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Gravitational resonances on $f(R)$-brane

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Abstract In this paper, we investigate various $f(R)$-brane models and compare their gravitational resonance structures with the corresponding general relativity (GR)-branes. Starting from some known GR-brane solutions, we derive thick $f(R)$-brane solutions such that the metric, scalar field, and scalar potential coincide with those of the corresponding GR-branes. We find that for the branes generated by a single or several canonical scalar fields, there is no obvious distinction between the GR-branes and the corresponding $f(R)$-branes in terms of gravitational resonance structure. Then we discuss the branes generated by a K-field. In this case, there could exist huge differences between GR-branes and $f(R)$-branes.

1 Introduction

Since general relativity (GR) was established by Einstein in 1915, a wide range of new theories of gravity have been proposed in the past 100 years. The simplest way to modify GR is to enlarge the dimensions of the space-time. The Kaluza–Klein (KK) theory as one of the pioneering five-dimensional space-time theories was proposed to unify electromagnetism with GR in the 1920s by Kaluza and Klein [1,2]. In order to generate effective four-dimensional theories, the extra spatial dimensions in the KK types of theories are assumed to be compacted to the Planck scale. About 60 years later, physicists found that if our world is some kind of four-dimensional domain wall in a higher-dimensional space-time, then the matter fields can be dynamically trapped on the wall. As a consequence, our world is effectively four-dimensional, even

for infinite extra dimensions [3–7]. String theory also predicts the existence of TeV scale extra dimensions [8].

But what really triggered the modern revolution of extra dimension theories is the work done by Arkani-Hamed, Dimopoulos, and Dvali (ADD) [9], which provides an alternative solution to the gauge hierarchy problem. The idea of ADD model is that all the matter fields are confined on a four-dimensional brane embedded in a higher-dimensional space-time, while gravity can propagate in the bulk. In ADD model, the fundamental scale is assumed to be the electroweak scale $M_* = m_{\mathrm{EW}} \sim 1$ TeV, and the large hierarchy between the Planck scale M_{Pl} and the fundamental scale is ascribed to the large volume of the extra dimensions. Due to the compactification of the extra dimensions, the four-dimensional Newton law appears at a distance much larger than the radius of the extra dimensions.

Soon after the ADD model, Randall and Sundrum (RS) found that by using a nonfactorizable warped geometry, one can also solve the hierarchy problem [10]. More interestingly, they have shown that it is possible to generate the four-dimensional Newtonian law at large distances in a warped space-time even when the extra dimension is infinitely large [11]. Now, these two models are known as RS-I and RS-II models, respectively. In the RS-II model, the four-dimensional Newtonian law is generated by a gravitational zero mode localized near the brane. Later, Gregory, Rubakov, and Sibiryakov (the GRS model) [12] have shown that four-dimensional gravity can also be reproduced at an intermediate scale even without a localizable gravitational zero mode. In fact, the four-dimensional Newtonian gravity is generated by a quasilocalized graviton (or gravitational resonance) in the GRS model [13,14]. The GRS model also shed light on solving the cosmological constant problem [14,15]. But on the other hand, it suffers from problems such as violating the positivity of energy and containing ghost modes [14–19].

Another interesting and widely studied model which reproduces four-dimensional Newton's law at intermediate scale is the DGP model proposed by Dvali, Gabadadze, and

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Porrati [20]. The DGP model offers us a new way to understand the acceleration of the universe [21]. It is interesting to note that in higher-dimensional realization of the DGP model, the four-dimensional Newtonian gravity is reproduced by massive graviton resonances [22–24]. In addition to the GRS and DGP models where the bulk is asymptotically flat, some works have considered graviton resonances in asymptotical AdS$_5$ space [25–28]. Gravitational resonance also appears in other topics of gravity research. For instance, in black hole perturbation theory it is possible to find gravitational resonances which are known as quasinormal modes [29]. No doubt, gravitational resonances deserve further consideration.

In this paper we focus on the appearance of gravitational resonances in thick RS-II type brane models, where the branes can be generated by, for example, a single canonical scalar field in an AdS$_5$ space [30–33] (more complicated thick brane models can be found in [34]). The possibility of the existence of gravitational resonances in thick brane models was first pointed out in Ref. [31]. But unfortunately, the solution therein does not support any gravitational resonance. In thick brane models constructed in general relativity, gravitational resonances can be found in cases with either a single canonical background scalar field [35,36], or two canonical scalar fields (the Bloch brane [37]) [38,39], or a single noncanonical scalar field (also known as K-field [40–42]; see [43–48] for thick branes generated by K-fields) [49].

However, the spectrum of graviton is determined by the equation of the gravitational mode (the spin-2 transverse and traceless tensor modes of the metric perturbations [11]). In general relativity, the equation of the gravitational mode is independent from the background scalar fields and only depends on the warp factor. But this is not always true in modified gravity theories. Thus, in order to find gravitational resonances, it is interesting to go beyond general relativity. For example, it is possible to find gravitational resonances in thick brane models in scalar-tensor gravity [50].

Another simply modified gravity is $f(R)$ gravity, where the Lagrangian of gravity is an arbitrary function of the scalar curvature. $f(R)$ gravity was first introduced in cosmology in 1970 [51], and nowadays it has been applied in a wide range of cosmology studies [52–59] (see [60–62] for reviews, and [48,63–76] for $f(R)$ braneworld). The graviton equations for a class of thick $f(R)$-brane models were derived in Ref. [77]. It is shown that the graviton equations are determined by both the background solution and the function $f(R)$. Using the results of Ref. [77], some of us found a series of gravitational resonances in Ref. [78], where the gravity Lagrangian density is $f(R) = R + \alpha R^2$, and the brane is generated by a single canonical scalar field. However, there exist some singularities in the model of [78].

The aim of this paper is to construct thick $f(R)$-brane models which support gravitational resonances but are free

of singularities. By starting with some well-studied GR-brane solutions, we construct new $f(R)$-brane solutions (either analytically or numerically) such that the metric and the scalar configuration remain the same. Then we compare the gravitational resonance spectra between our $f(R)$-branes and the corresponding GR-branes. Since gravitons satisfy different equations in different gravity theories, there might be some differences in the resonance spectra.

Note that in addition to the tensor modes, there are also spin-1 vector modes and spin-0 scalar modes. These modes evolve independently, no matter the thick brane is constructed in general relativity [79–83], or in $f(R)$ gravity [65,84]. The gravitational resonances only relate to the tensor sector. Usually, the vector modes only contain a nonnormalizable zero mode and therefore can be neglected when one only considers static sources [79,80,83,84]. The scalar sector will also contribute to the four-dimensional Newtonian gravity [82]. However, for $f(R)$-brane generated by a background scalar field, the scalar sector of the linear perturbations has two degrees of freedom after fixing the gauge. In general, these two scalar modes are governed by two coupled second-order differential equations, which are too complicated to be considered here and deserve an independent work elsewhere.

