

$\Phi - \Psi$ model for electrodynamics in dielectric media: exact quantisation in the Heisenberg representation

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Abstract We investigate the quantisation in the Heisenberg representation of a model which represents a simplification of the Hopfield model for dielectric media, where the electromagnetic field is replaced by a scalar field ϕ and the role of the polarisation field is played by a further scalar field ψ . The model, which is quadratic in the fields, is still characterised by a non-trivial physical content, as the physical particles correspond to the polaritons of the standard Hopfield model of condensed matter physics. Causality is also taken into account and a discussion of the standard interaction representation is also considered.

1 Introduction

In recent years, the investigations on possible revelations of the Hawking effect in analogues realised in dielectric media, [1–9], have raised the necessity of disposing of a model describing the quantum electromagnetic field interacting with a dispersive medium, reproducing the typical phenomenological dispersion relations.

We recall that in the Hopfield model, a purely phenomenological quantisation of the electromagnetic field in the dielectric medium is replaced by a picture where the electromagnetic field interacts with a set of oscillators reproducing sources for dispersive properties of the electromagnetic field in matter [10–13]. We stress that we do not take into account absorption in our paper, which is reasonable as far as the phenomena one is interested in are not too close to the absorption

region. Including absorption would imply a much more tricky approach (cf. e.g. [14–16]), which is far beyond the scope of the present paper.

Our model satisfies the requirement to be fully relativistic covariant. This is in agreement with the necessity to properly simulate electrodynamics of moving media [17–24], where phenomenological electrodynamics is adopted. Also in our case, in order to get a complete analysis, one needs to change the inertial frame passing, for example, from the frame where the medium is at rest, to the frame where a given signal is at rest, or to the lab frame if it does not coincide with one of them. With this in mind, a set of models taking into account the dispersion relations have been developed, based on a covariant reformulation of the Hopfield model [25–28]. In particular, in [25] the Hopfield model has been presented in a simplified version, where, in a two dimensional model, the electromagnetic field has been replaced by a massless scalar field, linearly coupled to a polarisation field, represented by a field of oscillators with characteristic pulsation ω_0 . This had the aim of simplifying several technical complications, but keeping the main relevant characteristics, as Lorentz covariance and the right dispersion relation. In the latter reference, we were interested in the Hawking effect and we did not focus on a systematic quantisation of the model. A perturbative quantisation in a given gauge and in the lab frame of the whole homogeneous and isotropic relativistic Hopfield model has been presented in [27], and a Lorentz and gauge covariant perturbative quantisation has been provided in [28]. However, this model can be quantised exactly, the exact quantisation is involved with some significant physical characteristics of the given system, and in particular in its spectral properties, so quantizing non-perturbatively in the Heisenberg representation of quantum field theory is far more than a simple and straightforward exercise. The exact

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quantisation of the relativistic Hopfield model will be presented elsewhere [29]: its construction is quite involved and passes through several technical intricacies, which go beyond the ones characterising the present model.

Since the main steps characteristic of the model are present also in the scalar simplified analogue, without the intricacies due to gauge invariance and the presence of unphysical modes, we will present here the exact quantisation of the scalar Hopfield model, which we dub the $\Phi - \Psi$ model. Since our aim is to illustrate here the strategy of [29] without hiding it behind technical details, we will mainly present the results without proves, illustrating the main steps at an intuitive level, by mean of precise statements, which will be fully proved, in a more general form, in [29] and [30].

In Sect. 2 we present the model and its quantisation, by showing how the Fock representation for the whole interacting model can be realised. In Sect. 3 we show how the problem of causality is related to Lorentz covariance and show how it can be realised in our model, which is only covariant and not invariant under Lorentz transformations. In Sect. 4 we compute the propagator both directly from the Fock representation and with the path integral method, and state their equivalence. In Sect. 5 we discuss the interaction representation and the Fano diagonalisation method. In Sect. 6 we present some further discussion. All statements are simple consequences of the ones proved in [29] and [30].

