Eur. Phys. J. C (2016) 76:267
DOI 10.1140/epjc/s10052-016-4109-6

THE EUROPEAN PHYSICAL JOURNAL C

CrossMark

Regular Article - Theoretical Physics

Cavity evolution and instability constraints of relativistic interiors

Z. Yousaf[a], **M. Zaeem ul Haq Bhatti**[b]

Department of Mathematics, University of the Punjab, Quaid-i-Azam Campus, Lahore 54590, Pakistan

Received: 9 April 2016 / Accepted: 19 April 2016 / Published online: 12 May 2016

Abstract In this manuscript, we have identified the dynamical instability constraints of a self-gravitating cylindrical object within the framework of the $f(R, T)$ theory of gravity. We have explored the modified field equations and the corresponding dynamical equations for the systematic constructions of our analysis. We have imposed the linear perturbations on the metric and material variables with some known static profile up to first order in the perturbation parameter. The role of the expansion scalar is also examined in this scenario. The instability regimes have been discussed against the background of the Newtonian and post-Newtonian limits. We found that the dark source terms due to the influence of the modification in the gravity model is responsible for the instability of the system.

1 Introduction

The global properties of our universe have been studied extensively by the researchers against the background of general theory of relativity (GR). Many new insights of astrophysics and cosmology have revealed an unexpected picture of the universe. The latest data of some reliable sources such as supernovae surveys and cosmic microwave background radiation have put forth the dark side of the universe. Therefore, one has to accept that the current matter-energy contents and the evolutionary picture of the universe is astonishing and requires some explanation. During the last couple of decades, the discussions in the standard GR due to the observational evidence is unable to describe the key features of the present low energy universe without imposing certain assumptions. Particularly, it requires the inclusion of some exotic contribution in the matter contents of the cosmos in order to study the dynamics of stars and their clusters as well as the current accelerated expansion of the universe.

[a] e-mail: zeeshan.math@pu.edu.pk

[b] e-mail: mzaeem.math@pu.edu.pk

The modified gravity approach indicates that the accelerated expansion is due to the influence of a modification in gravity for late/early time universe. Some generalizations have been proposed for GR since its inception, most of them have not passed the test of time. It is worth mentioning that any reasonable gravity theory should reduce to Newtonian (N) gravity for a slowly moving weak source. An additional degree of freedom exists generically in any modification of GR. There exist different modifications in the Einstein–Hilbert (EH) action describing the dark components (i.e., dark energy and dark matter) of the current accelerated expanding universe. Dark energy is used to explain the cosmic speed up and dark matter is used to explain the emergence of large scale structures in the universe. Thus, a wide class of gravity theories exist to study the dark side of the universe via the enhancement of the gravitational force. The $f(R, T)$ gravity theory is one of such generalizations of Einstein's theory of gravity, which rests on the matter and geometry coupling. In this theory, the Lagrangian for the EH action includes the extra degrees of freedom along with the trace of stress-energy tensor.

The exploration of the instability regimes for a collapse process strengthened the study of astronomical and astrophysical theories. The gravitational test through pulsar-timing experiments have motivated one to study the stability issue. Harada [1] presented the stability analysis in scalar tensor theory for a spherically symmetric star configuration and extracted the range of instability from the first order derivative of the coupling function. In the study of gravitational instability theory, the amplifications in the density perturbations are responsible for the generation of cosmic structures in the early universe. For a time dependent mass density, the instability for a gravitating system has been discussed in the framework of $f(R)$ theory [2]. Bamba et al. [3] have studied the matter instability describing the curvature inside the sphere in $f(R)$ gravity theory, which is one of the most important criteria to check the validity of any modified gravity theory. It has been the matter of interest for relativists that the dynamical equations are developed due to the Bianchi identities and the Einstein equations. Also, an exact evolution equation from the Lagrangian can be obtained through Bianchi identities representing the gravitational tidal field. Sharif and Yousaf [4–10] have explored the instability conditions of stellar systems at both the N and the post-Newtonian (pN) eras with different backgrounds in modified gravity theories.

Nojiri and Odintsov [11] studied the behavior of modified gravity on some solar system tests for Newton law corrections and matter instabilities with the effective cosmological constant regime in the late and early universe. Tiret and Combes [12] presented the dynamical evolution of spiral galaxies in the modified dynamics, which is compared to gravity with dark matter with numerical simulations. Bogdanos and Saridakis [13] explored the perturbative instabilities by imploding scalar and tensor perturbations within a flat background in Hořava gravity. Some astrophysical tests for modified gravity theories have been presented by Jain et al. [14], using low-redshift distance indicators to carry out tests, mainly focusing on particular stages on the evolution of supergiants and giants to observe distinct observational signatures.

The motion of the matter can be characterized by the fluid parameters like four-acceleration, shear tensor, expansion scalar, and vorticity tensor (which is zero for spherical stars). The significance of shear scalar and its vanishing have been put forth by many researchers for self-gravitating stars. The expansion scalar measures the change in the volume element of the fluid configuration during the evolution and its absence leads to the formation of a cavity within the system. This is based on the reason that during the evolution the system is expanding, leading to an increase in the volume element due to the increase in the external boundary. The increase in the volume is compensated by the formation of a cavity inside the system by imposing the expansion-free condition and the innermost shell would be away from the center.

Initially, Skripkin [15] analyzed the appearance of such kinds of vacuum cavity during the evolution of spherically symmetric models, which has significance in the modeling of voids. Later, it was found [16] that the Skripkin model has no compatibility with the Darmois matching conditions [17]. In the same paper, they also examined that spherical stars evolving under the expansion-free condition must have an inhomogeneous energy density. The formation of a cavity using the kinematical quantities different from the zero expansion condition has also been investigated in the literature [18]. Herrera et al. [19] also explored the instability eras for a spherically symmetric collapsing model due to zero expansion in the fluid configuration. Sharif and Bhatti [20] explored instability conditions of cylindrically symmetric self-gravitating systems coupled with a charged expansion-free anisotropic matter distribution.

In the study of stellar structures, it is common to model the star interior with a perfect fluid, which implies the same pressure in the interior of the compact object. Some theoretical advances in recent decades indicate the deviation from an isotropic pressure, particularly in the high density regimes, in the study of their properties. Weber [21] proposed that strong magnetic fields play a role in generating a pressure anisotropy inside a compact star. It is also observed that anisotropy of pressures is present in wormholes [22] or gravastars [23,24], so-called exotic solutions of the field equations. General relativistic stellar models gain crucial significance due to the existence of pressure anisotropy in the matter distribution [25,26]. Sharif et al. [27–32] investigated different effects of physical parameters on the dynamical instability of self-gravitating collapsing stars. The existence of various compact objects in the realm of $f(R)$ gravity has been investigated [33,34]. Recently, Yousaf et al. [35] explored the importance of energy density inhomogeneities in the study of stellar collapse.

The current paper presents a full analytical approach as has been initiated to understand the instability regimes of a cylindrical object within the physical background of $f(R,T)$ dark sources. The organization of this paper is as follows. The next section explores some basics of the $f(R,T)$ theory of gravity, including the modified field equations and kinematical quantities. In Sect. 3, we have provided the perturbation scheme up to first order to analyze the stability of our gravitating source. In Sect. 4, we have found the collapse equation by exploring the dynamical equation in the c.g.s unit systems. Section 5 investigates the instability ranges in the N limit with the zero expansion condition. The last section concludes our main findings.

2 The $f(R,T)$ gravity and cylindrical systems

The notion of $f(R,T)$ gravity as a possible modification in the gravitational framework of GR received much attention. This theory provides numerous interesting results in the field of physics and cosmology, like a plausible explanation of the accelerating cosmic expansion [36–39]. The main theme of this theory is to use an algebraic general function of Ricci as well as the trace of the energy-momentum tensor in the standard EH action. It can be written as [40]

$$S_{f(R,T)} = \int d^4x\sqrt{-g}[f(R,T) + L_M], \qquad (1)$$

where g, T are the traces of the metric as well as the standard GR energy-momentum tensors, respectively, while R is the Ricci scalar. There exists a variety L_M in the literature which corresponds to particular configurations of relativistic matter distributions. Choosing $L_M = \mu$ (where μ is the system's energy density) and varying the above action with respect to

$g_{\alpha\beta}$, the corresponding $f(R,T)$ field equations are given as follows:

$$G_{\alpha\beta} = T_{\alpha\beta}{}^{\text{eff}}, \tag{2}$$

where

$$T_{\alpha\beta}{}^{\text{eff}} = \Bigg[(1+f_T(R,T))T^{(m)}_{\alpha\beta} - \mu g_{\alpha\beta} f_T(R,T) - \left(\frac{f(R,T)}{R} - f_R(R,T)\right)\frac{R}{2} + + \left(\nabla_\alpha\nabla_\beta + g_{\alpha\beta}\Box\right) f_R(R,T)\Bigg]\frac{1}{f_R(R,T)}$$

is the effective energy-momentum tensor representing the modified version of gravitational contribution coming from the $f(R,T)$ extra degrees of freedom, while $G_{\alpha\beta}$ is the Einstein tensor. Further, ∇_α represents a covariant derivation, while $f_T(R,T)$, $\Box$, and $f_R(R,T)$ indicate the $\frac{\mathrm{d}f(R,T)}{\mathrm{d}T}$, $\nabla_\alpha\nabla^\alpha$, and $\frac{\mathrm{d}f(R,T)}{\mathrm{d}R}$ operators, respectively.

The system under consideration is modeled as a cylindrical stellar object whose relativistic motion is characterized by a three-dimensional timelike surface represented by $\Sigma^{(e)}$. This boundary demarcated our manifold into two different interior and exterior portions. These regions are denoted by $\mathcal{V}^-$ and $\mathcal{V}^+$, respectively. The $\mathcal{V}^-$ region can be described with the help of the following non-rotating diagonal spacetime [41]:

$$\mathrm{d}s_-^2 = A^2(t,r)\mathrm{d}t^2 - B^2(t,r)\mathrm{d}r^2 - C^2(t,r)\mathrm{d}\phi^2 - \mathrm{d}z^2, \tag{3}$$

while the spacetime for $\mathcal{V}^+$ is [43]

$$\mathrm{d}s_+^2 = \left(-\frac{2M}{r}\right)\mathrm{d}\nu^2 + 2\mathrm{d}\nu\mathrm{d}r - r^2(\mathrm{d}\phi^2 + \zeta^2\mathrm{d}z^2), \tag{4}$$

where M is a cylindrical gravitating mass, ν is the retarded time, and ζ indicates an arbitrary constant. The mathematical formula describing the fluid distribution within the cylindrical relativistic interior is [44]

$$T^-_{\alpha\beta} = (\mu + P_r)V_\alpha V_\beta - P_r g_{\alpha\beta} + (P_z - P_r)S_\alpha S_\beta + (P_\phi - P_r)K_\alpha K_\beta, \tag{5}$$

where P_ϕ, P_r, and P_z are stresses corresponding to the ϕ, r, and z directions, respectively. Here V_β and K_β, S_β are four-velocity and four-vectors, respectively, which under the comoving coordinate system

$$V_\beta = A\delta^0_\beta, \quad S_\beta = \delta^3_\beta, \quad K_\beta = C\delta^2_\beta,$$

obey some relations. These are given as follows:

$$V^\beta V_\beta = -1, \quad K^\beta K_\beta = S^\beta S_\beta = 1,$$
$$V^\beta K_\beta = S^\beta K_\beta = V^\beta S_\beta = 0.$$

The scalar variable controlling expansion and contraction of matter distribution is known as the expansion scalar. This can be obtained through mathematical expression $\Theta = V^\alpha_{;\alpha}$. The expansion scalar associated with cylindrically symmetric relativistic interior is