In our first $f(R)$-brane model, we briefly discuss the similar model considered in Ref. [78] and add the condition $f_R = \mathrm{d}f(R)/\mathrm{d}R > 0$, which guarantees that the graviton is not a ghost and the normalization of the zero mode. In this model, we find that there is nothing special about gravitational resonances in $f(R)$-brane model in contrast to GR-brane model. In order to get a different behavior, naturally, one can generalize this model to the case with multiple scalar fields. Based on this motivation, we study the brane generated by two scalar fields in second model and we obtain similar results as the case with a single scalar field. Our results show that for the brane generated by a canonical scalar field (there is no gravitational resonance) or two canonical scalar fields (there are gravitational resonances), there is no striking difference between $f(R)$-brane and GR-brane in terms of gravitational resonance. Under the circumstances, noncanonical scalar fields may be a breakthrough. In our third $f(R)$-brane model, indeed, we find that even for the case with only one single noncanonical scalar field, gravitational resonances on $f(R)$-brane can be significantly different from the GR-brane.

Therefore our work is organized as follows. In Sect. 2, we review the tensor perturbations of $f(R)$-brane and the condition of normalizable zero mode. In Sect. 3, we construct some thick $f(R)$-brane models in cases with, first, a single canonical scalar, and then two canonical scalars, and finally a single noncanonical scalar field or K-field. These $f(R)$-brane models are all based on the corresponding GR-models which have been extensively studied in the literature. Then we compare, via both analytical and numerical meth-

ods, gravitational resonances between our $f(R)$-branes and the corresponding GR-branes. In Sect. 4, we analyze how the $f(R)$-brane models impact gravitational resonances. Finally, in Sect. 5, we come to the conclusions and discussions.

2 Tensor perturbations of $f(R)$-brane

Now we consider the five-dimensional $f(R)$-brane model. Our action is taken in the general form

$$S = \int \mathrm{d}^4 x \mathrm{d}y \sqrt{-g} \left(\frac{1}{2\kappa_5^2} f(R) + L(\phi_i, X_i) \right), \tag{1}$$

where $f(R)$ is an arbitrary function of the scalar curvature R and $\kappa_5^2 = 8\pi G_5$ with G_5 the five-dimensional Newton constant. For convenience, we take $\kappa_5^2 = 2$. Capital Latin indices $M, N, \ldots = 0, 1, 2, 3, 4$ denote the bulk coordinates, and Greek indices $\mu, \nu, \ldots = 0, 1, 2, 3$ denote the brane coordinates. The coordinate $y = x^4$ stands for the extra dimension. ϕ_i is the ith scalar field, and its kinetic term X_i is given by

$$X_i = -\frac{1}{2} \partial^M \phi_i \partial_M \phi_i. \tag{2}$$

We consider a static flat brane, for which the line element can be written as

$$\mathrm{d}s^2 = a^2(y)\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}y^2, \tag{3}$$

where $a^2(y) = e^{2A(y)}$ is the warp factor, and $\eta_{\mu\nu}$ is the induced metric on the brane. For this background space-time, the scalar fields only depend on the extra dimension, i.e., $\phi_i = \phi_i(y)$. The scalar curvature is given by

$$R(y) = -4(5A'^2 + 2A''). \tag{4}$$

In this paper the prime denotes the derivative with respect to y. The Einstein equations and the equations of motion for the scalar fields are

$$f + 2f_R \left(4A'^2 + A''\right) - 6f_R' A' - 2f_R'' + 4L = 0, \tag{5a}$$

$$f + 8f_R \left(A'' + A'^2\right) - 8f_R' A' $$
$$+ 4\sum_i L_{X_i}\phi_i'^2 + 4L = 0, \tag{5b}$$

and

$$\phi_i''(L_{X_i} + 2X_i L_{X_i X_i}) + L_{\phi_i} = 2X_i L_{X_i \phi_i} - 4L_{X_i}\phi_i' A', \tag{6}$$

respectively, where $L_{X_i} \equiv \partial L / \partial X_i$, $L_{\phi_i} \equiv \partial L / \partial \phi_i$, and $f_R \equiv \mathrm{d}f(R)/\mathrm{d}R$.

In this paper, we only consider the minimal coupling between ϕ_i and X_i. Therefore, the Lagrangian density $L(\phi_i, X_i)$ is given by

$$L(\phi_i, X_i) = \sum_i \left[F(X_i) - V(\phi_i) \right], \tag{7}$$

where $F(X_i)$ is an arbitrary function of X_i, and $V(\phi_i)$ is the scalar potential of the scalar field ϕ_i.

Next, we consider the tensor perturbations of the background metric (3):

$$\mathrm{d}s^2 = e^{2A(y)}(\eta_{\mu\nu} + h_{\mu\nu})\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}y^2, \tag{8}$$

where $h_{\mu\nu}$ satisfies the transverse and traceless (TT) conditions $\eta^{\mu\nu}h_{\mu\nu} = 0 = \partial_\mu h_\nu^\mu$. With the coordinate transformation $\mathrm{d}z = a^{-1}\mathrm{d}y$, the perturbed Einstein equations read [77]

$$\left[\partial_z^2 + \left(3\frac{\partial_z a}{a} + \frac{\partial_z f_R}{f_R}\right)\partial_z + \Box^{(4)}\right] h_{\mu\nu} = 0, \tag{9}$$

where $\Box^{(4)} = \eta^{\mu\nu}\partial_\mu\partial_\nu$. Note that although in Ref. [77] only one scalar field with canonical dynamics was considered, Eq. (9) remains valid for our case because the TT tensor modes are decoupled from other modes of the perturbations. Then, with the KK decomposition [77]

$$h_{\mu\nu}(x^\rho, z) = \epsilon_{\mu\nu}(x^\rho)a^{-3/2} f_R^{-1/2} \psi(z), \tag{10}$$

where $\epsilon_{\mu\nu}(x^\rho)$ satisfies the TT conditions $\eta^{\mu\nu}\epsilon_{\mu\nu} = 0 = \partial_\mu\epsilon_\nu^\mu$, Eq. (9) can be simplified as a Schrödinger-like equation for $\psi(z)$ [77]:

$$\left[-\partial_z^2 + W(z)\right]\psi(z) = m^2\psi(z). \tag{11}$$

Here the effective potential $W(z)$ is given by [77]

$$W(z) = \frac{3}{4}\frac{(\partial_z a)^2}{a^2} + \frac{3}{2}\frac{\partial_z^2 a}{a} + \frac{3}{2}\frac{\partial_z a \partial_z f_R}{a f_R}$$
$$- \frac{1}{4}\left(\frac{\partial_z f_R}{f_R}\right)^2 + \frac{1}{2}\frac{\partial_z^2 f_R}{f_R}. \tag{12}$$

In the next section we will discuss the effective potential $W(z(y))$ in the coordinate y. Using the relation $\partial_z = a\,\partial_y$, Eq. (12) can be rewritten in the coordinate y:

$$W(z(y)) = \frac{3}{4}a'^2 + \frac{3}{2}(a'^2 + aa'') + \frac{3aa'}{2}\frac{f_R'}{f_R}$$
$$- \frac{1}{4}\left(\frac{af_R'}{f_R}\right)^2 + \frac{1}{2}\frac{aa'f_R' + a^2 f_R''}{f_R}. \tag{13}$$

From the Schrödinger-like equation (11), it is easy to give the solution of the gravitational zero mode

$$\psi^{(0)}(z) = N_0 a^{3/2}(z) f_R^{1/2}(z), \tag{14}$$

where N_0 is the normalization constant. To get four-dimensional Newtonian gravity on the brane, the zero mode should satisfy the following normalization condition:

$$\int_{-\infty}^{\infty} |\psi^{(0)}(z)|^2 \mathrm{d}z < \infty. \tag{15}$$

Apparently, in order to ensure the normalization of the zero mode, f_R should satisfy $f_R > 0$ in the whole space, which also guarantees that the graviton is not a ghost [61].