2 The $\Phi - \Psi$ model and its quantisation

We consider the $D + 1$ dimensional $\Phi - \Psi$ model whose classical dynamics is defined by the action

$$S[\phi, \psi] = \int d^{D+1}x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} v^\mu \partial_\mu \psi v^\nu \partial_\nu \psi - \frac{\omega_0^2}{2} \psi^2 - g \phi v^\mu \partial_\mu \psi \right] = \int d^{D+1}x \mathcal{L}_c, \tag{1}$$

where \mathbf{v} is the spacetime velocity of the rest frame for the ψ field. The conjugate momenta are

$$\pi_\phi = \partial_t \phi, \tag{2}$$

$$\pi_\psi = v^0 v^\mu \partial_\mu \psi - g v^0 \phi, \tag{3}$$

so that the Hamiltonian is

$$H = \int_{\mathbb{R}^D} d^D \mathbf{x} \left[\frac{\pi_\phi^2}{2} + \frac{\pi_\psi^2}{2v_0^2} + \frac{\pi_\psi}{v_0} \left(g\phi - \frac{\mathbf{v}}{v^0} \cdot \nabla \psi \right) + \frac{1}{2} \nabla \phi \cdot \nabla \phi + \frac{g^2}{2} \phi^2 + \frac{\omega_0^2}{2} \psi^2 \right]. \tag{4}$$

The classical equations of motion in the Fourier space are

$$\mathcal{M}V \equiv \begin{pmatrix} -k^2 & -ig\omega \\ ig\omega & -\omega^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5}$$

where $\omega := k^\mu v_\mu$. The dispersion relation is given by $\det \mathcal{M} = 0$, that is

$$DR(\mathbf{k}) := k^2 - \frac{g^2 \omega^2}{\omega^2 - \omega_0^2} = 0. \tag{6}$$

This defines the support of the solutions in the momentum space, with two positive branches, corresponding to the two solutions having positive ω , which we will indicate with $k_{(a)}^0$, $a = 1, 2$. Everywhere the suffix (a) will mean ‘‘evaluated at $k^0 = k_{(a)}^0(\mathbf{k})$ ’’. We will also use the symbol \mathbf{k} for the spatial component of a spacetime vector \mathbf{k} , and similar for all vectors, whereas $k^2 := \mathbf{k} \cdot \mathbf{k}$.

The classical solutions of the equations of motion are

$$\phi(\mathbf{x}) = \sum_{a=1}^2 \int_{\mathbb{R}^D} d\mu(\mathbf{k}) \left[a_{(a)}(\mathbf{k}) e^{-i\mathbf{k}_{(a)} \cdot \mathbf{x}} + a_{(a)}^\dagger(\mathbf{k}) e^{i\mathbf{k}_{(a)} \cdot \mathbf{x}} \right], \tag{7}$$

$$\pi_\phi(\mathbf{x}) = \sum_{a=1}^2 \int_{\mathbb{R}^D} d\mu(\mathbf{k}) (-ik_{(a)}^0) \left[a_{(a)}(\mathbf{k}) e^{-i\mathbf{k}_{(a)} \cdot \mathbf{x}} - a_{(a)}^\dagger(\mathbf{k}) e^{i\mathbf{k}_{(a)} \cdot \mathbf{x}} \right], \tag{8}$$

$$\psi(\mathbf{x}) = \sum_{a=1}^2 \int_{\mathbb{R}^D} d\mu(\mathbf{k}) \frac{ig\omega_{(a)}}{\omega_{(a)}^2 - \omega_0^2} \left[a_{(a)}(\mathbf{k}) e^{-i\mathbf{k}_{(a)} \cdot \mathbf{x}} - a_{(a)}^\dagger(\mathbf{k}) e^{i\mathbf{k}_{(a)} \cdot \mathbf{x}} \right], \tag{9}$$

$$\pi_\psi(\mathbf{x}) = \sum_{a=1}^2 \int_{\mathbb{R}^D} d\mu(\mathbf{k}) \frac{gv^0 \omega_0^2}{\omega_{(a)}^2 - \omega_0^2} \left[a_{(a)}(\mathbf{k}) e^{-i\mathbf{k}_{(a)} \cdot \mathbf{x}} + a_{(a)}^\dagger(\mathbf{k}) e^{i\mathbf{k}_{(a)} \cdot \mathbf{x}} \right], \tag{10}$$

where

$$d\mu(\mathbf{k}) := \frac{d^D \mathbf{k}}{(2\pi)^D} \frac{1}{DR'_{(a)}} \tag{11}$$

and

$$DR'_{(a)}(\mathbf{k}) := \frac{dDR}{dk^0}(\mathbf{k}_{(a)}) = 2k_{(a)}^0 + 2 \frac{g^2 \omega_0^2 \omega_{(a)} v^0}{(\omega_{(a)}^2 - \omega_0^2)^2}. \tag{12}$$

The set of such functions

$$\Psi = \begin{pmatrix} \phi \\ \psi \\ \pi_\phi \\ \pi_\psi \end{pmatrix} \tag{13}$$

is endowed with the conserved scalar product

$$(\Psi_1 | \Psi_2) = i \int_{\mathbb{R}^D} d^D \mathbf{x} \Psi_1^*(\mathbf{x}) \Omega \Psi_2(\mathbf{x}), \tag{14}$$

with

$$\Omega = \begin{pmatrix} \mathbb{O}_2 & \mathbb{I}_2 \\ -\mathbb{I}_2 & \mathbb{O}_2 \end{pmatrix}. \tag{15}$$

