

# Embeddings of the “New Massive Gravity”

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**Abstract** Here we apply different types of embeddings of the equations of motion of the linearized “New Massive Gravity” in order to generate alternative and even higher-order (in derivatives) massive gravity theories in  $D = 2 + 1$ . In the first part of the work we use the Weyl symmetry as a guiding principle for the embeddings. First we show that a Noether gauge embedding of the Weyl symmetry leads to a sixth-order model in derivatives with either a massive or a massless ghost, according to the chosen overall sign of the theory. On the other hand, if the Weyl symmetry is implemented by means of a Stueckelberg field we obtain a new scalar-tensor model for massive gravitons. It is ghost-free and Weyl invariant at the linearized level around Minkowski space. The model can be nonlinearly completed into a scalar field coupled to the NMG theory. The elimination of the scalar field leads to a nonlocal modification of the NMG. In the second part of the work we prove to all orders in derivatives that there is no local, ghost-free embedding of the linearized NMG equations of motion around Minkowski space when written in terms of one symmetric tensor. Regarding that point, NMG differs from the Fierz–Pauli theory, since in the latter case we can replace the Einstein–Hilbert action by specific  $f(R, \square R)$  generalizations and still keep the theory ghost-free at the linearized level.

## 1 Introduction

Massive spin-2 particles can be covariantly described by means of a symmetric rank-2 tensor. Although this is not the only possible tensor structure, it is very convenient. It is closely connected with a geometrical point of view (fluctuation about some metric) and it is a minimal description in the sense that we need just one auxiliary field, i.e., the trace of the tensor which vanishes on shell. If we further

require a second-order theory, in derivatives, we end up with a unique answer: the Fierz–Pauli (FP) theory [1]. Almost all developments in massive gravity, from earlier works [2–5] until recent developments, see [6,7] for review articles, are built up on top of the free FP theory. It is remarkable that the absence of ghosts [4] and of mass discontinuity [2,3] have been both achieved in recent theories with one [8,9] and two [10] dynamic metrics. A good question concerns the uniqueness of those massive gravities; see for instance [11–14]. Moreover, which features of those theories are model independent? Can we still dream of a renormalizable massive gravity? One way of addressing those questions is to abandon the FP paradigm as a starting point and allow for higher-derivative kinetic terms. In fact, the reader can find higher-derivative massive gravities and discussions regarding their physical consequences in [15–19].

At the linearized level, those higher-derivative models are specific generalizations of the FP theory which, however, have the same spectrum: massive spin-2 particles without ghosts. This is remarkable, since usually higher-derivatives introduce ghosts. The above result requires of course a systematic investigation of all possible higher-derivative generalizations of the FP model. In [20] we have addressed this problem.

One way of producing higher-derivative models dual to some “lower-derivative” theory is by means of the embedding of its Euler tensor (equations of motion). The method consists of adding to the action of some lower-derivative starting theory, quadratic terms in its equations of motion with coefficients which are functions of  $\square = \partial_\mu \partial^\mu$ . This guarantees that the equations of motion of the original theory also minimize the new action. We say that the equations of motion of the starting theory are embedded into the new theory. The particle content of the lower-order theory is inside the new theory. Then the coefficients are fixed such that the new theory contains exactly the same particle content of the lower-order one without extra propagating modes. Such method produces alternative dual models of higher order which by

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themselves are not very useful as free models; however, in some cases those free models correspond to quadratic truncations of nonlinear models which are not equivalent in general to nonlinear versions of the starting lower-order theory. In other words, the embedding procedure and the addition of nonlinear terms do not commute. As a successful application of this method we can mention the “New Massive Gravity” (NMG) theory of [21] which is of fourth-order in derivatives and can be obtained from the usual FP theory (of second order) by the addition of quadratic terms in the FP equations of motion with specific constant coefficients as explained in [22]. Another example is the higher-derivative topologically massive gravity of [22, 23] which is obtained from the topologically massive gravity (TMG) of [24]. The TMG itself, of third order, can be obtained at the linearized level from consecutive embeddings of the first-order spin-2 self-dual model of [25].

In [20] we have investigated, to all orders in derivatives, all possible embeddings of the equations of motion of the FP theory which are ghost-free at the linearized level. We have found a system of equations allowing for several solutions for the coefficients. Those theories are in general of higher order in derivatives but still ghost-free at the linearized approximation. They can all be nonlinearly completed with the help of a fiducial metric. Most of them are  $f(R, \square R)$  modifications of the Einstein–Hilbert kinetic term plus the FP mass term. Although a complete analysis has not yet been carried out, at least a subset of such theories can be modified with an appropriate non-derivative potential of the type suggested in [8, 9] and become, apparently, ghost-free beyond the linearized approximation, as shown in [17].

It turns out that all higher-order models obtained in [20] in arbitrary  $D$ -dimensions correspond to modifications of the FP theory in the spin-0 sector of the propagator. Any modification in the spin-2 sector leads to ghosts. There is however, one exception in  $D = 2 + 1$ . Namely, the “New Massive Gravity” (NMG) theory<sup>1</sup> of [21]. Although of fourth order in derivatives, the NMG is ghost-free, though it is apparently still not renormalizable [27]. Here we take the NMG model as our “lower-order” starting point and try different types of embeddings in order to produce alternative massive gravity models in  $D = 2 + 1$ . In this sense the present work complements the work [20] by taking care of the special case of  $D = 2 + 1$ .

Sometimes the embedding of equations of motion leads to new theories with gauge symmetries not present in the lower-order starting theory. In those cases, as in [22], one may call it a Noether gauge embedding (NGE). We have found useful

to split the embeddings here into two categories. Namely, the ones based on requiring Weyl symmetry of the new model (Sect. 2) and the other ones (Sect. 3). In the first part of Sect. 2 we require Weyl symmetry without introducing any extra field. In the second part we introduce a scalar Stueckelberg field and obtain a new scalar–tensor theory for massive gravitons in  $D = 2 + 1$ . In Sect. 3 we look at rather general local embeddings where quadratic terms in the NMG equations of motion are added to the NMG theory with coefficients which are arbitrary functions of  $\square$ . We examine the propagator of the final higher-order model and require equivalence of the particle content, i.e., the new dual theory must describe massive spin-2 particles and nothing else. Differently from the FP case we show here (Sect. 3) to all orders in derivatives that there is no local ghost-free embedding of the NMG equations of motion. In particular, the NGE embedding of the Weyl symmetry of the first part of Sect. 2 also leads to a ghost. Only the Stueckelberg approach leads to a ghost-free model which, after elimination of the scalar field, becomes nonlocal. The work in Sect. 3 is based on the analytic structure of the propagator. In Sect. 4 we present some final comments and our conclusions explaining that the negative results of Sect. 3 are related with the linearized reparametrization invariance of the NMG theory which makes it hard to be embedded.