3 Gravity resonances in various $f(R)$-brane models

In this section we study braneworld solutions and gravitational resonances in three types of thick $f(R)$-brane models. In our first model, the brane is generated by a single canonical scalar field, which is one of the most widely studied thick brane models. Next, we consider the brane generated by two canonical scalar fields (known as the Bloch brane) in the second model. The main property of the Bloch brane is that the brane is split into two sub-branes. Finally, we study the brane generated by a K-field, whose most important feature is that its kinetic term is noncanonical, which will produce some different results compared to the previous two cases.

3.1 $f(R)$-brane model with $L = X - V(\phi)$

First of all, we study the $f(R)$-brane generated by a single canonical scalar field, whose Lagrangian density reads $L = X - V(\phi) = -\frac{1}{2}\partial^M\phi\partial_M\phi - V(\phi)$. This kind of $f(R)$-brane model is one of the simplest and the most widely studied models. Therefore, one reason why we study it is to review and illustrate the research processes of gravitational resonances on $f(R)$-brane. The second and also more important reason is to give a direct comparison with our third model generated by a single noncanonical scalar field. The dynamical equations (5) and (6) become

$$f + 2f_R\left(4A'^2 + A''\right) - 6f_R'A' - 2f_R'' = 2(\phi'^2 + 2V),\tag{16a}$$

$$-8f_R\left(A'' + A'^2\right) + 8f_R'A' - f = 2(\phi'^2 - 2V),\tag{16b}$$

$$\phi'' + 4A'\phi' = V_\phi.\tag{16c}$$

For the case of $f(R) = R$, we consider the solution given in [31]:

$$\phi(y) = \sqrt{6b}\arctan\left[\tanh\left(\frac{ky}{2}\right)\right],\tag{17a}$$

$$A(y) = -b\ln\left[\cosh(ky)\right],\tag{17b}$$

$$V(\phi) = \frac{3bk^2}{8}\left[(1 - 4b) + (1 + 4b)\cos\left(\sqrt{\frac{8}{3b}}\phi\right)\right],\tag{17c}$$

where b and k are positive parameters. For convenience, we set $\hat{y} = ky$ in the next content, which is a dimensionless parameter.

Now we construct the $f(R)$-brane model that shares the same background solution (17a) of the GR-brane. Keeping the above background solution of the GR-brane unchanged, and incorporating Eqs. (16a) and (16b), the function f_R should satisfy

$$-\ddot{f}_R + \dot{f}_R\dot{A} - 3f_R\ddot{A} = 2\dot{\phi}^2 \geq 0,\tag{18}$$

where $\dot{A} = -b\tanh(\hat{y})$, $\ddot{A} = -b\operatorname{sech}^2(\hat{y})$, $\dot{\phi}^2 = \frac{3}{2}b\operatorname{sech}^2(\hat{y})$, and the dot denotes the derivative with respect to $\hat{y}$.

It is interesting to give the explicit expressions for f_R and $f(R)$. The procedures are described as follows. First, we solve Eq. (18) to give the analytic solution for f_R as a function of $\hat{y}$:

$$f_R = 1 + \alpha\operatorname{sech}^{\frac{b}{2}}(\hat{y})\left[P_{K_-}^{\frac{b}{2}}\left(\tanh(\hat{y})\right) - \beta Q_{K_-}^{\frac{b}{2}}\left(\tanh(\hat{y})\right)\right],\tag{19}$$

where α is an arbitrary constant, P and Q are, respectively, associated Legendre functions of the first and second kinds, $\beta = P_{K_+}^{b/2}(0)/Q_{K_+}^{b/2}(0)$ is given by the condition $\dot{f}_R|_{\hat{y}=0} = 0$, and $K_\pm = \frac{1}{2}\sqrt{(b-14)b+1} \pm 1/2$. Second, we define $\hat{R} \equiv \frac{R}{k^2} = 4b\left[(5b+2)\operatorname{sech}^2(\hat{y}) - 5b\right]$, which is obtained from Eqs. (4) and (17b), to give the expression of $\hat{y}$ in terms of the curvature scalar $\hat{R}$:

$$\hat{y} = \operatorname{arccosh}\left(\frac{2\sqrt{2b + 5b^2}}{\sqrt{20b^2 + \hat{R}}}\right).\tag{20}$$

Thirdly, defining $\hat{f} \equiv f/k^2$, which results in $\hat{f}_{\hat{R}} = f_R$, and substituting $\hat{y}(\hat{R})$ into (19) yield the expression of $\hat{f}_{\hat{R}}$ as a function of $\hat{R}$. Finally, integrating $\hat{f}_{\hat{R}}$ with respect to $\hat{R}$ gives the expression of $\hat{f}(\hat{R})$. In the following, we will first consider some special values of the parameter b, which will result in some simple solutions of $\hat{f}(\hat{R})$, and then general b.

3.1.1 $b = 1$

For the case of $b = 1$, we obtain the following solution:

$$\hat{f}_{\hat{R}} = 1 + \alpha_1\cos\left(\mathcal{H}(\hat{R})\right),\tag{21}$$

where $\mathcal{H}(\hat{R})$ is given by

$$\mathcal{H}(\hat{R}) = \sqrt{3}\ln\left(\frac{\sqrt{\hat{R} - 8} + \sqrt{\hat{R} + 20}}{2\sqrt{7}}\right).\tag{22}$$

Note that $\hat{f}_{\hat{R}}$ satisfies $\hat{f}_{\hat{R}}|_{\hat{y}=0} = 1 + \alpha_1$. Then the function $\hat{f}(\hat{R})$ can be integrated from Eq. (21) as

$$\hat{f}(\hat{R}) = \hat{R} + \frac{2\alpha_1}{7}\left[2(\hat{R} + 6)\cos\left(\mathcal{H}(\hat{R})\right)\right.$$
$$\left. + \sqrt{3(\hat{R} - 8)(\hat{R} + 20)}\sin\left(\mathcal{H}(\hat{R})\right)\right],\tag{23}$$

where the integration constant is absorbed into the scalar potential $V(\phi)$.

Table 1 The values of the parameters c_i in Eq. (24)

c_i	Value
c_1	$\dfrac{240936850868416}{149442323051763}$
c_2	$-\dfrac{38202658243}{16604702561307}$
c_3	$\dfrac{2065663433}{4605037510335808}$
c_4	$-\dfrac{18231247}{3831391208599392256}$
c_5	$\dfrac{351657}{3187717485554694356992}$
c_6	$-\dfrac{6507}{2652180947981505705017344}$
c_7	$-\dfrac{459}{1544630184104428922602 1011456}$

3.1.2 $b = \frac{52}{3}$

For the special case of $b = \frac{52}{3}$, we have

$$\hat{f}(\hat{R}) = \hat{R} + \alpha_2 \sum_{n=1}^{7} c_n \hat{R}^n, \tag{24}$$

where the values of the coefficients c_i are given in Table 1. The advantage of taking those values is that we can guarantee $\hat{f}_{\hat{R}}|_{\hat{y}=0} = 1 + \alpha_2$, which is similar to the case $b = 1$.