A basis of positive norm plane waves is

$$\zeta_{(a)}(\mathbf{k}; \mathbf{x}) = e^{-i\mathbf{k}^{(a)} \cdot \mathbf{x}} \begin{pmatrix} 1 \\ ig \frac{\omega_{(a)}}{\Omega_{(a)}^2 - \omega_0^2} \\ -ik_{(a)}^0 \\ gv^0 \frac{\omega_0^2}{\omega_{(a)}^2 - \omega_0^2} \end{pmatrix}. \tag{16}$$

Notice that

$$(\zeta_{(a)}(\mathbf{k})|\zeta_{(b)}(\mathbf{q})) = \delta_{ab}\delta^D(\mathbf{k} - \mathbf{q})(2\pi)^D DR'_{(a)}(\mathbf{k}), \tag{17}$$

which gives

$$a_{(a)}(\mathbf{k}) = (\zeta_{(a)}(\mathbf{k})|\Psi) \tag{18}$$

and then

$$[a_{(a)}(\mathbf{k}), a_{(b)}^\dagger(\mathbf{q})] = (\zeta_{(a)}(\mathbf{k})|\zeta_{(b)}(\mathbf{q})) \\ = \delta_{ab}\delta^D(\mathbf{k} - \mathbf{q})(2\pi)^D DR'_{(a)}(\mathbf{k}), \tag{19}$$

and all other commutators vanish. This result has been obtained by imposing the requirement that the quantum fields satisfy the equal time canonical commutation relations, which, as a consequence, are satisfied. This way, one can proceed in the usual way in constructing the Fock space, starting from the vacuum state Ω , the unique normalised state that is annihilated by all $a_{(a)}(\mathbf{k})$, we can realise the Fock space as the completion of the set of states generated by all polynomial actions of the creator fields $a_{(a)}^\dagger(\mathbf{k})$. This is standard and free of particular difficulties, apart from the fact that the vacuum state is unique only after fixing a choice of \mathbf{v} , since the theory is not invariant under the whole Poincaré group, but only under the subgroup leaving \mathbf{v} invariant.

Here, we simply notice that the fact that the CCR are satisfied, together with Lorentz covariance, allows one to prove that the principle of causality is satisfied. Since the explicit presence of \mathbf{v} breaks the Lorentz invariance, the question of the covariance is a little bit delicate and requires a careful analysis.

3 Causality and covariance

The algebra of quantum fields is generated by the canonical commutation relations (CCR):

$$[\phi(t, \mathbf{x}), \pi_\phi(t, \mathbf{y})] = i\delta^D(\mathbf{x} - \mathbf{y}), \\ [\psi(t, \mathbf{x}), \pi_\psi(t, \mathbf{y})] = i\delta^D(\mathbf{x} - \mathbf{y}),$$

where we indicated only the non-zero contributions. Causality conditions are apparently a little bit stronger:

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\phi(\mathbf{x}), \psi(\mathbf{y})] = [\phi(\mathbf{x}), \pi_\psi(\mathbf{y})] \\ = [\psi(\mathbf{x}), \psi(\mathbf{y})] = 0, \\ [\psi(\mathbf{x}), \pi_\phi(\mathbf{y})] = [\pi_\phi(\mathbf{x}), \pi_\phi(\mathbf{y})] = [\pi_\phi(\mathbf{x}), \pi_\psi(\mathbf{y})] \\ = [\pi_\psi(\mathbf{x}), \pi_\psi(\mathbf{y})] = 0, \\ [\phi(\mathbf{x}), \pi_\phi(\mathbf{y})] = [\psi(\mathbf{x}), \pi_\psi(\mathbf{y})] = 0, \tag{20}$$