## 2 Weyl embeddings of the “New Massive Gravity”

By a systematic Lagrangian procedure, called Noether gauge embedding (NGE), one can deduce a gauge invariant massive theory out of a non gauge invariant one [28]. The gauge symmetry of part of the initial Lagrangian is extended to the whole final theory. However, there is no guarantee that the particle spectrum is preserved. In [22] we have shown that the linearized NMG theory can be obtained via NGE from the usual Fierz–Pauli theory via embedding of linearized reparametrizations. We have also shown that a linearized higher-derivative topologically massive gravity is obtained from the usual linearized topologically massive gravity of [24] via NGE of the Weyl symmetry. In all those cases the particle content is preserved. One could wonder what would be the gauge invariant action obtained from the linearized New Massive Gravity theory. Since part of the action of NMG, the curvature square term, is invariant under Weyl transformation  $\delta_W h_{\mu\nu} = \phi \eta_{\mu\nu}$ , one might try to embed this symmetry into a new theory. This is what we next do. The linearized NMG theory can be written, up to an overall constant, as

$$\begin{aligned} S_{\text{NMG}} &= 2 \int d^3x \sqrt{-g} \left[ -R + \frac{1}{m^2} \left( R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2 \right) \right]_{hh} \\ &= \int d^3x \left[ h_{\mu\nu} G^{\mu\nu}(h) + \frac{2}{m^2} G_{\mu\nu}(h) S^{\mu\nu}(h) \right] \quad (1) \end{aligned}$$

<sup>1</sup> In the FP model there is only one massive pole in the spin-2 sector while in the NMG there is both a massive and a massless pole, see (38); however, the massless pole does not correspond to a propagating particle [26].

where  $G_{\mu\nu}(h)$  is the usual linearized self-adjoint Einstein tensor and  $S^{\mu\nu}(h)$  is the linearized Schouten tensor in  $D = 3$  defined as  $[R_{\mu\nu}(h) - \eta_{\mu\nu}R(h)/4]_{hh}$  which has the useful property of “commutativity” with the Einstein tensor in the sense that inside integrals  $G_{\mu\nu}(h)S^{\mu\nu}(f) = G_{\mu\nu}(f)S^{\mu\nu}(h)$ . In the NGE procedure an important ingredient is the Euler tensor,

$$\begin{aligned}
 K^{\mu\nu} &\equiv \frac{\delta S_{\text{NMG}}}{\delta h_{\mu\nu}} = 2 G^{\mu\nu}(h) + \frac{4}{m^2} G^{\mu\nu}[S(h)] \\
 &= -\square h^{\mu\nu} - \partial^\mu \partial^\nu h + \partial^\mu \partial_\alpha h^{\alpha\nu} + \partial^\nu \partial_\alpha h^{\alpha\mu} \\
 &\quad - \eta^{\mu\nu} [\partial_\alpha \partial_\beta h^{\alpha\beta} - \square h] \\
 &\quad - \frac{\square}{m^2} \left[ \square h^{\mu\nu} - \partial^\mu \partial_\alpha h^{\alpha\nu} - \partial^\nu \partial_\alpha h^{\alpha\mu} + \frac{\partial^\mu \partial^\nu h}{2} \right. \\
 &\quad \left. + \frac{\eta^{\mu\nu}}{2} (\partial_\alpha \partial_\beta h^{\alpha\beta} - \square h) \right] \\
 &\quad + \frac{1}{2m^2} \partial^\mu \partial^\nu (\partial_\alpha \partial_\beta h^{\alpha\beta}). \tag{2}
 \end{aligned}$$

With the help of an auxiliary field  $a_{\mu\nu}$  such that  $\delta_W a_{\mu\nu} = -\delta_W h_{\mu\nu}$  we implement a first iteration of the form

$$S^{(1)} = S_{\text{NMG}} + \int d^3x a_{\mu\nu} K^{\mu\nu}. \tag{3}$$

The Weyl variation of (3) can be written as

$$\delta_W S^{(1)} = - \int d^3x \delta_W [a_{\mu\nu} G^{\mu\nu}(a)]. \tag{4}$$

Therefore we end up with the Weyl invariant theory,

$$S_W = S_{\text{NMG}} + \int d^3x (a_{\mu\nu} K^{\mu\nu} + a_{\mu\nu} G^{\mu\nu}(a)). \tag{5}$$

Noticing that the Euler tensor (2) can be written in terms of the Einstein tensor, i.e.,  $K^{\mu\nu} = 2G^{\mu\nu}(H)$  with  $H^{\mu\nu} \equiv h^{\mu\nu} + 2 S^{\mu\nu}(h)/m^2$  we can rewrite  $S_W$  as

$$\begin{aligned}
 S_W &= S_{\text{NMG}} + \int d^3x (2 a_{\mu\nu} G^{\mu\nu}(H) + a_{\mu\nu} G^{\mu\nu}(a)) \\
 &= S_{\text{NMG}} + \int d^3x (-H_{\mu\nu} G^{\mu\nu}(H) + (a + H)_{\mu\nu} G^{\mu\nu}(a + H)). \tag{6}
 \end{aligned}$$

After the shift  $a_{\mu\nu} \rightarrow \tilde{a}_{\mu\nu} - H_{\mu\nu}$  in (6), the  $\tilde{a}_{\mu\nu}$  auxiliary field decouples. We can safely discard the last term,  $\tilde{a}_{\mu\nu} G^{\mu\nu}(\tilde{a})$ , which is a linearized Einstein–Hilbert term without particle content. Thus we have a sixth-order Weyl invariant action which turns out to have a nonlinear completion,

$$\begin{aligned}
 S_W &= \int d^3x \left\{ h_{\mu\nu} G^{\mu\nu}(h) + \frac{2}{m^2} G_{\mu\nu}(h) S^{\mu\nu}(h) \right. \\
 &\quad \left. - \left[ h + \frac{2}{m^2} S(h) \right]_{\mu\nu} G^{\mu\nu} \left[ h + \frac{2}{m^2} S(h) \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{2}{m^2} \int d^3x \left[ G_{\mu\nu}(h) S^{\mu\nu}(h) + \frac{2}{m^2} S_{\mu\nu}(h) G^{\mu\nu}[S(h)] \right] \\
 &= \frac{2}{m^4} \int d^3x \left[ \sqrt{-g} \left( R_{\mu\nu} - \frac{3}{8} g_{\mu\nu} R \right) (\square - m^2) R^{\mu\nu} \right]_{hh}. \tag{7}
 \end{aligned}$$

Since the tensor structure of (7) is the same as the curvature square term of the NMG theory, it is clear that  $S_W$  is invariant under Weyl transformations. The particle content of  $S_W$  will be examined in the next section. The theory  $S_W$  contains a ghost.