3.1.3 The general b

For general b, Eqs. (19) and (20) give the solution of $\hat{f}_{\hat{R}}$, which is a complex expression. In this case, we can give the analytic $\hat{f}(\hat{R})$ from Eqs. (19), (20), and (16a) or (16b), which is given by

$$
\begin{aligned}
\hat{f}(\hat{R}) = \hat{R} + \alpha \Bigg\{ & \frac{24b^2 + 2\hat{R} + 2b\hat{R}}{2 + 5b} \Big[P_{K_-}^{b/2}(\Psi) - \beta Q_{K_-}^{b/2}(\Psi) \Big] \\
& - 4(b^2 - 2bK_+)\Psi \Big[P_{K_+}^{b/2}(\Psi) - \Psi P_{K_-}^{b/2}(\Psi) \\
& + \beta \Psi \left(Q_{K_-}^{b/2}(\Psi) - Q_{K_+}^{b/2}(\Psi) \right) \Big] \Bigg\} \Theta^{b/2},
\end{aligned} \tag{25}
$$

where the parameters α, β, and $K_\pm$ are those appearing in (19) and Θ and Ψ are given by

$$\Psi = \sqrt{1 - \Theta^2}, \tag{26}$$

$$\Theta = \frac{\sqrt{20b^2 + \hat{R}}}{2\sqrt{2b + 5b^2}}. \tag{27}$$

To investigate the resonant modes of gravity we recall the definition of the relative probability [85]:

$$P(m^2) = \frac{\int_{-z_b}^{z_b} |\psi(z)|^2 \, dz}{\int_{-z_{max}}^{z_{max}} |\psi(z)|^2 \, dz}, \tag{28}$$

where $\psi(z)$ is the solution of Eq. (11), $2z_b$ is approximately the width of the brane, and $z_{max} = 10z_b$. Here $|\psi(z)|^2$ can be explained as the probability density [85,86]. A resonant mode with mass m_n exists, if the relative probability $P(m^2)$

has a peak around $m = m_n$ and this peak has a full width at half maximum (FWHM). So, the number of the peaks that have FWHM corresponds to the number of the resonant modes. For the sake of simplicity, we impose the following conditions on $\psi(z)$:

$$\psi_{\text{even}}(0) = 1, \quad \partial_z \psi_{\text{even}}(0) = 0; \tag{29a}$$

$$\psi_{\text{odd}}(0) = 0, \quad \partial_z \psi_{\text{odd}}(0) = 1, \tag{29b}$$

where ψ_{even} and ψ_{odd} correspond to the even and odd parity modes of $\psi(z)$, respectively. Since there is no essential difference between the above conditions for the relative probability $P(m^2)$, we will only display our results with one of the conditions (29) in the following examples. For the sake of brevity, we discuss gravitational resonance in the coordinate $\hat{y}$, but set $y = \hat{y}$, $R = \hat{R}$ and $f = \hat{f}$, which is equivalent to setting $k = 1$ in the coordinate y.

Note that in this $f(R)$-brane model there are two primary parameters b and α. The parameter b determines the brane solution and α affects the range of values of the function $f(R)$. Therefore, they may characterize the gravitational resonances that will be discussed in the following. It can be seen that the expression of $f(R)$ in (25) for general b is very complex while it becomes simple for two special values: $b = 1$ and $b = 52/3$. We find that the shapes of $f(R(y))$ as the functions of y for the two special and general values of b are similar. Therefore, in the following calculations, we only need consider the two special values, for which the expressions of $f(R)$ and the effective potential $W(y)$ are simple. As for the parameter α, it characterizes the deviation of the $f(R)$-brane from the GR-brane. This can be seen from (19).

For the case of $b = 1$, we substitute (23) into (13) to solve Schrödinger-like Eq. (11) with three typical values of α_1: $\alpha_1 = 0$ (GR-brane), $\alpha_1 = 100$ ($f(R)$-brane with $f_R > 1$), and $\alpha_1 = -0.13$ ($f(R)$-brane with $0 < f_R < 1$) (see Fig. 1a, b). Note that the functions $f_R(y)$ for $\alpha_1 = 100$ and $\alpha_1 = -0.13$ (both guarantee $f_R(y) > 0$ in the whole space) have already deviated from $f_R(y) = 1$ for $\alpha_1 = 0$ (the GR-brane) as far as possible. For $\alpha > 100$, the potential $W(z(y))$ can hardly change more compared with $\alpha_1 = 100$. For $\alpha_1 < -0.13$, f_R may be negative at some locations. The effective potentials $W(z(y))$ are shown in Fig. 1c. Our calculation shows that for all the values of α_1 (see Fig. 1d), there is no gravitational resonance.

For the case of $b = \frac{52}{3}$, the corresponding solution of $f(R)$ is given by Eq. (24). Compared with the case of $b = 1$, although the maximum of the potential $W(z(y))$ is enlarged, no resonant KK mode is found. The corresponding results are shown in Fig. 2. For other values of b, the results are similar: the $f(R)$-brane can hardly deviate from the corresponding GR-brane in terms of gravitational resonance and we do not find any gravitational resonance.

(a) $\alpha_1 = 0, 100$

(b) $\alpha_1 = 0, -0.13$

(c) $\alpha_1 = 0, 100, -0.13$

(d) $\alpha_1 = 0, 100, -0.13$
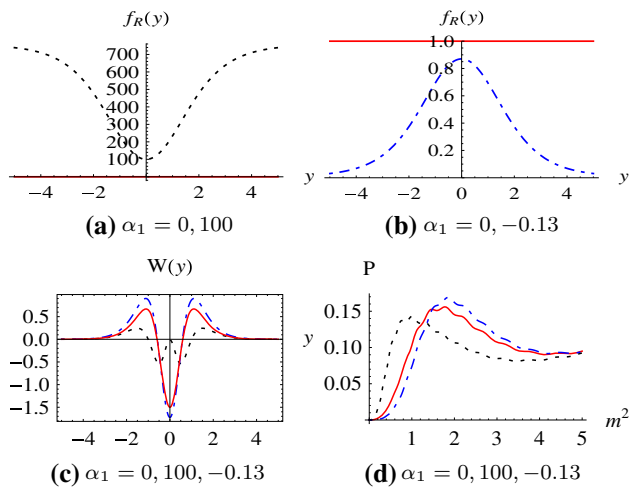
Fig. 1 Plots of the function f_R, effective potential $W(z(y))$, and relative probability $P(m^2)$ of the odd KK modes for the single scalar brane with standard kinetic term with $b = 1$. In the figures, the GR-brane ($\alpha_1 = 0$) is denoted by the *red solid lines*, the $f(R)$-branes with $\alpha_1 = 100$ and $\alpha_1 = -0.13$ are donated by the *black dotted lines* and *blue dot-dashed lines*, respectively (from now on, *red solid lines* always correspond to the GR-brane, *black dotted lines* correspond to $f_R > 1$, *blue dot-dashed lines* correspond to $f_R < 1$, and the rest of figures are denoted by *black solid lines* in all our plots)

