

A topological-like model for gravity in 4D space-time

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Abstract In this paper we consider a model for gravity in four-dimensional space-time originally proposed by Chamseddine, which may be derived by dimensional reduction and truncation from a five-dimensional Chern–Simons theory. Its topological origin makes it an interesting candidate for an easier quantization, e.g., in the loop quantization framework. The present paper is dedicated to a classical analysis of the model’s properties. Cosmological solutions as well as wave solutions are found and compared with the corresponding solutions of Einstein’s general relativity with cosmological constant.

1 Introduction

Gravitation as described by Einstein’s general relativity is notoriously difficult to reconcile with quantum theory, a task which is nevertheless necessary if one want to understand physics at the very small scale defined by the Planck length $l_P \sim 10^{-35}$ m, also in the hope that quantum mechanics will cure the singularities of the classical theory such as the Big Bang and Black Hole ones. Loop quantum gravity [1–3] (LQG) is one attempt to do it. It starts from Einstein’s classical general relativity (GR) in the Ashtekar–Barbero formalism where the dynamical variables are an $SU(2)$ Yang–Mills type connection together with its canonical conjugate momentum field. The dynamics is expressed as a set of constraints which correspond to the gauge invariances of the theory [4, 5], namely $SU(2)$ local invariance and the invariances under the space and time diffeomorphisms. The quantum theory is then defined by constructing a Hilbert space whose elements are certain wave functionals of the connection obeying the constraints. The latter should be well defined as self-

adjoint operators, and then solved in the sense that they select the physical wave functionals as those which are annihilated by them. The main difficulty is in the definition of the time diffeomorphism constraint and its solution. Much progress has been made by Thiemann et al. [2], and more recently by Rovelli et al. in the spin foam formalism [3], leading to very promising results.

General relativity is a “background invariant theory”, which means that no a priori geometric structure is given to the space-time manifold where the theory is defined: the metric belongs to the dynamical fields. Another class of background-independent theories is provided by topological theories such as the Chern–Simons (CS) theories.¹ Remarkably enough [6], gravity in three space-time dimensions can be written as a CS theory whose gauge group is the local Poincaré group $ISO(1,2)$, but also $SO(1,3)$ or $SO(2,2)$ if there is a positive or negative cosmological constant. The question is: Could one describe higher-dimension gravity as a CS theory [7–14]? An essential difference between gravity in 3D and gravities in more than three dimensions is that the former has no local degree of freedom, whereas the latter do. The same happens for the CS theories in 3D and in more than three dimensions. Since CS theories live in odd-dimensional space-times, the first one which admits local degrees of freedom is the one in 5D, with the gauge group $ISO(1,4)$, $SO(1,5)$ or $SO(2,4)$ (Poincaré, de Sitter or anti-de Sitter).

An advantage of topological theories—with the gauge groups mentioned above—is that (some of) the diffeomorphism invariances are consequences of the gauge invariances. For CS in 3D, all diffeomorphism invariances follow on shell from gauge invariance [15, 16], whereas, in 5D, only the invariance under the time diffeomorphisms follows on shell [17, 18]. In the latter case this means that the constraint associated with the time diffeomorphisms is a consequence

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¹ We could also mention the BF theories. However, unless some constraints are applied to them, they have to address local degrees of freedom in any space-time dimension.

of the other constraints. Thus the difficult task of defining and solving the diffeomorphism constraint in the 5D quantum gravity described by this CS theory would be avoided.

Being interested in 4D gravity, can we find a similar, topological-like theory? An answer has been provided by Chamseddine [13]: a theory which, beyond containing the gravitation fields is also containing a dilaton-type scalar field. It can be derived from the 5D CS theory by dimensional reduction and truncation of some of the component fields. As we will show, the set of solutions of the Chamseddine model is a subset of the solutions of the complete, non-truncated 5D CS theory reduced to 4D. As such, in view of the interesting properties concerning the constraints mentioned above, it is worthwhile to study the classical aspects of the Chamseddine model, which is the purpose of the present work. A study of the whole 5D CS theory, with or without Kaluza–Klein dimensional reduction, classically and quantum mechanically, will be reported elsewhere [19]. More specifically, the proposal of the present work is to investigate the dynamics of the Chamseddine model and compare some of its solutions with solutions of the conventional Einstein theory. We will in particular focus on solutions of the cosmological type and wave solutions.

The present paper begins in Sect. 2 with a review of Chamseddine’s derivation of his 4D model—whose gauge invariance is de Sitter SO(1,4) or anti-de Sitter SO(2,3)—from a SO(1,5) or SO(2,4) 5D Chern–Simons theory by dimensional reduction and truncation of some fields. We clarify some points of this truncation, and, moreover, show through a well-chosen gauge fixing that the model is a theory of a dilaton-like scalar field interacting with a gravitational field with torsion. We also show here that the field equations of the Chamseddine theory are special solutions of the full untruncated CS theory reduced to 4D. Linear approximations are studied in Sect. 3, leading to the Newtonian limit and to gravitational wave solutions. Section 4 is devoted to the study of cosmological solutions of the theory and their comparison with the Λ CDM model. Conclusions and outlooks are presented in Sect. 5. Conventions and notations are displayed in an appendix.

2 The Chamseddine model

2.1 5D (A)dS Chern–Simons theory as a theory of gravity in 5D

We start with a description of the five-dimensional Chern–Simons (CS) theory for the (anti-)de Sitter gauge group and its interpretation as a gravitation theory [9–11] (see [12] for a comprehensive review). Our notations and conventions are summarized in the appendix.

The gauge group transformations are those which leave invariant the metric $\eta_{MN} = \text{diag}(-1, 1, 1, 1, 1, s)$, with $M, N, \dots = 0, \dots, 5$ and where s takes the values ± 1 . The signatures $(-1, 1, 1, 1, 1, 1)$ and $(-1, 1, 1, 1, 1, -1)$ correspond, respectively, to the Minkowskian de Sitter group SO(1,5) and anti-de Sitter group SO(2,4) for 5D space-time. They will be collectively denoted² by (A)dS₆.

A basis of the Lie algebra (a)ds₆ of (A)dS₆ is given by the generators $M_{MN} = -M_{NM}$, realized as the 6×6 matrices $(M_{MN})^P{}_Q := -\delta_M^P \eta_{NQ} + \delta_N^P \eta_{MQ}$, obeying the commutation relations

$$[M_{MN}, M_{PQ}] = \eta_{MP} M_{NQ} - \eta_{MQ} M_{NP} - \eta_{NP} M_{MQ} + \eta_{NQ} M_{MP}. \tag{2.1}$$

The field variables³ of the theory are the components of a connection form $\hat{A} = \hat{A}_\alpha dx^\alpha$, with values in the Lie algebra (a)ds₆. In the basis (M_{MN}) , the connection form reads

$$\hat{A} = \frac{1}{2} \hat{A}^{MN} M_{MN} = \frac{1}{2} \hat{A}_\alpha^{MN} dx^\alpha M_{MN}, \tag{2.2}$$

and it transforms as

$$\delta \hat{A}^{MN} = \hat{d}\hat{\epsilon}^{MN} + \hat{A}^M{}_P \hat{\epsilon}^{PN} - \hat{A}^N{}_P \hat{\epsilon}^{PM} \tag{2.3}$$

under the infinitesimal (A)dS₆ gauge transformations. The gauge invariant CS action is given by⁴

$$S_{\text{CS}} = \frac{1}{24} \varepsilon_{MNPQRS} \times \int_{\mathcal{M}_5} \left(\hat{A}^{MN} \hat{d}\hat{A}^P Q \hat{d}\hat{A}^{RS} + \frac{3}{2} \hat{A}^{MN} (\hat{A}^2)^{PQ} \hat{d}\hat{A}^{RS} + \frac{3}{5} \hat{A}^{MN} (\hat{A}^2)^{PQ} (\hat{A}^2)^{RS} \right). \tag{2.4}$$

ε_{MNPQRS} is the Levi-Civita totally antisymmetric tensor with the normalization condition⁵ $\varepsilon_{012345} = 1$. The field equations obtained by varying the connection \hat{A} are

$$\frac{1}{4} \varepsilon_{MNPQRS} \hat{F}^{PQ} \hat{F}^{RS} = 0, \quad \text{where} \quad \hat{F}^{PQ} = \hat{d}\hat{A}^{PQ} + (\hat{A}^2)^{PQ}, \tag{2.5}$$

with $\hat{F}^{MN} = \hat{d}\hat{A}^{MN} + \hat{A}^N{}_P \hat{A}^{PN}$ the Yang–Mills curvature.

In the same way as one can interpret the three-dimensional CS theory for the pseudo-orthogonal gauge group SO(1,3) or SO(2,2) as a gravitation theory with cosmological constant [6], one can indeed do the same here making the following identifications of the generators M_{MN} with the five-

² The suffix N in (A)dS _{N} makes explicit the dimension of the defining representation space.

³ Fields and forms in five-dimensional space-time \mathcal{M}_5 are written with a hat, space-time indices being denoted by $\alpha, \beta, \dots = 0, \dots, 4$.

