$$

$$\begin{aligned} T^{(1)} &= \frac{1}{2} \delta_\omega^\sigma f + 2f_T T_{\nu\omega}^\beta S_\beta^{\nu\sigma} \\ & \quad + 2e^{-1} e_\omega^a \partial_\alpha \left[ e f_T e_a^\beta S_\beta^{\alpha\sigma} \right] + a_1 e^{-1} f_T \delta_\omega^\sigma \partial_\mu (e g^{\mu\beta} T_{\beta\nu}^v), \end{aligned} \tag{38}$$

$$\begin{aligned} T^{(2)} &= a_1 (\partial_\mu f_T) \left[ \delta_\omega^\sigma g^{\mu\beta} T_{\beta\nu}^v - \delta_\omega^\mu g^{\beta\sigma} T_{\beta\nu}^v \right. \\ & \quad \left. - g^{\mu\sigma} T_{\omega\nu}^v - g^{\mu\beta} T_{\beta\omega}^\sigma \right], \end{aligned} \tag{39}$$

$$T^{(3)} = -a_1 e^{-1} e_\omega^a \partial_\alpha \left[ e (\partial_\mu f_T) (g^{\mu\alpha} e_a^\sigma - g^{\mu\sigma} e_a^\alpha) \right]. \tag{40}$$

Developing the last term we have

$$\begin{aligned} T^{(3)} &= -a_1 \delta_\omega^\sigma \bar{\square} f_T + a_1 g^{\mu\sigma} \bar{\nabla}_\omega \bar{\nabla}_\mu f_T - a_1 g^{\nu\sigma} \bar{\Gamma}_{\omega\nu}^\mu \partial_\mu f_T \\ & \quad - a_1 e^{-1} e_\omega^a (\partial_\mu f_T) \left[ (g^{\mu\alpha} e_a^\sigma - g^{\mu\sigma} e_a^\alpha) \partial_\alpha e \right. \\ & \quad \left. + e (e_a^\sigma \partial_\alpha g^{\mu\alpha} + g^{\mu\alpha} \partial_\alpha e_a^\sigma - e_a^\alpha \partial_\alpha g^{\mu\sigma} - g^{\mu\sigma} \partial_\alpha e_a^\alpha) \right]. \end{aligned} \tag{41}$$

Using (2), (3), (4), (7), (35), and (36) in (39) and (41) we have the sum of terms  $T^{(2)}$  and  $T^{(3)}$  resulting in

$$T^{(2)} + T^{(3)} = -a_1 \delta_\omega^\sigma \bar{\square} f_T + a_1 g^{\mu\sigma} \bar{\nabla}_\omega \bar{\nabla}_\mu f_T. \tag{42}$$

Now we use the identity (16) in (38), then we can rewrite the equation of motion (37), using (42), as follows:

$$\begin{aligned} & -f_T G_\nu^\mu - a_1 \left[ \delta_\nu^\mu \bar{\square} - g^{\mu\alpha} \bar{\nabla}_\nu \bar{\nabla}_\alpha \right] f_T \\ & \quad + \frac{1}{2} [-T f_T + f] \delta_\nu^\mu + 2S_\nu^{\alpha\mu} \partial_\alpha f_T + \kappa^2 \Theta_\nu^\mu = 0. \end{aligned} \tag{43}$$



Considering  $\mathcal{T} \equiv \mathcal{T}(-T - a_1 B)$ , with  $B = 2\partial_\mu(eg^{\mu\beta}T_{\beta\nu}^v)$ , we see that GTT will only be equivalent to  $f(\bar{R})$  gravity in the limit  $a_1 \rightarrow 1$ , so  $\mathcal{T} \rightarrow \bar{R}$  and the term  $2S_{\nu}^{\alpha\mu}\partial_\alpha f_{\mathcal{T}}$  must be identically zero, as shown in Sect. III, subsection C, of [68]. When this term vanishes, we have exactly one theory invariant by local Lorentz transformations, which occurs only when  $a_1 \rightarrow 1$ , and thus Eq. (43) becomes identical to  $f(\bar{R})$  gravity, which is covariant and independent of the chosen set of tetrads.

In the next section we will specify a set of tetrads that explicitly show the equivalence between the two theories in the limit referred to above.

#### 4.1 Friedmann–Lemaître–Robertson–Walker case

In this section we explicitly show that the GTT equations of motion in (34) are exactly the same as  $f(\bar{R})$  gravity for the particular limit in which  $a_1 \rightarrow 1$ . We can then begin comparing the equations of motion for an easier symmetry of the metric, as the maximum symmetry for the cosmological Friedmann–Lemaître–Robertson–Walker (FLRW) flat metric,

$$dS_{FLRW}^2 = dt^2 - a^2(t) (dx^2 + dy^2 + dz^2). \tag{44}$$

Considering the case of cosmology, with line element FLRW flat (44), for a diagonal tetrad  $[e_\sigma^a] = \text{diag}[1, a(t), a(t), a(t)]$ , we see that Eq. (34) become

$$\begin{aligned} \kappa^2\Theta_0^0 &= \frac{1}{2a^2} \left\{ 6a_1a\dot{a} \left( \frac{d}{dt} f_{\mathcal{T}} \right) \right. \\ &\quad \left. + \left[ 12(1 - a_1)(\dot{a})^2 - 6a_1a\ddot{a} \right] f_{\mathcal{T}} - fa^2 \right\}, \tag{45} \end{aligned}$$

$$\begin{aligned} \kappa^2\Theta_1^1 &= \kappa^2\Theta_2^2 = \kappa^2\Theta_3^3 \\ &= -\frac{1}{2a^2} \left\{ 2a \left( a_1a \frac{d}{dt} + 2\dot{a} \right) \frac{d}{dt} f_{\mathcal{T}} + \left[ (4 - 6a_1)a\ddot{a} \right. \right. \\ &\quad \left. \left. + (8 - 12a_1)(\dot{a})^2 \right] f_{\mathcal{T}} - fa^2 \right\}, \tag{46} \end{aligned}$$

where  $\dot{a} = (d/dt)a$  and  $\ddot{a} = (d^2/dt^2)a$ .