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right), \tag{6}$$

where an over dot symbolizes temporal partial differentiation. The corresponding Ricci scalar is

$$R(t,r) = \frac{2}{B^2}\left[\frac{A''}{A} + \frac{C''}{C} + \frac{A'}{A}\left(\frac{C'}{C} - \frac{B'}{B}\right) - \frac{B'C'}{BC}\right] - \frac{2}{A^2}\left[\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \frac{\dot{B}\dot{C}}{BC}\right], \tag{7}$$

where a prime means radial partial differentiation. The $f(R,T)$ field equation (2) for the metric (3) gives the following set of equations:

$$G_{00} = \frac{A^2}{f_R}\left[\mu + \frac{R}{2}\left(\frac{f}{R} - f_R\right) + \psi_{00}\right], \quad G_{01} = \psi_{01}, \tag{8}$$

$$G_{11} = \frac{B^2}{f_R}\left[P_r(1+f_T) + \mu f_T - \frac{R}{2}\left(\frac{f}{R} - f_R\right) + \psi_{11}\right], \tag{9}$$

$$G_{22} = \frac{C^2}{f_R}\left[P_\phi(1+f_T) + \mu f_T - \frac{R}{2}\left(\frac{f}{R} - f_R\right) + \psi_{22}\right], \tag{10}$$

$$G_{33} = \frac{1}{f_R}\left[P_z(1+f_T) - \mu f_T - \frac{R}{2}\left(\frac{f}{R} - f_R\right) + \psi_{33}\right], \tag{11}$$

where

$$\psi_{00} = \frac{\partial_{rr} f_R}{B^2} - \frac{\partial_t f_R}{A^2}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\frac{\dot{f}_R}{A^2} - \frac{\partial_r f_R}{B^2}\left(\frac{B'}{B} - \frac{C'}{C}\right), \tag{12}$$

$$\psi_{01} = \frac{1}{f_R}\left(\partial_r\partial_t f_R - \frac{A'}{A}\partial_t f_R - \frac{\dot{B}}{B}\partial_r f_R\right), \tag{13}$$

$$\psi_{11} = \frac{\partial_t\partial_t f_R}{A^2} + \left(\frac{\dot{C}}{C} - \frac{\dot{A}}{A}\right)\frac{\partial_t f_R}{A^2} - \left(\frac{A'}{A} + \frac{C'}{C}\right)\frac{\partial_r f_R}{B^2}, \tag{14}$$

$$\psi_{22} = \frac{\partial_{tt} f_R}{A^2} + \frac{\partial_{rr} f_R}{B^2} + \left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right)\frac{\partial_t f_R}{A^2} + \left(\frac{B'}{B} - \frac{A'}{A}\right)\frac{\partial_r f_R}{B^2}, \tag{15}$$

$$\psi_{33} = \frac{\partial_{tt} f_R}{A^2} - \frac{\partial_{rr} f_R}{B^2} + \left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\frac{\partial_t f_R}{A^2} + \left(\frac{B'}{B} - \frac{A'}{A} - \frac{C'}{C}\right)\frac{\partial_r f_R}{B^2}. \tag{16}$$

Now, we are interested to formulate two equations describing the dynamical evolution of cylindrical relativistic interiors framed within $f(R,T)$ background. In $f(R,T)$ gravitational theory, the divergence of the stress-energy tensor is non-vanishing and is obtained thus:

$$\nabla^{\alpha}T_{\alpha\beta} = \frac{f_T}{1-f_T}\left[(\Theta_{\alpha\beta}+T_{\alpha\beta})\nabla^{\alpha}\ln f_T-\frac{1}{2}g_{\alpha\beta}\nabla^{\alpha}T+\nabla^{\alpha}\Theta_{\alpha\beta}\right]. \tag{17}$$

The divergence of $f(R,T)$ energy-momentum tensor yields the following couple of continuity equations:

$$\dot{\mu}\left(\frac{1+f_T+f_Rf_T}{f_R(1+f_T)}\right)-\frac{\mu}{f_R}\partial_t f_R - \frac{B\dot{B}}{A^2f_R}(1+f_T)(\mu+P_r)-\frac{C\dot{C}}{A^2f_R}(1+f_T)(\mu+P_\phi) + \frac{2\mu}{1+f_T}\partial_t f_T+\frac{\partial_t T}{2(1+f_T)}+D_0=0, \tag{18}$$

$$\frac{P_r'}{f_R}+\frac{P_r}{f_R}\left\{\partial_r f_T-\frac{(1+f_T)\partial_r f_R}{f_R}\right\} - \frac{AA'}{A^2f_R}(1+f_T)(\mu+P_r)+\frac{\mu'}{f_R}+\frac{\mu}{f_R} \times\left\{\partial_r f_T-\frac{f_T\partial_r f_R}{f_R}\right\}-\frac{(\mu-P_r)}{(1+f_T)}\partial_r f_T + \frac{f_T}{(1+f_T)}\left(\mu'+\frac{T'}{2}\right)+\frac{C'}{CB^2f_R} \times(1+f_T)(P_r-P_\phi)+D_1=0. \tag{19}$$

These are the required couple of dynamical equations obtained through the contracted Bianchi identities of the $f(R,T)$ effective stress-energy tensor. It is well known that these dynamical equations assist enough to help to analyze the dynamical evolution of the collapse of the stellar system with the passage of time. This also helps to explore the total energy variation within the collapsing celestial self-gravitating systems in regard with time and adjacent boundaries. MacCallum et al. [42] gave a nice way to discuss perturbed boundary conditions in joining a matter filled interior with the asymptotically flat vacuum exterior solution. In the above equations, the quantities D_0 and D_1 are functions of t and r and represent extra curvature dark source terms emerging from the $f(R,T)$ gravitational field. The quantities D_0 and D_1 describe $f(R)$ corrections in the energy variations of the collapsing cylindrical relativistic interior associated with time and adjacent surfaces, respectively. These corrections are given in Appendix.

The matter quantity of cylindrical collapsing stellar geometry can be defined through the gravitational C-energy, which was proposed by Thorne [45]. Thus we obtain

$$m(t,r)=\left\{1-\left(\frac{C'}{B}\right)^2-\left(\frac{\dot{C}}{A}\right)^2\right\}\frac{l}{8}, \tag{20}$$

where l indicates the specific cylindrical length. Before calculating its variations among adjacent surfaces of a cylindrical anisotropic fluid distribution, we shall define some operators. The proper and radial derivative operators are defined as follows:

$$D_T=\frac{1}{A}\frac{\partial}{\partial t},\quad D_C=\frac{1}{C'}\frac{\partial}{\partial r}. \tag{21}$$

The relativistic velocity of the collapsing stellar interior can be obtained with the help of the proper derivative operator:

$$U=D_TC=\frac{\dot{C}}{A}. \tag{22}$$

Using Eqs. (20) and (22), we obtain

$$\tilde{\mathrm{E}}\equiv\frac{C'}{B}=\left[1-\frac{8}{l}m(t,r)+U^2\right]^{1/2}. \tag{23}$$

Next, from Eqs. (20), (21), and (23), it follows that

$$D_Cm=\frac{l}{4f_R}\left[\mu+\frac{R}{2}\left(\frac{f}{R}-f_R\right)+\psi_{00}-\frac{\psi_{01}}{BA}\frac{U}{\tilde{\mathrm{E}}}\right]C. \tag{24}$$

This equation provides us with the total energy variation between adjacent surfaces within the matter configuration. From the above equation it is evident that the first part of the right hand side is due to the energy density and $f(R)$ higher curvature quantities. The last term in the above equation is $(\overset{(H)}{T_{01}}/BA)(U/\tilde{\mathrm{E}})<0$. In this term, the quantity U represents the collapsing matter velocity. It is well known that U is less than zero for the collapsing fluid models. Thus, the last term (containing U and non-attractive $f(R)$ corrections) lessens the fluid energy and influences the evolutionary system phases. Equation (16) upon integration yields

$$m=\frac{1}{4}\int_0^C\left[\frac{lC}{f_R}\left(\mu+\frac{R}{2}\left(\frac{f}{R}-f_R\right)+\psi_{00}-\frac{\psi_{01}}{BA}\frac{U}{\tilde{\mathrm{E}}}\right)\right]dC. \tag{25}$$

It is well known that zero expansion leads to the emergence of boundary surfaces in which one (outer) demarcates the relativistic interior fluid from the exterior spacetime, while the second (inner) separates the Minkowskian core from the matter distribution. Under a null expansion scalar framework, the relativistic fluid evolves without being compressed. For example, during expansion of the self-gravitating system, the increase in volume of the matter configuration leads to expansion of the external boundary surface, which can be counterbalanced by a similar expansion of the internal surface to make Θ zero. Thus, a zero expansion scalar triggers evolution of the relativistic system in such a way that the inner most shell moves away from the central point, thereby causing the emergence of a vacuum core. Due to this zero expansion, matter sources could be effective for the explanation of the voids. Voids are, roughly speaking, underdense areas incorporating a substantial amount of information on the cosmological environment [46]. Voids provide a reliable guide to study the large scale cosmic structure formation. They are richer in modified gravity [47] as compared to GR, as this extended gravity theory is more likely to host large structures with smaller radii.

The continuity of Eqs. (3) and (4) over $\Sigma^{(e)}$ can be obtained by using the Darmois matching conditions [48]

$$m(t,r)-M\overset{\Sigma^{(e)}}{=}\frac{l}{8},\quad l\overset{\Sigma^{(e)}}{=}4C, \tag{26}$$

$$P_r\overset{\Sigma^{(e)}}{=}\left[\mu f_T-\frac{f-Rf_R}{2}\right]\left[1+\frac{(1+f_T)}{f_R}\right]^{-1}, \tag{27}$$

where $\overset{\Sigma^{(e)}}{=}$ means that measurements are performed over an outer hypersurface. The matching conditions over $\Sigma^{(i)}$ lead to

$$m(t,r)\overset{\Sigma^{(i)}}{=}0,$$

$$P_r\overset{\Sigma^{(i)}}{=}\left(\mu f_T-\frac{f-Rf_R}{2}\right)\left(1+\frac{(1+f_T)}{f_R}\right)^{-1}. \tag{28}$$

To present $f(R,T)$ gravity as a cosmologically and theoretically consistent theory, the selection of its models is very important. Here, we consider a particular class of models as follows:

$$f(R,T)=f_1(R)+f_2(R)f_3(T). \tag{29}$$

This model involves the explicit non-minimal curvature matter coupling. We now consider the $f(R,T)$ power law type model given by

$$f(R)=R+\lambda R^2T^2. \tag{30}$$

Such functional $f(R,T)$ configurations match the Lagrangian form mentioned in Eq. (29). Here, we take $f_1(R)=R,\ f_2(R)=R^2$, and $f_3(T)=T^2$. All GR solutions can be obtained by taking the limit $\lambda\to 0$.