⁴ We do not write explicitly the wedge symbol \wedge for the external products of forms. E.g., the product $(\hat{A}^2)^{PQ} = -(\hat{A}^2)^{QP}$ stems from $\hat{A}^{PT} \wedge \hat{A}^T Q$.

⁵ Indices are lowered and raised using the metric η_{MN} .

dimensional Lorentz generators M_{AB} and “translation” generators P_A , with $A, B, \dots = 0, \dots, 4$:

$$M_{AB} = M_{AB}, \quad P_A := \lambda M_{A5}, \tag{2.6}$$

where $\lambda > 0$ is a parameter with the dimension of a mass or of the inverse of a length in the system of units where $c = \hbar = 1$. The commutation relations (2.1) take the explicit form of the (A)dS₆ commutation relations (A.4), with η_{AB} being the five-dimensional Lorentz metric $(-1, 1, 1, 1, 1)$ and $s|\lambda^2|$ playing the role of the “cosmological constant” (see Appendix A.2). We define accordingly the spin connection form $\hat{\omega}^{AB}$ and the 5-bein form \hat{e}^A as

$$\hat{\omega}^{AB} := \hat{A}^{AB}, \quad \hat{e}^A := \frac{1}{\lambda} \hat{A}^{A5}. \tag{2.7}$$

We can thus write the infinitesimal (A)dS₆ gauge transformations (2.3) and the CS action (2.4) as⁶

$$\begin{aligned} \delta \hat{\omega}^{AB} &= \hat{d} \hat{e}^{AB} + \hat{\omega}^A_C \hat{e}^{CB} - \hat{\omega}^B_C \hat{e}^{CA} \\ &\quad + \lambda \left(\hat{e}^A_5 \hat{e}^B - \hat{e}^B_5 \hat{e}^A \right), \\ \delta \hat{e}^A &= \hat{e}^C \hat{e}_C^A + \frac{1}{\lambda} \left(\hat{d} \hat{e}^{A5} + \hat{\omega}^A_C \hat{e}^{C5} \right), \end{aligned} \tag{2.8}$$

$$\begin{aligned} S^{(5D)} &= \frac{\lambda}{8} \int_{\mathcal{M}_5} \varepsilon_{ABCDE} \\ &\quad \times \left(\hat{e}^A \hat{R}_{(5)}^{BC} \hat{R}_{(5)}^{DE} - \frac{2s\lambda^2}{3} \hat{e}^A \hat{e}^B \hat{e}^C \hat{R}_{(5)}^{DE} + \frac{\lambda^4}{5} \hat{e}^A \hat{e}^B \hat{e}^C \hat{e}^D \hat{e}^E \right). \end{aligned} \tag{2.9}$$

We recognize in the second and third terms the standard Einstein–Hilbert and cosmological terms of 5D gravity, respectively. The novelty of the Chern–Simons action is the appearance of the first term, which is of the form $e \hat{R}_{(5)} \hat{R}_{(5)}$; this term does not enter through an arbitrary coupling constant, but instead through a rational number which is prefixed by the requirement of the theory to be (A)dS₆ invariant, although the action is written in a manifestly Lorentz SO(1,4) invariant form.

Since later on we will proceed to a dimensional reduction from 5D to 4D, we will need the decomposition of the (A)dS₆ group in terms of 4D Lorentz SO(1,3) representations, as displayed in (A.6, A.7). The connection components are accordingly decomposed as

⁶ A subscript will be used to distinguish the (A)dS₅ curvature from the Lorentzian SO(1,3) curvature. We opt not to write the subscript in the latter case. Thus we have

$$\begin{aligned} \hat{R}_{(5)}^{AB} &= \hat{d} \hat{\omega}^{AB} + \hat{\omega}^A_C \hat{\omega}^{CB}, \\ \hat{R}^{IJ} &= \hat{d} \hat{\omega}^{IJ} + \hat{\omega}^I_K \hat{\omega}^{KJ}. \end{aligned}$$

Let us note that the latter is not just obtained by restricting the former to the indices I, J , instead we have

$$\hat{R}_{(5)}^{IJ} = \hat{R}^{IJ} - \hat{\omega}^{4I} \hat{\omega}^{4J}.$$

$$\hat{\omega}^{AB} = \{\hat{\omega}^{IJ}, \hat{\omega}^{I4} =: \lambda \hat{b}^I\}, \quad \hat{e}^A = \{\hat{e}^I, \hat{e}^4\}.$$

With these definitions, the action (2.9) is written

$$\begin{aligned} S^{(5D)} &= \frac{1\lambda}{8} \int_{\mathcal{M}_5} \varepsilon_{IJKL} \left(\hat{e}^4 (\hat{R}^{IJ} - \lambda^2 (\hat{b}^I \hat{b}^J + s \hat{e}^I \hat{e}^J)) \right. \\ &\quad \times (\hat{R}^{KL} - \lambda^2 (\hat{b}^K \hat{b}^L + s \hat{e}^K \hat{e}^L)) \\ &\quad + 2\lambda \hat{D} \hat{e}^I \hat{b}^J \left(\hat{R}^{KL} - \frac{2\lambda^2}{3} \hat{b}^K \hat{b}^L \right) \\ &\quad \left. - 2\lambda \hat{D} \hat{b}^I \hat{e}^J \left(\hat{R}^{KL} - \frac{2s\lambda^2}{3} \hat{e}^K \hat{e}^L \right) \right), \end{aligned} \tag{2.10}$$

with \hat{D} the covariant external derivative: $\hat{D} \hat{e}^I = \hat{d} \hat{e}^I + \hat{\omega}^I_J \hat{e}^J$, etc., and $\hat{R}^{IJ} = d \hat{\omega}^{IJ} + \hat{\omega}^I_K \hat{\omega}^{KJ}$. Let us note that the fields \hat{e}^I and \hat{b}^I play a symmetrical role in the action, so in principle we can use any of the two to define a four-dimensional soldering form. A qualitative difference between these quantities will show up after a suitable truncation.

In view of the announced dimensional reduction, we make explicit a split of the D=5 space-time coordinates in D=4 space-time coordinates x^μ , $\mu = 0, \dots, 3$ and the fifth coordinate $\chi := x^4$ by writing the form fields as

$$\begin{aligned} \hat{e}^I &= e^I_\mu dx^\mu + e^I_\chi d\chi, \\ \hat{b}^I &= b^I_\mu dx^\mu + b^I_\chi d\chi, \\ \hat{e}^4 &= e^4_\mu dx^\mu + e^4_\chi d\chi, \\ \hat{\omega}^{IJ} &= \omega^{IJ}_\mu dx^\mu + \omega^{IJ}_\chi d\chi. \end{aligned} \tag{2.11}$$

The corresponding splits for the curvature components read

$$\hat{F}^{MN} = F^{MN} + F_\chi^{MN} d\chi \tag{2.12}$$

where

$$F^{MN} = \frac{1}{2} F_{\mu\nu}^{MN} dx^\mu dx^\nu, \quad F_\chi^{MN} = F_{\mu\chi}^{MN} dx^\mu.$$

We also have to split each of these forms in terms of their Lorentz SO(1,3) components,

$$\begin{aligned} F^{MN} &= (F^{IJ}, F^{4I}, F^{5I}, F^{45}), \\ F_\chi^{MN} &= (F_\chi^{IJ}, F_\chi^{4I}, F_\chi^{5I}, F_\chi^{45}). \end{aligned}$$

Then with the relabeling (2.11), (2.12), the curvature components take the form

$$\begin{aligned} F^{IJ} &= R^{IJ} - \lambda^2 b^I b^J - s\lambda^2 e^I e^J, \\ F^{I4} &= \lambda D b^I - s\lambda^2 e^I e^4, \\ F^{I5} &= \lambda D e^I + \lambda^2 b^I e^4, \\ F^{45} &= \lambda d e^4 - \lambda^2 b_I e^I, \\ F_\chi^{IJ} &= R_\chi^{IJ} + \lambda^2 (b^I_\chi b^J - b^I b^J_\chi) + s\lambda^2 (e^I_\chi e^J - e^I e^J_\chi), \\ F_\chi^{I4} &= \lambda (D b^I_\chi + \omega^I_J b^J) + s\lambda^2 (e^I_\chi e^4 - e^I e^4_\chi) \\ F_\chi^{I5} &= \lambda (D e^I_\chi + \omega^I_J e^J) - \lambda^2 (b^I_\chi e^4 - b^I e^4_\chi), \\ F_\chi^{45} &= \lambda d e^4_\chi + \lambda^2 (b^I_\chi e^I - b^I e^I_\chi), \end{aligned} \tag{2.13}$$

where D , and R^{IJ} represent the covariant exterior derivative and the curvature 2-form associated to the Lorentz connection ω^{IJ} . The field equations (2.5) are then split into 4-forms equations and 3-form equations (the χ components). The 4-form equations are

$$\begin{aligned} \varepsilon_{IJKL}(F^{45}F^{KL} - 2F^{K4}F^{L5}) &= 0, \\ \varepsilon_{IJKL}F^{J5}F^{KL} &= 0, \\ \varepsilon_{IJKL}F^{J4}F^{KL} &= 0, \\ \varepsilon_{IJKL}F^{IJ}F^{KL} &= 0, \end{aligned} \tag{2.14}$$

and the 3-form equations are

$$\begin{aligned} \varepsilon_{IJKL}(F_{\chi}^{45}F^{KL} + F^{45}F_{\chi}^{KL} - 2F_{\chi}^{K4}F^{L5} - 2F^{K4}F_{\chi}^{L5}) &= 0, \\ \varepsilon_{IJKL}(F_{\chi}^{J5}F^{KL} + F^{J5}F_{\chi}^{KL}) &= 0, \\ \varepsilon_{IJKL}(F_{\chi}^{J4}F^{KL} + F^{J4}F_{\chi}^{KL}) &= 0, \\ \varepsilon_{IJKL}F^{IJ}F_{\chi}^{KL} &= 0, \end{aligned} \tag{2.15}$$

with the curvature components given by (2.13).