Another way to embed the Weyl symmetry in the “New Massive Gravity” is to introduce a scalar Stueckelberg field in (1) by substituting  $h_{\mu\nu} \rightarrow h_{\mu\nu} + \eta_{\mu\nu} \phi$ . Since the fourth-order term of  $S_{\text{NMG}}$  is Weyl invariant we end up with

$$\begin{aligned}
 S_\phi^L[h, \phi] &= \int d^3x \left[ h_{\mu\nu} G^{\mu\nu}(h) + \frac{2}{m^2} G_{\mu\nu}(h) S^{\mu\nu}(h) \right. \\
 &\quad \left. + \frac{1}{2} \phi \square \phi + \frac{1}{2} \phi (\square h - \partial^\mu \partial^\nu h_{\mu\nu}) \right]. \tag{8}
 \end{aligned}$$

By construction, the linear theory  $S_\phi^L[h, \phi]$  is invariant under linearized Weyl transformations:  $\delta_W h_{\mu\nu} = \eta_{\mu\nu} \Lambda$ ;  $\delta_W \phi = -\Lambda$ . We can easily find a nonlinear version of  $S_\phi^L[h, \phi]$ , namely,

$$\begin{aligned}
 S_\phi^{NL}[g_{\mu\nu}, \phi] &= 2 \int d^3x \sqrt{-g} \left[ -R + \frac{1}{m^2} \left( R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2 \right) \right. \\
 &\quad \left. + \frac{1}{2} \phi \square \phi - \frac{1}{2} \phi R \right]. \tag{9}
 \end{aligned}$$

It is clear that the equations of motion  $\delta S_\phi^{NL} = 0$  contain the trivial solution  $g_{\mu\nu} = \eta_{\mu\nu}$ ,  $\phi = 0$ . Expanding about such vacuum up until quadratic terms in the fluctuations we recover  $S_\phi^L$ . Therefore, the particle content of  $S_\phi^{NL}$  consists, at tree level, of one massive spin-2 particle just like the NMG of [21]. However, as in the K-model (massless limit of NMG) studied in detail in [29,30], we might have problems at nonlinear level since the linearized Weyl symmetry is probably broken at nonlinear level and consequently the scalar field stops being pure gauge in the full model (51). In particular, the phenomenon of bifurcation of constraints found in [30] might also be present here. A detailed study of the constraint structure should be carried out.

### 3 Generalized Euler tensor embedding of “New Massive Gravity”

Our starting point is the linearized NMG theory with the addition of quadratic terms in its equations of motion:

$$\mathcal{L}_G[h_{\mu\nu}] = \frac{1}{2} h_{\mu\nu} K^{\mu\nu} + \frac{1}{2} K_{\mu\nu} d(\square) K^{\mu\nu} + \frac{1}{2} K f(\square) K. \tag{10}$$

The first term in (10) is the linearized NMG theory. The NMG Euler tensor  $K^{\mu\nu}$  is given in (2). The coefficients  $d(\square)$  and  $f(\square)$  are so far arbitrary functions of  $\square = \partial_\mu \partial^\mu$  such that the Lagrangian  $\mathcal{L}_G$  remains local. Due to the conservation law  $\partial^\mu K_{\mu\nu} = 0$  which holds identically due to the linearized reparametrization invariance of the NMG ( $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ ), other terms which might show up in (10) like  $(\partial_\mu K^{\mu\nu})^2$  and  $\partial_\mu K^{\mu\nu} \partial_\nu K$  do not contribute.<sup>2</sup> In terms of the original field  $h_{\mu\nu}$  we can rewrite  $\mathcal{L}_G$  as

$$\mathcal{L}_G = \partial^\mu h_{\mu\nu} c_1(\square) \partial_\alpha h^{\alpha\nu} + \partial^\mu h c_2(\square) \partial^\nu h_{\nu\mu} + h c_3(\square) h + h^{\mu\nu} c_4(\square) h_{\mu\nu} + \partial^\mu \partial^\nu h_{\mu\nu} c_5(\square) \partial^\alpha \partial^\beta h_{\alpha\beta}. \tag{11}$$

The coefficients  $c_i(\square)$  are given by<sup>3</sup>

$$c_1 = \square d - 1 + \frac{\square}{m^2} \left[ 1 - 2\square d + \frac{\square^2}{m^2} d \right], \tag{12}$$

$$c_2 = 1 - \frac{\square}{2m^2} + \square f + \frac{\square d}{2} - \frac{\square d}{2} \left( \frac{\square}{m^2} - 1 \right)^2, \tag{13}$$

$$c_3 = \frac{\square}{2} c_2, \tag{14}$$

$$c_4 = \frac{\square}{2} c_1, \tag{15}$$

$$c_5 = \frac{f+d}{2} + \frac{1}{4m^2} + \frac{d\square}{4m^2} \left( \frac{\square}{m^2} - 2 \right). \tag{16}$$

The Lagrangian  $\mathcal{L}_G$  can be further rewritten in terms of a four index differential operator  $\mathcal{L}_G = h^{\mu\nu} G_{\mu\nu\alpha\beta} h^{\alpha\beta}$  whose inverse  $G^{-1}$  does not exist due to the linearized reparametrization symmetry. We choose the de Donder gauge fixing term:

$$\mathcal{L}_{GF} = \lambda (\partial^\mu h_{\mu\nu} - \partial_\nu h/2)^2, \tag{17}$$

which amounts to the shift  $c_1 \rightarrow c_1 + \lambda$ ,  $c_2 \rightarrow c_2 - \lambda$  in Eqs. (12) and (13) but not in (14) and (15). In (14) we make  $c_3 \rightarrow c_3 - \lambda\square/4$ . Before we display  $G^{-1}$  we take a closer look at the coefficients  $c_i(\square)$  in (12)–(16). A local Lagrangian density in terms of  $h_{\mu\nu}$  requires

$$d = \frac{a}{\square} + \tilde{d}(\square), \tag{18}$$

$$f = -\frac{a}{\square} + \tilde{f}(\square), \tag{19}$$

where  $\tilde{d}(\square)$  and  $\tilde{f}(\square)$  are analytic functions of  $\square$  while  $a$  is an arbitrary real constant. In terms of the spin- $s$  projection operators  $P_{IJ}^{(s)}$  given in the appendix A and suppressing the four indices we have

<sup>2</sup> We have restricted ourselves here to parity invariant theories, thus avoiding terms like  $\epsilon^{\mu\nu\alpha} K_{\mu\nu} \partial_\nu K_\alpha^\gamma$ .