We can now compare these equations with those obtained from  $f(\bar{R})$  gravity, whose equations of motion are [5–9]

$$\kappa^2\Theta_\nu^\mu = f_{\bar{R}}\bar{R}_\nu^\mu - \frac{1}{2}\delta_\nu^\mu f + (\delta_\nu^\mu \bar{\square} - g^{\mu\beta}\bar{\nabla}_\beta\bar{\nabla}_\nu) f_{\bar{R}}. \tag{47}$$

Considering the flat FLRW metric (44), Eq. (47) provide us with

$$\kappa^2\Theta_0^0 = \frac{1}{2a} \left[ 6\dot{a} \frac{d}{dt} f_{\bar{R}} - 6\ddot{a}f_{\bar{R}} - af \right], \tag{48}$$

$$\begin{aligned} \kappa^2\Theta_1^1 &= \kappa^2\Theta_2^2 = \kappa^2\Theta_3^3 \\ &= -\frac{1}{2a^2} \left[ \left( 2a^2 \frac{d}{dt} + 4a\dot{a} \right) \frac{d}{dt} f_{\bar{R}} \right. \\ &\quad \left. - \left( 2a\ddot{a} + 4(\dot{a})^2 \right) f_{\bar{R}} - fa^2 \right]. \tag{49} \end{aligned}$$

Subtracting (45) from (48) we have

$$\begin{aligned} 0 &= \frac{3}{a^2} \left\{ 2(1 - a_1)\dot{a}^2 f_{\mathcal{T}} + a\dot{a} \left( a_1 \frac{df_{\mathcal{T}}}{dt} - \frac{df_{\bar{R}}}{dt} \right) \right. \\ &\quad \left. + a\ddot{a}(f_{\bar{R}} - a_1 f_{\mathcal{T}}) + \frac{a^2}{6} [f(\bar{R}) - f(\mathcal{T})] \right\}. \tag{50} \end{aligned}$$

Subtracting (46) from (49) we obtain

$$\begin{aligned} 0 &= \frac{1}{a^2} \left\{ a^2 \left( a_1 \frac{d^2 f_{\mathcal{T}}}{dt^2} - \frac{d^2 f_{\bar{R}}}{dt^2} \right) + 2a\dot{a} \left( \frac{df_{\mathcal{T}}}{dt} - \frac{df_{\bar{R}}}{dt} \right) \right. \\ &\quad \left. + a\ddot{a}[(2 - 3a_1)f_{\mathcal{T}} + f_{\bar{R}}] + \dot{a}^2[(4 - 6a_1)f_{\mathcal{T}} + 2f_{\bar{R}}] \right. \\ &\quad \left. + \frac{a^2}{2} [f(\bar{R}) - f(\mathcal{T})] \right\}. \tag{51} \end{aligned}$$

Now we clearly see that in the limit  $a_1 \rightarrow 1$  we have  $\{\mathcal{T} \rightarrow \bar{R}, f(\mathcal{T}) \rightarrow f(\bar{R}), f_{\mathcal{T}} \rightarrow f_{\bar{R}}\}$ , then (50) and (51) are identically null, showing the equivalence of equations of motion between GTT and  $f(\bar{R})$  for this limit. The conclusion is that GTT is only invariant under local Lorentz transformations and at the same time invariant by general coordinates transformations in the limit at which  $a_1 \rightarrow 1$ .

#### 4.2 Spherically symmetric case

We have demonstrated in general that GTT is equivalent to  $f(R)$  gravity, but in addition to explain this through a metric with specific symmetry, we want to leave the equations of motion open for further analysis of this theory.

Let us now consider the case of a spherically symmetric and static line element,

$$dS^2 = e^{a(r)} dt^2 - e^{b(r)} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2); \tag{52}$$

we can choose the following diagonal tetrad:  $[e_\sigma^a] = \text{diag}[e^{a(r)/2}, e^{b(r)/2}, r, r \sin\theta]$ , which, taking into account (34), provides us with the following equations of motion:

$$\begin{aligned} \kappa^2\Theta_0^0 &= -\frac{e^{-b}}{4r^2} \left\{ 4a_1r^2 \frac{d^2}{dr^2} f_{\mathcal{T}} + (8r - 2a_1r^2b') \frac{d}{dr} f_{\mathcal{T}} \right. \\ &\quad \left. + \left[ (a_1r^2a' + 4(a_1 - 1)r) b' \right. \right. \\ &\quad \left. \left. + 4(a_1 - 1)e^b - a_1r^2(2a'' + (a')^2) \right. \right. \\ &\quad \left. \left. - 4(2a_1 - 1)ra' - 8(a_1 - 1) \right] f_{\mathcal{T}} + 2f r^2 e^b \right\}, \tag{53} \end{aligned}$$

$$\begin{aligned} \kappa^2 \Theta_1^1 = & \frac{e^{-b}}{4r^2} \left\{ a_1 \left( 2r^2 a' + 8r \right) \frac{d}{dr} f_T \right. \\ & + \left[ a_1 r (ra' + 4)b' + 4(a_1 - 1)e^b \right. \\ & - a_1 \left( 2r^2 a'' + r^2 (a')^2 \right) + 8(1 - a_1)ra' \\ & \left. \left. + 8(1 - a_1) \right] f_T + 2 f r^2 e^b \right\}, \end{aligned} \tag{54}$$