(a) $\alpha_2 = 0, 100$

(b) $\alpha_2 = 0, -0.012$

(c) $\alpha_2 = 0, 100, -0.012$
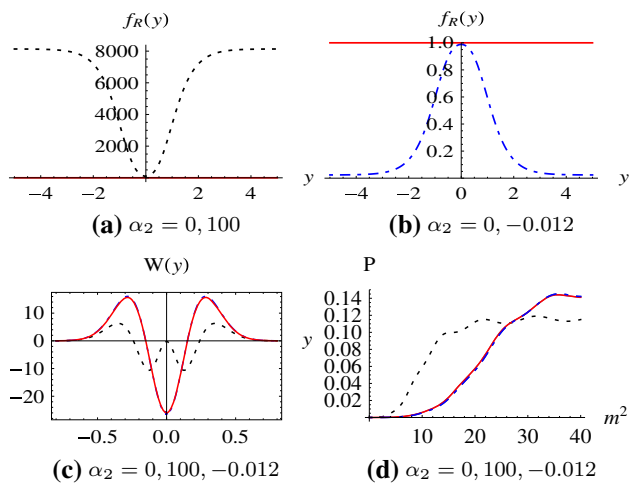
(d) $\alpha_2 = 0, 100, -0.012$

Fig. 2 Plots of the function $f_R(y)$, effective potential $W(z(y))$, and relative probability $P(m^2)$ of the odd KK modes for the canonical single scalar brane with $b = 52/3$ and $\alpha_2 = 0, 100, -0.012$, respectively

3.2 $f(R)$-brane model with $L = X_1 + X_2 - V(\phi_1, \phi_2)$

Next let us discuss the Bloch brane, which is a kind of important brane with inner structure generated by two scalar fields. It was first obtained in Ref. [37] and then developed and discussed in Refs. [39,87–92]. In this subsection, we explore gravitational resonances on the Bloch brane in the $f(R)$ gravity for several reasons. First, there exist gravitational resonances on the Bloch brane [39]. Therefore, we can compare

the difference between the GR-brane and $f(R)$-brane in the presence of gravitational resonances, which is expected to be a complement to the former $f(R)$-brane model. Second, the Bloch brane is generated by two canonical scalar fields. Investigation of the Bloch brane in this subsection could pave the way for Sect. 4, where we will generalize the conclusion of the first two models to the cases with multiple canonical scalar fields.

The Lagrangian density (7) for the Bloch brane is given by $L = -\frac{1}{2}\partial^M \phi \partial_M \phi - \frac{1}{2}\partial^M \chi \partial_M \chi - V(\phi, \chi)$, where ϕ and χ are two interacted real scalar fields.

The dynamical equations (5) and (6) are modified to

$$f + 2f_R \left(4A'^2 + A''\right) - 6f'_R A' - 2f''_R = 2(\phi'^2 + \chi'^2 + 2V), \tag{30a}$$

$$-f - 8f_R \left(A'' + A'^2\right) + 8f'_R A' = 2(\phi'^2 + \chi'^2 - 2V), \tag{30b}$$

$$\phi'' + 4A'\phi' = V_\phi, \tag{30c}$$

$$\chi'' + 4A'\chi' = V_\chi. \tag{30d}$$

For $f(R) = R$, there are many symmetric and asymmetric solutions [37,87,87]. We consider one of the solutions given in Refs. [37,87], where the scalar potential $V(\phi)$ takes the following form:

$$V(\phi) = \frac{1}{2} \left[\left(\tilde{b}v^2 - \tilde{b}\phi^2 - d\chi^2\right)^2 + 4d^2\phi^2\chi^2 \right] - \frac{4}{3} \left(\tilde{b}\phi v^2 - \frac{1}{3}\tilde{b}\phi^3 - d\phi\chi^2\right)^2. \tag{31}$$

For the case of $\tilde{b} > 2d > 0$, the general symmetric Bloch-brane solution reads [37,87]

$$\phi(y) = v\tanh(2dvy), \tag{32a}$$

$$\chi(y) = v\sqrt{\frac{\tilde{b}-2d}{d}} \, \text{sech}(2dvy), \tag{32b}$$

$$A(y) = \frac{v^2}{9d} \left[(\tilde{b}-3d)\tanh^2(2dvy) - 2\tilde{b}\ln\cosh(2dvy) \right]. \tag{32c}$$

Analogously, selecting certain values of the parameters v and $\tilde{b}$, we can calculate the explicit expression of $f(R)$. For $v = \sqrt{3/2}$ and $\tilde{b} = 3d$, we have

$$f_R = 1 + \gamma \cos\left(\mathcal{Y}(R)\right), \tag{33}$$

where γ is a constant, $R = 24d^2 \left[7\,\text{sech}^2\left(\sqrt{6}dy\right) - 5 \right]$, and $\mathcal{Y}(R)$ reads

$$\mathcal{Y}(R) = \sqrt{3}\ln\left(\frac{\sqrt{R - 48d^2} + \sqrt{120d^2 + R}}{2\sqrt{42}d}\right). \tag{34}$$

Fig. 3 Plots of the function $f_R(y)$, effective potential $W(z(y))$, and relative probability $P(m^2)$ of odd resonance KK modes on the Bloch brane with $d = 1, v = 1, c_0 = -2-7.0 \times 10^{-16}$, and $f_R|_{y=0} = 1, 50, 0.92$, respectively

(a) $f_R|_{y=0} = 1, 50$

(b) $f_R|_{y=0} = 1, 0.92$

(c) $f_R|_{y=0} = 1, 50, 0.92$

(d) $f_R|_{y=0} = 1, 50, 0.92$
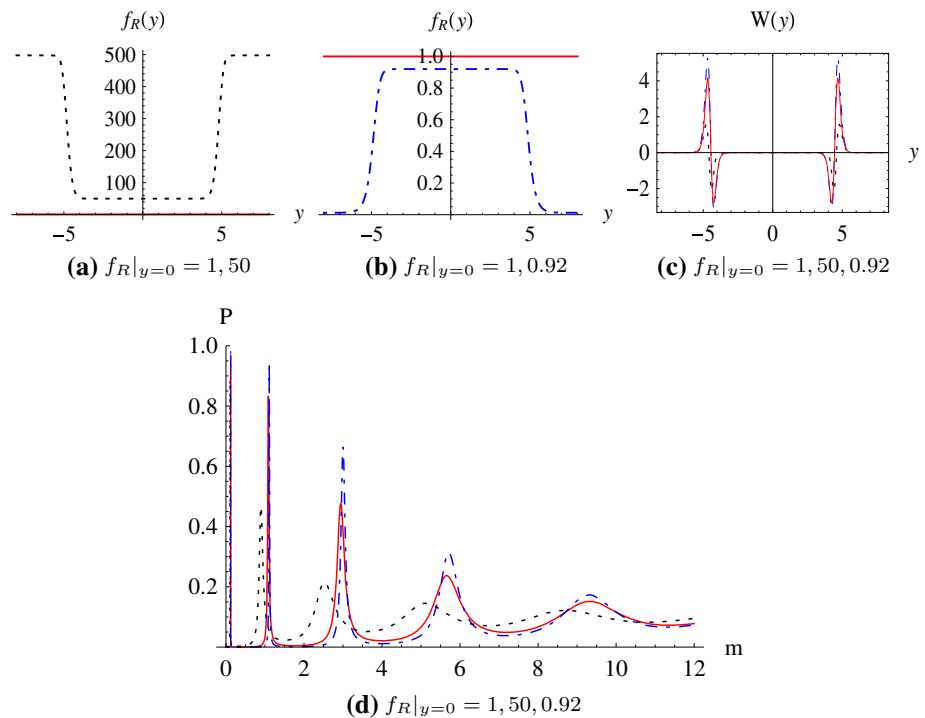
Therefore, integrating the function f_R over R we get