<sup>3</sup> Henceforth, unless otherwise stated, we replace  $d(\square)$ ,  $f(\square)$  by  $d$ ,  $f$  respectively, though they are still arbitrary functions of  $\square$ .

$$G^{-1} = \frac{2m^4 P_{SS}^{(2)}}{\square(\square - m^2) \left[ m^2(1-a) + a\square + \square(\square - m^2) \tilde{d} \right]} - \frac{2 P_{SS}^{(1)}}{\lambda \square} + \frac{2 P_{SS}^{(0)}}{\square \left[ 1 - a + \square(\tilde{d} + 2\tilde{f}) \right]} + \frac{2\sqrt{2} \left( P_{SW}^{(0)} + P_{WS}^{(0)} \right)}{\square \left[ 1 - a + \square(\tilde{d} + 2\tilde{f}) \right]} - 4 \frac{\left[ 1 - a + \square(\tilde{d} + 2\tilde{f}) - \lambda \right] P_{WW}^{(0)}}{\lambda \square \left[ 1 - a + \square(\tilde{d} + 2\tilde{f}) \right]}. \tag{20}$$

Regarding the local symmetries of  $\mathcal{L}_G$  there is one special case,

$$a = 1 \text{ and } \tilde{d}(\square) = -2\tilde{f}(\square), \tag{21}$$

since we have a zero in the denominator of the spin-0 sector which indicates a spin-0 symmetry. Indeed, under a Weyl transformation  $\delta_W h_{\mu\nu} = \Lambda \eta_{\mu\nu}$  we have from (10) after integrations by parts

$$\delta_W \mathcal{L}_G = h(2c_4 + 6c_3 - \square c_2)\Lambda + \partial^\mu \partial^\nu h_{\mu\nu} [2\square c_5 - 2c_1 - 3c_2]\Lambda = (\square h - \partial^\mu \partial^\nu h_{\mu\nu}) [1 - a + \square(\tilde{d} + 2\tilde{f})]\Lambda. \tag{22}$$

In the special case (21) we need to add another (Weyl) gauge fixing term, we may choose  $\mathcal{L}_{GF}^W = \xi h^2$ . This implies the shift  $c_3 \rightarrow c_3 + \xi$ . Consequently, in the Weyl symmetric case we have

$$G_W^{-1} = \frac{2m^4 P_{SS}^{(2)}}{\square^2(\square - m^2) [1 - 2\tilde{f}(\square - m^2)]} - \frac{2 P_{SS}^{(1)}}{\lambda \square} + \left( \frac{1}{8\xi} - \frac{1}{2\lambda \square} \right) P_{SS}^{(0)} + \left( \frac{1}{4\xi} - \frac{1}{\lambda \square} \right) P_{WW}^{(0)} + \frac{\sqrt{2}}{8} + \left( \frac{1}{\xi} + \frac{4}{\lambda \square} \right) \times \left[ P_{SW}^{(0)} + P_{WS}^{(0)} \right]. \tag{23}$$

Next we analyze the particle content of  $\mathcal{L}_G$  from the analytic structure of  $G^{-1}$  and  $G_W^{-1}$ . In momentum space we can calculate the gauge invariant two-point amplitude  $\mathcal{A}(k)$  by saturating  $G^{-1}$  or  $G_W^{-1}$  with external sources. For instance,

$$\mathcal{A}(k) = -\frac{i}{2} T_{\mu\nu}^*(k) (G^{-1})^{\mu\nu\alpha\beta}(k) T_{\alpha\beta}(k). \tag{24}$$

Here  $G^{-1}(k) = G^{-1}(\partial_\mu \rightarrow i k_\mu)$ . Due to the linearized reparametrization symmetry, the source must be transverse  $k^\mu T_{\mu\nu} = 0$ , consequently,

$$\mathcal{A}(k) = i \left[ \frac{S^{(0)}}{k^2 [1 - a - k^2(\bar{d} + 2\bar{f})]} - \frac{m^4 S^{(2)}}{k^2(k^2 + m^2)[m^2(1 - a) - ak^2 + k^2\bar{d}(k^2 + m^2)]} \right],$$

$$\equiv i \left[ \frac{S^{(0)}}{k^2 P(k^2)} - \frac{m^4 S^{(2)}}{k^2(k^2 + m^2)Q(k^2)} \right], \tag{25}$$

where  $\bar{d} = \bar{d}(-k^2)$  and  $\bar{f} = \bar{f}(-k^2)$  are analytic functions of  $k^2 = k_\mu k^\mu$  and

$$S^{(0)} = T_{\mu\nu}^* (P_{SS}^{(0)})^{\mu\nu\alpha\beta} T_{\alpha\beta} = \frac{|T|^2}{2}, \tag{26}$$

$$S^{(2)} = T_{\mu\nu}^* (P_{SS}^{(2)})^{\mu\nu\alpha\beta} T_{\alpha\beta} = T_{\mu\nu}^* T^{\mu\nu} - \frac{|T|^2}{2}. \tag{27}$$

The quantity  $T = \eta_{\mu\nu} T^{\mu\nu} = -T_{00} + T_{ii}$  is the trace of the external source in momentum space. If the two conditions for Weyl invariance (21) hold we must have  $T = 0$ .