3 Perturbation scheme

Perturbation theory gives us a mathematical technique that assists enough to find an approximate solution of a differential equation. After applying a perturbation scheme, one can break the corresponding equations into "solvable/static" and "perturbed" parts. The impact of the perturbed terms in the equation keeps on decreasing, controlled by the perturbation parameter. Here, we consider the perturbation scheme that was proposed by Herrera et al. [49]. In this scheme, we take α to be perturbation parameter with $\alpha\in[0,1]$ and consider effects up to first order. We assume that the system initially is in a state of hydrostatic equilibrium. Due to this, the cylindrical scale factors as well as fluid variables are independent of temporal coordinate. We shall represent static configurations of corresponding variables by zero subscript. Upon perturbations, all these structural variables depend upon the same time dependence $\eta(t)$, which eventually gives the same time dependence to Ricci scalar. The perturbation scheme is

$$A(t,r)=A_0(r)+\alpha\eta(t)a(r), \tag{31}$$
$$B(t,r)=B_0(r)+\alpha\eta(t)b(r), \tag{32}$$
$$C(t,r)=C_0(r)+\alpha\eta(t)c(r), \tag{33}$$
$$R(t,r)=R_0(r)+\alpha T(t)e(r), \tag{34}$$
$$P_\phi(t,r)=P_{\phi 0}(r)+\alpha\bar{P}_\phi(t,r), \tag{35}$$
$$P_r(t,r)=P_{r0}(r)+\alpha\bar{P}_r(t,r), \tag{36}$$
$$\mu(t,r)=\mu_0(r)+\alpha\bar{\mu}(t,r), \tag{37}$$
$$P_\perp(t,r)=P_{\perp 0}(r)+\alpha\bar{P}_\perp(t,r), \tag{38}$$
$$m(t,r)=m_0(r)+\alpha\bar{m}(t,r), \tag{39}$$
$$f(t,r)=R_0(1+\lambda T_0^2R_0)+\alpha\eta(t)e(r)(1+2\lambda R_0T_0^2), \tag{40}$$
$$f_R(t,r)=(1+2\lambda R_0T_0^2)+2\alpha\eta(t)\lambda e(r)T_0^2, \tag{41}$$
$$\Theta(t,r)=\alpha\bar{\Theta}(t,r), \tag{42}$$

where R_0 is the static form of the Ricci invariant whose value is

$$R_0(r)=\frac{2}{B_0^2}\left[\frac{A_0''}{A_0}+\frac{A_0'}{A_0}\left(\frac{1}{r}-\frac{B_0'}{B_0}\right)-\frac{B_0'}{B_0 r}\right],$$

while its perturbed form is

$$-\eta e=\frac{2\ddot{\eta}}{A_0^2}\left(\frac{b}{B_0}+\frac{c}{r}\right)+\eta\left[\frac{4b}{B_0^3}R_0-\frac{2}{B_0^2}\left\{\frac{a''}{A_0}-\frac{aA_0''}{A_0^2}\right.\right.$$
$$+\frac{c''}{r}-\frac{A_0'}{A_0}\left\{\left(\frac{b}{B_0}\right)'\left(\frac{c}{r}\right)'\right\}$$
$$\left.\left.+\left(\frac{a}{A_0}\right)'\left(\frac{1}{r}-\frac{B_0'}{B_0}\right)-\frac{B_0'}{B_0}\left(\frac{c}{r}\right)'-\frac{1}{r}\left(\frac{b}{B_0}\right)'\right\}\right].$$

The $f(R,T)$ field equations (8)–(11) against a static background with $C_0=r$ turn out to be

$$\frac{1}{rB_0^2}\times\frac{B_0'}{B_0}=\frac{1}{(1+2\lambda R_0T_0^2)}\left[\mu_0-\frac{\lambda}{2}R_0^2T_0^2+\overset{(S)}{\psi}_{00}\right], \tag{43}$$

$$\frac{1}{rB_0^2}\frac{A_0'}{A_0}=\frac{1}{(1+2\lambda R_0T_0^2)}\times\left[(P_{r0}+\mu_0)(1+2\lambda R_0^2T_0)+\frac{\lambda}{2}R_0^2T_0^2+\overset{(S)}{\psi}_{11}\right], \tag{44}$$

$$\frac{A_0'}{B_0A_0}\left(\frac{B_0'}{B_0^2}+\frac{A_0''}{B_0A_0'}\right)=\frac{1}{(1+2\lambda R_0T_0^2)}\times\left[(P_{\phi0}+\mu_0)(1+2\lambda R_0^2T_0)+\frac{\lambda}{2}R_0^2T_0^2+\overset{(S)}{\psi}_{22}\right], \quad (45)$$

$$\left(\frac{A_0'}{A_0r}+\frac{A_0''}{A_0}\right)\frac{1}{B_0^2}-\frac{B_0'}{B_0^3}\left(\frac{1}{r}+\frac{A_0'}{A_0}\right)=\frac{1}{(1+2\lambda R_0T_0^2)}\times\left[(P_{z0}-\mu_0)(1+2\lambda R_0^2T_0)+\frac{\lambda}{2}R_0^2T_0^2+\overset{(S)}{\psi}_{33}\right], \quad (46)$$

where $\overset{(S)}{\psi}_{ii}$ indicate static configurations of the corresponding dark source components and are given by

$$\overset{(S)}{\psi}_{00}=\frac{2\lambda T_0}{B_0^2}\left[2R_0T_0'+T_0R_0''+2R_0T_0''+2R_0\frac{T_0'^2}{T_0}+\left(\frac{1}{r}-\frac{B_0'}{B_0}\right)(T_0R_0'+2R_0T_0')\right],$$

$$\overset{(S)}{\psi}_{11}=-\frac{2\lambda T_0}{B_0^2}\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)(T_0R_0'+2R_0T_0'),$$

$$\overset{(S)}{\psi}_{22}=\frac{2\lambda T_0}{B_0^2}\left[2R_0T_0'+T_0R_0''+2R_0T_0''+2R_0\frac{T_0'^2}{T_0}+\left(\frac{B_0'}{B_0}-\frac{A_0'}{A_0}\right)(T_0R_0'+2R_0T_0')\right],$$

$$\overset{(S)}{\psi}_{33}=\frac{2\lambda T_0}{B_0^2}\left[(T_0R_0'+2R_0T_0')\left(\frac{B_0'}{B_0}-\frac{A_0'}{A_0}-\frac{1}{r}\right)-2R_0T_0'-T_0R_0''-2R_0T_0''-2R_0\frac{T_0'^2}{T_0}\right].$$

Equations (8)–(10) in a non-hydrostatic state take the following form:

$$\bar{\mu}=\chi_2\eta, \quad (47)$$

$$\bar{\mu}+\bar{P}_r=\eta\chi_3-\frac{\overset{(P_1)}{\psi}_{00}\dot{\eta}}{(1+2\lambda R_0^2T_0)}-\frac{\ddot{\eta}}{(1+2\lambda R_0^2T_0)}\times\left\{\overset{(P_2)}{\psi}_{00}+\frac{c}{rA_0^2}(1+2\lambda R_0T_0^2)\right\}\left(\frac{2b}{B_0^2}+2e\lambda T_0^2\right), \quad (48)$$

$$\bar{\mu}+\bar{P}_\phi=\eta\chi_5-\frac{\ddot{\eta}}{(1+2\lambda R_0^2T_0)}\times\left[\frac{b}{A_0^2B_0}(1+2\lambda R_0T_0^2)+\overset{(P_2)}{\psi}_{22}\right], \quad (49)$$

where the χ_i are dark source terms due to $f(R,T)$ gravity. These terms contain static combinations of the metric variables and are addressed in Appendix **A**. After applying our radial perturbation approach, we have seen that the first dynamical equation (18) is trivially obeyed, while the second dynamical equation (19) turns out to be

$$\frac{P_{r0}'}{(1+2\lambda R_0T_0^2)}+\frac{2\lambda P_{r0}}{(1+2\lambda R_0T_0^2)}\times\left[R_0(2T_0R_0'+R_0T_0')-\frac{T_0(T_0R_0'+2R_0T_0')}{(1+2\lambda R_0T_0^2)}\times(1+2\lambda R_0^2T_0)\right]\left[\frac{b}{B_0}+\frac{a}{A_0}+\frac{2e\lambda T_0^2}{(1+2\lambda R_0T_0^2)}\right]$$
$$+\frac{\mu_0'}{(1+2\lambda R_0T_0^2)}+\frac{2\lambda\mu_0}{(1+2\lambda R_0T_0^2)}\times\left[R_0(2T_0R_0'+R_0T_0')-\frac{2\lambda T_0^2R_0^2(T_0R_0'+2R_0T_0')}{(1+2\lambda R_0T_0^2)}\right]$$
$$-2\lambda R_0\frac{(2T_0R_0'+2R_0T_0')}{(1+2\lambda R_0^2T_0)}(\mu_0-P_{r0})+\frac{2\lambda R_0^2T_0}{(1+2\lambda R_0T_0^2)}\left(\mu_0'+\frac{T_0'}{2}\right)$$
$$+\frac{r(1+2\lambda R_0^2T_0)}{B_0^2(1+2\lambda R_0T_0^2)}(P_{r0}-P_{\phi0})-\frac{A_0A_0'}{B_0^2(1+2\lambda R_0T_0^2)}(\mu_0+P_{r0})(1+2\lambda R_0^2T_0)+D_{1S}=0, \quad (50)$$

where D_{1S} is the static form of D_1 and is found to be

$$D_{1S}=\overset{(S)}{\psi}_{11,1}-\frac{A_0A_0'}{B_0^2(1+2\lambda R_0T_0^2)}\left(\overset{(S)}{\psi}_{00}+\overset{(S)}{\psi}_{11}\right)+\frac{\lambda}{2}(R_0^2T_0^2)'+\frac{r\left(\overset{(S)}{\psi}_{11}+\overset{(S)}{\psi}_{22}\right)}{B_0^2(1+2\lambda R_0T_0^2)}.$$

The static as well as non-static portions of the cylindrical relativistic C-energy function turn out to be

$$m_0=\frac{l}{8}\left(1-\frac{1}{B_0^2}\right),\quad \bar{m}=\frac{l}{4}\left(\frac{b}{B_0}-c'\right)\frac{\eta}{B_0^2}, \quad (51)$$

the expansion scalar in its perturbed formulation is

$$\bar{\Theta}=\left(\frac{b}{B_0}+\frac{c}{r}\right)\frac{\dot{\eta}}{A_0}. \quad (52)$$

The equation relating b and B_0 is found after perturbing Eq. (8) as

$$\frac{b}{B_0}=r\left(\psi_{01}-\frac{cA_0'}{rA_0}+\frac{c'}{r}\right).$$

The first and second conservation laws (18) and (19) after using perturbation scheme provide the following non-static perturbed distributions:

$$\bar{\mu} + \chi_1(r)\dot{\eta} = 0, \tag{53}$$

$$\frac{\bar{P}_r'}{(1+2\lambda R_0^2T_0)} + \frac{2\lambda \bar{P}_r}{(1+2\lambda R_0^2T_0)} \times \left[R_0(2T_0R_0' + R_0T_0') - T_0(T_0R_0' + 2R_0T_0') \times \frac{(1+2\lambda R_0^2T_0)}{(1+2\lambda R_0T_0^2)}\right] - (\bar{\mu}+\bar{P}_r)\frac{A_0A_0'}{B_0^2} + \frac{\bar{\mu}'}{(1+2\lambda R_0^2T_0)} + \frac{2\lambda\bar{\mu}}{(1+2\lambda R_0^2T_0)}\left[R_0(2T_0R_0'+R_0T_0') - \frac{2\lambda R_0^2T_0^2}{(1+2\lambda R_0T_0^2)}\right] + \frac{2\lambda R_0^2T_0\bar{\mu}'}{(1+2\lambda R_0^2T_0)} - 2\lambda R_0\frac{(2T_0R_0' + R_0T_0')}{(1+2\lambda R_0^2T_0)} \times (\bar{\mu} - \bar{P}_r) + \frac{r}{B_0^2}(\bar{P}_r - \bar{P}_\phi)\frac{(1+2\lambda R_0^2T_0)}{(1+2\lambda R_0T_0^2)} - \frac{A_0A_0'}{B_0^2(1+2\lambda R_0T_0^2)}\left(\dot{\eta}\overset{(P_1)}{\psi_{11}} + \ddot{\eta}\overset{(P_2)}{\psi_{11}}\right) + \frac{r}{B_0^2(1+2\lambda R_0T_0^2)}\left\{\ddot{\eta}\left(\overset{(P_2)}{\psi_{11}} - \overset{(P_2)}{\psi_{22}}\right) + \dot{\eta}\overset{(P_1)}{\psi_{11}}\right\} + \frac{2e\lambda T_0\eta P_{r0}'}{(1+2\lambda R_0T_0^2)^2} + \eta D_3 = 0, \tag{54}$$

where χ_1 constitutes gravitational effects coming from effective energy density of the cylindrical stellar objects, while D_3 contains $f(R, T)$ higher curvature corrections. These terms are addressed in the appendix. Equations (53) and (54), known as dynamical equations, would be very useful in the discussion of collapsing behavior of relativistic stellar interiors.