for any pair of points \mathbf{x}, \mathbf{y} spatially separated, $(\mathbf{x} - \mathbf{y})^2 < 0$. If the theory is Lorentz invariant, then (20) follow from the CCRs, since we can change frame into the one where $x^0 = y^0 = t$ and then employ the CCRs in order to prove their vanishing. Our model Lagrangian is associated with equations of motion which are covariant with respect to the Lorentz group. This fact does not correspond to a full Lorentz invariance, due to the fact that the Lorentzian metric is not the only absolute object of the theory, but a further absolute object [31] appears: the velocity \mathbf{v} of the medium. This implies that our theory, and any covariant theory of a dielectric medium (cf. e.g. [32]), is involved with a preferred frame, which corresponds to the rest frame of the medium. The explicit presence of the vector \mathbf{v} in the Lagrangian implies that, in general, boosts are no more symmetries. This breaks the Poincaré symmetry group down to the subgroup leaving \mathbf{v} invariant. This behaviour should not be a surprise, as it is common to all the cases where e.g. Klein–Gordon equation is studied in the presence of an external potential. Loss of Lorentz invariance is evident, but, at the same time, the field equations are covariant, and solutions are transformed into solutions of the Klein–Gordon equation by the Lorentz group, provided that external potential is transformed too. See e.g. [33] (p. 516). In this sense, also our theory remains Lorentz covariant, as the equations of motions are, and the above argument proving causality applies again if the covariance is respected at the level of the representation of the quantum theory. Covariance ensures that different inertial observers perceive the same physics, i.e. are involved with the same processes with the same probability. In particular, the number of polaritons in the process remain the same. Unitary maps between Fock spaces of different inertial observer are a natural consequence of Poincaré covariance, and is a consequence of the fact that the Lagrangian is invariant under simultaneous transformations of the fields and of the vector \mathbf{v} under Poincaré group. So, at the quantum level there exists a Fock space $\mathcal{F}_{\mathbf{v}}$ for any \mathbf{v} and a set of unitary maps

$$U : G \longrightarrow \mathcal{U}, \tag{21}$$

$$\Lambda \longmapsto U(\Lambda) : \mathcal{F}_{\mathbf{v}} \longrightarrow \mathcal{F}_{U(\Lambda)\mathbf{v}}, \quad \forall \mathbf{v} \text{ timelike}, \tag{22}$$

where \mathcal{U} is the set of all possible isometric maps among Fock spaces $\mathcal{F}_{\mathbf{v}}$ and $\mathcal{F}_{\mathbf{w}}$, and $U(\Lambda)$ is defined by

$$U_{\mathbf{v}}(\Lambda)|0\rangle_{\mathbf{v}} = |0\rangle_{\Lambda\mathbf{v}}, \tag{23}$$

and

$$U_{\mathbf{v}}(\Lambda)o(\mathbf{k}; \mathbf{v})U_{\mathbf{v}}(\Lambda)^{-1} = o(U(\Lambda)\mathbf{k}; U(\Lambda)\mathbf{v}), \tag{24}$$

where o is intended to be any one among the operators $a_{(a)}(\mathbf{k}), a_{(a)}^\dagger(\mathbf{k}), a = 1, 2$.

4 The propagator

Since the theory is Gaussian, it is completely determined by the two point functions. It is given by the matrix

$$iG_{\mathbf{v}}^{IJ}(\mathbf{x}, \mathbf{y}) = \mathbf{v}\langle 0|T(\Phi^I(\mathbf{x})\Phi^J(\mathbf{y}))|0\rangle_{\mathbf{v}}, \quad I, J = 1, 2, \tag{25}$$

where

$$\Phi^1 = \phi, \quad \Phi^2 = \psi. \tag{26}$$

We will also write

$$G_{\mathbf{v}}^{IJ}(\mathbf{x}, \mathbf{y}) = G_{\mathbf{v}}^{IJ}(\mathbf{x}, \mathbf{y})_+ \theta(x^0 - y^0) + G_{\mathbf{v}}^{IJ}(\mathbf{x}, \mathbf{y})_- \theta(y^0 - x^0). \tag{27}$$

Since $G_{\mathbf{v}}^{IJ}(\mathbf{x}, \mathbf{y})_-$ is easily obtained from $G_{\mathbf{v}}^{IJ}(\mathbf{x}, \mathbf{y})_+$, we will write down only the latter:

$$iG_{\mathbf{v}}^{11}(\mathbf{x}, \mathbf{y})_+ = \mathbf{v}\langle 0|\phi(\mathbf{x})\phi(\mathbf{y})|0\rangle_{\mathbf{v}} = \sum_{a=1}^2 \int_{\mathbb{R}^D} d\mu(\mathbf{k}) e^{-i\mathbf{k}_{(a)} \cdot (\mathbf{x}-\mathbf{y})}; \tag{28}$$

$$iG_{\mathbf{v}}^{12}(\mathbf{x}, \mathbf{y})_+ = \mathbf{v}\langle 0|\phi(\mathbf{x})\psi(\mathbf{y})|0\rangle_{\mathbf{v}} = \sum_{a=1}^2 \int_{\mathbb{R}^D} d\mu(\mathbf{k}) \frac{-ig\omega_{(a)}}{\omega_{(a)}^2 - \omega_0^2} e^{-i\mathbf{k}_{(a)} \cdot (\mathbf{x}-\mathbf{y})}; \tag{29}$$