This theory is still invariant under the full (A)dS₆ transformations, which now read, in terms of the 5D ‘‘hat’’ quantities:

$$\begin{aligned} \delta\hat{\omega}^{IJ} &= \hat{D}\hat{\epsilon}^{IJ} + \lambda_s(\hat{\epsilon}^{I5}\hat{e}^J - \hat{\epsilon}^{J5}\hat{e}^I) + \lambda(\hat{\epsilon}^{I4}\hat{b}^J - \hat{\epsilon}^{J4}\hat{b}^I), \\ \delta\hat{e}^I &= (1/\lambda)\hat{D}\hat{\epsilon}^{I5} + \hat{e}^J\hat{\epsilon}_J^I + \hat{b}^I\hat{\epsilon}^{45} - \hat{e}_{\chi}^4\hat{\epsilon}^{I4}, \\ \delta\hat{b}^I &= (1/\lambda)\hat{D}\hat{\epsilon}^{I4} + \hat{b}^J\hat{\epsilon}_J^I - s\hat{e}^I\hat{\epsilon}^{45} + s\hat{e}_{\chi}^4\hat{\epsilon}^{I5}, \\ \delta\hat{e}^4 &= (1/\lambda)\hat{d}\hat{\epsilon}^{45} - \hat{b}_I\hat{\epsilon}^{I5} + \hat{e}_I\hat{\epsilon}^{I4}. \end{aligned} \tag{2.16}$$

A partial gauge fixing

The action (2.9) or (2.10) and the field equations (2.14) and (2.15) may be simplified by a partial gauge fixing consisting of the eight conditions

$$b_{\chi}^I = 0, \quad e_{\chi}^I = 0, \quad I = 0, \dots, 3, \tag{2.17}$$

which fix the gauge symmetries generated by $M_{I5} = P_I/\lambda$ and $M_{I4} = Q_I/\lambda$, respectively, as can be inferred from the transformation laws (2.16) for the χ components of \hat{e}^I and \hat{b}^I , the field \hat{e}_{χ}^4 being assumed not to vanish. This reduces the explicit gauge symmetry to the group $SO(1,3) \times (A)dS_2$, where $SO(1,3)$ is the 4D Lorentz group and $(A)dS_2 = U(1)$ if $s > 0$ (theory with positive cosmological constant) or the dilatation group if $s < 0$ (theory with negative cosmological constant). Of course (2.17) is only a gauge fixing: the theory remains a full (A)dS₆ gauge theory.

2.2 The Chamseddine action

A 4D theory may be obtained through a Kaluza–Klein dimensional compactification in which ‘‘matter-like’’ fields are realized as fifth-dimension components of the 5D fields. In our context we may assume that the fifth dimension, of the coordinate $x^4 = \chi$, is compact and ‘‘microscopic’’, and the

fields are expanded in Kaluza–Klein modes. In the present paper, we restrict the study to the zero-mode sector—which amounts to considering all fields as constant in χ —leaving a complete discussion involving all modes for future work [19]. This means

$$\partial_{\chi}f(x) = 0, \quad \forall \text{ field } f. \tag{2.18}$$

The Chamseddine model has been obtained [13] by a truncation consisting in setting some fields to zero:

$$e_{\chi}^I = 0, \quad \omega_{\chi}^{IJ} = 0, \quad e_{\mu}^4 = 0, \quad b_{\mu}^I = 0. \tag{2.19}$$

We may observe that the first condition is in fact nothing but a gauge fixing condition, the second of (2.17). The other three truncation conditions do indeed break (A)dS₆, apparently to $SO(1,3)$. However, a reordering of the remaining fields in new multiplets allows one to show that the resulting theory actually has a hidden (A)dS₅ gauge invariance [13]. In order to see this, one does not apply for the moment the first of the gauge fixing conditions (2.17), and one reorders the fields in (A)dS₅ multiplets as

$$\mathbb{A}^{AB} = \{\mathbb{A}^{IJ}, \mathbb{A}^{I4}\} := \{\omega^{IJ}, \lambda e^I\}, \tag{2.20}$$

$$\Phi^A = \{\Phi^I, \Phi^4\} := \{-b_{\chi}^I, e_{\chi}^4\}. \tag{2.21}$$

Using these definitions together with the truncation conditions (2.19), the action (2.9) or (2.10) reduces to the obviously (A)dS₅ invariant expression⁷

$$S^{(4D)} = \frac{1}{8} \int_{\mathcal{M}_4} \varepsilon_{ABCDE} \Phi^A \mathbb{F}^{BC} \mathbb{F}^{DE}, \tag{2.22}$$

with the (A)dS₅ curvature

$$\mathbb{F}^{AB} = d\mathbb{A}^{AB} + \mathbb{A}^A_C \mathbb{A}^{CB}, \tag{2.23}$$

which in terms of the $SO(1,3)$ components reads

$$\begin{aligned} \mathbb{F}^{IJ} &= R^{IJ} - \lambda^2 e^I e^J, \quad (R^{IJ} = d\omega^{IJ} + \omega^I_K \omega^{KJ}) \\ \mathbb{F}^{I4} &= \lambda D e^I \quad (D e^I = d e^I + \omega^I_J e^J). \end{aligned}$$

The infinitesimal (A)dS₅ gauge transformations which leave the Chamseddine action invariant may be written as

$$\begin{aligned} \delta\mathbb{A}^{AB} &= d\epsilon^{AB} + \mathbb{A}^A_C \epsilon^{CB} - \mathbb{A}^B_C \epsilon^{CA}, \\ \delta\Phi^A &= \Phi_B \epsilon^{BA}. \end{aligned} \tag{2.24}$$

The equations of motion from (2.22) are

$$\frac{\delta S^{(4D)}}{\delta \Phi^A} = \frac{1}{8} \varepsilon_{ABCDE} \mathbb{F}^{BC} \mathbb{F}^{DE} = 0, \tag{2.25}$$

$$\frac{\delta S^{(4D)}}{\delta \mathbb{A}^{AB}} = \frac{1}{2} \varepsilon_{ABCDE} \mathbb{D} \Phi^C \mathbb{F}^{DE} = 0. \tag{2.26}$$

⁷ No parameter is needed in front of the action, since any such parameter may be absorbed in a redefinition of the scalar field Φ^A .

Or, in terms of the SO(1,3) components,

$$\begin{aligned} \frac{\delta S}{\delta e^I} &= -\frac{\lambda}{2}\epsilon_{IJKL}(D\Phi^J + s\lambda e^J\Phi^4)(R^{KL} - s\lambda^2 e^K e^L) = 0, \\ \frac{\delta S}{\delta \omega^{IJ}} &= \frac{1}{2}\epsilon_{IJKL} \\ &\times \left((d\Phi^4 - \lambda e^I\Phi^4)(R^{KL} - s\lambda^2 e^K e^L) \right. \\ &\left. + \lambda(D\Phi^K + s\lambda e^K\Phi^4)De^L \right) = 0, \\ \frac{\delta S}{\delta \Phi^4} &= \frac{1}{8}\epsilon_{IJKL}(R^{IJ} - s\lambda^2 e^I e^J)(R^{KL} - s\lambda^2 e^K e^L) = 0, \\ \frac{\delta S}{\delta \Phi^I} &= \frac{\lambda}{2}\epsilon_{IJKL}De^J(R^{KL} - s\lambda^2 e^K e^L) = 0. \end{aligned} \tag{2.27}$$

2.2.1 Introducing matter: continuity equation

We may introduce matter adding to the action (2.22) a matter term S_m which we will suppose (A)dS₅ gauge invariant and independent of the scalar field Φ^A . (A)dS₅ gauge invariance of the total action,

$$S = S^{(4D)}[e, \omega, \Phi] + S_m[e, \omega], \tag{2.28}$$

can be expressed through the local ‘‘Ward identity’’

$$W_{AB}S := -\mathbb{D}\frac{\delta S}{\delta \Phi^A} + \Phi_A\frac{\delta S}{\delta \Phi^B} - \Phi_B\frac{\delta S}{\delta \Phi^A} = 0.$$