The matching condition (27) accounts for the perturbation and provides us with

$$P_{r0} \overset{\Sigma^{(e)}}{=} -\left(1+\frac{(1+2\lambda R_0^2T_0)}{(1+2\lambda R_0T_0^2)}\right)^{-1} \times \left[\mu_0(1+2\lambda R_0^2T_0) + \frac{\lambda}{2}R_0^2T_0^2\right], \tag{55}$$

$$\bar{P}_r \overset{\Sigma^{(e)}}{=} \left(1+\frac{(1+2\lambda R_0^2T_0)}{(1+2\lambda R_0T_0^2)}\right)^{-1} \times [(2e\lambda R_0^2\mu_0 + e\lambda R_0T_0^2)\eta + \bar{\mu}(1+2\lambda R_0^2T_0)] + \frac{2e\lambda P_{r0}\eta}{(1+2\lambda R_0T_0^2)}\left[R_0^2 + \frac{T_0^2(1+2\lambda R_0^2T_0)}{(1+2\lambda R_0T_0^2)}\right] \times \left[1+\frac{(1+2\lambda R_0^2T_0)}{(1+2\lambda R_0T_0^2)}\right]. \tag{56}$$

Using the (00) field equation in the second of the above equations, we get

$$\bar{P}_r \overset{\Sigma^{(e)}}{=} \chi_4\eta, \tag{57}$$

where

$$\chi_4 \overset{\Sigma^{(e)}}{=} e\lambda R_0\left(1+\frac{(1+2\lambda R_0^2T_0)}{(1+2\lambda R_0T_0^2)}\right)^{-1} \times (2\mu_0R_0 + T_0^2) + \frac{2e\lambda P_{r0}}{(1+2\lambda R_0T_0^2)} \times \left[R_0^2 + \frac{T_0^2(1+2\lambda R_0^2T_0)}{(1+2\lambda R_0T_0^2)}\right] \times \left[1+\frac{(1+2\lambda R_0^2T_0)}{(1+2\lambda R_0T_0^2)}\right] + \chi_2(1+2\lambda R_0^2T_0) \times \left(1+\frac{(1+2\lambda R_0^2T_0)}{(1+2\lambda R_0T_0^2)}\right)^{-1}.$$

Equations (48) and (57), after some manipulation, give the following second order partial differential equation:

$$\gamma_1\ddot{\eta} + \gamma_2\dot{\eta} + \gamma_3\eta \overset{\Sigma^{(e)}}{=} 0, \tag{58}$$

where

$$\gamma_1 = \frac{1}{(1+2\lambda R_0^2T_0)}\left\{\overset{(P_2)}{\psi_{11}} + \frac{c(1+2\lambda R_0T_0^2)}{rA_0^2}\right\},$$

$$\gamma_2 = \frac{\overset{(P_1)}{\psi_{11}}}{(1+2\lambda R_0^2T_0)}, \quad \gamma_3 = \chi_4 - \chi_3 + \chi_2.$$

Equation (58) has two solutions with two different behaviors and these behaviors are totally independent of each other. In this paper, we are interested to find the unstable constraints of an evolving cylindrical compact object in modified gravity. Further, it was mentioned earlier that our system was initially in a complete hydrostatic equilibrium. Then it enters into the collapsing phase by reducing its areal radius. Therefore, we now restrict our perturbations in such a way that all radially perturbed functions, i.e., a, b, c, and e, are positive definite, eventually making $\omega^2_{\Sigma^{(e)}} > 0$. The solution associated with Eq. (58) is obtained as follows:

$$\eta = -\exp(\omega_{\Sigma^{(e)}}t), \quad \text{where} \quad \omega_{\Sigma^{(e)}} = \frac{-\gamma_2+\sqrt{\gamma_2^2-4\gamma_1\gamma_3}}{2\gamma_1}. \tag{59}$$

4 N & pN terms and collapse equation

In this section, we shall express the second dynamical equation into centimeter-gram-second (c.g.s.) units and then indicate terms relating to the N, pN and parameterized post Newtonian (ppN) epochs. This would be done by expanding c.g.s.

second dynamical equation up to $O(\frac{1}{\mathcal{C}^4})$, where $\mathcal{C}$ indicates the light speed. For the N and pN epochs, we shall consider the following approximations:

$$\mu_0 \gg P_{j0}, \quad A_0 = 1 - \frac{m_0\mathcal{G}}{r\mathcal{C}^2}, \quad B_0 = 1 + \frac{m_0\mathcal{G}}{r\mathcal{C}^2}, \tag{60}$$

where $\mathcal{G}$ is the gravitational constant. Equation (45) gives us following peculiar form of the double derivative of A_0 with respect to the radial coordinate:

$$\frac{A_0''}{A_0} = -\frac{A_0'B_0'}{A_0B_0} + \frac{B_0^2}{(1+2\lambda R_0T_0^2)} \times \left[(\mu_0 + P_{\phi 0})(1+2\lambda R_0^2T_0) + \frac{\lambda}{2}R_0^2T_0^2 + \overset{(S)}{\psi_{22}}\right]. \tag{61}$$

Equations (44) and (51) provide the following first radial derivatives of A_0 and B_0:

$$\frac{B_0'}{B_0} = \frac{4m_0'}{(l-8m_0)}, \tag{62}$$

$$\frac{A_0'}{A_0} = \frac{2r^2l(\mu_0+P_{r0})(1+2\lambda R_0^2T_0) + \lambda R_0^2T_0^2r^2l - 4(l-8m_0)\lambda T_0^2}{2r(l-8m_0)(1+2\lambda R_0T_0^2+2\lambda rT_0)} = \varphi(\text{say}). \tag{63}$$

We use Eqs. (60), (61), and (63) in the static form of second dynamical equation (50) which after converting into the c.g.s. system turns out to be

$$\begin{aligned}
P_{r0}' &= (1+2\lambda\mathcal{C}^{-4}\mathcal{G}R_0^2T_0)(P_{\phi 0} - P_{r0}) \\
&\times \frac{1}{r}\frac{\mathcal{G}}{\mathcal{C}^2}\left(\frac{r}{\mathcal{G}\mathcal{C}^2} - 2m_0\right) - 2\lambda P_{r0}\left[\frac{R_0\mathcal{G}}{\mathcal{C}^4}(2T_0R_0' + R_0T_0')\right. \\
&\left.-\frac{\mathcal{C}^4}{\mathcal{G}}T_0(T_0R_0' + 2R_0T_0')\frac{(1+2\lambda\mathcal{C}^{-4}\mathcal{G}R_0^2T_0)}{(1+2\lambda\mathcal{C}^4\mathcal{G}^{-1}R_0T_0^2)}\right] \\
&- \mu_0'\mathcal{C}^2 - 2\lambda\mu_0\mathcal{G}\mathcal{C}^{-2} \\
&\times\left[R_0(2T_0R_0' + R_0T_0') + \frac{2\lambda R_0^2T_0^2(R_0'T_0 + 2R_0T_0')}{(1+2\lambda\mathcal{C}^4\mathcal{G}^{-1}R_0T_0^2)}\right. \\
&+2\lambda R_0\left(\frac{\mathcal{G}\mu_0}{\mathcal{C}^2} - \frac{P_{r0}\mathcal{G}}{\mathcal{C}^4}\right)(2T_0R_0' + R_0T_0') \\
&\left.-2\lambda R_0^2T_0\left(\frac{\mathcal{G}\mu_0'}{\mathcal{C}^2} + \frac{\mathcal{C}^4T_0'}{2\mathcal{G}}\right)\right] + \varphi|_{\mathcal{G}}\left[\left(\frac{r^2 - 4rm_0\mathcal{G}\mathcal{C}^{-2}}{r^2}\right)\right. \\
&\times\left\{\frac{2\lambda\mathcal{C}^8T_0}{\mathcal{G}^2r}\left(r - \frac{\mathcal{G}m_0}{\mathcal{C}^2}\right)\left(2R_0T_0' + T_0R_0'' + 2R_0T_0'' + 2R_0\frac{T_0'^2}{T_0}\right)\right. \\
&+\mathcal{C}^2\left(\mu_0 + \frac{P_{r0}}{\mathcal{C}^2}\right)(1+2\lambda\mathcal{C}^{-4}\mathcal{G}R_0^2T_0) + \frac{2\lambda\mathcal{C}^8T_0}{r\mathcal{G}^2}\left(r - \frac{2\mathcal{G}m_0}{\mathcal{C}^2}\right) \\
&\left.\times\left(\frac{4\mathcal{G}m_0\mathcal{C}^{-2}}{l - 8\mathcal{G}m_0\mathcal{C}^{-2}} - \varphi|_{\mathcal{G}}\right)(T_0R_0' + 2R_0T_0')\right\} \\
&-\frac{4\lambda\mathcal{C}^8T_0}{\mathcal{G}^2}\left(r - \frac{2\mathcal{G}m_0}{\mathcal{C}^2}\right)(T_0R_0' + 2R_0T_0')\Bigg] \\
&+\frac{2\lambda\mathcal{C}^4}{\mathcal{G}}(1+2\lambda\mathcal{C}^4\mathcal{G}^{-1}R_0T_0^2)\left[\left(1 \quad \frac{2\mathcal{G}m_0}{r\mathcal{C}^2}\right) - (T_0R_0' + 2R_0T_0')\right. \\
&\left.\times\left(\frac{1}{r} + \varphi|_{\mathcal{G}}\right)\right]_{,1} + 2(r - 4\mathcal{G}m_0\mathcal{C}^{-2})\lambda\mathcal{C}^8\mathcal{G}^{-2}T_0\left[\frac{1}{r}\right. \\
&\left.+\frac{4\mathcal{G}\mathcal{C}^{-2}m_0'}{(l-8\mathcal{G}m_0\mathcal{C}^{-2})} + 4R_0T_0' + T_0R_0'' + 2R_0T_0'' + \frac{2R_0T_0'^2}{T_0}\right].
\end{aligned} \tag{64}$$

It is worthy to mention that expansion of the above equation provides terms relating to some specific eras with details as follows:

Terms of $O(\mathcal{C}^0)$ indicates contribution at N era,

Terms of $O\left(\frac{1}{\mathcal{C}^2}\right)$ indicates contribution at N era,

Terms of $O\left(\frac{1}{\mathcal{C}^4}\right)$ indicates contribution at N era.

The complete expansion of the above equation up to $O(\mathcal{C}^{-4})$ is described in Appendix **A**.