$$iG_{\mathbf{v}}^{22}(\mathbf{x}, \mathbf{y})_+ = \mathbf{v}\langle 0|\psi(\mathbf{x})\psi(\mathbf{y})|0\rangle_{\mathbf{v}} = \sum_{a=1}^2 \int_{\mathbb{R}^D} d\mu(\mathbf{k}) \frac{g^2\omega_{(a)}^2}{(\omega_{(a)}^2 - \omega_0^2)^2} e^{-i\mathbf{k}_{(a)} \cdot (\mathbf{x}-\mathbf{y})}. \tag{30}$$

The propagator can be determined also by means of the path integral formulation. After introducing the currents J_ϕ and J_ψ , we can define the functional generating the propagators:

$$Z[J_\phi, J_\psi] = \int [D\phi D\psi] \exp \left\{ i \int_{\mathbb{R}^{D+1}} \mathcal{L}_c d^{D+1}\mathbf{x} + i \int_{\mathbb{R}^{D+1}} J_\phi \phi d^{D+1}\mathbf{x} + i \int_{\mathbb{R}^{D+1}} J_\psi \psi d^{D+1}\mathbf{x} \right\}. \tag{31}$$

From this we can formally compute the propagator, which as a result is found to be

$$G_{\Phi^I \Phi^J}(\mathbf{x}, \mathbf{y}) = -i \frac{\delta Z[J_\phi, J_\psi]}{\delta J_{\Phi^I}(\mathbf{x}) \delta J_{\Phi^J}(\mathbf{y})} \Big|_{J_K} = \int_{\mathbb{R}^2} \frac{d^{D+1}\mathbf{k}}{(2\pi)^{D+1}} e^{-i\mathbf{k} \cdot (\mathbf{x}-\mathbf{y})} \mathcal{M}^{-1}(\mathbf{k})_{IJ}, \tag{32}$$

where \mathcal{M} is defined in (5), so that

$$\mathcal{M}^{-1}(\mathbf{k}) = \frac{1}{k^2(\omega^2 - \omega_0^2) - g^2\omega^2} \begin{pmatrix} -\omega^2 + \omega_0^2 & ig\omega \\ -ig\omega & -k^2 \end{pmatrix}. \tag{33}$$

Naturally, this is not the complete story, since this expression requires a prescription avoiding the poles defined by the dispersion relation. Such a prescription must respect causality. As a result this can be accomplished by means of a $i\varepsilon$ Feynman prescription. Indeed, we have the following.

Theorem 1 *The propagator is*

$$G_{\Phi^I \Phi^J}(\mathbf{x}, \mathbf{y}) = \int_{\mathbb{R}^{D+1}} \frac{d^{D+1}\mathbf{k}}{(2\pi)^{D+1}} e^{-i\mathbf{k} \cdot (\mathbf{x}-\mathbf{y})} \mathcal{M}_{i\varepsilon}^{-1}(\mathbf{k})_{IJ}, \tag{34}$$

where $\mathcal{M}_{i\varepsilon}^{-1}(\mathbf{k})$ is obtained from (33) by taking the complex shifts $k^2 \rightarrow k^2 + i\varepsilon, \omega_0^2 \rightarrow \omega_0^2 - i\varepsilon$.

This proposition is a particular case of a more general one proved in [30].

5 The interaction representation and Fano diagonalisation

In this section we take into account the more standard Interaction representation, and perform the so-called Fano diagonalisation [11] of the full Hamiltonian operator in order to find its eigenmodes. As the Hamiltonian is quadratic in the fields and their conjugate momenta, we are able to obtain an exact result which leads again to polaritons as physical states of the system. It is interesting to stress that this approach, which is pursued both in the original paper by Hopfield [10] and in standard textbooks (see e.g. [12,13]), is in principle apt to perturbation theory and leads to the same result thanks to the diagonalisation process.