$$f(R) = R + \frac{2\gamma}{7}\Big[\sqrt{3(R-48d^2)(120d^2+R)}\sin\mathcal{Y}(R)$$
$$+ 2\left(36d^2 + R\right)\cos\mathcal{Y}(R)\Big]. \tag{35}$$

Note that when $d = \sqrt{\frac{1}{6}}$, this solution is identical with Eq. (23).

For the case of $\tilde{b} = d$, the authors in Ref. [87] obtained the degenerate Bloch-brane solution:

$$\phi(y) = v\frac{\sqrt{c_0^2 - 4}\sinh(2dvy)}{\sqrt{c_0^2 - 4}\cosh(2dvy) - c_0}, \tag{36a}$$

$$\chi(y) = \frac{2v}{\sqrt{c_0^2 - 4}\cosh(2dvy) - c_0}, \tag{36b}$$

$$e^{2A(y)} = \left(\frac{\sqrt{c_0^2 - 4} - c_0}{\sqrt{c_0^2 - 4}\cosh(2dvy) - c_0}\right)^{4v^2/9}$$

$$\times \exp\left[-\frac{4v^2\left(c_0^2 - 4 - c_0\sqrt{c_0^2 - 4}\right)}{9\left(\sqrt{c_0^2 - 4} - c_0\right)^2}\right]$$

$$\times \exp\left[\frac{4v^2\left(c_0^2 - 4 - c_0\sqrt{c_0^2 - 4}\cosh(2dvy)\right)}{9\left(\sqrt{c_0^2 - 4}\cosh(2dvy) - c_0\right)^2}\right], \tag{36c}$$

where $c_0 < -2$.

Next, we take the above Bloch-brane solution (36) as our background solution of $f(R)$-brane. Using the similar processes used in the previous section, the function f_R should satisfy

$$-f_R'' + f_R'A' - 3f_R A'' = 2(\phi'^2 + \chi'^2) \geq 0. \tag{37}$$

Since it is difficult to give the analytical solution of f_R, we can obtain f_R (also $f(R)$) and the potential $W(z(y))$ by means of numerical methods. Because $f(R(y))$ (and hence f_R) is an even function of the extra dimension y, we have $f_R'|_{y=0} = 0$. To construct a numerical f_R satisfying Eq. (37), we need one more initial condition, i.e., the value of $f_R|_{y=0}$. With the adjustable parameter $f_R|_{y=0}$, we can construct a different numerical f_R.

Here, we consider the following typical set of parameters reducing a double-brane: $a = d = 1$, $v = 2$, and $c_0 = -2-7.0 \times 10^{-16}$, which was used in Ref. [39] for the GR-brane model. It is convenient to compare the results of the $f(R)$-brane and the GR-brane.

The effective potential $W(z(y))$ and the relative probability $P(m^2)$ of the gravitational KK modes are shown, respectively, in Fig. 3c, d. We find that there are five resonance peaks in Fig. 3d for the GR-brane. For the $f(R)$-brane model, the potential $W(z(y))$ (see Fig. 3c) almost reaches the maximum deviation from the GR-brane when $f_R|_{y=0} = 50$ and 0.92 (the numerical solutions of f_R are shown in Fig. 3a, b). The corresponding resonance mass spectra are shown in Fig. 3d, from which we can see that there exist four peaks for

$f_R|_{y=0} = 50$ and five peaks for $f_R|_{y=0} = 0.92$, respectively. Therefore, there is no obvious difference between these spectra. For other values of the parameters we also have the same conclusion.

3.3 $f(R)$-brane model with $L = X - \lambda X^2 - V(\phi)$

In the previous two subsections, the $f(R)$-branes are all generated by one or two canonical scalar fields. For comparison, we try to investigate $f(R)$-brane generated by noncanonical scalar fields (K-fields) in this subsection.

K-fields were introduced to study inflation for the first time [40–42] and then were extended to many fields. In brane world models, the branes generated by K-fields are generally known as K-branes, which might present some new properties (such as the localization of bulk fermions and gravitons [47]) compared with the corresponding standard branes. Therefore, we expect there are some new results of gravitational resonances on K-branes.

We consider the simple case of one K-field with the Lagrangian density $L = X - \lambda X^2 - V(\phi)$ ($\lambda \neq 0$). The dynamical equations (5) and (6) are reduced to

$$f + 2f_R\left(4A'^2 + A''\right) - 6f'_R A' - 2f''_R = 2\left(\phi'^2 + \frac{1}{2}\lambda\phi'^4 + 2V\right),$$

$$-f - 8f_R\left(A'' + A'^2\right) + 8f'_R A' = 2\left(\phi'^2 + \frac{3}{2}\lambda\phi'^4 - 2V\right),$$

$$\phi'' + 4A'\phi' + \lambda(3\phi'' + 4A'\phi')\phi'^2 = V_\phi. \quad (38)$$

For $f(R) = R$, the authors in Ref. [47] obtained one analytic solution. For general $f(R)$, in the vast majority of cases, the dynamical equations remain hard to solve except in some specific cases (for example, if $f(R)$ is a polynomial). As an example, we consider the following scalar field and warp factor:
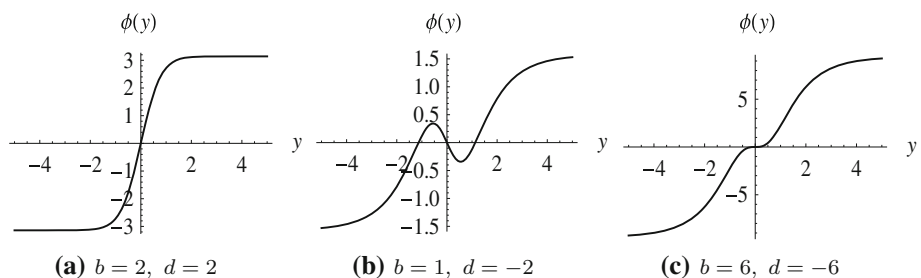
$$\phi(y) = \omega \arctan\left[\tanh\left(\frac{y}{2}\right)\right], \quad (39)$$

$$A(y) = -\ln\left[\cosh(y)\right]. \quad (40)$$

For the case of $\omega = 1$, we get the solution

$$f(R) = R + \frac{\lambda}{3136}R^2 + \frac{(29\lambda - 980)}{1176}R, \quad (41)$$

where $R = 28\,\text{sech}^2(y) - 20$ and $\lambda > -\frac{10976}{1955}$ to guarantee $f_R > 0$.