The $f(R,T)$ field equations are filled with complicated derivatives of the radial and temporal coordinates, the exploration of their generic solutions is a painstaking task that yet has not been accomplished. However, certain restrictions with some physical background could assist enough to find their solution. In this perspective, we consider the expansion-free evolution of cylindrical compact objects against a linear perturbation. The expansion-free condition can be obtained from Eq. (52) as follows:

$$\frac{b}{B_0} = -\frac{c}{r}. \tag{65}$$

We are now interested in a calculation of the $f(R,T)$ expansion-free cylindrical collapse equation. This would be furnished by considering the second perturbed dynamical equation along with the supposition that cylindrical evolution is supported by a null expansion scalar background. Thus, using Eq. (54), junction conditions (55), (56), (59), and the above expansion-free constraint, we obtain the following modified collapse equation over the exterior boundary surface:

$$\begin{aligned}
&\frac{\chi_4'\eta}{(1+2\lambda R_0^2T_0)} + \frac{2\lambda\chi_4\eta}{(1+2\lambda R_0^2T_0)} \\
&\times\left[R_0(2T_0R_0' + R_0T_0') - T_0(T_0R_0' + 2R_0T_0')\frac{(1+2\lambda R_0^2T_0)}{(1+2\lambda R_0T_0^2)}\right] \\
&-(\chi_2+\chi_4)\eta\frac{A_0A_0'}{B_0^2} + \frac{\chi_2'\eta}{(1+2\lambda R_0^2T_0)} + \frac{2\lambda\chi_2}{(1+2\lambda R_0^2T_0)} \\
&\times\left[R_0(2T_0R_0' + R_0T_0') - \frac{2\lambda R_0^2T_0^2}{(1+2\lambda R_0T_0^2)}\right]\eta \\
&+\frac{2\lambda R_0^2T_0\chi_2'\eta}{(1+2\lambda R_0^2T_0)} - \frac{(2T_0R_0' + R_0T_0')}{(1+2\lambda R_0^2T_0)}2\lambda R_0(\chi_2 - \chi_4)\eta
\end{aligned}$$

$$+\frac{r}{B_0^2}\left\{(\chi_4-\chi_5-\chi_2)-\frac{\omega^2}{(1+2\lambda R_0^2T_0)}\left(\frac{b}{B_0^2r^2}(1+2\lambda\right.\right.$$
$$\left.\left.\times\ R_0T_0^2)+\overset{(P_2)}{\psi_{22}}\right)\right\}\frac{(1+2\lambda R_0^2T_0)}{(1+2\lambda R_0T_0^2)}\eta=(\Phi+\Omega_e)\eta, \qquad (66)$$

where the quantities Φ and Ω_e are described in Appendix **A**. It is worthy to stress that in the above equation, the term Φ describes the gravitational contribution in expansion-free systems. This contribution is the same as in the expansion of cylindrical systems evolution. However, the quantity Ω_e is responsible for the emergence of a cylindrical central Minkowskian core during evolution of the system. The gravitational effects coming from Ω_e causes the naked singularity appearance during cylindrical collapse. This comes from the fact that the quantity Ω_e contains all those terms that have been evaluated by imposing the expansion-free condition. When the system fluid moves inward, during the collapsing phenomenon, with null expansion rate, there will be a blowup of the shearing scalar at the center. It is well known from the work of Joshi et al. [50] that a strong shear could produce hindrances in the appearance of apparent horizon, thus producing a platform for naked singularity formation. Thus, Ω_e gave a way to discuss naked singularity emergence in a simple way. On making $\frac{b}{B_0}\neq-\frac{c}{r}$, one can remove all of the expansion-free effects in the above collapse equation.

5 Instability constraints at both N & pN epochs

In this section, we explore unstable regions of collapsing cylindrical stellar model at both the N and the pN eras. In the pN limits, Eq. (66) in relativistic units yields

$$\eta\left[\frac{\chi_4'}{(1+2\lambda R_0T_0^2)}+\frac{2\lambda\chi_4}{(1+2\lambda R_0T_0^2)}\right.$$
$$\times\left\{R_0(2T_0R_0'+R_0T_0')-T_0(T_0R_0'+2R_0T_0')\frac{(1+2\lambda R_0^2T_0)}{(1+2\lambda R_0T_0^2)}\right\}$$
$$+\frac{\chi_2'}{(1+2\lambda R_0T_0^2)}+\frac{2\lambda\chi_2}{(1+2\lambda R_0T_0^2)}$$
$$\times\left\{R_0(2T_0R_0'+R_0T_0')-\frac{2\lambda R_0^2T_0^2}{(1+2\lambda R_0T_0^2)}\right\}$$
$$\times+2\lambda R_0(\chi_4-\chi_2)\frac{(2T_0R_0'+R_0T_0')}{(1+2\lambda R_0^2T_0)}+\frac{2\lambda R_0^2T_0\chi_2'}{(1+2\lambda R_0^2T_0)}$$
$$\left.+(r-2m_0)\frac{(1+2\lambda R_0^2T_0)}{(1+2\lambda R_0T_0^2)}\{(\chi_4-\chi_5-\chi_2)\}\right]\eta$$
$$=\frac{\omega^2}{(1+2\lambda R_0^2T_0)}\left(\frac{b}{r^2}(1+2\lambda R_0T_0^2)\right.$$
$$\left.\times\ (r+2m_0)(r-2m_0)+\overset{(P_2)}{\psi_{22}}\right)$$
$$+\eta(\Phi+\Omega_e)|_{pN}+(\chi_2+\chi_4)\frac{(r-2m_0)^2}{r^2}\varphi\eta. \qquad (67)$$

For the onset of instability, one needs to satisfy the above relation. Since the majority of the above terms are positive, instabilities will appear because of negative terms in the above equation. For that reason we consider the following constraints to be obeyed:

$$r>2m_0,\ \chi_4>\chi_2+\chi_5,\ 2T_0R_0'+R_0T_0')$$
$$>T_0(T_0R_0'+2R_0T_0')\frac{(1+2\lambda R_0^2T_0)}{(1+2\lambda R_0T_0^2)}.$$

These are the required instability constraints that the system must satisfy in order to enter in the unstable window during evolution. In other words, the cylindrical relativistic anisotropic interior will be unstable as long as it obeys the above relations. It is worthy to stress that in order to consider Eq. (67) as an instable constraint, we need to consider that all the terms on both sides of the equation are definite positive. Thus, we take $\Phi>0$ and $\Omega_e>0$. It is seen that this constraint depends merely on the static configurations of fluid as well as $f(R,T)$ variables. Now, we consider N order effects; then Eq. (67) reduces to

$$\frac{2\lambda\chi_4}{(1+2\lambda R_0T_0^2)}\left\{R_0(2T_0R_0'+R_0T_0')-\frac{2\lambda R_0^2T_0^2}{(1+2\lambda R_0T_0^2)}\right\}$$
$$+r\left(1-\frac{2m_0}{r}\right)\times\frac{(1+2\lambda R_0^2T_0)}{(1+2\lambda R_0T_0^2)}(\chi_4-\chi_5-\chi_2)$$
$$=(\chi_2+\chi_4)\left(1-\frac{2m_0}{r}\right)^2\varphi+\left(1-\frac{2m_0}{r}\right)$$
$$\times\frac{\omega^2r}{(1+2\lambda R_0^2T_0)}\left\{b(1+2\lambda R_0T_0^2)\left(1+\frac{m_0}{r}\right)+\overset{(P_2)}{\psi_{22}}\right\}$$
$$+\Phi_N+\Omega_{e_N}. \qquad (68)$$

Now, we use constant curvature condition first in the above equation and then in Eq. (24). After using simultaneously these equations, we get

$$\left[1-\frac{2}{(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}\int_{r_{\Sigma^{(i)}}}^r\mu_0r^2\mathrm{d}r+\frac{\lambda\tilde{R_0}^2\tilde{T_0}^2}{r(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}\int_{r_{\Sigma^{(i)}}}^r r^2\mathrm{d}r\right]$$
$$\times\frac{(1+2\lambda\tilde{R_0}^2\tilde{T_0})}{(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}(\tilde{\chi}_4-\tilde{\chi}_5-\tilde{\chi}_2)$$
$$=\tilde{\varphi}\left[1-\frac{2}{(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}\int_{r_{\Sigma^{(i)}}}^r\mu_0r^2\mathrm{d}r+\frac{\lambda\tilde{R_0}^2\tilde{T_0}^2}{r(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}\right.$$
$$\left.\times\int_{r_{\Sigma^{(i)}}}^r r^2\mathrm{d}r\right]^2(\tilde{\chi}_2+\tilde{\chi}_4)+\frac{\tilde{\omega}^2r}{(1+2\lambda\tilde{R_0}^2\tilde{T}_0)}$$
$$\times\left[1-\frac{2}{(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}\int_{r_{\Sigma^{(i)}}}^r\mu_0r^2\mathrm{d}r\right.$$
$$\left.+\frac{\lambda\tilde{R_0}^2\tilde{T_0}^2}{r(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}\int_{r_{\Sigma^{(i)}}}^r r^2\mathrm{d}r\right]$$

$$\times\left[\overset{(P_2)}{\tilde{\psi}_{22}}+b\left\{1-\frac{2}{(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}\int_{r_{\Sigma^{(i)}}}^{r}\mu_0 r^2\mathrm{d}r\right.\right.$$
$$\left.\left.-\frac{\lambda\tilde{R_0}^2\tilde{T_0}^2}{r(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}\int_{r_{\Sigma^{(i)}}}^{r}r^2\mathrm{d}r\right\}(1+2\lambda\tilde{R}_0\tilde{T_0}^2)\right]$$
$$+\frac{4\lambda^2\tilde{R_0}^2\tilde{T_0}^2\tilde{\chi}_4}{(1+2\lambda\tilde{R_0}^2\tilde{T}_0)^2}+\tilde{\Phi}_N+\tilde{\Omega}_{e_N}, \tag{69}$$

where a tilde shows that the terms are evaluated under the constant curvature condition. Now, we use the ansatz $\mu_0=\xi r^n$ in which $n\in(-\infty,\infty)$, while ξ is any positive real number constant. This type of ansatz is well justified because any function of a single variable can be expanded in a series form of that variable. Considering $n\neq-3$ and $n\neq-4$, we get from the above equation

$$\left[1-\frac{2\xi(r^{n+3}-r_{\Sigma^{(i)}}^{n+3})}{3(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}+\frac{2\xi(r^{n+4}-r_{\Sigma^{(i)}}^{n+4})}{3(n+4)(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}\right.$$
$$\left.+\frac{\lambda\tilde{R_0}^2\tilde{T_0}^2(r^3-r_{\Sigma^{(i)}}^3)}{3r(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}\right](\tilde{\chi}_4-\tilde{\chi}_5-\tilde{\chi}_2)\frac{(1+2\lambda\tilde{R_0}^2\tilde{T}_0)}{(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}$$
$$=\left[1-\frac{2\xi(r^{n+3}-r_{\Sigma^{(i)}}^{n+3})}{3(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}+\frac{2\xi(r^{n+4}-r_{\Sigma^{(i)}}^{n+4})}{3(n+4)(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}\right.$$
$$\left.+\frac{\lambda\tilde{R_0}^2\tilde{T_0}^2(r^3-r_{\Sigma^{(i)}}^3)}{3r(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}\right]^2\tilde{\varphi}(\tilde{\chi}_2+\tilde{\chi}_4)$$
$$+\frac{\tilde{\omega}^2 r}{(1+2\lambda\tilde{R_0}^2\tilde{T}_0)}\left[1-\frac{2\xi(r^{n+3}-r_{\Sigma^{(i)}}^{n+3})}{3(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}\right.$$
$$\left.+\frac{2\xi(r^{n+4}-r_{\Sigma^{(i)}}^{n+4})}{3(n+4)(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}+\frac{\lambda\tilde{R_0}^2\tilde{T_0}^2(r^3-r_{\Sigma^{(i)}}^3)}{3r(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}\right]$$
$$\times\left[b\left\{1-\frac{2\xi(r^{n+3}-r_{\Sigma^{(i)}}^{n+3})}{3r(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}\right.\right.$$
$$+\frac{2\xi(r^{n+4}-r_{\Sigma^{(i)}}^{n+4})}{3r(n+4)(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}$$
$$\left.\left.-\frac{\lambda\tilde{R_0}^2\tilde{T_0}^2(r^3-r_{\Sigma^{(i)}}^3)}{r(1+2\lambda\tilde{R}_0\tilde{T_0}^2)}\right\}(1+2\lambda\tilde{R}_0\tilde{T_0}^2)+\overset{(P_2)}{\tilde{\psi}_{22}}\right]$$
$$+\frac{4\lambda^2\tilde{R_0}^2\tilde{T_0}^2\tilde{\chi}_4}{(1+2\lambda\tilde{R_0}^2\tilde{T}_0)^2}+\tilde{\Phi}_N+\tilde{\Omega}_{e_N}. \tag{70}$$

This is the required instability constraint at the N era for the cylindrically symmetric collapsing systems framed within $f(R,T)$ gravity. This indicates that our relativistic instability constraint depends on the radially dependent fluid and $f(R,T)$ model variables. It is worthy to stress that this instability constraint is independent of the stiffness parameter, which generally has utmost relevance in the discussion of the dynamical instability of any stellar object. Here, we also need to suppose that all quantities occurring on both sides of the above equation are non-zero and non-negative.