$$\kappa^2 \Theta_2^2 = \frac{(a_1 - 1) \cos \theta \frac{d}{dr} f_T}{r^2 \sin \theta} = 0, \tag{55}$$

$$\begin{aligned} \kappa^2 \Theta_2^2 = \kappa^2 \Theta_3^3 = & \frac{e^{-b}}{4r^2} \left\{ 4a_1 r^2 \frac{d^2}{dr^2} f_T \right. \\ & - 2r (a_1 r b' - ra' - 2) \frac{d}{dr} f_T \\ & + \left[ \left( (a_1 - 1)r^2 a' + (4a_1 - 2)r \right) b' + 4a_1 e^b \right. \\ & + 2(1 - a_1)r^2 a'' + (1 - a_1)r^2 (a')^2 + (6 - 8a_1)ra' \\ & \left. \left. - 4(2a_1 - 1) \right] f_T + 2 f r^2 e^b \right\}, \end{aligned} \tag{56}$$

where ' denotes a derivation in relation to radial coordinate  $r$ . Taking the metric (52) to the equations of  $f(\bar{R})$  gravity in (47), we obtain

$$\begin{aligned} \kappa^2 \Theta_0^0 = & -\frac{e^{-b}}{4r} \left\{ 4r \frac{d^2}{dr^2} f_{\bar{R}} + (8 - 2rb') \frac{d}{dr} f_{\bar{R}} \right. \\ & \left. + \left[ r(a'b' - 2a'' - (a')^2) - 4a' \right] f_{\bar{R}} + 2f r e^b \right\}, \end{aligned} \tag{57}$$

$$\begin{aligned} \kappa^2 \Theta_1^1 = & \frac{e^{-b}}{4r} \left\{ (2ra' + 8) \frac{d}{dr} f_{\bar{R}} \right. \\ & \left. + \left[ (ra' + 4)b' - 2ra'' - r(a')^2 \right] f_{\bar{R}} + 2f r e^b \right\}, \end{aligned} \tag{58}$$

$$\begin{aligned} \kappa^2 \Theta_2^2 = \kappa^2 \Theta_3^3 = & \frac{e^{-b}}{2r^2} \left\{ 2r^2 \frac{d^2}{dr^2} f_{\bar{R}} - r(rb' - ra' - 2) \frac{d}{dr} f_{\bar{R}} \right. \\ & \left. + (rb' + 2e^b - ra' - 2) f_{\bar{R}} + f r^2 e^b \right\}. \end{aligned} \tag{59}$$

Here first we notice that if  $a_1 \neq 1$ , there exists an equation, Eq. (55), outside the diagonal for GTT, resulting in the restriction of the functional form  $f(T) = c_1 T + c_0$ , where  $c_0, c_1 \in \mathfrak{R}$ . Then we have the same constraint to  $f(T)$  gravity in this case [67].

We also see that in the limit  $a_1 \rightarrow 1$ ,  $\{T \rightarrow \bar{R}, f(T) \rightarrow f(\bar{R}), f_T \rightarrow f_{\bar{R}}\}$ , all equations (53)–(56) for GTT are identical to  $f(\bar{R})$  given in (57)–(59).

Now choose a set of non-diagonal tetrads,

$$\{e^a_\mu\} = \begin{bmatrix} e^{a/2} & 0 & 0 & 0 \\ 0 & e^{b/2} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ 0 & e^{b/2} \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ 0 & e^{b/2} \cos \theta & -r \sin \theta & 0 \end{bmatrix}, \tag{60}$$

and the equations to GTT in (34) provide us with

$$\begin{aligned} \kappa^2 \Theta_0^0 = & -\frac{e^{-b}}{4r^2} \left\{ 4a_1 r^2 \frac{d^2}{dr^2} f_T - \left( 2a_1 r^2 b' - 8(a_1 - 1) r e^{b/2} \right. \right. \\ & - 8r \left. \right) \frac{d}{dr} f_T + \left[ (a_1 r^2 a' + 4(a_1 - 1)r) b' \right. \\ & + (4(a_1 - 1)ra' + 8(a_1 - 1)) e^{b/2} - a_1 r^2 (2a'' + (a')^2) \\ & \left. \left. - (8a_1 - 4)ra' - 8(a_1 - 1) \right] f_T + 2 f r^2 e^b \right\}, \end{aligned} \tag{61}$$

$$\begin{aligned} \kappa^2 \Theta_1^1 = & \frac{e^{-3b/2}}{4r^2} \left\{ 2a_1 r (ra' + 4) e^{b/2} \frac{d}{dr} f_T \right. \\ & + \left[ a_1 r (a'r + 4) e^{b/2} b' + [4(a_1 - 1)ra' + 8(a_1 - 1)] e^b \right. \\ & - (a_1 r^2 (2a'' + (a')^2) - 8(1 - a_1)ra' \\ & \left. \left. - 8(1 - a_1)) e^{b/2} \right] f_T + 2 f r^2 e^{3b/2} \right\}, \end{aligned} \tag{62}$$

$$\begin{aligned} \kappa^2 \Theta_2^2 = \kappa^2 \Theta_3^3 = & \frac{e^{-b}}{4r^2} \left\{ 4a_1 r^2 \frac{d^2}{dr^2} f_T \right. \\ & - \left( 2a_1 b' - 4(a_1 - 1) r e^{b/2} - 2r^2 a' - 4r \right) \frac{d}{dr} f_T \\ & + \left[ \left( (a_1 - 1)r^2 a' + (4a_1 - 2)r \right) b' + 4e^b \right. \\ & + (4(a_1 - 1)ra' + 8(a_1 - 1)) e^{b/2} \\ & - 2(a_1 - 1)r^2 a'' - (a_1 - 1)r^2 (a')^2 + (6 - 8a_1)ra' \\ & \left. \left. - (8a_1 - 4) \right] f_T + 2 f r^2 e^b \right\}. \end{aligned} \tag{63}$$

We can then see that in this case the equations of motion are diagonal. But equivalence of GTT with  $f(\bar{R})$  gravity is only given in the limit  $a_1 \rightarrow 1$ , when Eqs. (57)–(59) and (61)–(63) are identical.