If we require the coefficient of R to be 1, then we obtain

$$\phi(y) = \sqrt{\frac{\pm 14\sqrt{174\lambda + 49} - 98}{29\lambda}}\arctan\left[\tanh\left(\frac{y}{2}\right)\right], \quad (42)$$

$$A(y) = -\ln\left[\cosh(y)\right], \quad (43)$$

$$f(R) = R + \frac{\left(87\lambda \mp 7\sqrt{174\lambda + 49} + 49\right)}{6728\lambda}R^2. \quad (44)$$

For $+$ in Eq. (42), we require $-\frac{49}{174} < \lambda < \frac{5}{2}$ and $-$ corresponds to $-\frac{49}{174} < \lambda < -\frac{4}{25}$. This solution was also obtained in Ref. [76] recently with another method.

Here, to study the solution of ϕ with more abundant structure, we reconstruct one analytic solution of ϕ given as

$$\phi = b \arcsin\left[\tanh(y)\right] + d \tanh(y)\text{sech}(y), \quad (45)$$

where b and d are free parameters. Therefore, the warp factor is determined by

$$2\lambda\phi'^4 + 3A'' + 2\phi'^2 = 0, \quad (46)$$

with the boundary conditions $A(0) = 0$ and $A'(0) = 0$. Meanwhile, f_R should satisfy
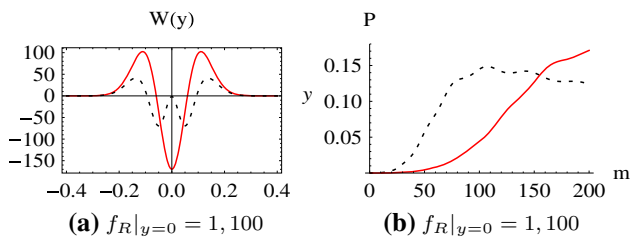
$$2\lambda\phi'^4 + 3f_R A'' + f''_R - f'_R A' + 2\phi'^2 = 0. \quad (47)$$

The value of $f_R|_{y=0}$ can be arbitrary so long as the corresponding f_R satisfies Eq. (47) and guarantees $f_R > 0$ in the whole space.
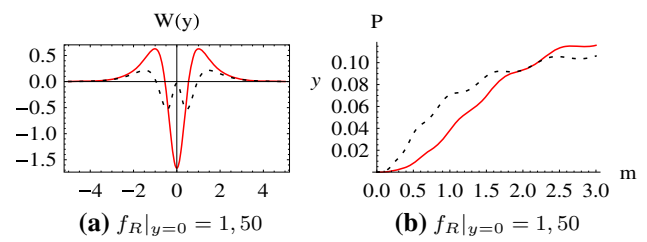
If the parameter λ is positive, the solution of the wrap factor will be lumplike and the constraint condition (47) will be the same as Eqs. (18) and (37). Our results show that it has no breakthrough comparing with the former cases discussed in the previous two subsections.

For $\lambda > 0$, we consider three types of solutions of the scalar field (see Fig. 4). Because when $0 < f_R|_{y=0} < 1$ the range for $f_R|_{y=0}$ is so narrow, we only consider the case $f_R|_{y=0} \geq 1$ in this $f(R)$-brane model. We find that when the scalar field ϕ is taken as the form of single kink or non-monotonic kink, the results are the same as the brane generated by a canonical scalar field (see Figs. 5, 6). For the case of double kink, the result is similar to the Bloch brane (see Fig. 7).

Fig. 4 Plots of the scalar field ϕ given by Eq. (45). The figures are, from *left* to *right*, kink-like solution, non-monotonic kink solution, and double kink-like solution

(a) $b = 2, \; d = 2$ **(b)** $b = 1, \; d = -2$ **(c)** $b = 6, \; d = -6$

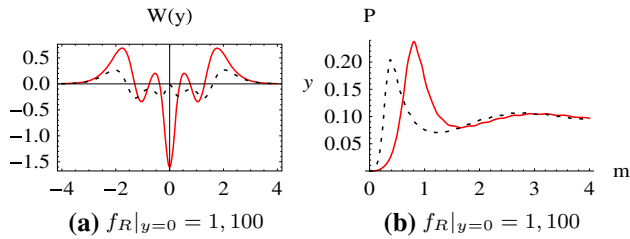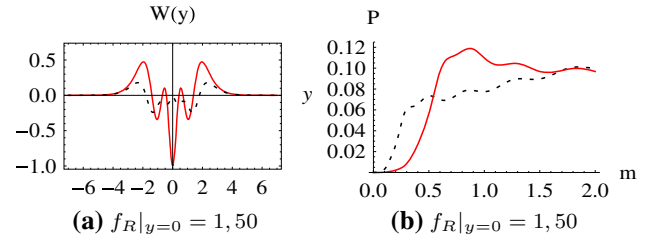**(a)** $f_R|_{y=0} = 1, 100$　　　　**(b)** $f_R|_{y=0} = 1, 100$

Fig. 5 Plots of the effective potential $W(z(y))$ and relative probability $P(m^2)$ of the odd resonance KK modes on the brane generated by a K-field with $b = 2, d = 2, \lambda = 0.6$, and $f_R|_{y=0} = 1, 100$
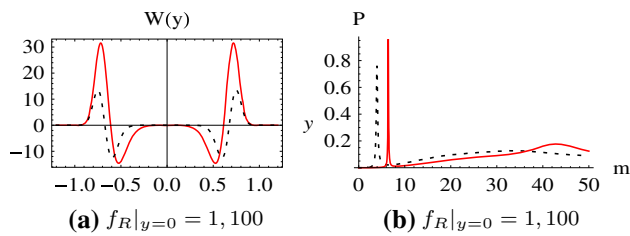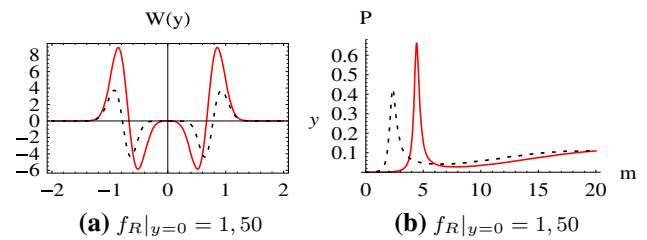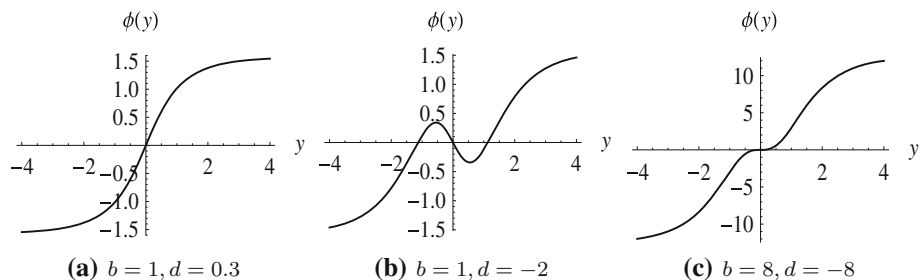