6 Summary and discussion

In this paper, we have explored some dynamical constraints which are essential for a cylindrical object to be in a physically stable state. We have constructed our analysis quite systematically by forming the modified field equations within the background of $f(R,T)$ gravity theory. The cylindrical system is chosen to be filled with anisotropic matter in the interior, while the exterior region is considered vacuum. The dynamical equations are obtained from the contracted Bianchi identities and some useful kinematical variables are also explored including the expansion scalar. We have presented the linear perturbation technique for metric and matter variables with some known static profile of a cylindrical object. Initially, the system is assumed to be at rest and then gradually enters into the non-static phase with the same time dependence on the metric coefficients.

We have perturbed all the relevant equations to construct the collapse equation. The general collapse equation is obtained by using the conservation laws and field equations up to first order in perturbation parameter. Also, we have constructed a real static solution describing a collapsing state at large past time. For this we have considered that all the metric functions in the static background are positive indicating the cylindrical line element to be Lorentz invariant. Moreover, we have categorized N, pN, and ppN eras by expanding the collapse equation up to order of $\mathcal{C}^{-4}$.

For this purpose we have converted our system to the c.g.s unit systems because we know that every term related with some power of speed of light has some physical interpretation and describes some useful regimes. Since the terms associated with the zeroth order of speed of light, i.e., $\mathcal{C}^0$ corresponds to the N era and the terms linked with $\mathcal{C}^{-2}$ provides the information of the pN regime. Similarly, the terms appearing with the order of $\mathcal{C}^{-4}$ present the era of ppN. Further, we have imposed the expansion-free condition on the collapse equation due to the physical significance of this constraint. Such systems are consistent with those astronomical objects which have an inner cavity after the central explosion. A physical application of our study is possible in those astrophysical objects which have cavity in the interior region. This is due to the fact that cavity formation in the expansion-free case is compensated by the increment in the boundary surface during the overall expansion.

Generally, the adiabatic index (Γ) which describes the rigidity in the fluid distribution indicates the instability regimes for a gravitating source in the presence of the expansion scalar. Particular values of Γ (i.e., $\Gamma<\frac{4}{3}$ for spherical systems and $\Gamma<1$ for cylindrical objects with perfect matter in the interior) exist in the literature, indicating the unstable phase of the relativistic body. However, in the absence of the expansion scalar, this factor Γ does not have such importance to evaluate the unstable regions of the relativistic system. To examine the unstable regions of a self-gravitating cylindrical object with zero expansion, it should satisfy the requirements (67) and (70). The violation of these constraints describes the stable configuration of the collapsing system. We found that the instability range depends upon the dark source terms originating due the $f(R,T)$ theory of gravity as well as on the length of the cylinder. Moreover, the material profile like the anisotropic pressure and energy density also controls the stability of the object during the evolution.

The exploration of the zero expansion condition could be closely linked with the study of voids which are sponge like structures and can be explained with the Minkowskian cavity inside it. Thus, the zero expansion condition asserts the existence of a Minkowskian cavity at the center. The potential applications of our dynamical analysis are present in those astronomical objects which are carrying a central Minkowskian cavity. Finally, we would like mention here that all our results correspond to the instability constraints obtained in GR in the particular limit, i.e., $f(R,T)=R$ [20].

Acknowledgments We would like to thank Professor Malcolm A. H. MacCallum for his many valuable suggestions and comments that significantly improved the paper.

Funded by SCOAP^3.

Appendix

The dark source terms D_0 and D_1 of Eqs. (18) and (19) are given as follows:

$$D_0=\frac{\psi_{01,1}}{A^2}-\frac{\psi_{01}}{A^2}\left(\frac{A'}{A}+\frac{B'}{B}+\frac{2AA'}{B^2}-\frac{CC'}{B^2}\right)-\frac{\psi_{00}}{A^2f_R}(B\dot{B}+C\dot{C})-\frac{\psi_{11}}{A^2f_R}B\dot{B}-\frac{\psi_{22}}{A^2f_R}C\dot{C}, \tag{A1}$$

$$D_1=\frac{\psi_{01,0}}{B^2}-\frac{\psi_{01}}{B^2}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{2B\dot{B}}{A^2}+\frac{C\dot{C}}{A^2}\right)-\frac{AA'}{B^2f_R}(\psi_{00}+\psi_{11})+\frac{CC'}{B^2f_R}\times(\psi_{11}-\psi_{22})-\left\{\frac{f-Rf_R}{2}-\psi_{11}\right\}_{,1}. \tag{A2}$$

The components of extra curvature terms mentioned in Eqs. (47)–(49) are

$$\chi_2=\frac{1}{B_0^2(1+2\lambda R_0T_0^2)}\left\{\left(\frac{c}{r}\right)'\frac{B_0'}{B_0}+\frac{1}{r}\left(\frac{b}{B_0}\right)'-\frac{c''}{r}\right\}-\left(\overset{(P)}{\psi}_{00}-e\lambda R_0T_0^2\right)-\left(\mu_0-\lambda R_0^2T_0^2+\overset{(S)}{\psi}_{00}\right)\times\left(\frac{2b}{B_0^2}+\frac{2\lambda T_0^2}{(1+2\lambda R_0T_0^2)^3}\right),$$

$$\chi_3=\frac{1}{(1+2\lambda R_0^2T_0)}\left[(1+2\lambda R_0T_0^2)\left\{\frac{A_0'}{A_0}\left(\frac{c}{r}\right)'\frac{1}{r}\left(\frac{a}{A_0}\right)\right\}-\left\{\frac{\lambda}{2}R_0^2T_0^2+\overset{(S)}{\psi}_{11}+(\mu_0+P_{r0})(1+2\lambda R_0^2T_0)\right\}\times\left\{\frac{2b}{B_0^2}+2e\lambda T_0^2\right\}-\left[2e\lambda R_0^2(\mu_0+P_{r0})+e\lambda R_0T_0^2\right]\right],$$

$$\chi_5=-\frac{(1+2\lambda R_0T_0^2)}{A_0B_0^3(1+2\lambda R_0^2T_0)}\times\left\{b'A_0'+a'B_0'-\frac{2b}{B_0}(A_0'B_0')+B_0a''-bA_0''\right\}-\left(2R_0P_{\phi0}+2\mu_0R_0^2+T_0^2+\frac{\overset{(P)}{\psi}_{22}}{e\lambda R_0}\right)\frac{e\lambda R_0}{(1+2\lambda R_0^2T_0)}-(1+2\lambda R_0^2T_0)\left[\frac{\lambda}{2}R_0^2T_0^2+\overset{(S)}{\psi}_{22}+(\mu_0+P_{\phi0})(1+2\lambda R_0^2T_0)\right]\times\left[\frac{b}{B_0}+\frac{a}{A_0}+\frac{2e\lambda T_0^2}{(1+2\lambda R_0T_0^2)}\right].$$

The mathematical expressions mentioned in Eqs. (53) and (54) are

$$\chi_1(r)=\left[2\lambda T_0\mu_0\frac{(eT_0+2R_0z)}{(1+2\lambda R_0T_0^2)}+\frac{(1+2\lambda R_0^2T_0)}{(1+2\lambda R_0T_0^2)}\times(\mu_0+P_{r0})\left(\frac{c}{A_0^2}+\frac{bB_0}{A_0^2}\right)-4\lambda\mu_0R_0\frac{(eT_0R_0z)}{(1+2\lambda R_0^2T_0)}-\frac{z}{(1+2\lambda R_0^2T_0)}-\frac{1}{A_0^2(1+2\lambda R_0T_0^2)}\left\{bB_0\overset{(S)}{\psi}_{11}+(bB_0+rc)\overset{(S)}{\psi}_{00}+rc\overset{(S)}{\psi}_{22}\right\}-\overset{(P_1)}{\psi}_{01}\frac{1}{A_0^2}-\overset{(S)}{\psi}_{01}\frac{1}{A_0^2}\left(\frac{A_0'}{A_0}+\frac{B_0'}{B_0}-\frac{r}{B_0^2}+\frac{2A_0A_0'}{B_0^2}\right)\right]\times\left[\frac{(1+2\lambda R_0T_0^2)(1+2\lambda R_0^2T_0)}{(1+4\lambda R_0^2T_0)}\right], \tag{A3}$$

$$
\begin{aligned}
D_3 =& \frac{2\lambda R_0 P_{r0}}{(1+2\lambda R_0^2 T_0)}(2T_0e'+2eR_0'+R_0z')-\frac{4e\lambda^2 T_0 P_{r0}}{(1+2\lambda R_0^2 T_0)^2}\left[P_{r0}(2T_0R_0'+R_0T_0')-T_0(T_0R_0'+2R_0T_0')\frac{(1+2\lambda R_0^2 T_0)}{(1+2\lambda R_0 T_0^2)}\right]\\
&-\frac{4e\lambda^2 T_0 P_{r0}}{(1+2\lambda R_0 T_0^2)^2}(T_0R_0'+2R_0T_0')\left\{\frac{(1+2\lambda R_0^2 T_0)}{(1+2\lambda R_0 T_0^2)}-R_0^2\right\}+\frac{2\lambda T_0 P_{r0}}{(1+2\lambda R_0 T_0^2)^2}\left(T_0e'+2eR_0\frac{T_0'}{T_0}+2R_0z'\right)\\
&\times (1+2\lambda R_0^2 T_0)-(\mu_0+P_{r0})\frac{A_0A_0'}{B_0^2(1+2\lambda R_0 T_0^2)}\left\{\frac{a}{A_0}+\frac{a'}{A_0'}-\frac{2b}{B_0}-\frac{2e\lambda T_0}{(1+2\lambda R_0 T_0^2)}\right\}(1+2\lambda R_0^2 T_0)\\
&-\frac{2e\lambda T_0\mu_0'}{(1+2\lambda R_0 T_0^2)^2}-\frac{2e\lambda R_0^2 A_0A_0'}{B_0^2(1+2\lambda R_0^2 T_0)}(\mu_0+P_{r0})+\frac{4e\lambda^2 T_0 P_{r0}}{(1+2\lambda R_0 T_0^2)^2}\left[2\lambda R_0^2 T_0^2\frac{(T_0R_0'+2R_0T_0')}{(1+2\lambda R_0 T_0^2)}-\mu_0(2T_0R_0'\right.\\
&\left.+R_0T_0')\right]+2\lambda R_0\mu_0\frac{(2T_0e'+2eR_0'+R_0z')}{(1+2\lambda R_0 T_0^2)}+\frac{4\lambda^2 T_0^2 R_0^2\mu_0}{(1+2\lambda R_0 T_0^2)^2}\left(2R_0z'+T_0e'+2eR_0\frac{T_0'}{T_0}\right)\\
&-4e\lambda^2 T_0\mu_0 R_0^2\frac{(T_0R_0'+2R_0T_0')}{(1+2\lambda R_0 T_0^2)^2}\left\{\frac{2\lambda T_0}{(1+2\lambda R_0 T_0^2)}-R_0^2\right\}+\frac{2e\lambda R_0^2}{(1+2\lambda R_0^2 T_0)}\left(\mu_0'+\frac{T_0'}{2}\right)\left[1-\frac{2\lambda R_0^2 T_0}{(1+2\lambda R_0^2 T_0)}\right]\\
&+\frac{2\lambda R_0^2 T_0\bar{\mu}'}{(1+2\lambda R_0^2 T_0)}-\frac{\lambda R_0^2 T_0 z'}{(1+2\lambda R_0^2 T_0)}+\frac{2\lambda R_0}{(1+2\lambda R_0^2 T_0)}(\mu_0-P_{r0})\left[\frac{2e\lambda R_0^2}{(1+2\lambda R_0^2 T_0)}(2T_0R_0'\right.\\
&\left.+R_0T_0')-2T_0e'-2eR_0'-R_0z'\right]+r\frac{(P_{r0}-P_{\phi 0})}{B_0^2(1+2\lambda R_0 T_0^2)}(1+2\lambda R_0^2 T_0)\left\{\frac{c}{r}+c'-\frac{2e\lambda T_0}{(1+2\lambda R_0 T_0^2)}-\frac{2b}{B_0}\right\}\\
&-\frac{2er\lambda R_0^2}{(1+2\lambda R_0 T_0^2)}(P_{r0}-P_{\phi 0})-\left\{\left(\overset{(S)}{\psi}_{00}+\overset{(S)}{\psi}_{11}\right)\left(\frac{a}{A_0}+\frac{a'}{A_0'}-\frac{2b}{B_0}-\frac{2e\lambda T_0}{(1+2\lambda R_0 T_0^2)}\right)+\overset{(P)}{\psi}_{00}+\overset{(P)}{\psi}_{11}\right\}\\
&\times\frac{A_0A_0'}{B_0^2(1+2\lambda R_0 T_0^2)}\lambda(eR_0T-0^2)'+\overset{(P)}{\psi}_{00,1}+\frac{r}{B_0^2(1+2\lambda R_0 T_0^2)}\left(\overset{(P)}{\psi}_{11}-\overset{(P)}{\psi}_{22}\right).
\end{aligned}
\tag{A4}
$$