### 5 Equivalence between GTT and a particular case of the $f(T, B)$ gravity

In this section we make an important observation. When we were finishing the calculation of the non-diagonal tetrads case of the previous subsection, we noted that a collaboration have submitted exactly the same idea as our work here. The so-call  $f(T, B)$  gravity [68], with  $B = -2\bar{\nabla}^\mu T_{\mu\nu}^\nu$ , is a more general theory than presented here, where the algebraic function is

contained in the action; it may be any analytic function of the variables  $T$  and  $B$ . We noted then that the equivalence of this theory with the  $f(\bar{R})$  gravity is given only for the specific functional form  $f(T, B) \equiv f(-T + B) = f(\bar{R})$ . Compared to our theory, we see that GTT is a particular case of  $f(T, B)$  gravity, when  $f(T, B) \equiv f(-T + a_1 B) = f(T)$ . We can show this again explicitly using the equations of motion.

The equation of motion for  $f(T, B)$  gravity is given by

$$2\delta^\lambda_\nu \bar{\square} f_B - 2\bar{\nabla}^\lambda \bar{\nabla}_\nu f_B + B f_B \delta^\lambda_\nu + 4\partial_\mu (f_B + f_T) S_\nu^{\mu\lambda} + 4e^{-1} e^\alpha_\nu \partial_\mu (e e^\beta_\alpha S_\beta^{\mu\lambda}) f_T - 4f_T T^\sigma_{\mu\nu} S^\lambda_{\sigma} - f \delta^\lambda_\nu = 2\kappa^2 \Theta_\nu^\lambda. \tag{64}$$

The first observation here is that this theory does not fall under  $f(T)$  gravity in general, as well as our GTT, as mentioned at the end of Sect. 3. Taking  $f(T, B) \equiv f(T)$ , thus  $f_B = 0$ , the equation of motion (64), using the identity (16), becomes

$$4(\partial_\mu f_T) S_\nu^{\mu\lambda} - 2f_T G_\nu^\lambda + \delta^\lambda_\nu (T f_T - f) = 2\kappa^2 \Theta_\nu^\lambda. \tag{65}$$

This equation is not equal to (17) for  $f(T)$  gravity, and cannot fall under GR when  $f(T) \equiv T - 2\Lambda$ , due to the sign. This shows that the  $f(T, B)$  gravity also does not return to  $f(T)$  gravity in general.

Now we can show that in the particular case  $f(-T + a_1 B)$  this theory falls under our GTT. We take the FLRW metric (44) with diagonal tetrads  $[e^a_\mu] = \text{diag}[1, a, a, a]$ , and the equations of motion (64) provide us with

$$\kappa^2 \Theta_0^0 = -\frac{1}{2a^2} \left\{ 12(\dot{a})^2 f_T - 6a\dot{a} \frac{d}{dt} f_B + 2 \left[ 3a\ddot{a} + 6(\dot{a})^2 \right] f_B + f a^2 \right\}, \tag{66}$$

$$\kappa^2 \Theta_1^1 = \kappa^2 \Theta_2^2 = \kappa^2 \Theta_3^3 = \frac{1}{2a^2} \left\{ 4a(\dot{a}) \left( \frac{d}{dt} f_T \right) + \left[ 4a\ddot{a} + 8(\dot{a})^2 \right] f_T - 2a^2 \left( \frac{d^2}{dt^2} f_B \right) + \left[ 6a\ddot{a} + 12(\dot{a})^2 \right] f_B + f a^2 \right\}. \tag{67}$$

Now identifying  $f(-T + a_1 B) = f(T)$ , recalling that  $T$  is given in (19),

$$f_T = \frac{\partial f}{\partial T} = \frac{\partial T}{\partial T} \frac{df}{dT} = -\frac{df}{dT},$$

$$f_B = \frac{\partial f}{\partial B} = \frac{\partial T}{\partial B} \frac{df}{dT} = a_1 \frac{df}{dT}. \tag{68}$$

We see that Eqs. (66) and (67) are identical with GTT (45) and (46), thus showing the equivalence between the theories.