In line of principle, the interaction representation is constructed by assuming that g is small and allows a perturbation theory in powers of g . For simplicity, we consider only the case where $\mathbf{v} = (c, \mathbf{0})$, and we put $c = 1$. The Hamiltonian

is then characterised by three contributions: two free-field contributions H_ϕ^0, H_ψ^0 , and an interaction one H_{int} as in the following equations:

$$H_\phi^0 = \int_{\mathbb{R}^D} d^D \mathbf{x} \left[\frac{\pi_\phi^2}{2} + \frac{1}{2} \nabla \phi \cdot \nabla \phi \right], \tag{35}$$

$$H_\psi^0 = \int_{\mathbb{R}^D} d^D \mathbf{x} \left[\frac{\pi_\psi^2}{2v_0^2} + \frac{\omega_0^2}{2} \psi^2 \right], \tag{36}$$

$$H_{int} = \int_{\mathbb{R}^D} d^D \mathbf{x} \left[g\phi \frac{1}{v_0} \pi_\psi + \frac{g^2}{2} \phi^2 \right]. \tag{37}$$

Note that the conjugate momentum π_ψ is now a free-field one (i.e. it is the one in (3) with $g = 0$), and that the dispersion relation for the free field ϕ is $(k^0)^2 - \mathbf{k} \cdot \mathbf{k} = 0$, whereas the dispersion relation for ψ is $\omega^2 - \omega_0^2 = 0$. We now write the free fields in terms of creation and annihilation operators. In order to allow for a more direct comparison with the existing literature, we choose

$$\phi(\mathbf{x}) = \int_{\mathbb{R}^D} \frac{d^D \mathbf{k}}{(2\pi)^{D/2}} \frac{1}{\sqrt{2k^0}} \left[b(\mathbf{k}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} + h.c. \right], \tag{38}$$

$$\pi_\phi(\mathbf{x}) = \int_{\mathbb{R}^D} \frac{d^D \mathbf{k}}{(2\pi)^{D/2}} \sqrt{\frac{k^0}{2}} \left[-ib(\mathbf{k}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} + h.c. \right], \tag{39}$$

$$\psi(\mathbf{x}) = \int_{\mathbb{R}^D} \frac{d^D \mathbf{k}}{(2\pi)^{D/2}} \frac{1}{\sqrt{2\omega_0}} \left[d(\mathbf{k}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} + h.c. \right], \tag{40}$$

$$\pi_\psi(\mathbf{x}) = \int_{\mathbb{R}^D} \frac{d^D \mathbf{k}}{(2\pi)^{D/2}} \sqrt{\frac{\omega_0}{2}} \left[-id(\mathbf{k}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} + h.c. \right], \tag{41}$$

with

$$[b(\mathbf{k}, t), b^\dagger(\mathbf{q}, t)] = \delta^D(\mathbf{k} - \mathbf{q}), \tag{42}$$

$$[d(\mathbf{k}, t), d^\dagger(\mathbf{q}, t)] = \delta^D(\mathbf{k} - \mathbf{q}), \tag{43}$$

and all the remaining CCRs equal to zero. We obtain

$$H_\phi^0 = \int d^D \mathbf{p} p_0 b^\dagger(\mathbf{p}, t) b(\mathbf{p}, t), \tag{44}$$

$$H_\psi^0 = \int d^D \mathbf{p} \omega_0 d^\dagger(\mathbf{p}, t) d(\mathbf{p}, t), \tag{45}$$

$$H_{int} = \int d^D \mathbf{p} \left[-i \frac{g}{2} \sqrt{\frac{\omega_0}{p_0}} \left(d(\mathbf{p}, t) b^\dagger(\mathbf{p}, t) - d^\dagger(\mathbf{p}, t) b(\mathbf{p}, t) \right) \right. \\ \left. + d(\mathbf{p}, t) b(-\mathbf{p}, t) - d^\dagger(\mathbf{p}, t) b^\dagger(-\mathbf{p}, t) \right) \\ \left. + \frac{g^2}{4p_0} \left(b(\mathbf{p}, t) b^\dagger(\mathbf{p}, t) + b^\dagger(\mathbf{p}, t) b(\mathbf{p}, t) \right) \right. \\ \left. + b(\mathbf{p}, t) b(-\mathbf{p}, t) + b^\dagger(\mathbf{p}, t) b^\dagger(-\mathbf{p}, t) \right]. \tag{46}$$

The diagonalisation process consists in finding normal mode annihilation operators,

$$\alpha(\mathbf{p}, t) = w b(\mathbf{p}, t) + x d(\mathbf{p}, t) + y b^\dagger(-\mathbf{p}, t) + z d^\dagger(-\mathbf{p}, t) \tag{47}$$

such that [10]

$$[\alpha(\mathbf{p}, t), H] = E(\mathbf{p})\alpha(\mathbf{p}, t). \tag{48}$$

The former eigenvalue problem amounts to the following one:

$$\det[A - EI] = 0, \tag{49}$$

where A is the matrix

$$A = \begin{bmatrix} p_0 + \frac{g^2}{2p_0} & i \frac{g}{2} \sqrt{\frac{\omega_0}{p_0}} & -\frac{g^2}{2p_0} & i \frac{g}{2} \sqrt{\frac{\omega_0}{p_0}} \\ -i \frac{g}{2} \sqrt{\frac{\omega_0}{p_0}} & \omega_0 & i \frac{g}{2} \sqrt{\frac{\omega_0}{p_0}} & 0 \\ \frac{g^2}{2p_0} & i \frac{g}{2} \sqrt{\frac{\omega_0}{p_0}} & -p_0 - \frac{g^2}{2p_0} & i \frac{g}{2} \sqrt{\frac{\omega_0}{p_0}} \\ i \frac{g}{2} \sqrt{\frac{\omega_0}{p_0}} & 0 & -i \frac{g}{2} \sqrt{\frac{\omega_0}{p_0}} & -\omega_0 \end{bmatrix}. \tag{50}$$

From (49) one obtains the equation

$$E^4 - E^2(|\mathbf{k}|^2 + g^2 + \omega_0^2) + |\mathbf{k}|^2 \omega_0^2 = 0, \tag{51}$$

which amounts to

$$|\mathbf{k}|^2 = E^2 \left(1 + \frac{g^2}{\omega_0^2 - E^2} \right), \tag{52}$$

which again gives the same eigenmodes as in the previous sections. In particular, two positive branches E_\pm can be obtained, with associated eigenvectors. One can obtain α_a , with $a = \pm$, which correspond to the $a_{(a)}$ given in the previous analysis.

6 Final comments

In the appendix it is shown that $DR'_{(a)}$, $a = 1, 2$, are always positive, apart in $\mathbf{k} = 0$ for the lower branch. In this case, DR' vanishes linearly in $|\mathbf{k}|$, and the integrals defining the fields (in a distributional sense) are well posed for $D > 1$. Thus, the fields and propagators are well defined as tempered distributions.

On the opposite, in a two dimensional spacetime, the integrals diverge unless the oscillator modes vanish quickly enough at the origin. This divergence at $k = 0$ cannot be interpreted as an infrared divergence, since it does not occur only in the propagator, but also in the definition of the fields, so that it needs to be eliminated by a suitable choice of the

space of test functions. Assuming that these must be chosen inside the set of smooth rapidly decreasing functions $S(\mathbb{R}^2)$, we must consider functions whose Fourier transform vanishes in $\mathbf{k} = 0$ when k^0 is evaluated on $k_{(2)}^0(k)$. Since also k^0 vanishes, by employing smoothness, it is sufficient to consider functions in $S(\mathbb{R}^2)$ whose Fourier transform vanish in $\mathbf{k} = \mathbf{0}$. These are the rapidly decreasing smooth functions having null mean. So, the test functions must be chosen in

$$S_0(\mathbb{R}^2) = \left\{ f \in S(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} d^2\mathbf{x} f(\mathbf{x}) = 0 \right\}. \tag{53}$$

Thus, it is exactly the same as for any massless free field.

After having defined the fields, we have been able to treat them exactly, being the action quadratic. However, despite the theory is Gaussian and, then, essentially a free theory, in a sense the interaction manifests itself through a highly non-trivial dispersion relation.

The second difficulty, related to causality, is covariance, which is realised in a non-trivial way. We have just sketched how this can be done: in practice the quantum algebra is represented on a bundle of Fock spaces over the homogeneous space¹

$$\mathcal{B} = \mathcal{P}/G, \tag{54}$$

where \mathcal{P} is the Poincaré group and G its subgroup that leaves \mathbf{v} invariants (the little group of \mathbf{v}).

Finally, we have computed the two point function, which characterises the whole theory, being Gaussian. We have done it both starting from the canonical representation and with the path integral method. Here, the $i\epsilon$ Feynman–Stückelberg prescription has been introduced and stated to be equivalent to the causal propagator computed in the oscillator representation. The proof can be found in [30].

In the relativistic Hopfield model, the true target of all our efforts, all these difficulties are present and amplified by the presence of a higher number of field components, including non-physical ones, a larger number of spectral branches, the

necessity of taking under control the gauge symmetry, the presence of constraints and of the dipole ghost, and a major involution of all explicit formulae. However, all these additive complications, which one is required to overcome, are not peculiar of the specific model and ends up in hiding the specific ones, which are transparent in the simplified $\Psi - \Phi$ model we have presented here.