The quantities Φ and Ω_e appearing in Eq. (66) are

$$
\begin{aligned}
\Phi =& \frac{(r-2m_0)\omega^2}{(1+2\lambda R_0 T_0^2)}\left(\overset{(P_2)}{\psi}_{11}-\overset{(P_2)}{\psi}_{22}\right)-\frac{\omega\varphi(r-2m_0)^2}{r^2(1+2\lambda R_0 T_0^2)}+\frac{\omega(r-2m_0)}{(1+2\lambda R_0 T_0^2)}\overset{(P_1)}{\psi}_{11}+\frac{2e\lambda T_0 P_{r0}'}{(1+2\lambda R_0 T_0^2)^2}+\frac{2e\lambda^2 T_0 P_{r0}}{(1+2\lambda R_0 T_0^2)^2}\\
&\times\left[P_{r0}(2T_0R_0'+R_0T_0')-T_0(T_0R_0'+2R_0T_0')\frac{(1+2\lambda R_0^2 T_0)}{(1+2\lambda R_0 T_0^2)}\right]+\frac{2\lambda R_0 P_{r0}}{(1+2\lambda R_0 T_0^2)}(2T_0e'+2eR_0'+R_0z')-(T_0R_0'\\
&+2R_0T_0')\frac{4e\lambda^2 T_0 P_{r0}}{(1+2\lambda R_0 T_0^2)^2}\left\{\frac{(1+2\lambda R_0^2 T_0)}{(1+2\lambda R_0 T_0^2)}-R_0^2\right\}+\frac{2\lambda T_0 P_{r0}}{(1+2\lambda R_0 T_0^2)^2}\left(T_0e'+2R_0z'+2eR_0\frac{T_0'}{T_0}\right)(1+2\lambda R_0^2 T_0)\\
&-\frac{(r-2m_0)^2}{r^2(1+2\lambda R_0 T_0^2)}(\mu_0+P_{r0})(1+2\lambda R_0^2 T_0)\left\{\frac{a}{r}(r+m_0)+\frac{m_0a'}{r^2}-\frac{2e\lambda T_0}{(1+2\lambda R_0 T_0^2)}\right\}-\frac{2e\lambda R_0^2(r-2m_0)}{(1+2\lambda R_0^2 T_0)}(\mu_0,\\
&+P_{r0})-\frac{2e\lambda T_0\mu_0'}{(1+2\lambda R_0 T_0^2)}-\frac{4e\lambda^2 T_0 P_{r0}}{(1+2\lambda R_0 T_0^2)}[\mu_0(2T_0R_0'+R_0T_0')]-2\lambda R_0^2 T_0^2\\
&\times\frac{(T_0R_0'+2R_0T_0')}{(1+2\lambda R_0 T_0^2)}+\frac{2\lambda R_0\mu_0}{(1+2\lambda R_0 T_0^2)}(2T_0e'+2eR_0'+R_0z')+\frac{4\lambda^2 T_0^2 R_0^2\mu_0}{(1+2\lambda R_0 T_0^2)^2}\\
&\times\left(T_0e'+2R_0z'+2eR_0\frac{T_0'}{T_0}\right)-4e\lambda^2 T_0\mu_0 R_0^2\frac{(T_0R_0'+2R_0T_0')}{(1+2\lambda R_0 T_0^2)^2}\left\{\frac{2\lambda T_0}{(1+2\lambda R_0 T_0^2)}\right.\\
&\left.-R_0^2\right\}+\frac{2e\lambda R_0^2}{(1+2\lambda R_0^2 T_0)}\left(\mu_0'+\frac{T_0'}{2}\right)\left(1-\frac{2\lambda R_0^2 T_0}{(1+2\lambda R_0^2 T_0)}\right)-\frac{\lambda T_0 R_0^2 z'}{(1+2\lambda R_0^2 T_0)}
\end{aligned}
$$

$$+\frac{2\chi_2'\lambda R_0^2T_0}{(1+2\lambda R_0^2T_0)}-\frac{2\lambda R_0(\mu_0-P_{r0})}{(1+2\lambda R_0^2T_0)}\left[2T_0e'+2eR_0'+2R_0z'-\frac{2e\lambda R_0^2}{(1+2\lambda R_0^2T_0)}\times(2T_0R_0'+R_0T_0')\right]$$
$$+\frac{(P_{r0}-P_{\phi0})}{(1+2\lambda R_0T_0^2)}(r-2m_0)(1+2\lambda R_0^2T_0)\left\{\frac{c}{r}+c'-\frac{2e\lambda T_0}{(1+2\lambda R_0T_0^2)}\right\}-\frac{2e\lambda R_0^2}{(1+2\lambda R_0T_0^2)}(r-2m_0)(P_{r0}-P_{\phi0})$$
$$-\frac{\varphi(r-2m_0)}{r^2(1+2\lambda R_0T_0^2)}\left\{\left(\overset{(S)}{\psi}_{00}+\overset{(S)}{\psi}_{11}\right)\left(\frac{a}{r}(r+m_0)+\frac{m_0a'}{r^2}-\frac{2e\lambda T_0}{(1+2\lambda R_0T_0^2)}\right)+\overset{(P)}{\psi}_{00}+\overset{(P)}{\psi}_{11}\right\}$$
$$\times\lambda(eR_0T_0^2)'+\overset{(P)}{\psi}_{00,11}+\frac{(r-2m_0)}{(1+2\lambda R_0T_0^2)}\left(\overset{(P)}{\psi}_{00}+\overset{(P)}{\psi}_{22}\right), \tag{A5}$$

$$\Omega_e=\frac{2c}{r}\left[\frac{(P_{r0}-P_{\phi0})}{(1+2\lambda R_0T_0^2)}(r-2m_0)(1+2\lambda R_0^2T_0)\left\{\frac{c}{r}+c'-\frac{2e\lambda T_0}{(1+2\lambda R_0T_0^2)}\right\}+\frac{\varphi(r-2m_0)}{(1+2\lambda R_0T_0^2)}\left(\overset{(P)}{\psi}_{11}-\overset{(P)}{\psi}_{22}\right)\right.$$
$$\left.+\frac{\varphi(r-2m_0)}{r^2(1+2\lambda R_0T_0^2)}(\mu_0+P_{r0})(1+2\lambda R_0^2T_0)\right]. \tag{A6}$$

The expansion of Eq. (64) provides

$$P_{r0}'=\frac{1}{\mathcal{C}^0}\left[2\lambda R_0^2T_0\frac{(P_{\phi0}-P_{r0})}{r}+2\lambda P_{r0}T_0(T_0R_0'+2R_0T_0')(1-2\lambda R_0T_0^2)-256\right.$$
$$\times\mathcal{G}m_0^2T_0m_0'-\left[\frac{2\lambda m_0\mathcal{G}}{r^2l}(T_0R_0'+2R_0T_0')\left\{2r^2l\mu_0-8\lambda^2r^2lR_0^3T_0^3\mu_0-8\lambda^2r^3R_0^2T_0^2\right.\right.$$
$$\left.\left.\times\ \mu_0l-32\lambda r^2P_{r0}R_0T_0^2-32r^3P_{r0}\lambda m_0T_0\right\}\right]_{,1}-4\lambda^2\left[\frac{R_0T_0^2}{r^2l}(T_0R_0'+2R_0T_0')\left\{4\lambda r^2\right.\right.$$
$$\left.\left.\times\ lR_0^2T_0\mu_0\mathcal{G}+16r^2P_{r0}m_0\mathcal{G}^2-64\lambda^2R_0^3T_0^3P_{r0}\mathcal{G}^2r^2-64\lambda^2r^3R_0^2T_0^2P_{r0}\mathcal{G}^2\right\}\right]_{,1}+\lambda^2T_0^3$$
$$\times\left(2R_0T_0'+T_0R_0''+2R_0T_0''+2R_0\frac{T_0'^2}{T_0}\right)4rR_0P_{r0}\mathcal{G}-\frac{\lambda T_0m_0}{rl\mathcal{G}}-\frac{4m_0\lambda T_0}{rl}\left(2R_0T_0'\right.$$
$$\left.+\ T_0R_0''+2R_0T_0''+2R_0\frac{T_0'^2}{T_0}\right)\left(4r^2\lambda lR_0^2T_0\mu_0\mathcal{G}+16r^2m_0\mathcal{G}^2P_{r0}-64\lambda^2R_0^3T_0^3P_{r0}r^2\mathcal{G}^2\right.$$
$$\left.-\ 64\lambda^2r^3(\mathcal{G}R_0T_0)^2P_{r0}\right)+\frac{4m_0^2T_0\lambda}{r^2l}(16\mathcal{G}r^2\mu_0m_0-64\lambda^2R_0^3T_0^3r^2\mu_0\mathcal{G}-64m_0r^3(T_0$$
$$\times\ R_0\lambda)^2\mu_0\mathcal{G}+2r^2lP_{r0}-8\lambda^2R_0^3T_0^4r^2lP_{r0}-8\lambda^2r^3T_0^2R_0^2lP_{r0})+\frac{\mathcal{G}\mu_0}{2rl}(2r^2l\mu_0$$
$$-\ 8\lambda^2r^2lR_0^3T_0^3\mu_0-8\lambda^2r^3R_0^2T_0^2\mu_0l-32\lambda r^2P_{r0}R_0T_0^2-32r^3P_{r0}\lambda m_0T_0)+\frac{\mathcal{G}}{2rl}$$
$$\times\left(\frac{P_{r0}}{\mathcal{G}}+2\lambda R_0^2T_0\mu_0+2\lambda R_0^2T_0P_{r0}\mathcal{G}-\frac{4m_0\mu_0}{r}\right)\left(\frac{\lambda R_0^2T_0^2r^2l}{\mathcal{G}}-32r^2\mu_0\lambda R_0T_0^2\right.$$
$$\left.\left.-\ 32\mu_0\lambda m_0rT_0-\frac{4\lambda lP_{r0}R_0T_0^2}{\mathcal{G}}-\frac{4r^3\lambda lP_{r0}T_0}{\mathcal{G}}+8\lambda(R_0T_0r)^2m_0\right)\right]$$
$$+\frac{\mathcal{G}}{\mathcal{C}^2}\left[\frac{2m_0}{r}(P_{r0}-P_{\phi0})-8\lambda^3P_{r0}T_0^4(T_0R_0'+2R_0T_0')+2\lambda\mu_0R_0(2T_0R_0'+R_0T_0')\right.$$
$$+\ 4\lambda\mu_0R_0^2T_0^2(R_0'T_0+2R_0T_0')-256m_0^2\mathcal{G}m_0'-\left[\frac{2\lambda m_0}{r^2l}(T_0R_0'+2R_0T_0')\{16r^2\mathcal{G}\mu_0m_0\right.$$