We can also confirm this by choosing the spherical symmetry for the metric (52), first for diagonal tetrads  $[e^a_\mu] = \text{diag}[e^{a/2}, e^{b/2}, r, r \sin \theta]$ , thus, Eq. (64) provide us with

$$\kappa^2 \Theta_0^0 = \frac{e^{-b}}{4r^2} \left\{ 8r \frac{d}{dr} f_T - 2 \left[ 2rb' + 2e^b - 2ra' - 4 \right] f_T - 2r^2 \left[ 2 \frac{d}{dr} - b' \right] \frac{d}{dr} f_B + \left[ -r(ra' + 4)b' - 4e^b + r \left( 2ra'' + r(a')^2 + 8a' \right) + 8 \right] f_B - 2fr^2 e^b \right\}, \tag{69}$$

$$\kappa^2 \Theta_1^1 = \frac{e^{-b}}{4r^2} \left\{ 2 \left[ 2e^b - 4ra' - 4 \right] f_T + 2r \left[ ra' + 4 \right] \frac{d}{dr} f_B + \left[ r \left( ra' + 4 \right) b' + 4e^b - r \left( 2ra'' - r(a')^2 - 8a' \right) - 8 \right] f_B + 2r^2 f e^b \right\}, \tag{70}$$

$$\kappa^2 \Theta_2^2 = -\frac{1}{r^2 \sin \theta} \left[ \cos \theta \frac{d}{dr} f_T + \cos \theta \frac{d}{dr} f_B \right] = 0, \tag{71}$$

$$\kappa^2 \Theta_2^2 = \kappa^2 \Theta_3^3 = \frac{e^{-b}}{4r^2} \left\{ 2r \left( ra' + 2 \right) \frac{d}{dr} f_T + \left[ -r \left( ra' + 2 \right) b' + r^2 \left( 2a'' + (a')^2 \right) + 6ra' + 4 \right] f_T - 2r^2 \left[ 2 \frac{d}{dr} - b' \right] \frac{d}{dr} f_B + \left[ -r \left( ra' - 4 \right) b' - 4e^b + r^2 \left( 2a'' + (a')^2 \right) + 8ra' + 8 \right] f_B - 2fr^2 e^b \right\}. \tag{72}$$

Again we have the equivalence of the equations of motion (69)–(72) with (53)–(56), for the identifications  $f(T, B) = f(T)$  and (68).

By taking the choice of non-diagonal tetrads (60), the equations of motion from  $f(T, B)$  gravity (64) provide us with

$$\kappa^2 \Theta_0^0 = -\frac{e^{-5b/2}}{4r^2} \left\{ 8r \left[ e^{2b} - e^{3b/2} \right] \frac{d}{dr} f_T + \left[ 4r e^{3b/2} b' + 2(2ra' + 4)e^{2b} - 4(ra' + 2)e^{3b/2} \right] f_T + \left[ 4r^2 e^{3b/2} \frac{d}{dr} + \left( 8r e^{2b} - 2r^2 e^{3b/2} \right) \right] \frac{d}{dr} f_B + \left[ r \left( ra' + 4 \right) e^{3b/2} b' + 4 \left( ra' + 2 \right) e^{2b} - \left( 2r^2 a'' + r(a')^2 + 8ra' + 8 \right) e^{3b/2} \right] f_B + 2fr^2 e^{5b/2} \right\}, \tag{73}$$

$$\kappa^2 \Theta_1^1 = \frac{e^{-5b/2}}{4r^2} \left\{ \left[ 4(ra' + 2)e^{2b} - 8(ra' + 1)e^{3b/2} \right] f_T + 2r \left( ra' + 4 \right) e^{3b/2} \frac{d}{dr} f_B + \left[ r \left( ra' + 4 \right) e^{3b/2} b' + 4 \left( ra' + 2 \right) e^{2b} - \left( 2r^2 a'' + r^2 (a')^2 + 8ra' + 8 \right) e^{3b/2} \right] f_B + 2fr^2 e^{5b/2} \right\}, \tag{74}$$



$$\begin{aligned} \kappa^2 \Theta_2^2 = \kappa^2 \Theta_3^3 = \frac{e^{-5b/2}}{4r^2} & \left\{ \left[ 4re^{2b} - 2r(ra' + 2)e^{3b/2} \right] \frac{d}{dr} f_T \right. \\ & + \left[ r(ra' + 2)e^{3b/2}b' - 4e^{5b/2} + 4(ra' + 2)e^{2b} \right. \\ & \left. - (2r^2a'' + r^2(a')^2 + 6ra' + 4)e^{3b/2} \right] f_T \\ & + \left[ 4r^2e^{3b/2} \frac{d}{dr} + 4re^{2b} - 2r^2e^{3b/2}b' \right] \frac{d}{dr} f_B \\ & + \left[ r(ra' + 4)e^{3b/2}b' + 4(ra' + 2)e^{2b} \right. \\ & \left. - (2r^2a'' + r^2(a')^2 + 8ra' + 8)e^{3b/2} \right] f_B \\ & \left. + 2fr^2e^{5b/2} \right\}. \end{aligned} \tag{75}$$

Just as before, making the identifications  $f(T, B) = f(T)$  and (68), Eqs. (73)–(75) are identical to GTT in (61)–(63), confirming again the equivalence of these theories.

### 6 Applications to GTT

#### 6.1 Reconstruction for de Sitter Universe

A method to obtain the functional form of the algebraic function  $f(T)$  is the so-called reconstruction. This method consists of specifying a model that fixes the matter content of the theory in terms of the scalar  $\mathcal{T}$ , allowing one to reconstruct the functional form of  $f(T)$  through the equations of motion of the theory.