We shall be interested in particular in the Ward identity linked to invariance along the generators M_{I4} :

$$W_I S := -\lambda e^J\frac{\delta S}{\delta \omega^{IJ}} - \frac{1}{\lambda}D\frac{\delta S}{\delta e^I} + \Phi_I\frac{\delta S}{\delta \Phi^4} - s\Phi^4\frac{\delta S}{\delta \Phi^I} = 0. \tag{2.29}$$

Note that these identities hold separately for both actions, $S^{(4D)}$ and S_m . Defining

$$\mathcal{T}_I := \frac{\delta S_m}{\delta e^I}, \quad \mathcal{T}_{IJ} := \frac{\delta S_m}{\delta \omega^{IJ}},$$

we may rewrite (2.29) as

$$\begin{aligned} \lambda e^J\mathcal{T}_{IJ} + \frac{1}{\lambda}D\mathcal{T}_I &= -\lambda e^J\frac{\delta S^{(4D)}}{\delta \omega^{IJ}} - \frac{1}{\lambda}D\frac{\delta S^{(4D)}}{\delta e^I} \\ &+ \Phi_I\frac{\delta S^{(4D)}}{\delta \Phi^4} - s\Phi^4\frac{\delta S^{(4D)}}{\delta \Phi^I} = W_I S^{(4D)} = 0, \end{aligned} \tag{2.30}$$

the last equality expressing the invariance of $S^{(4D)}$. This leads to the general continuity equation

$$\lambda 2e^J\mathcal{T}_{IJ} + D\mathcal{T}_I = 0. \tag{2.31}$$

The 3-form \mathcal{T}_I is related to the energy-momentum tensor components $\mathcal{T}^N{}_I$ in the tetrad frame by

$$\mathcal{T}_I = \frac{1}{6}\epsilon_{NJKL}\mathcal{T}^N{}_I e^J e^K e^L. \tag{2.32}$$

If $\mathcal{T}_{IJ} = 0$, i.e., if the matter action S_m does not depend on the spin connection ω , (2.31) is interpreted as the continuity equation for energy and momentum.

2.2.2 Partial gauge fixing of the Chamseddine theory

From the gauge transformations (2.24) leaving the action (2.22) invariant, one sees that a possible partial gauge fixing is given by the four conditions

$$\Phi^I = 0, \quad I = 0, \dots, 3. \tag{2.33}$$

The total action, including matter, then reduces to

$$\begin{aligned} \bar{S} &= \frac{1}{8}\int_{\mathcal{M}_4}\epsilon_{IJKL}\Phi^4\mathbb{F}^{IJ}\mathbb{F}^{KL} + S_m \\ &= \frac{1}{8}\int_{\mathcal{M}_4}\epsilon_{IJKL}\Phi^4(R^{IJ} - s\lambda^2 e^I e^J) \\ &\quad (R^{KL} - s\lambda^2 e^K e^L) + S_m, \end{aligned} \tag{2.34}$$

where the matter action S_m is supposed to be independent of Φ^A , as above, but will also be assumed not to depend on the spin connection ω from now on. The field equations derived from the latter action—to which we may add a matter action S_m , supposed to obey the same (A)dS₅ gauge invariance as the pure Chamseddine part—are

$$\begin{aligned} \frac{\delta \bar{S}}{\delta e^I} &= -\frac{1}{2}s\lambda^2\Phi^4\epsilon_{IJKL}(e^J R^{KL} - s\lambda^2 e^J e^K e^L) + \mathcal{T}_I = 0, \\ \frac{\delta \bar{S}}{\delta \omega^{IJ}} &= \frac{1}{2}\epsilon_{IJKL}\left(d\Phi^4(R^{KL} - s\lambda^2 e^K e^L) + 2s\lambda^2\Phi^4 e^K De^L\right) = 0, \\ \frac{\delta \bar{S}}{\delta \Phi^4} &= \frac{1}{8}\epsilon_{IJKL}(R^{IJ} - s\lambda^2 e^I e^J)(R^{KL} - s\lambda^2 e^K e^L) = 0, \end{aligned} \tag{2.35}$$

where $T^I := De^I$ is the torsion and \mathcal{T}_I the energy-momentum 3-form (2.32).

Comparison of the first of Eq. (2.35) with the standard Einstein equation in the first order formalism,

$$\begin{aligned} \epsilon_{IJKL}\left(e^J R^{KL} - \frac{\Lambda}{3}e^J e^K e^L\right) &= -16\pi G\mathcal{T}_I, \\ T^I = De^I &= 0, \end{aligned} \tag{2.36}$$

suggests one to identify $3s\lambda^2$ with the cosmological constant,

$$\Lambda := 3s\lambda^2, \tag{2.37}$$

and to define the function

$$G(x) := -\frac{3}{8\pi\Lambda\Phi^4(x)}, \tag{2.38}$$

as a variable ‘‘Newton parameter’’, proportional to the inverse of the dilation field Φ^4 . With this, the field equations (2.35) take the form

$$\begin{aligned}
 \epsilon_{IJKL} \left(e^J R^{KL} - \frac{\Lambda}{3} e^J e^K e^L \right) &= -16\pi G(x) \mathcal{T}_I, \\
 \epsilon_{IJKL} \left(dG(x) \left(R^{KL} - \frac{\Lambda}{3} e^K e^L \right) - 2 \frac{\Lambda}{3} G(x) e^K D e^L \right) &= 0, \\
 \epsilon_{IJKL} \left(R^{IJ} - \frac{\Lambda}{3} e^I e^J \right) \left(R^{KL} - \frac{\Lambda}{3} e^K e^L \right) &= 0,
 \end{aligned}
 \tag{2.39}$$

where we have emphasized the x -dependence of the Newton parameter G .

Let us finish this subsection with some comments:

1. The theory is clearly singular in $\Lambda = 0$. This value would correspond to a vanishing s in the (A)dS₅ metric (A.2), which thus would become singular.
2. The first field equation in (2.35) has the form of the usual Einstein equation in the first order formalism, but with varying Newton coupling parameter $G(x)$. The second one determines the torsion T^I in terms of the basic fields ω^{IJ} , e^I , and G . In particular, the torsion is zero if G is constant. The third equation is a new constraint. One must emphasize that the torsion here is not an independent field.
3. A canonical analysis [19] shows that the number of physical degrees of freedom of the theory is three: this corresponds to the two degrees of freedom of the gravitational field plus one corresponding to the scalar dilaton field Φ^4 —the Newton coupling parameter G .
4. In the absence of matter, an obvious trivial solution is the constant curvature and torsion free (anti-)de Sitter space: $R^{IJ} = \frac{\Lambda}{3} e^I e^J$.
5. The last equation, which clearly admits the constant curvature solution, is also compatible with non-trivial solutions, as the examples treated below do show.
6. It is interesting to note that the gauge fixing condition (2.33) is nothing but the first of the gauge fixing conditions (2.17) [see Definition (2.21)].

2.2.3 Energy-momentum continuity

In Einstein theory, the continuity equation for the energy-momentum tensor reads, in the first order formalism used here,

$$D\mathcal{T}_I = 0, \tag{2.40}$$

where D is the exterior derivative with respect to the spin connection ω^{IJ} and the 3-form \mathcal{T}_I is related to the energy-momentum tensor by (2.32). The continuity equation (2.40) follows from the Einstein field equations (2.36) and the Bianchi identity $DR^{IJ} = 0$. As we saw in Sect. (2.2.1), it turns out that it still holds in our case, as a consequence of

the (A)dS₅ invariance expressed by the identity (2.30) and of the hypothesis we have made that the matter action is independent not only from the scalar fields Φ , but also from the spin connection ω .

It is interesting to look at the identity (2.30) with the gauge fixing condition $\Phi^I = 0$ being applied. Taking into account the hypothesis that S_m only depends on the tetrad e , this leads to the identity

$$D\mathcal{T}_I = \frac{\Lambda}{3} e^J \frac{\delta \bar{S}}{\delta \omega^{IJ}} + \sqrt{\frac{|\Lambda|}{3}} \Phi^4 \frac{\delta S}{\delta \Phi^I} \Big|_{\Phi^I=0},$$

where \bar{S} is the total gauge fixed action (2.34), and S the total action (2.28) before gauge fixing. Since $D\mathcal{T}_I = 0$, the latter identity shows that the equation

$$\frac{\delta S}{\delta \Phi^I} \Big|_{\Phi^I=0} = 0 \tag{2.41}$$

is valid “on shell”, i.e., if the field equations (2.35) of the gauge fixed theory are satisfied. This is just the equation of the non-gauge fixed theory obtained by varying Φ^I , taken at $\Phi^I = 0$. In fact, the on-shell validity of (2.41) can be derived directly from the Ward identity (2.29) taken at $\Phi^I = 0$, as one can easily check.