$$
\begin{aligned}
&- 64\lambda^2 R_0^3 T_0^3 r^2 \mu_0 \mathcal{G} - 64 m_0 r^3 T_0^2 \lambda^2 R_0^2 \mu_0 \mathcal{G} + 2r^2 l P_{r0} - 8\lambda^2 R_0^3 T_0^3 P_{r0} r^2 T_0 l - 8r^3 \\
&\times \lambda^2 T_0^2 R_0^2 l P_{r0}\}\Big]' - 16\lambda^3 R_0^2 T_0^4 (T_0 R_0' + 2R_0 T_0') P_{r0} \mathcal{G} + 4\lambda^2 R_0 P_{r0} T_0^3 m_0 \big(2R_0 T_0' \\
&+ 2R_0 T_0'' + T_0 R_0'' + 2R_0 \frac{T_0'^2}{T_0}\bigg) + \frac{4m_0^2 T_0 \lambda}{r^2 l \mathcal{G}} (4r^2 l R_0^2 T_0 \mu_0 \mathcal{G} + 16 r^2 P_{r0} m_0 P_{r0} \mathcal{G}^2 \\
&- 64\lambda^2 R_0^3 T_0^3 r^2 \mathcal{G}^2 P_{r0} - 64\lambda^2 r^3 R_0^2 T_0^2 P_{r0} \mathcal{G}^2) + \frac{\mu_0}{2rl}\Big\{16 r^2 \mathcal{G} \mu_0 m_0 - 64\lambda^2 R_0^3 T_0^3 r^2 \mathcal{G} \mu_0 \\
&- 64 m_0 r^3 T_0^2 \lambda^2 R_0^2 \mu_0 \mathcal{G} + 2r^2 l P_{r0} - 8\lambda^2 R_0^3 T_0^3 r^2 l T_0 P_{r0} - 8\lambda^2 r^3 T_0^2 R_0^2 l P_{r0}\Big\} + \frac{1}{2r^2 l} \\
&\times \left(\frac{P_{r0}}{\mathcal{G}} + 2\lambda R_0^2 T_0 \mu_0 + 2\lambda R_0^2 T_0 P_{r0} \mathcal{G} - \frac{4m_0 \mu_0}{r}\right) (2r^2 l \mu_0 - 8\lambda^2 r^2 l R_0^3 T_0^3 \mu_0 - 8\lambda^2 \\
&\times r^3 R_0^2 \mu_0 T_0^2 l - 32\lambda r^2 P_{r0} R - 0 T_0^2 - 32 r^3 \lambda P_{r0} m_0 T_0) + \frac{1}{r^2 l}(-4\lambda R_0^2 T_0 m_0 - 2m_0 \\
&\times P_{r0} - 4m_0 R_0^2 T_0 \lambda P_{r0} \mathcal{G})(-32 r^2 \mu_0 \lambda R_0 T_0^2 - 32 \mu_0 \lambda m_0 r T_0 - 4\lambda l P_{r0} R_0 T_0^2 \mathcal{G}^{-1} \\
&- 4r^3 \lambda l P_{r0} T_0 \mathcal{G}^{-1} + \lambda R_0^2 T_0^2 r^2 l \mathcal{G}^{-1} + 8\lambda R_0^2 T_0^2 r^2 m_0)\Big] \\
&+ \frac{\mathcal{G}}{\mathcal{C}^4}\Big[(P_{\phi 0} - P_{r0})\mathcal{G}^{-1} - 2\lambda P_{r0} R_0 (2T_0 R_0' + R_0 T_0') + 2\lambda P_{r0} T_0 (T_0 R_0' + 2R_0 T_0') \\
&- 4\lambda^2 P_{r0} T_0^2 (T_0 R_0' + 2R_0 T_0') R_0^2 + 4\lambda^2 \mu_0^2 R_0 \mathcal{G} (2T_0 R_0' + R_0 T_0') - 4\lambda^2 R_0^2 T_0 \mu_0' \mathcal{G} \\
&- \bigg\{\frac{2\lambda m_0}{r^2 l}(T_0 R_0' + 2R_0 T_0')(4\lambda r^2 l R_0^2 T_0 \mu_0 \mathcal{G} + 16 r^2 l P_{r0} \mathcal{G}^2 m_0 l^{-1} - 64\lambda^2 R_0^3 T_0^3 r^2 \\
&\times P_{r0} \mathcal{G}^2 - 64\lambda^2 r^3 R_0^2 T_0^2 \mathcal{G}^2 P_{r0})\bigg\}' + 16 m_0^2 T_0^3 \lambda^2 \mathcal{G} R_0 P_{r0} + \frac{1}{2rl}(P_{r0}\mathcal{G}^{-1} + 2\lambda R_0^2 \\
&\times T_0 \mu_0 + 2\lambda R_0^2 T_0 P_{r0} \mathcal{G} - 4m_0 \mu_0 r^{-1})(16 \mathcal{G} r^2 \mu_0 m_0 - 64\lambda^2 (R_0 T_0)^3 r^2 \mu_0 \mathcal{G} - 64 \\
&\times m_0 r^3 T_0^2 \lambda^2 R_0^2 \mu_0 \mathcal{G} + 2r^2 l P_{r0} - 8\lambda^2 (R_0 T_0)^3 r^2 l T_0 P_{r0} - 8\lambda^2 r^3 T_0^2 l T_0^2 P_{r0}) + \frac{1}{r^2 l} \\
&\times \big(-4\lambda R_0^2 T_0 m_0 - 2m_0 P_{r0} - 4m_0 R_0^2 + \lambda T_0 P_{r0} \mathcal{G})(2r^2 l \mu_0 - 8\lambda^2 r^2 l (R_0 T_0)^3 \mu_0 \\
&- 8\lambda^2 r^3 (T_0 R_0)^2 \mu_0 l - 32\lambda r^2 P_{r0} R_0 T_0^2 - 32 r^3 P_{r0} \lambda m_0 T_0)\Big].
\end{aligned}
$$

References

1. T. Harada, Prog. Theor. Phys. **98**, 359 (1997)
2. E.V. Arbuzova, A.D. Dolgov, Phys. Lett. B **700**, 289 (2011)
3. K. Bamba, S. Nojiri, S.D. Odintsov, Phys. Lett. B **698**, 451 (2011)
4. M. Sharif, Z. Yousaf, Phys. Rev. D **88**, 024020 (2013)
5. M. Sharif, Z. Yousaf, Mon. Not. R. Astron. Soc. **440**, 3479 (2014)
6. M. Sharif, Z. Yousaf, J. Cosmol, Astropart. Phys. **06**, 019 (2014)
7. M. Sharif, Z. Yousaf, Astrophys. Space Sci. **354**, 431 (2014)
8. M. Sharif, Z. Yousaf, Astrophys. Space Sci. **354**, 471 (2014)
9. M. Sharif, Z. Yousaf, Astrophys. Space Sci. **354**, 481 (2014)
10. M. Sharif, Z. Yousaf, Eur. Phys. J. C **75**, 194 (2014). arXiv:1504.04367 [gr-qc]
11. S. Nojiri, S.D. Odintsov, Phys. Lett. B **652**, 343 (2007)
12. O. Tiret, F. Combes, Astron. Astrophys. **464**, 517 (2007)
13. C. Bogdanos, E.N. Saridakis, Class. Quantum Grav. **27**, 075005 (2010)
14. B. Jain, V. Vikram, J. Sakstein, Astron. J. **779**, 39 (2013)
15. V.A. Skripkin, Soviet Phys. Doklady **135**, 1183 (1960)
16. L. Herrera, G. Le Denmat, N.O. Santos, Phys. Rev. D **79**, 087505 (2009)
17. G. Darmois, *Memorial des Sciences Mathematiques* (Gautheir-Villars, Paris, 1927), p. 25
18. L. Herrera, G. Le Denmat, N.O. Santos, Class. Quantum Grav. **27**, 135017 (2010)
19. L. Herrera, G. Le Denmat, N.O. Santos, Gen. Relativ. Gravit. **44**, 1143 (2012)
20. M. Sharif, M.Z. Bhatti, J. Cosmol. Astropart. Phys. **10**, 056 (2013)
21. F. Weber, *Pulsars as astrophysical observatories for nuclear and particle physics* (Institute of Physics, 1999)
22. M.S. Morris, K.S. Thorne, Am. J. Phys. **56**, 395 (1988)
23. C. Cattoen, T. Faber, M. Visser, Class. Quantum Grav. **22**, 4189 (2005)
24. A. DeBenedictis, D. Horvat, S. Ilijić, S. Kloster, K.S. Viswanathan, Class. Quantum Grav. **23**, 2303 (2006)
25. L. Herrera, J. Ospino, A. Di Prisco, Phys. Rev. D **77**, 027502 (2008)
26. L. Herrera, N.O. Santos, A. Wang, Phys. Rev. D **78**, 084026 (2008)
27. M. Sharif, M.Z. Bhatti, J. Cosmol. Astropart. Phys. **11**, 014 (2013)
28. M. Sharif, M.Z. Bhatti, Phys. Lett A **378**, 469 (2014)
29. M. Sharif, M.Z. Bhatti, Astropart. Phys. **56**, 35 (2014)
30. M. Sharif, M.Z. Bhatti, Astrophys. Space Sci. **355**, 2174 (2014)
31. M. Sharif, M.Z. Bhatti, Int. J. Mod. Phys. D **23**, 1450085 (2014)
32. M. Sharif, M.Z. Bhatti, Mon. Not. R. Astron. Soc. **450**, 1015 (2015)
33. Z. Yousaf, M.Z. Bhatti, Mon. Not. R. Astron. Soc. **458**, 1785 (2016)

34. M.Z. Bhatti, Z. Yousaf, Eur. Phys. J. C **76**, 219 (2016). arXiv:1604.01395 [gr-qc]
35. Z. Yousaf, K. Bamba, M.Z. Bhatti, Phys. Rev. D **93**, 064059 (2016). arXiv: 1603.03175
36. S. Nojiri, S.D. Odintsov, eConf C **0602061**, 06 (2006)
37. S. Nojiri, S.D. Odintsov, Int. J. Geom. Meth. Mod. Phys. **4**, 115 (2007). arXiv:hep-th/0601213
38. S. Capozziello, M.D. Laurentis, Phys. Rep. **509**, 167 (2011)
39. S. Nojiri, S.D. Odintsov, Phys. Rep. **505**, 59 (2011)
40. T. Harko, F.S.N. Lobo, S. Nojiri, S.D. Odintsov, Phys. Rev. D **84**, 024020 (2011)
41. M. Sharif, Z. Yousaf, Can. J. Phys. **90**, 865 (2012)
42. M.A.H. MacCallum, M. Mars, R. Vera, Phys. Rev. D **75**, 024017 (2007)
43. H. Chao-Guang, Acta Phys. Sin. **4**, 617 (1995)
44. A. Di Prisco, L. Herrera, M.A.H. MacCallum, N.O. Santos, Phys. Rev. D **80**, 064031 (2009)
45. K.S. Thorne, Phys. Rev. B **138**, 251 (1965)
46. R. Van de Weygaert, E. Platen, Int. J. Mod. Phys. Conf. Ser. **1**, 41 (2011)
47. B. Li, G.B. Zhao, K. Koyama, Mon. Not. R. Astron. Soc. **421**, 3481 (2012)
48. G. Darmois, Memorial des Sciences Mathematiques (Gautheir-Villars, 1927), p. 25
49. L. Herrera, N.O. Santos, G. Le Denmat, Mon. Not. R. Astron. Soc. **237**, 257 (1989)
50. P. Joshi, N. Dadhich, R. Maartens, Phys. Rev. D **65**, 101501 (2002)