We will choose the particular case of a flat FLRW metric in which  $a(t) = a_0 \exp[H_0(t - t_0)]$ ,  $a_0, H_0, t_0 \in \mathfrak{R}_+$ ; it provides us with the model of de Sitter universe, where  $H(t) = \dot{a}/a = H_0$ . In this case, using (19), we have  $H_0(T) = \sqrt{\mathcal{T}/[6(1 - 3a_1)]}$ ,  $\dot{H} \equiv 0$  and  $(d/dt)f_T = f_{TT}(d/dt)\mathcal{T} \equiv 0$ . Knowing that  $\kappa^2 \Theta_0^0 = \kappa^2 \rho = 3H_0^2$ , Eq. (45) provides us with

$$3[H_0(T)]^2 = 3[H_0(T)]^2(2 - 3a_1)f_T(T) - \frac{1}{2}f(T), \tag{76}$$

and integrating with respect to that  $\mathcal{T}$  results in

$$f(T) = \mathcal{T} + [(2 - 3a_1)\mathcal{T}]^{(1-3a_1)/(2-3a_1)} c_1, \quad c_1 \in \mathfrak{R}. \tag{77}$$

#### 6.2 Spherially symmetric type-de Sitter solution

We here take the limit  $a_1 \rightarrow 0$  in (34), which after using the identity (16) and considering  $\mathcal{T} \rightarrow -T$  results in

$$f_T(-T)G_\omega^\sigma + \frac{1}{2}\delta_\omega^\sigma [f(-T) - Tf_T(-T)] = -\kappa^2 \Theta_\omega^\sigma. \tag{78}$$

As in  $f(\bar{R})$  gravity [69], we can consider the very specific case where  $\bar{R} \equiv \bar{R}_0 = -T_0 + B_0$ ,  $\mathcal{T} \equiv T_0 = -T_0 + a_1 B_0 = -T_0$ , with  $\bar{R}_0, T_0, B_0 \in \mathfrak{R}$  and  $B_0$  is defined by (19). In the case of a perfect fluid  $\Theta_v^\lambda = \text{diag}[\rho_0, -p_0, -p_0, -p_0]$ , and  $\partial_\mu f_T = f_{TT} \partial_\mu T_0 \equiv 0$ , which results in the equations

$$\bar{R}_\omega^\sigma = -\frac{\kappa^2}{f_{T_0}(-T_0)} \Theta_\omega^\sigma + \frac{1}{2} \delta_\omega^\sigma \left[ \bar{R}_0 + T_0 - \frac{f(-T_0)}{f_{T_0}(-T_0)} \right], \tag{79}$$

which taking the trace results in

$$B_0 = \frac{\kappa^2}{f_{T_0}(-T_0)} (\rho_0 - 3p_0) + 2 \frac{f(-T_0)}{f_{T_0}(-T_0)} - T_0. \tag{80}$$

Considering now the line element (52), for  $b(r) = -a(r)$  and  $p_0 = -\rho_0$  (type dark energy), we can integrate the equations of motion (79), where we get the following solution:

$$\begin{aligned} a(r) = -b(r) = \log & \left[ 1 + \frac{c_1}{2f_{T_0}(-T_0)r} \right. \\ & \left. + \frac{f(-T_0) - f_{T_0}(-T_0)T_0 + 2\kappa^2 \rho_0 r^2}{6f_{T_0}(-T_0)} \right]. \end{aligned} \tag{81}$$

This is a static de Sitter type solution where we can identify the effective cosmological constant  $(-\Lambda_{\text{eff}}/3) = [f(-T_0) - f_{T_0}(-T_0)T_0 + 2\kappa^2 \rho_0]/[6f_{T_0}(-T_0)]$ . A de Sitter type solution was also previously obtained in  $f(\bar{R})$  gravity for  $a(r) = -b(r)$  and  $\bar{R} = \bar{R}_0$  [69]. We emphasize here that this solution boils down to a different theory from  $f(T)$  gravity, because GTT does not fall under  $f(T)$  gravity for  $a_1 \rightarrow 0$ , except for the special case where  $f(T)$  is an odd analytic function, that is,  $f(-T) = -f(T)$ .

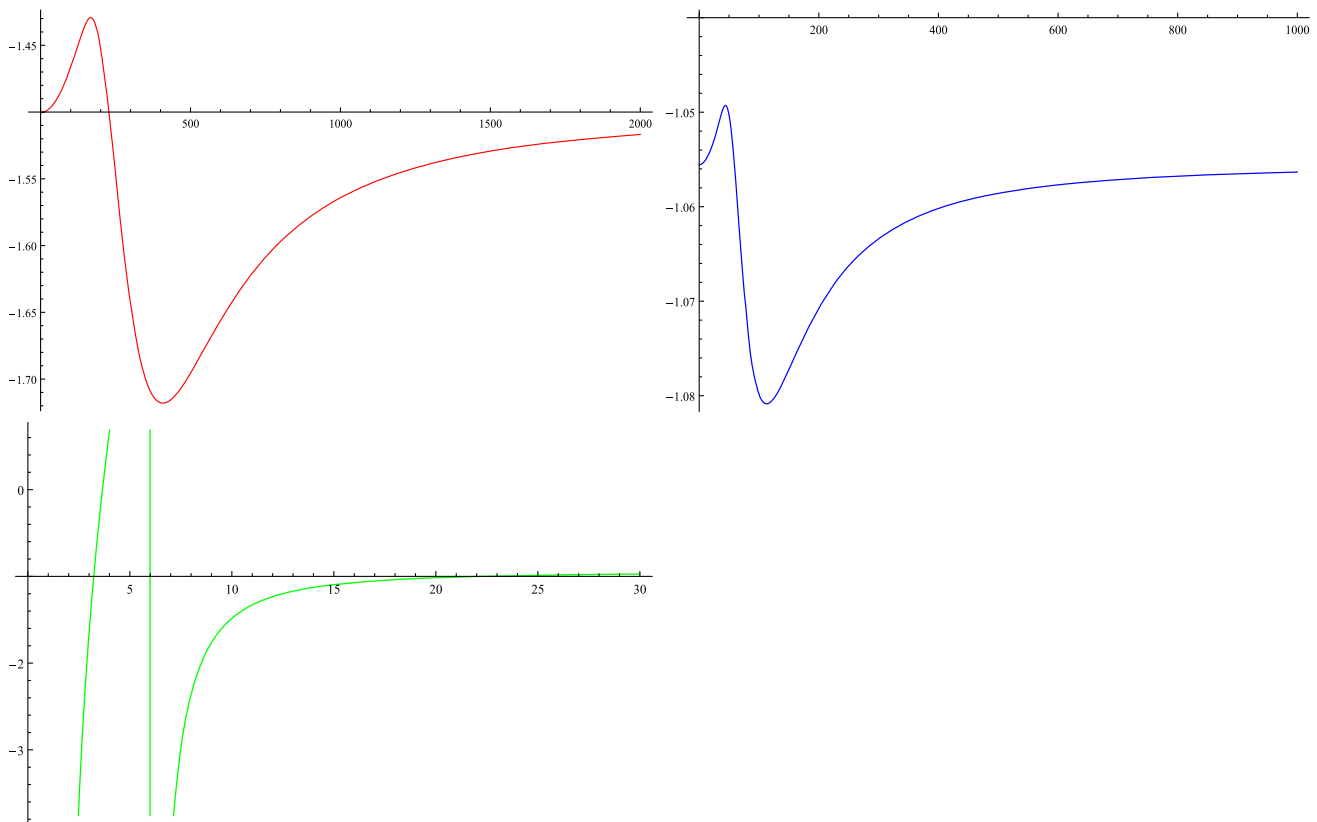
#### 6.3 Evolution for the state parameter of the dark energy