2.2.4 The field equations of the Chamseddine model as particular field equations of the dimensionally reduced 5D Chern–Simons theory

One may ask if the equations of motion derived from the truncated theory, namely the Chamseddine model equations (2.25) and (2.26), together with the truncation equations (2.19) and the χ -independence conditions (2.18), are also solutions of the equations of motion (2.5) of the full original (A)dS₆ Chern–Simons theory. In the following we show the answer is positive.

The field equations of the full CS theory reduced in four dimensions are given by (2.14) and (2.15) together with (2.13). After imposing the truncation (2.19) together with the restriction (2.18) and the relabeling (2.20), (2.21), the curvature components take the form

$$\begin{aligned}
 F^{IJ} &= R^{IJ} - s\lambda^2 e^I e^J, \quad F^{I4} = 0, \quad F^{I5} = \lambda D e^I, \quad F^{45} = 0, \\
 F_\chi^{IJ} &= 0, \quad F_\chi^{I4} = -\lambda D \Phi^I - s\lambda^2 e^I \Phi^4, \quad F_\chi^{I5} = 0 \\
 F_\chi^{45} &= \lambda d\Phi^4 - \lambda^2 e_I \Phi^I.
 \end{aligned}
 \tag{2.42}$$

Inserting the expressions (2.42) in the eight equations (2.14) and (2.15), we obtain four trivial equations $0 = 0$, and four non-trivial ones which are identical to those obtained from the action of the (A)dS₅ Chamseddine model, Eq. 2.27. We conclude that the set of solutions of the equations of motion of the (A)dS₅ is a particular subset of the solutions of the general (A)dS₆ Chern–Simons theory.

It is noteworthy that the four trivial equations are those derived from the CS action by varying the four fields destined for truncation. Had we performed any other sort of truncation in the field equations, we would have obtained more independent equations than what one obtains directly from the truncated action.

It is also enlightening to see that the effect of the truncation, when applied directly to the original (A)dS₆ connection, leads to

$$\hat{A} = \frac{1}{2} \mathbb{A}^{AB} M_{AB} dx^\mu + \Phi^A P_A d\chi, \tag{2.43}$$

where \mathbb{A}^{AB} and Φ^A were introduced in (2.20) and (2.21). We see that the effect of the truncation is to confine the (A)dS₅ symmetry to a four-dimensional connection, whereas the “translational” sector of the group is restricted to the χ dimension. Then it is clear why the truncation, which may seem not obvious at first sight, results after some simplifications in a four-dimensional (A)dS₅ gauge theory with Φ a four-dimensional scalar transforming as a vector under (A)dS₅ transformations.

The curvature associated with the truncated connection (2.43) is

$$\hat{F} = \frac{1}{4} \mathbb{F}^{AB} M_{AB} dx^\mu dx^\nu + \mathbb{D}_\mu \Phi^A P_A dx^\mu d\chi. \tag{2.44}$$

A straightforward calculation shows that by replacing the above result in (2.5) we obtain (2.25), (2.26), the field equations of the truncated action (2.22).

3 Linear approximations

In order to investigate the Newtonian limit of the Chamseddine theory or to look for the presence of wave-like solutions of the theory in the vacuum, we split the field variables between background ones, marked with an index ⁰ on the top, and perturbations as follows:

$$\omega^{IJ} = \hat{\omega}^{IJ} + a^{IJ}, \quad e^I = \hat{e}^I + h^I, \quad G = \hat{G} + \phi. \tag{3.1}$$

Up to terms of order higher than one in the perturbation, the curvature $R = d\omega + \omega^2$ and the torsion $T^I = De^I$ read

$$R^{IJ} = \hat{R}^{IJ} + \hat{D}a^{IJ}, \quad T^I = \hat{T}^I + \hat{D}h^I + a^I{}_J \hat{e}^J,$$

where \hat{D} is the covariant derivative corresponding to the background connection $\hat{\omega}$. The background considered here is a constant curvature de Sitter space-time, solution of

$$\hat{R}^{IJ} - \frac{\Lambda}{3} \hat{e}^I \hat{e}^J = 0,$$

hence the expression $F^{IJ} = R^{IJ} - \frac{\Lambda}{3} e^I e^J$ is of first order:

$$F^{IJ} = f^{IJ} \quad (+ \text{orders} > 1),$$

$$f^{IJ} = \hat{D}a^{IJ} - \frac{\Lambda}{3} (\hat{e}^I h^J - \hat{e}^J h^I).$$

We shall also assume that the zeroth order Newton parameter \hat{G} is a (non-zero) constant. The second field equation then implies that the zeroth order torsion is vanishing: $\hat{T}^I = 0$. The first field equation shows that the energy-momentum 3-form \mathcal{T}_I must be considered to be of first order, and the third equation is identically solved up to and including the first order.

The first and second field equations read, at first order,

$$\varepsilon_{IJKL} \hat{e}^J \left(\hat{D}a^{KL} - \frac{\Lambda}{3} (\hat{e}^K h^L - \hat{e}^L h^K) \right) = -8\pi G_0 \mathcal{T}_I, \tag{3.2}$$

$$\varepsilon_{IJKL} G_0 \hat{e}^K (\hat{D}h^I + a^I{}_J \hat{e}^J) = 0,$$

where $G_0 = \hat{G}$ is the Newton parameter at zeroth order, interpreted as the actual Newton constant. The second of these equations implies a vanishing torsion at first order, too:

$$\hat{D}h^I + a^I{}_J \hat{e}^J = 0.$$

We are thus left with the first of equations (3.2), where the first order connection a^{IJ} may be solved in terms of the vierbein perturbation components h^I_μ and their derivatives through the null torsion condition. This is just Einstein GR with cosmological constant at first order of the perturbation, in a de Sitter background.

A first implication is that the theory admits a Newtonian limit like Einstein’s does. A second implication concerns the theory with cosmological constant in the vacuum. Since at first order the theory coincides with Einstein’s, we can rely on the results of an extensive study made by the authors of [20, 21], where they show that, beyond the constant curvature solution, there are propagating wave solutions. We refer to their paper for more details.

4 Cosmological solutions

In order to explore the physical content of the Chamseddine model, we look in this section for solutions of the cosmological type and compare them with the known Λ CDM results [22].

4.1 Isotropy and homogeneity

We examine the solutions of the field equations (2.35) considering a space-time foliated by a family of isotropic and homogeneous three-dimensional spatial slices, as described by the standard Big Bang cosmology. The metric that describes this is the Friedmann–Lemaître–Robertson–Walker (FLRW) metric, given by

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right],$$

depending on the time-dependent scale factor $a(t)$ and the space curvature parameter $k = 0, \pm 1$. The space-time coordinates are the time coordinate t and the spatial spherical coordinates r, θ, φ . The FLRW metric admits six isometries generated by six global Killing vectors associated with three spatial translation $\xi_{(a)}$ and three rotation $\xi_{[ab]}$ invariances—i.e., such that $\mathcal{L}_{\xi_{[ab]}}g_{\mu\nu} = \mathcal{L}_{\xi_{(a)}}g_{\mu\nu} = 0$ —which read in Cartesian coordinates $x^a, a = 1, 2, 3$,

$$\xi_{(a)} = \sqrt{1 - kr^2}\partial_a, \quad \text{and} \quad \xi_{[ab]} = x_a\partial_b - x_b\partial_a.$$

We assume that the torsion and the scalar field (the Newton parameter G) have the same isometries as the metric, i.e., $\mathcal{L}_{\xi}T^\rho_{\mu\nu} = 0$ and $\mathcal{L}_{\xi}G = 0$. These conditions imply $G = G(t)$, and the non-vanishing components of $T^\alpha_{\mu\nu}$ are⁸ [23]

$$T^r_{\theta\varphi} = 2f(t)a(t)r^2\sqrt{1 - kr^2}\sin\theta, \quad T^\varphi_{r\theta} = \frac{2f(t)a(t)}{\sqrt{1 - kr^2}\sin\theta},$$

$$T^\theta_{r\varphi} = -\frac{2f(t)a(t)\sin\theta}{\sqrt{1 - kr^2}}, \quad T^r_{rt} = T^\theta_{\theta t} = T^\varphi_{\varphi t} = h(t)$$

where $f(t)$ and $h(t)$ are functions of time to be determined by the field equations.

Working in the first order formalism, we have to choose a corresponding parametrization of the vierbein. A convenient choice [23] is⁹:

$$e^0 = dt, \quad e^1 = \frac{a(t)}{\sqrt{1 - kr^2}}dr,$$

$$e^2 = a(t)r d\theta, \quad e^3 = a(t)r \sin\theta d\varphi.$$

In this basis the torsion 2-form becomes

$$T^0 = 0, \quad T^i = h(t)e^i e^0 + f(t)\varepsilon^i_{jk}e^j e^k.$$

(The indices $i, j \dots$ take the values 1,2,3.) The spin connection ω which gives rise to this torsion reads

$$\omega^{0i} = (H + h)e^i, \quad \omega^{12} = -\frac{\sqrt{1 - kr^2}}{ar}e^2 - fe^3,$$

$$\omega^{31} = \frac{\sqrt{1 - kr^2}}{ar}e^3 - fe^2, \quad \omega^{23} = -\frac{\cot\theta}{ar}e^3 - fe^1,$$

where

$$H := \dot{a}(t)/a(t) \tag{4.1}$$

is the Hubble parameter. The Riemann curvature is given by

$$R^{0i} = ((\dot{H} + \dot{h}) + H(H + h))e^0 e^i + f(H + h)\varepsilon^i_{jk}e^j e^k,$$

$$R^{ij} = \left((H + h)^2 + \frac{k}{a^2} - f^2 \right) e^i e^j + (\dot{f} + Hf)\varepsilon^{ij}_k e^k e^0.$$

⁸ Torsion $T^I_{\mu\nu}$ is defined by $T^I = De^I$, whereas $T^\rho_{\mu\nu} = e^\rho_I T^I_{\mu\nu}$.