A key role is played by the imaginary part of the residue of  $\mathcal{A}(k)$  at each pole. For instance, at  $k^2 = -m^2$  we have

$$I_m \equiv \Im \lim_{k^2 \rightarrow -m^2} (k^2 + m^2)\mathcal{A}(k). \tag{28}$$

If and only if  $I_m > 0$  we have a physical particle. If  $I_m = 0$  we have a non-propagating mode while  $I_m < 0$  or no definite sign for  $I_m$  signalizes the presence of ghost. In order to verify the sign of  $I_m$  we fix a convenient coordinate frame splitting the cases of massless and massive poles. In the massless case we fix a frame such that  $k^\mu = (k_0, \epsilon, k_0)$ , thus  $k^\mu k_\mu = \epsilon^2$ . We will take  $\epsilon \rightarrow 0$  at the end. This caution is necessary for the analysis of double poles. From the three conditions  $k^\mu T_{\mu\nu} = 0$  we have in this frame

$$T_{01} = -T_{12} - \frac{\epsilon}{k_0} T_{11}, \tag{29}$$

$$T_{02} = -T_{22} - \frac{\epsilon}{k_0} T_{12}, \tag{30}$$

$$T_{00} = T_{22} + 2\frac{\epsilon}{k_0} T_{12} + \frac{\epsilon^2}{k_0^2} T_{11}. \tag{31}$$

Consequently,

$$T_{\mu\nu}^* T^{\mu\nu} = |T_{11}|^2 - 2\frac{\epsilon}{k_0} (T_{12}T_{11}^* + T_{12}^*T_{11}) + \frac{\epsilon^2}{k_0^2} [2|T_{12}|^2 + (T_{11}T_{22}^* + T_{11}^*T_{22}) - 2|T_{11}|^2], \tag{32}$$

$$|T|^2 = |T_{11}|^2 - 2\frac{\epsilon}{k_0} (T_{12}T_{11}^* + T_{12}^*T_{11}) + 2\frac{\epsilon^2}{k_0^2} (2|T_{12}|^2 - |T_{11}|^2). \tag{33}$$

In the case of massive poles we choose the frame  $k^\mu = (m, \epsilon, 0)$  such that  $k^2 + m^2 = \epsilon^2$ . From  $k^\mu T_{\mu\nu} = 0$  we have

$$T_{01} = -\frac{\epsilon}{m} T_{11}; \quad T_{02} = -\frac{\epsilon}{m} T_{12}; \quad T_{00} = \frac{\epsilon^2}{m^2} T_{11}. \tag{34}$$

Thus,

$$T_{\mu\nu}^* T^{\mu\nu} = |T_{22}|^2 + |T_{11}|^2 \left(1 - \frac{\epsilon^2}{m^2}\right)^2 + 2|T_{12}|^2 \left(1 - \frac{\epsilon^2}{m^2}\right), \tag{35}$$

$$|T|^2 = |T_{22}|^2 + |T_{11}|^2 \left(1 - \frac{\epsilon^2}{m^2}\right)^2 + \left(1 - \frac{\epsilon^2}{m^2}\right) (T_{11}T_{22}^* + T_{11}^*T_{22}). \tag{36}$$

Since  $\bar{d}(k^2)$  and  $\bar{f}(k^2)$  are arbitrary analytic functions, there might be double poles in the denominator of  $\mathcal{A}(k)$ . We first examine those poles. From (26), (27), (35), and (36) we see that is impossible to take linear combinations of  $S^{(0)}$  and  $S^{(2)}$  in order to end up only with terms of order  $\epsilon^2$ . Therefore a massive double pole  $1/(k^2 + m^2)^2 = 1/\epsilon^4$  cannot be reduced to a simple pole by any fine tuning of the functions  $\bar{d}$  and  $\bar{f}$ . So henceforth we assume that all massive poles must be simple poles. The conclusion remains the same for the Weyl symmetric case. In the latter case  $T_{00} = T_{11} + T_{22}$  and (34) imply  $T_{22} = -T_{11} + (\epsilon^2/m^2)T_{11}$ , which does not help canceling the term  $|T_{12}|^2$  in (35).

The massless case is a bit different. From (26), (27), (32), and (33) we see that we do have one special combination of order  $\epsilon^2$  which may turn double poles into simple ones, namely,

$$S^{(2)} - S^{(0)} = T_{\mu\nu}^* T^{\mu\nu} - |T|^2 = \frac{\epsilon^2}{k_0^2} \times [(T_{11}T_{22}^* + T_{11}^*T_{22}) - 2|T_{12}|^2]. \tag{37}$$

However, the term  $(T_{11}T_{22}^* + T_{11}^*T_{22})$  has no definite sign. So we end up with a ghost. In the special case of Weyl symmetry, using (31) in  $T_{00} = T_{11} + T_{22}$  we have  $T_{11} = 2(\epsilon/k_0)T_{12} + \mathcal{O}(\epsilon^2)$ . Thus, the dangerous term of (37) becomes of order  $\epsilon^3$  and will not contribute to the residue. So we may hope to turn a double massless pole  $1/k^4$  into a physical pole only in the Weyl invariant case after a specific fine tuning of  $\bar{d}$  and  $\bar{f}$ . In particular, this is the mechanism behind the fourth-order K-term which describes a physical massless particle as explained in [29] via decomposition of  $h_{\mu\nu}$  in orthogonal modes and in [31] via the analytic structure of the propagator.

In summary, multiple poles lead us to ghosts in general except for the double massless pole in the Weyl symmetric case which will be examined later on.

Henceforth we split our analysis in four cases:

$a \neq 1$  (Case I)

- $a = 1$  and  $\bar{d} + 2\bar{f} \neq 0$  (Case II)
- $a = 1$  and  $\bar{d} + 2\bar{f} = 0$  (Case III)
- $a = 0 = \bar{d}$  and  $\bar{f} = 1/(2k^2)$  (Case IV)

In the cases III and IV we have Weyl symmetry. The case IV corresponds to the  $S_\phi$  model of (8) after elimination of  $\phi$ .

### 3.1 Case I: $a \neq 1$

As a warm up we start reproducing the results of [26] for the NMG theory. We take  $a = 0 = \bar{d} = \bar{f}$ . We have one massless and one massive pole,

$$\mathcal{A}(k) = i \left[ \frac{S^{(0)}}{k^2} - \frac{m^2 S^{(2)}}{k^2(k^2 + m^2)} \right]. \tag{38}$$

Taking  $\epsilon \rightarrow 0$  in (32) and (33) we have a vanishing residue at the massless pole and consequently a non-propagating mode:

$$\begin{aligned} I_0 &= \Im \lim_{k^2 \rightarrow 0} k^2 \mathcal{A}(k) = S^{(0)} - S^{(2)} \\ &= \frac{|T|^2}{2} - \left( T_{\mu\nu}^* T^{\mu\nu} - \frac{|T|^2}{2} \right) = 0. \end{aligned} \tag{39}$$