A good test for our theory is the evolution of a model of the universe. This can discard or keep a theory depending on whether it is in agreement with the observational data.

Let us follow the procedure found in [70] to determine the state parameter  $\omega_{\text{DE}}$ . For a universe permeated by a perfect fluid, of which the equation of state is governed by  $p = \omega\rho$ , we can rewrite the equations of motion (45) and (46) as

$$\begin{aligned} 3H^2 &= \kappa^2 G_{\text{eff}} (\rho_m + \rho_{\text{DE}}), \\ H &= \frac{\dot{a}}{a}, \\ G_{\text{eff}} &= \frac{1}{(3a_1 - 2)f_T}, \\ \rho_{\text{DE}} &= \frac{1}{\kappa^2} \left[ 3a_1 (\dot{H} f_T - H \dot{f}_T) - \frac{1}{2} f \right], \end{aligned} \tag{82}$$

$$\begin{aligned} \dot{H} &= -\frac{\kappa^2}{2} G_{\text{eff}} (p_m + p_{\text{DE}} + \rho_m + \rho_{\text{DE}}), \\ p_{\text{DE}} &= \frac{1}{\kappa^2} \left[ a_1 \ddot{f}_T + H \dot{f}_T - \frac{1}{2} f \right]. \end{aligned} \tag{83}$$



**Fig. 1** Representation of the temporal evolution of  $\omega_{DE}(t)$

Now we can define the state parameter of dark energy by

$$\omega_{DE} = \frac{p_{DE}}{\rho_{DE}} = \frac{a_1 \ddot{f}_{\mathcal{T}} + H \dot{f}_{\mathcal{T}} - \frac{1}{2} f}{3a_1 (\dot{H} f_{\mathcal{T}} - H \dot{f}_{\mathcal{T}}) - \frac{1}{2} f}. \tag{84}$$

We now assume an exponential model [70] defined by

$$f(\mathcal{T}) = \mathcal{T} - \beta \mathcal{T}_s \left( 1 - \exp \left[ \frac{\mathcal{T}}{\mathcal{T}_s} \right] \right). \tag{85}$$

We will now test for a solution of power law type,

$$a(t) = t^\alpha, \quad H(t) = \frac{\alpha}{t}. \tag{86}$$

We can show that (86) is a solution of the equations of motion (82) and (83) if the matter part is given by the expressions

$$\begin{aligned} \rho_{mt} = \frac{1}{2t^4 \mathcal{T}_s \kappa^2} \left\{ e^{\frac{6\alpha[a_1 + \alpha(1-2a_1)]}{\mathcal{T}_s t^2}} \left[ \mathcal{T}_s^2 t^4 + 6\alpha \mathcal{T}_s t^2 (a_1 \right. \right. \\ \left. \left. + \alpha(3a_1 - 2)) + 72a_1 \alpha^2 (a_1(2\alpha - 1) - \alpha) \right] \beta \right. \\ \left. + \mathcal{T}_s t^2 \left[ -6\alpha^2 + 6\alpha a_1(2 + \alpha) - \beta \mathcal{T}_s t^2 \right] \right\}, \tag{87} \end{aligned}$$

$$\begin{aligned} p_{mt} = \frac{1}{2\mathcal{T}_s^2 \kappa^2 t^6} \left\{ e^{\frac{6\alpha[a_1 + \alpha(1-2a_1)]}{\mathcal{T}_s t^2}} \left[ -\mathcal{T}_s^3 t^6 \right. \right. \\ \left. \left. + 2\mathcal{T}_s^2 \alpha t^4 (-4 + 3a_1(2 - 3\alpha) + 6\alpha) \right. \right. \\ \left. \left. + 288a_1 \alpha^2 (a_1 + \alpha(1 - 2a_1))^2 \right] \right\} \end{aligned}$$

$$\begin{aligned} - 24\mathcal{T}_s t^2 \alpha(3a_1 - \alpha)(-\alpha + a_1(2\alpha - 1)) \beta \\ + \mathcal{T}_s^2 t^4 \left[ 2\alpha(-4 - 3a_1(\alpha - 1) + 3\alpha) + \beta \mathcal{T}_s t^2 \right] \Big\}. \tag{88} \end{aligned}$$