⁹ This choice amounts to a gauge fixing of the local Lorentz invariance.

Consequently

$$F^{0i} = \left((\dot{H} + \dot{h}) + H(H + h) - \frac{\Lambda}{3} \right) e^0 e^i$$

$$+ f(H + h)\varepsilon^i_{jk}e^j e^k,$$

$$F^{ij} = \left((H + h)^2 + \frac{k}{a^2} - f^2 - \frac{\Lambda}{3} \right) e^i e^j$$

$$+ (\dot{f} + Hf)\varepsilon^{ij}_k e^k e^0.$$

4.2 Field equations

We assume matter to consist of a perfect fluid of density ρ_m and pressure p_m , with an energy-momentum tensor $T^I_J = \text{diag}(-\rho_m, p_m, p_m, p_m)$. Substituting in the field equations (2.35), with $dG = \dot{G}e^0$, we get the system of differential equations

$$U^2 + \frac{k}{a^2} - f^2 - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho_m, \tag{4.2}$$

$$U^2 + \frac{k}{a^2} - f^2 - \Lambda + 2(\dot{U} + HU) = -8\pi G p_m, \tag{4.3}$$

$$\dot{G} \left(U^2 + \frac{k}{a^2} - f^2 - \frac{\Lambda}{3} \right) - \frac{2\Lambda}{3}Gh = 0, \tag{4.4}$$

$$f \left(\dot{G}U - \frac{\Lambda}{3}G \right) = 0, \tag{4.5}$$

$$\left(U^2 + \frac{k}{a^2} - f^2 - \frac{\Lambda}{3} \right) \left(\dot{U} + HU - \frac{\Lambda}{3} \right)$$

$$- 2fU(\dot{f} + Hf) = 0, \tag{4.6}$$

where $U := H + h$ and $G = G(t)$ is the Newton coupling parameter (2.38).

4.3 Continuity equations

A first continuity equation for the energy and pressure of matter follows directly from the energy-momentum continuity equation (2.40). Calculating the components of the energy-momentum 3-form \mathcal{T}_I , from (2.32) one finds

$$\mathcal{T}_0 = -\frac{\rho_m(t)}{6}\varepsilon_{ijk}e^i e^j e^k,$$

$$\mathcal{T}_i = -\frac{p_m(t)}{2}\varepsilon_{ijk}e^0 e^j e^k,$$

consequently

$$\mathcal{T}_0 = -\rho_m(t)e^1 e^2 e^3 = -\frac{\rho_m(t)a(t)^3 r^2 \sin\theta}{\sqrt{1 - kr^2}}dr \wedge d\theta \wedge d\varphi,$$

$$\mathcal{T}_1 = -p_m(t)e^0 e^2 e^3 = -p_m(t)a(t)^2 r^2 \sin\theta dt \wedge d\theta \wedge d\varphi,$$

$$\mathcal{T}_2 = -p_m(t)e^0 e^3 e^1 = -p_m(t)\frac{a(t)^2 r \sin\theta}{\sqrt{1 - kr^2}}dt \wedge d\varphi \wedge dr,$$

$$\mathcal{T}_3 = -p_m(t)e^0 e^1 e^2 = -p_m(t)\frac{a(t)^2 r}{\sqrt{1 - kr^2}}dt \wedge dr \wedge d\theta.$$

The equation $D\mathcal{T}_0 = 0$ yields the density–pressure–torsion continuity equation

$$\dot{\rho}_m + 3H(\rho_m + p_m) + 3h p_m = 0. \tag{4.7}$$

The equations $D\mathcal{T}_i = 0$ for $i = 1, 2, 3$ are trivially satisfied, being of the form $0 = 0$.

Note the torsion dependence in the last term of (4.7). However, for a matter with zero pressure (cold matter, dust), this continuity equation takes the usual form [24]:

$$\frac{d}{dt} (\rho_m a^3) = 0, \quad \text{if } p_m = 0. \tag{4.8}$$

A second continuity equation can be found in the following way: One notes that, substituting $U = H + h$ in Eqs. (4.2) and (4.3) leads to analogs of the standard Friedmann equations:

$$H^2 = \frac{8\pi G_0}{3} \rho_{\text{tot}}, \quad 2\dot{H} + 3H^2 = -8\pi G_0 p_{\text{tot}}, \tag{4.9}$$

where G_0 is the Newton constant, taken as the present value of $G(t)$, and $\rho_{\text{tot}}, p_{\text{tot}}$ are the “total density and pressure”,

$$\rho_{\text{tot}} = \frac{G}{G_0} (\rho_m + \rho_k + \rho_T + \rho_\Lambda),$$

$$p_{\text{tot}} = \frac{G}{G_0} (p_m + p_k + p_T + p_\Lambda),$$

with

$$\begin{aligned} \rho_k &= -\frac{3}{8\pi G} \frac{k}{a^2}, \quad \rho_T = \frac{3}{8\pi G} (f^2 - 2Hh - h^2), \quad \rho_\Lambda = \frac{\Lambda}{8\pi G} \\ p_k &= -\rho_k/3, \quad p_T = \frac{1}{8\pi G} (2\dot{h} + 4Hh + h^2 - f^2), \quad p_\Lambda = -\rho_\Lambda. \end{aligned}$$

ρ_T and p_T may be interpreted as the contributions of the torsion to the total density and pressure ρ_{tot} and p_{tot} . As a consequence of the Friedmann-like equations (4.9), the total density and pressure satisfy the continuity equation

$$\dot{\rho}_{\text{tot}} + 3H(\rho_{\text{tot}} + p_{\text{tot}}) = 0.$$

4.4 Pressure-less matter with $\Lambda > 0$ and $k = 0$

In this subsection we present the general solution of Eqs. (4.2)–(4.6) in the case of pressure-less matter (cold matter or dust), with $p_m = 0$, with a positive cosmological constant Λ and a null curvature parameter k , as favored by the observational results [22]. From Eq. (4.5) follows

$$\text{either } f(t) = 0, \quad \text{or } \dot{G}U - \frac{\Lambda}{3}G = 0. \tag{4.10}$$

We have first checked that the former case leads to the “trivial” solution of a null torsion de Sitter space with cosmological constant Λ , the vierbein or the metric being defined by the scale parameter $a(t) = \exp(\sqrt{\Lambda/3}t)$.

We hence assume the function $f(t)$ to be non-vanishing. The equations to be solved are Eqs. (4.2)–(4.4), (4.6), and the second of (4.10), together with the Hubble parameter

definition (4.1) in terms of the scale $a(t)$. The general solution is given by the following expressions¹⁰, where the time coordinate has been redefined by

$$\tau(t) := \sqrt{\frac{\Lambda}{3}} t.$$

Scale parameter:

$$a(t) = C_4 (3e^{\tau(t)} + C_3 e^{-\tau(t)})^{1/3} (\cosh(\tau(t) - C_1))^{2/3}. \tag{4.11}$$

Torsion parameter $f(t)$:

$$\begin{aligned} f(t) &= \frac{\sqrt{\Lambda}}{3} \left[\left(-9e^{2\tau(t)} - 3C_3 + (6e^{2\tau(t)} - 2C_3) \tanh(\tau(t) - C_1) \right. \right. \\ &\quad \left. \left. + (3e^{2\tau(t)} + C_3) \tanh^2(\tau(t) - C_1) \right) / (3e^{2\tau(t)} + C_3) \right]^{1/2}. \end{aligned} \tag{4.12}$$

Torsion parameter $h(t)$:

$$h(t) = \sqrt{\frac{\Lambda}{3}} \frac{(-3e^{2\tau(t)} + C_3 + (3e^{2\tau(t)} + C_3) \tanh(\tau(t) - C_1))}{9e^{2\tau(t)} + 3C_3}. \tag{4.13}$$

Hubble parameter $H = \dot{a}/a$:

$$H(t) = \sqrt{\frac{\Lambda}{3}} \tanh(\tau(t) - C_1) - h[t]. \tag{4.14}$$

Newton parameter $G(t) = -3/(8\pi \Lambda \Phi(t))$ [cf. (2.38)]:

$$G(t) = C_2 \sinh(\tau(t) - C_1). \tag{4.15}$$

Cold matter density:

$$\rho_m(t) = \frac{3}{8\pi G(t)} \left((H(t) + h(t))^2 - f^2(t) - \frac{\Lambda}{3} \right). \tag{4.16}$$