The residue at the massless pole vanishes for the very same reason as it does in the Maxwell–Chern–Simons theory of [24], namely, the lowest-order term (in derivatives) of the theory (linearized Einstein–Hilbert) has no particle content. Taking  $\epsilon \rightarrow 0$  in (35) and (36) we have a positive residue at the massive pole, a physical massive spin-2 particle,

$$\begin{aligned} I_m &= \Im \lim_{k^2 \rightarrow -m^2} (k^2 + m^2) \mathcal{A}(k) = S^{(2)} \\ &= 2|T_{12}|^2 + \frac{|T_{11} - T_{22}|^2}{2} > 0. \end{aligned} \tag{40}$$

Now we go back to the general case  $a \neq 1$ . Except for the NMG case  $a = 0 = \bar{d} = \bar{f}$ , which will not be treated here anymore, we have in general extra massive poles stemming from the polynomial  $Q(k^2)$ . Requiring that no tachyons show up we can write

$$\mathcal{A}(k) = \frac{-i m^2 S_A(k^2) \prod_{i=1}^{N_Q} m_i^2}{(a - 1)k^2(k^2 + m^2)(k^2 + m_1^2) \cdots (k^2 + m_{N_Q}^2)}, \tag{41}$$

where  $N_Q \geq 1$  is the number of extra massive poles coming from  $Q(k^2)$ . Since we are specially interested in the massive poles, we can write from (26), (27) and (35), (36) at  $\epsilon \rightarrow 0$ ,

$$\begin{aligned} S_A(k^2) &\equiv S^{(2)} + A(k^2) S^{(0)} = 2|T_{12}|^2 + |T_{11}|^2 + |T_{22}|^2 \\ &\quad + \frac{(A - 1)}{2} |T_{11} + T_{22}|^2. \end{aligned} \tag{42}$$

The quantity  $A(k^2)$  is an analytic real function of  $k^2$  whose specific form is not important, it is defined by comparing (41)

with (25). Defining the polynomial of degree  $N_Q + 1$ :

$$\mathcal{P}(k^2) = (k^2 + m_0^2)(k^2 + m_1^2) \cdots (k^2 + m_{N_Q}^2), \tag{43}$$

where  $m_0^2 \equiv m^2$  is the mass squared already present in the NMG theory, it is clear that the sign of the residue  $I_{m_j}$  at some pole  $k^2 = -m_j^2$  depends essentially upon the sign of the ratio  $S_A/\mathcal{P}'$  calculated at  $k^2 = -m_j^2$ , where  $\mathcal{P}' = d\mathcal{P}/dk^2$ . Since the derivative of a polynomial has alternating signs at its consecutive simple zeros, the only hope of having positive residues at the different massive poles is to require that  $S_A$  also has alternating signs at such points. However, it is easy to prove that  $S_A(-m_j^2)$  either has no definite sign or is definite positive. The point is that if  $A(-m_j^2) \geq 0$  we can guarantee that the last three terms of (42) add up to a non-negative number, so in those cases  $S_A(-m_j^2) \geq 0$ . On the other hand, since the three complex numbers  $T_{12}, T_{11}, T_{22}$  are totally unconstrained, even if we take  $A(-m_j^2) < 0$ , depending on the relative strength of those three complex numbers, the sign of (42) may change. Thus, we cannot guarantee that  $S_A(-m_j^2) < 0$ . In conclusion, whenever we have more than one massive pole we have ghosts and only the NMG case is safe at  $a \neq 1$ .

### 3.2 Case II: $a = 1$ and $\bar{d} + 2\bar{f} \neq 0$

In this case we have in principle a double massless pole and massive poles:

$$\mathcal{A}(k) = -i \left\{ \frac{m^4 S^{(2)}}{k^4(k^2 + m^2)[(k^2 + m^2)\bar{d} - 1]} + \frac{S^{(0)}}{k^4(\bar{d} + 2\bar{f})} \right\}. \tag{44}$$

We can choose  $\bar{d}(0) + \bar{f}(0) = \frac{1}{2m^2}$  and turn the double massless pole into a simple one. However, since we have no Weyl symmetry, as explained in the paragraph of Eq. (37), we are doomed to have a massless ghost. Therefore we go to the next case.

### 3.3 Case III: $a = 1$ and $\bar{d} + 2\bar{f} = 0$

If we set  $a = 1$  and  $\bar{d} = -2\bar{f}$ , or  $\bar{d} = -2\bar{f}$ , in Eqs. (12)–(16) and plug those results in (11) we have

$$\begin{aligned} \mathcal{L}_W &= \frac{1}{m^4} \partial_\mu h^{\mu\nu} \square H(\square) \partial^\alpha h_{\alpha\nu} - \frac{1}{2m^4} \partial^\mu h \square H(\square) \partial^\alpha h_{\alpha\mu} \\ &\quad - \frac{1}{4m^4} h \square^2 H(\square) h \\ &\quad + \frac{1}{2m^4} h_{\mu\nu} \square^2 H(\square) h^{\mu\nu} + \frac{1}{4m^4} \partial^\mu \partial^\nu h_{\mu\nu} H(\square) \partial^\alpha \partial^\beta h_{\alpha\beta}, \end{aligned} \tag{45}$$

where  $H(\square) = (\square - m^2)[1 - 2(\square - m^2)\bar{f}(\square)]$ . The above Lagrangian can be nonlinearly completed in terms of the

square of curvatures in the form of a K-term of the NMG theory, i.e.,

$$\mathcal{L}_W^{NL} = \frac{2\sqrt{-g}}{m^4} \left( R^{\mu\nu} - \frac{3}{8}g^{\mu\nu} R \right) (\square - m^2) \times [1 - 2(\square - m^2)\tilde{f}(\square)]R_{\mu\nu}. \tag{46}$$

In the case  $\tilde{f} = 0$  we recover the Weyl embedding of the previous section; see (7).