Figure 1 is the temporal evolution of the state parameter  $\omega_{DE}$  of the dark energy. The red curve is obtained with a constant given by  $\{\alpha = 2, \beta = 1, \mathcal{T}_s = H_0 \Omega_m^{(0)} / \beta, H_0 = 0.75, \Omega_m^{(0)} = 0.23, a_1 = 1000\}$ , where we can see that the fluid is always phantom  $\omega_{DE} < -1$ . The blue curve is obtained with a constant given by  $\{\alpha = 20, \beta = 1, \mathcal{T}_s = H_0 \Omega_m^{(0)} / \beta, H_0 = 0.75, \Omega_m^{(0)} = 0.23, a_1 = 1\}$ , where we see that the fluid is always phantom  $\omega_{DE} < -1$ , but it fluctuates approximately between the values  $-1.05$  and  $-1.08$ . The most interesting case is the green curve obtained for the constants  $\{\alpha = 2, \beta = 1, \mathcal{T}_s = H_0 \Omega_m^{(0)} / \beta, H_0 = 0.75, \Omega_m^{(0)} = 0.23, a_1 = 0.1\}$ . In this case we see that the fluid begins in a rather phantom phase, going through another quintessence type phase, heading toward behavior of baryonic matter ( $\omega > 0$ ), and finally returning to the phantom phase. The result is that the current accelerated expansion of the universe and the crossing of the phantom divide from the phantom phase to the non-phantom (quintessence) one can be realized as well as in [70].

### 6.4 Thermodynamics for an apparent horizon

A further application is for thermodynamics of the apparent horizon in cosmology with the FLRW metric. We can follow the formulation given in [71].

We can establish a similar equation of continuity, deriving over time (82) and using (83)

$$\begin{aligned} \dot{\rho}_m + \dot{\rho}_{DE} + 3H(\rho_m + \rho_{DE} + p_m + p_{DE}) \\ = 3H^2 \frac{d}{dt} \left( \frac{1}{G_{\text{eff}}} \right). \end{aligned} \tag{89}$$

Whereas the baryonic matter is conserved ( $\dot{\rho}_m + 3H(\rho_m + p_m) \equiv 0$ ), we see that dark energy is not conserved, yielding the interpretation that it is a system out of equilibrium with entropy production (non-equilibrium thermodynamics). Following exactly the same steps as in [71], we can establish the first law of thermodynamics,

$$\begin{aligned} T_A dS_A + T_A dS_p &= -dE_{MS} + W dV, \\ T_A dS_p &= -\frac{1}{2} \hat{r}_A (1 + 2\pi \hat{r}_A T_A) d \left( \frac{1}{G_{\text{eff}}} \right), \end{aligned} \tag{90}$$

where  $T_A$  the temperature of the apparent horizon,  $dS_A$  is the entropy of the apparent horizon,  $dS_p$  is the produced entropy,  $dE_{MS}$  is the Misner–Sharp energy,  $W$  the work, and  $dV$  the volume element of the apparent horizon. Here it is clearly seen that the first law of thermodynamics is consistent for entropy production associated with an effective Newton constant  $G_{\text{eff}}$ , given in (82), for which for the linear case of  $f(T)$  the entropy production vanishes and the system goes back to equilibrium.

If we take the same model as the previous section, i.e. (85) and (86), we can explicitly show the time dependence of the effective Newton constant in (82),

$$G_{\text{eff}} = (3a_1 - 2)^{-1} \left\{ 1 + \beta e^{\frac{6\alpha[a_1 + \alpha(1-2a_1)]}{T_s t^2}} \right\}^{-1}. \tag{91}$$

Here two important observations are in order. The first is that it gets an explicit dependence of the first law of thermodynamics corresponding to the specific choice of the value on  $a_1$  in (91). The second is that by taking the particular value  $\beta \equiv 0$  in (91), clearly we have  $G_{\text{eff}} = (3a_1 - 2)^{-1}$ , which again shows the dependence of the theory in relation to the specific value of  $a_1$ , and from (91), (85) and (90) we return to the linear theory, where there is no entropy production.

## 7 Conclusion

We construct a theory that describes the gravitational interaction through effects of torsion of space-time. This theory generalizes the teleparallel theory, keeping the invariance by

both local Lorentz transformations as general coordinates transformations for a particular case.

The action of our theory is described by a general algebraic function that depends on a tensorial scalar  $\mathcal{T}$ , which is classified by a real parameter  $a_1$ . Our theory falls exactly under  $f(\tilde{R})$  gravity when we take the limit  $a_1 \rightarrow 1$ . This is shown from the equations of motion of the two theories.

We show explicitly through the equations of motion of our theory that it is also equivalent to the recently found  $f(T, B)$  gravity, when  $f(T, B) = f(-T + a_1 B)$ .

We make two small applications of our theory, reconstructing the action for the particular case of a de Sitter universe for the flat FLRW metric, with a set of diagonal tetrads, and for obtaining a static de Sitter type solution. We also analyze the evolution of the state parameter of the dark energy and the first law of thermodynamics for the apparent horizon.

Our theory is a good scenario for an attempt to explain the accelerated expansion of our universe, by modifying the usual teleparallel gravitation, or by analogy to Einstein gravity. The real parameter  $a_1$  which classifies which theory GTT describes is crucial to any consideration of cosmological phenomena. We also expect new solutions of black holes to arise through our theory, in which one may also suggest to shed some light on the so-called dark matter explanation of local effects of gravitation.

Another perspective is to show the stability of the three solutions discussed here. This should be a topic for future work.

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