The four integration constants C_1, C_2, C_3, C_4 and the cosmological constant Λ have to be determined by five physical conditions, which we choose to be

$$\begin{aligned} a(0) &= 0 : \quad \text{hypothesis of a Big Bang,} \\ a(t_0) &= 1 : \quad t_0 = \text{present age of the Universe,} \\ H(t_0) &= H_0 : \quad \text{present value of the Hubble parameter,} \\ G(t_0) &= G_0 : \quad \text{present value of the Newton parameter,} \\ \rho_m(t_0) &= \rho_0 : \quad \text{present value of the cold matter mass} \\ &\quad \text{density,} \end{aligned} \tag{4.17}$$

with the present observational [22] and experimental data given by

$$\begin{aligned} t_0 &= 13.8 \times 10^9 \text{ Gy} \quad (1 \text{ Gy} = 10^9 \text{ years} = 3.1558^{16} \text{ s}), \\ H_0 &= 0.0693 \text{ Gy}^{-1}, \\ \rho_0 &= 2.664 \times 10^{-27} \text{ Kg m}^{-3}, \end{aligned}$$

¹⁰ The solution is obtained using the program Mathematica [25].

$$G_0 = 6.674 \times 10^{-11} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1}.$$

For comparison with the standard Λ CDM results, we need the Λ CDM formula for the scale parameter $a(t)$, for a Universe dominated by cold dark matter of the present relative density [22] $\Omega_m = 0.309$. With the contribution of radiation neglected, the normalized Λ CDM scale parameter reads [24]

$$a_{\Lambda\text{CDM}}(t) = \left(\sinh \left(\frac{3}{2} H_0 \sqrt{1 - \Omega_{m0}} t \right) / \sinh \left(\frac{3}{2} H_0 \sqrt{1 - \Omega_{m0}} t_0 \right) \right)^{2/3}. \tag{4.18}$$

Figure 1 shows the time evolution of the scale parameter a , of the Hubble constant H , of the deceleration parameter $q = -\ddot{a}/(\dot{a}a^2)$, of the mass density ρ_m , and of the normalized Newton parameter G/G_0 , each one being compared with the corresponding Λ CDM quantity. Except for the deceleration q , the deviations are rather small. The Newton parameter, which has to be equal to the actual Newton constant G_0 at the present time, shows a slight decrease toward the past, growing to $\sim 85\%$ of its present value near of the Big Bang. The deceleration q differs notably from the Λ CDM one, but the time of the transition between the deceleration and the acceleration era almost coincides. The present value $q(t_0) = -0.25$ is, however, only half of the Λ CDM value.

Time evolutions of the torsion parameters h and f , as well as the relative densities $\Omega_m(t)$, $\Omega_\Lambda(t)$, and $\Omega_T(t)$ for matter, cosmological constant, and torsion, respectively, are shown in Fig. 2a, b.

We observe from Fig. 2c that the end of the cold matter dominance area, at $t \sim 10.2$ Gy for Λ CDM, occurs at $t \sim 8.5$ Gy for our model, matter dominance being defined, in the latter case, as the dominance of Ω_m over the sum $\Omega_\Lambda + \Omega_T$.

Finally, the present values of the concentrations are:

- $\Omega_m(t_0) = 0.308$ (which belongs to the input data),
- $\Omega_\Lambda(t_0) = 0.289$,
- $\Omega_T(t_0) = 0.403$.

This has to be contrasted with the Λ CDM values $\Omega_m(t_0) = 0.308$ and $\Omega_\Lambda(t_0) = 0.692$: in our model the torsion contributes together with the cosmological constant to the acceleration.

Finally, as a matter of verification, we have checked that our solution of the field equations does satisfy the continuity equation (4.8).

4.5 Search for other solutions

Since torsion may contribute to the acceleration, as in the solution studied above, one could expect solutions presenting a positive present acceleration even with a negative cosmological constant. This occurs for instance for the class of models investigated in [23]. In our case, we have checked that there is no solution with $\Lambda < 0$ and a positive acceleration fulfilling the physical boundary conditions represented by the present values of the cold matter density and of the Hubble and Newton parameters. Another class of solutions with a bounce at some time in the past do exist, but none of

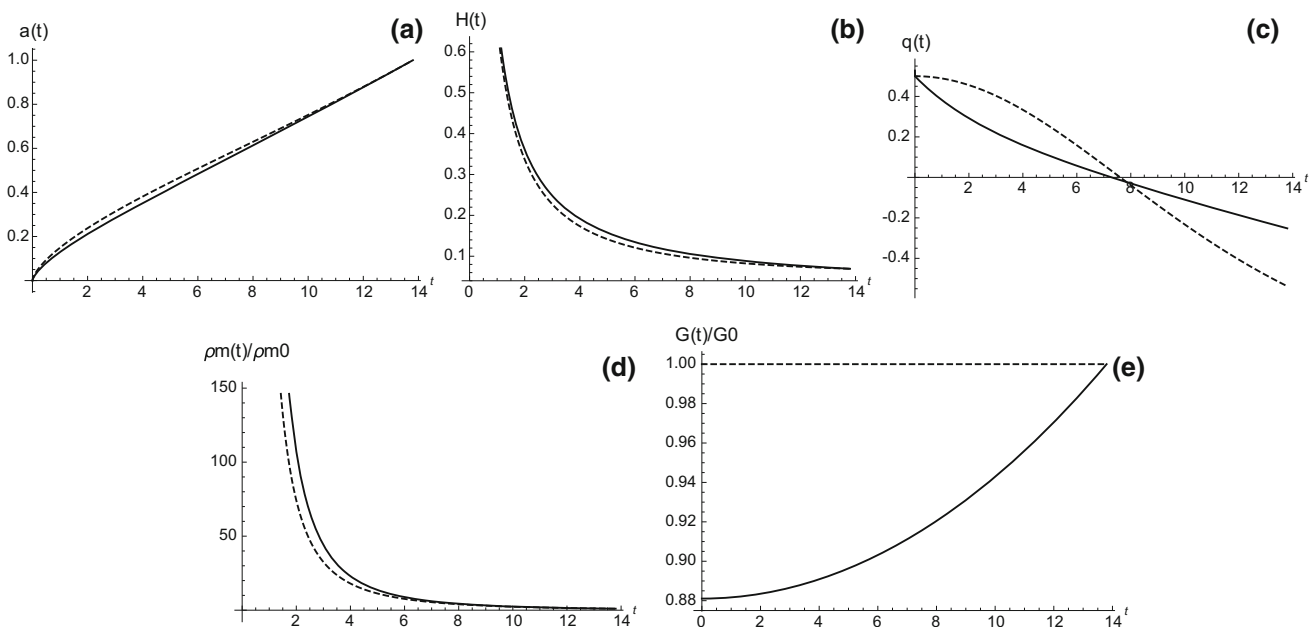


Fig. 1 **a** Normalized scale parameter $a(t)$; **b** Hubble parameter $H(t)$; **c** deceleration parameter $q(t)$; **d** cold matter density $\rho_m(t)$; **e** time-dependent gravitation coupling parameter $G(t)$; *Solid lines* model predictions; *dashed lines* standard Λ CDM results

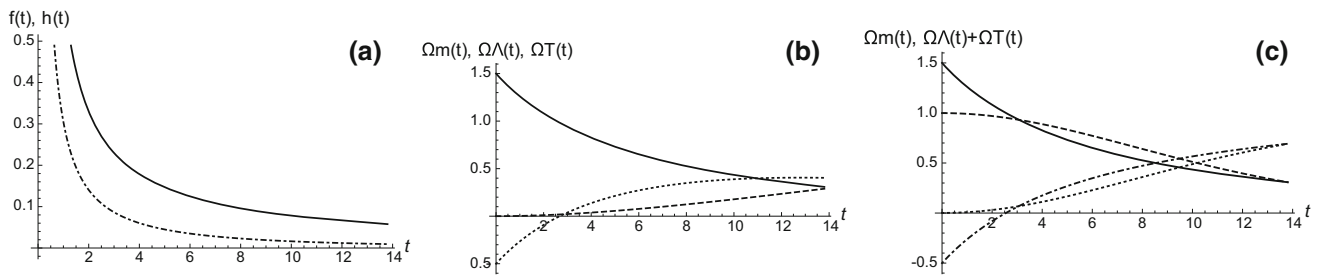


Fig. 2 **a** Torsion parameters $f(t)$ (solid line) and $h(t)$ (dashed line); **b** relative densities $\Omega_m(t)$ (solid line), $\Omega_\Lambda(t)$ (dashed line) and $\Omega_T(t)$ (dotted line); **c** $\Omega_m(t)$ (solid line) and $\Omega_\Lambda(t) + \Omega_T(t)$ (dot-dashed line); Λ CDM results are shown for Ω_m (dashed line) and Ω_Λ (dotted line)

them is compatible even roughly with the physical boundary conditions.