Back to the two-point amplitude  $\mathcal{A}(k)$ : we have again a double massless pole and massive poles in general. Assuming that the analytic function  $\bar{f}(k^2) = \tilde{f}(\square \rightarrow -k^2)$  is such that we have no tachyons, we can write

$$\mathcal{A}(k) = i \frac{m^4 S^{(2)}}{k^4(k^2 + m^2)[1 + 2\bar{f}(k^2 + m^2)]} = i \frac{m^2 S^{(2)} \prod_{i=0}^{N_Q} m_i^2}{k^4 \mathcal{P}(k^2)}, \tag{47}$$

where  $\mathcal{P}(k^2)$  is defined in (43). From (27), (35), and the fact that  $T = 0$ , due to the Weyl symmetry, we have at each massive pole

$$I_{m_j} = \Im \lim_{k^2 \rightarrow -m_j^2} (k^2 + m_j^2) \mathcal{A}(k) = \frac{2(|T_{12}|^2 + |T_{11}|^2)}{\mathcal{P}'(-m_j^2)}. \tag{48}$$

Due to the alternating signs of  $\mathcal{P}'$  at its consecutive single zeros, it is impossible to have  $I_{m_j} > 0$  for all  $j = 0, \dots, N_Q$ . We are forced to assume  $N_Q = 0$ , i.e.,  $\bar{f} = 0$ . In this subcase  $\mathcal{P}(k^2) = k^2 + m^2$ , so  $\mathcal{P}' = 1$  and the massive pole is a physical one  $I_m = I_{m_0} = 2(|T_{12}|^2 + |T_{11}|^2) > 0$ . Regarding the massless double pole, we have already seen that due to the Weyl symmetry we have  $T_{11} = 2(\epsilon/k_0)T_{12} + \mathcal{O}(\epsilon^2)$ , substituting back in (32) we obtain [31]

$$T^{\mu\nu} T_{\mu\nu}^* = -2 \frac{\epsilon^2}{k_0^2} |T_{12}|^2 + \mathcal{O}(\epsilon^3). \tag{49}$$

Consequently, although the apparent double pole has become a simple pole, we still have a ghost due to the negative sign of the residue:

$$I_0 = \Im \lim_{k^2 \rightarrow 0} k^2 \mathcal{A}(k) = \lim_{\epsilon \rightarrow 0} \frac{m^2}{\epsilon^2} T^{\mu\nu} T_{\mu\nu}^* = -\frac{2m^2}{k_0^2} |T_{12}|^2 < 0. \tag{50}$$

Therefore, the Weyl invariant theory (7) of the previous section will unavoidably contain a ghost.

### 3.4 Case IV: $a = 0 = \bar{d}$ and $\bar{f} = 1/(2k^2)$

In this last case we have a nonlocal theory corresponding to the Weyl invariant action  $S_\phi$  given in (8) after the elimination of  $\phi$ :

$$S_\phi^L[g_{\mu\nu}, \phi] = 2 \int d^3x \sqrt{-g} \left[ -R + \frac{1}{m^2} \left( R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2 \right) - \frac{1}{8} R \frac{1}{\square} R \right]_{hh}. \tag{51}$$

After adding a gauge fixing term like (17) plus another one for the Weyl symmetry  $\mathcal{L}_{GF}^W = \zeta h^2$  the action acquires the form (11) with the coefficients

$$c_1 = \frac{\square}{m^2} - 1 + \lambda; \quad c_2 = \frac{1}{2} - \frac{\square}{2m^2} - \lambda; \tag{52}$$

$$c_3 = -\frac{\square^2}{4m^2} + \frac{\square^2}{4}(1 - c),$$

$$c_4 = \frac{\square}{2m^2}(\square - m^2); \quad c_5 = \frac{1}{4m^2\square}(\square - m^2). \tag{53}$$

The propagator, suppressing indices, is given by

$$G^{-1} = \frac{2m^2 P_{SS}^{(2)}}{\square(\square - m^2)} - \frac{2P_{SS}^{(1)}}{\lambda\square} + \frac{(P_{WW}^{(0)} + P_{SS}^{(0)})}{8} \left( \frac{1}{\zeta} - \frac{4}{\lambda\square} \right) + \frac{(4\zeta + \lambda\square)}{4\zeta\lambda\square} (P_{SW}^{(0)} + P_{WS}^{(0)}). \tag{54}$$

After saturating the propagator with transverse and traceless sources as in (24) we are left with two simple poles, one massive and one massless which come both from the pure spin-2 sector:

$$\mathcal{A}(k) = -i \frac{m^2 S^{(2)}}{k^2(k^2 + m^2)}. \tag{55}$$

As expected, the dependence on the gauge parameters  $\lambda$  and  $\zeta$  disappear which guarantees gauge invariance of the two-point amplitude. When we look closer at the massless pole using  $k_\mu = (k_0, \epsilon, k_0)$ , it is easy to see that its residue vanishes with power  $\epsilon^2$ . From (32), (33), and  $T = \eta_{\mu\nu} T^{\mu\nu} = 0$  we have

$$I_0 = \Im \lim_{k^2 \rightarrow 0} k^2 \mathcal{A}(k) = -\lim_{\epsilon \rightarrow 0} T_{\mu\nu}^* T^{\mu\nu} = -\lim_{\epsilon \rightarrow 0} \frac{\epsilon^2}{k_0^2} \left[ -2|T_{12}|^2 + (T_{11} T_{22}^* + T_{11}^* T_{22}) \right] = 0. \tag{56}$$

The massive pole is a physical one (positive residue). From (35) we have

$$I_m = \Im \lim_{k^2 \rightarrow -m^2} (k^2 + m^2) \mathcal{A}(k) = S^{(2)} = |T_{22}|^2 + 2|T_{12}|^2 > 0. \tag{57}$$

Therefore, the linearized version of the model (51) is unitary and contains only massive gravitons in the spectrum.

## 4 Conclusion

Recent work on massive gravity [8–10] has shown how to overcome longstanding problems like the appearance of

ghosts at the nonlinear level [4] and the vDVZ [2, 3] mass discontinuity. Those models can be described in terms of one [8, 9] or two [10] dynamic metrics. In the linearized limit those theories reduce, respectively, to the old paradigmatic Fierz–Pauli (FP) theory [1] and to the addition of the linearized Einstein–Hilbert plus the FP theory. In fact, those new massive gravity models are built up on top of the second-order FP theory which describes free massive spin-2 particles. It is expected that if we change the starting point (underlying free theory) we might end up with alternative massive gravities. Those alternative massive gravities might help us to understand which physical features of the new theories are really model independent and eventually we might still hope of finding a renormalizable massive gravity model.

In [20] we have shown that there are several higher-derivative modifications of the Fierz–Pauli theory which describe free massive spin-2 particles and are still ghost-free at linear level. So they could be used as new starting points for alternative massive gravities. Indeed, they can be identified with quadratic truncations of  $f(R, \square R)$  modifications of the Einstein–Hilbert theory plus the Fierz–Pauli (FP) mass term; see [20]. Some of those models have been further changed, see e.g. [19], by the addition of a convenient non-derivative nonlinear potential of the type found in [8, 9] in order to account for the absence of ghosts at nonlinear level. They define massive gravity theories with interesting cosmological properties.