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Appendix A: On the dispersion relation

It may be useful to analyse some properties of the dispersion relations. They can be exactly solved in the lab frame (defined by $\mathbf{v} \equiv (1, \mathbf{0})$). There are four real solutions for k^0 , which are $k_{(1)}^0 = \omega_+(\mathbf{k})$, $k_{(2)}^0 = \omega_-(\mathbf{k})$, $k_{(3)}^0 = -\omega_-(-\mathbf{k})$, $k_{(4)}^0 = -\omega_+(-\mathbf{k})$, with

$$\omega_{\pm}(\mathbf{k}) = \frac{1}{2}\sqrt{(\omega_0 + |\mathbf{k}|)^2 + g^2} \pm \frac{1}{2}\sqrt{(\omega_0 - |\mathbf{k}|)^2 + g^2}. \tag{A.1}$$

These four branches, two positive and two negative, are represented in Fig. 1 (in the $D = 1$ case), from which it is also evident that $DR'_{(a)}$ is positive for $a = 1, 2$ and negative for $a = 3, 4$, and vanishes only in the origin, thus for $\mathbf{k} = 0$ in the branches 2, 3. In this limit we see that

$$\omega_- = \frac{\omega_0}{\omega_+} |\mathbf{k}| \approx \frac{\omega_0}{\sqrt{\omega_0^2 + g^2}} |\mathbf{k}|, \tag{A.2}$$

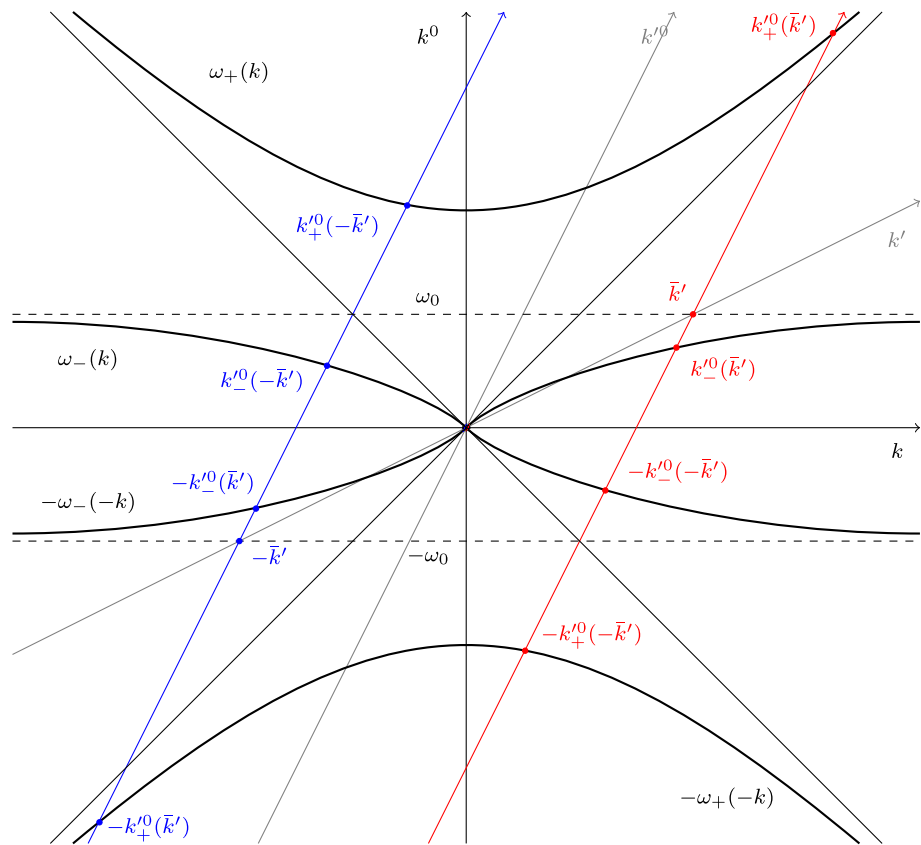
and then DR' vanishes linearly in $|\mathbf{k}|$.

¹ Since \mathbf{v} is assumed to be a future directed timelike vector of norm 1, and P acts transitively on the set of such vectors,

$$\mathcal{B} \simeq \{ \mathbf{x} \in \mathbb{R}^{1,3} \mid x^2 = 1, x^0 > 0 \},$$

which is the future paraboloid of mass 1 in the Minkowski space $\mathbb{R}^{1,3}$.

Fig. 1 The *thick black lines* are the solutions of dispersion relations in the lab frame. Above the *upper* one $DR(\mathbf{k})$ is positive. Below it is negative until reaching the asymptote $k^0 = \omega_0$. Below it DR is again positive and it changes sign in crossing the line $k^0 = \omega_-(k)$. This behaviour continues symmetrically in the *lower* half plane. The *grey lines* represent the axes of a boosted frame and the *red line* is at fixed k' . It cuts the curves $DR = 0$ from increasing values of DR (when k'^0 increases) in the *upper* half plane and with decreasing sign in the *lower* half plane



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