5 Conclusion and outlooks

We have seen in rather great detail how the dimensional reduction and truncation of the (A)dS₆ Chern–Simons theory in 5D to the 4D Chamseddine model is working. In particular, we have shown that the field equations of the latter form a subset of the field equations of the former, which is a non-trivial result. Chamseddine’s theory involves a scalar dilaton-like field which we have interpreted as a varying Newton coupling parameter. We have explored the solutions of the field equations, focusing on two examples. In the first one we have shown the existence, in a linear approximation, of a Newtonian limit and of gravitational waves of the same type as the ones of standard GR. The Newton parameter is supposed to be constant in the zeroth order and turns out to remain undetermined at first order. The wave solutions confirm the canonical result [19] of three degrees of freedom: two for the “graviton” and one for the Newton parameter field. The second example is of the FLRW cosmological type. We have found a solution with boundary conditions corresponding to the present values of the physical parameters: the Newton and Hubble parameters, and the cold matter mass density. It shows a behavior fitting rather well that of the standard Λ CDM model, at least qualitatively. The cosmological constant of this solution turns out to be positive, however, smaller than that of the Λ CDM model, the torsion contributing substantially to the present acceleration of the expansion. A similar but different model has been studied by the authors of [23]. The main difference is that, in their action, a scalar field appears as a factor only in the supplementary term, quadratic in the curvature. In our case, the scalar field appears as a common factor of the whole Lagrangian density and, moreover, the term quadratic in the curvature is not independent due to the constraint of the (A)dS₅ gauge symmetry SO(1,4) or SO(2,3).

A study of the full 5D Chern–Simons theory is under way, with one spatial dimension being compactified [19]. It will allow one to explore a larger domain of solutions, this theory possessing 13 degrees of freedom as shown in [17, 18].

Concerning the quantization, the prospect [19] is for a Loop Quantization [1, 2] of the 5D CS theory. Indeed, the latter is generic in the sense of the authors of [17, 18], i.e., the scalar or “Hamiltonian” constraint is a consequence of the other constraints, which are easier to solve [1, 2].

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Appendix: Conventions and notation

A.1 Conventions

4D and 5D space-time indices: $\mu, \dots = 0, \dots, 3$ and $\alpha, \dots = 0, \dots, 4$.

3D and 4D space indices: $a, \dots = 1, \dots, 3$ and $m, \dots = 1, \dots, 4$.

The de Sitter or anti-de Sitter groups SO(n,N-n) are collectively denoted by (A)dS_N. Their indices and corresponding invariant metrics are denoted by

$$(A)dS_6 : M, N, \dots = 0, \dots, 5, \quad \eta_{MN} = \text{diag}(-1, 1, 1, 1, 1, s), \tag{A.1}$$

$$(A)dS_5 : A, B, \dots = 0, \dots, 4, \quad \eta_{AB} = \text{diag}(-1, 1, 1, 1, s), \tag{A.2}$$

where s takes the values ± 1 for dS or AdS, respectively. 4D Lorentz SO(1,3) indices are denoted by $I, \dots = 0, \dots, 3$, the corresponding metric being $\eta_{IJ} = \text{diag}(-1, 1, 1, 1)$. These

metrics and their inverses allow one to lower and raise the various group indices.

The respective Levi-Civita symbols are defined as

$$\begin{aligned} \varepsilon_{MNPQRS} &= \begin{cases} \varepsilon_{012345} := 1, \\ \varepsilon_{ABCDE4} := \varepsilon_{ABCDE}, \end{cases} \\ \varepsilon_{ABCDE} &= \begin{cases} \varepsilon_{01234} := 1, \\ \varepsilon_{IJKL4} := \varepsilon_{IJKL}, \end{cases} \\ \varepsilon_{IJKL} &= \begin{cases} \varepsilon_{0123} := 1, \\ \varepsilon_{0ijk} := \varepsilon_{ijk}, \end{cases} \end{aligned}$$

for the internal spaces, and

$$\begin{aligned} \varepsilon^{\alpha\beta\gamma\delta\varepsilon} &= \begin{cases} \varepsilon^{01234} := 1, \\ \varepsilon^{\mu\nu\rho\sigma 4} := \varepsilon^{\mu\nu\rho\sigma}, \end{cases} \\ \varepsilon^{\mu\nu\rho\sigma} &= \begin{cases} \varepsilon^{0123} := 1, \\ \varepsilon^{0abc} := \varepsilon^{abc}, \end{cases} \end{aligned}$$

for the 5D and 4D space-times.

A.2 Lie algebra basis

A basis of the Lie algebra (a)ds₆ of the group (A)dS₆ may be given by the 15 matrices $M_{PQ} = -M_{QP}$:

$$(M_{PQ})^M{}_N := -(\delta^A_P \eta_{NQ} - \eta_{PN} \delta^M_Q)$$

satisfying the (a)ds₆ commutation relations

$$\begin{aligned} [M_{MN}, M_{PQ}] &= -\eta_{MQ} M_{NP} - \eta_{NP} M_{MQ} \\ &\quad + \eta_{MP} M_{NQ} + \eta_{NQ} M_{MP}. \end{aligned} \tag{A.3}$$

One can decompose this basis according to representations of the 5D Lorentz group SO(1,4) as

$$M_{MN} = \begin{cases} M_{AB} \\ P_A := \lambda M_{A5} \end{cases}$$

where a positive dimensionful parameter λ has been introduced, related to a cosmological constant $\Lambda \sim s\lambda^2$ ($s = \eta_{55}$) of a 5D gravitation theory. The commutation relations read now

$$\begin{aligned} [M_{AB}, M_{CD}] &= -\tilde{\eta}_{AD} M_{BC} - \tilde{\eta}_{BC} M_{AD} + \tilde{\eta}_{AC} M_{BD} \\ &\quad + \tilde{\eta}_{BD} M_{AC}, \\ [M_{AB}, P_C] &= \tilde{\eta}_{AC} P_B - \tilde{\eta}_{BC} P_A, \\ [P_A, P_B] &= s\lambda^2 M_{AB}, \end{aligned} \tag{A.4}$$

with $\tilde{\eta}_{AB} = \text{diag}(-1, 1, 1, 1, 1)$. The ten generators M_{AB} generate the 5D Lorentz group, and together with the five generators P_A , generate the (A)dS₆ group for 5D space-time. The M_{AB} may be represented by the 5×5 matrices

$$(M_{CD})^A{}_B := -(\delta^A_C \tilde{\eta}_{BD} - \tilde{\eta}_{CB} \delta^A_D).$$

The first line of A.4, namely

$$\begin{aligned} [M_{AB}, M_{CD}] &= -\eta_{AD} M_{BC} - \eta_{BC} M_{AD} \\ &\quad + \eta_{AC} M_{BD} + \eta_{BD} M_{AC}, \end{aligned} \tag{A.5}$$

but this time with the metric $\eta_{AB} = \text{diag}(-1, 1, 1, 1, s)$, gives the commutation rules of the Lie algebra of (A)dS₅. Its decomposition according to representations of the 4D Lorentz group reads

$$M_{AB} = \begin{cases} M_{IJ} \\ P_I := \lambda M_{I4} \end{cases}.$$

In the same way as above we have introduced the dimensionful parameter λ related now to the cosmological constant of a 4D gravitation theory. Thus

$$\begin{aligned} [M_{IJ}, M_{KL}] &= -\eta_{IL} M_{JK} - \eta_{JK} M_{IL} \\ &\quad + \eta_{IK} M_{JL} + \eta_{JL} M_{IK}, \\ [M_{IJ}, P_K] &= \eta_{IK} P_J - \eta_{JK} P_I, \\ [P_I, P_J] &= s\lambda^2 M_{IJ}. \end{aligned}$$

We are also interested in the full decomposition of the (A)dS₆ algebra according to representations of the Lorentz group SO(1,3):

$$M_{IJ}, \quad P_I := \lambda M_{I5}, \quad Q_I := \lambda M_{I4}, \quad R := M_{45}, \tag{A.6}$$

$$\begin{aligned} [M_{IJ}, M_{KL}] &= -\eta_{IL} M_{JK} - \eta_{JK} M_{IL} \\ &\quad + \eta_{IK} M_{JL} + \eta_{JL} M_{IK}, \\ [M_{IJ}, P_K] &= \eta_{IK} P_J - \eta_{JK} P_I, \\ [M_{IJ}, Q_K] &= \eta_{IK} Q_J - \eta_{JK} Q_I, \\ [M_{IJ}, R] &= 0, \\ [P_I, P_J] &= s\lambda^2 M_{IJ}, \quad [Q_I, Q_J] = \lambda^2 M_{IJ}, \\ [P_I, Q_J] &= \lambda^2 \eta_{IJ} R, \\ [P_I, R] &= s Q_I, \quad [Q_I, R] = -P_I. \end{aligned} \tag{A.7}$$

A.3 Dimensions

The dimensions of the fields and the parameters of the theory, given in mass units, are

$\frac{ds}{\dim}$	ω^{IJ}	e^I	λ	Λ	M_{IJ}	P_I
-1	1	0	1	2	0	1

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