All modifications of the FP model found in [20] occur in the spin-0 sector of the theory. Modifications in the spin-2 sector lead in general to a ghost, with the exception of the so-called “New Massive Gravity” of [21] which only exists in  $D = 2 + 1$ . In section III we have used the embedding procedure used in [20] in order to search for arbitrary higher-order modifications of the NMG model similarly to what has been done in [20] for the case of the FP theory. Although the embedding procedure does not guarantee full equivalence of the initial and final models, it ensures that the particle content of the initial theory is contained in the new theory. At the end one always has to check both particle contents and fix the embedding coefficients, see (10), in order to achieve full equivalence.

Contrary to the FP case, we conclude here that there is no local embedding of the NMG theory which remains ghost-free. We have carried out a thorough calculation of the residues at all possible massive and massless poles in the propagator. There is always a pole with negative residue (ghost). A key technical point is the fact that the Euler tensor  $K_{\mu\nu}$ , which defines the NMG equations of motion  $K_{\mu\nu} = 0$ , satisfies  $\partial^\mu K_{\mu\nu} = 0$  identically due to the linearized reparametrization invariance  $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ . Consequently, the quadratic terms  $(\partial^\mu K_{\mu\nu})^2$ ,  $(\partial^\mu \partial^\nu K_{\mu\nu})^2$ , and  $\partial^\mu K_{\mu\nu} \partial^\nu K$  identically vanish too and strongly restrict the embedding Ansatz (10), contrary to the embedding of the

FP theory where we have five independent quadratic terms in the equations of motion which can be added to the FP theory. The more local symmetries we have, the more difficult is to embed it into a higher-order model.

In Sect. 2 we have used the Weyl symmetry as a guiding principle for the embedding of the NMG. First we have looked at the Noether gauge embedment of the Weyl symmetry, which leads to the sixth-order model (7). In this case we have one massive and one massless pole. Unfortunately, their residues have opposite signs. If we reverse the overall sign of (7), we have a physical massless graviton and a massive ghost. In this case we might use the Weyl invariant model (7) with reversed sign as a phenomenological toy model along the lines of [32]. Namely, we can have a consistent unitary theory if the ghost mass stays above the energy cut-off of the theory. Although the model (7) is of sixth order in derivatives, the would-be double massless pole is reduced to a simple pole and the analytic structure of the propagator is similar to some curvature square modifications of general relativity in  $D = 4$ .

At the end of Sect. 2 we have obtained a promising candidate for a consistent massive gravity different from the NMG model. From the introduction of a scalar Stueckelberg field in the linearized version of the NMG theory we have derived the linearized Weyl invariant model given in (8). As in the case of (7) we have one massless and one massive pole but differently from (7) both poles are simple poles. The Weyl symmetry now kills the residue at the massless pole such that the “would-be” massless ghost does not propagate at all as shown at the end of Sect. 3 (Case IV). The residue at the massive pole is positive and we are left with physical massive gravitons, at least in the linearized approximation. The model can be nonlinearly completed leading to the scalar-tensor theory (51) which might be an alternative to the usual NMG model. However, as in the case of the pure K-term (massless limit of NMG) analyzed in [30], the Weyl symmetry is probably broken beyond the linearized level which might lead to a ghost in the full theory. This is probably true also for the higher-derivative topologically massive gravity of [22, 23]. Those examples require detailed investigations of the constraints structure which are beyond the scope of the present work.

Finally, we mention Ref. [33] where stability and unitarity of fourth-order (quadratic in curvatures) three dimensional gravities, including the parity breaking gravitational Chern–Simons term of [24], have been investigated around maximally symmetric spaces. Whenever our results overlap (fourth-order theories around Minkowski space without parity breaking terms) we have agreement.

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### Appendix

Here we display the operators  $P_{IJ}^{(s)}$ , the coefficients  $A_{ij}(\square)$ . They make use, as building blocks, of the spin-0 and spin-1 projection operators acting on vector fields, respectively,

$$\omega_{\mu\nu} = \frac{\partial_\mu \partial_\nu}{\square}, \quad \theta_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\square}, \tag{58}$$

we define the spin- $s$  operators  $P_{IJ}^{(s)}$  acting on symmetric rank-2 tensors in  $D$  dimensions:

$$\left(P_{SS}^{(2)}\right)_{\alpha\beta}^{\lambda\mu} = \frac{1}{2} \left(\theta^\lambda_\alpha \theta^\mu_\beta + \theta^\mu_\alpha \theta^\lambda_\beta\right) - \frac{\theta^{\lambda\mu} \theta_{\alpha\beta}}{D-1}, \tag{59}$$

$$\left(P_{SS}^{(1)}\right)_{\alpha\beta}^{\lambda\mu} = \frac{1}{2} \left(\theta^\lambda_\alpha \omega^\mu_\beta + \theta^\mu_\alpha \omega^\lambda_\beta + \theta^\lambda_\beta \omega^\mu_\alpha + \theta^\mu_\beta \omega^\lambda_\alpha\right), \tag{60}$$

$$\left(P_{SS}^{(0)}\right)_{\alpha\beta}^{\lambda\mu} = \frac{1}{D-1} \theta^{\lambda\mu} \theta_{\alpha\beta}, \quad \left(P_{WW}^{(0)}\right)_{\alpha\beta}^{\lambda\mu} = \omega^{\lambda\mu} \omega_{\alpha\beta}, \tag{61}$$

$$\left(P_{SW}^{(0)}\right)_{\alpha\beta}^{\lambda\mu} = \frac{1}{\sqrt{D-1}} \theta^{\lambda\mu} \omega_{\alpha\beta}, \quad \left(P_{WS}^{(0)}\right)_{\alpha\beta}^{\lambda\mu} = \frac{1}{\sqrt{D-1}} \omega^{\lambda\mu} \theta_{\alpha\beta}. \tag{62}$$

They satisfy the symmetric closure relation

$$\left[P_{SS}^{(2)} + P_{SS}^{(1)} + P_{SS}^{(0)} + P_{WW}^{(0)}\right]_{\mu\nu\alpha\beta} = \frac{\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha}}{2}. \tag{63}$$

and the algebra

$$\left(P^{(s)}\right)_{IJ} \left(P^{(r)}\right)^{JK} = \delta^{rs} \left(P^{(s)}\right)_I^K. \tag{64}$$

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