(a) $f_R|_{y=0} = 1, 100$　　　　**(b)** $f_R|_{y=0} = 1, 100$

Fig. 6 Plots of the effective potential $W(z(y))$ and relative probability $P(m^2)$ of odd resonance KK modes on the brane generated by a K-field with $b = 1, d = -2, \lambda = 0.6$, and $f_R|_{y=0} = 1, 100$

(a) $f_R|_{y=0} = 1, 100$　　　　**(b)** $f_R|_{y=0} = 1, 100$

Fig. 7 Plots of the effective potential $W(z(y))$ and relative probability $P(m^2)$ of the odd resonance KK modes on the brane generated by a K-field with $b = 6, d = -6, \lambda = 0.6$, and $f_R|_{y=0} = 1, 100$

(a) $f_R|_{y=0} = 1, 50$　　　　**(b)** $f_R|_{y=0} = 1, 50$

Fig. 9 Plots of the effective potential $W(z(y))$ and relative probability $P(m^2)$ of the odd resonance KK modes on the brane generated by a K-field with $b = 1, d = 0.3, \lambda = -0.01$, and $f_R|_{y=0} = 1, 50$

(a) $f_R|_{y=0} = 1, 50$　　　　**(b)** $f_R|_{y=0} = 1, 50$

Fig. 10 Plots of the effective potential $W(z(y))$ and relative probability $P(m^2)$ of the odd resonance KK modes on the brane generated by a K-field with $b = 1, d = -2, \lambda = -0.01$, and $f_R|_{y=0} = 1, 50$

(a) $f_R|_{y=0} = 1, 50$　　　　**(b)** $f_R|_{y=0} = 1, 50$

Fig. 11 Plots of the effective potential $W(z(y))$ and relative probability $P(m^2)$ of the odd resonance KK modes on the brane generated by a K-field with $b = 8, d = -8, \lambda = -0.01$, and $f_R|_{y=0} = 1, 50$

For $\lambda < 0$, we also set three similar groups of parameters (see Fig. 8) and all of them have lumplike solutions for warp factor. Here we also only consider the values of parameters which result in $f_R|_{y=0} \geq 1$ in $f(R)$-brane model. Obviously, the consequences (see Figs. 9, 10, 11) are analogous to the case $\lambda > 0$.

It seems that we can draw the conclusion that the difference of the gravitational resonances is small between the $f(R)$-brane and GR-brane when the warp factor is monotonic in the area $y \in [0, +\infty)$.

Therefore, we consider another interesting scenario, i.e., the warp factor is a non-monotonic function of y in the area $y \in [0, +\infty)$. The parameters are taken as $b = 2, d =$

Fig. 8 Plots of the scalar field ϕ given by Eq. (45). The figures are, from *left* to *right*, a kink-like solution, a non-monotonic kink solution, and a double kink-like solution

(a) $b = 1, d = 0.3$　　　　**(b)** $b = 1, d = -2$　　　　**(c)** $b = 8, d = -8$
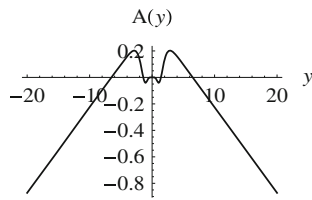
Fig. 12 Plot of the warp factor $A(y)$ given by Eq. (46) with $b = 2$, $d = -2$, and $\lambda = -0.6$. The parameters correspond to a double kink-like solution of the scalar field
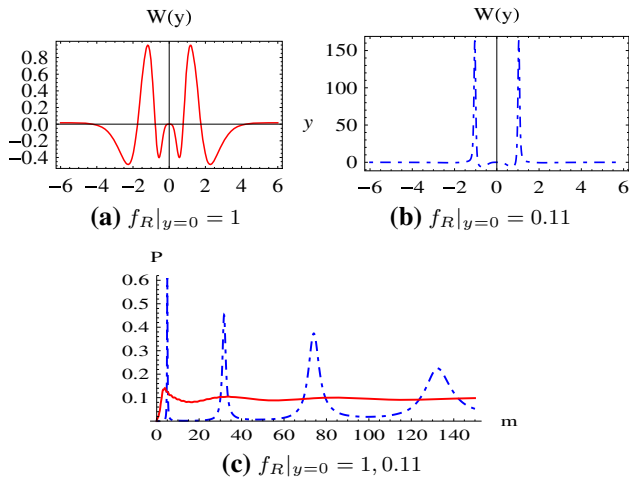
Fig. 13 Plots of the effective potential $W(z(y))$ and relative probability $P(m^2)$ of the odd resonance KK modes on the brane generated by a K-field with $b = 2$, $d = -2$, $\lambda = -0.6$, and $f_R|_{y=0} = 1, 0.11$

-2, and $\lambda = -0.6$ (see Fig. 12). The effective potential $W(z(y))$ and relative probability $P(m^2)$ of the GR-brane are shown in Fig. 13a, c. Apparently, there is no gravitational resonance on the GR-brane. Setting the same parameters for the $f(R)$-brane model, it can be seen that when $f_R|_{y=0} = 0.11$, in comparison to the GR-brane the effective potential and relative probability change greatly (see Fig. 13b, c).

For the set of parameters $b = 2$, $d = -2$, $\lambda = -0.6$, and $f_R|_{y=0} = 0.106$, we can see that f_R almost vanishes at $y = \pm 1$ (see Fig. 14a). We display the effective potential $W(z(y))$ in Fig. 14b and relative probability $P(m^2)$ in Fig. 14c to prepare for the next section.

4 The impact of $f(R)$-brane

Now, we analyze how $f(R)$-brane model impacts gravitational resonances. Focusing on Eq. (13), the different parts of the potential $W(z(y))$ between the GR-brane and $f(R)$-brane are denoted by ΔW:

$$\Delta W = \frac{3aa'}{2}\frac{f_R'}{f_R} - \frac{1}{4}\left(\frac{af_R'}{f_R}\right)^2 + \frac{1}{2}\frac{aa'f_R' + a^2f_R''}{f_R}. \tag{48}$$

First of all, we analyze the model with only one canonical scalar field. In this case, we have for the GR-brane
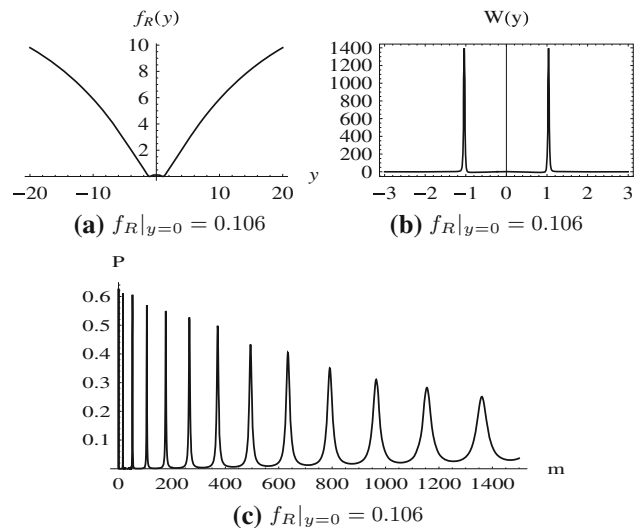
Fig. 14 Plots of the function $f_R(y)$, effective potential $W(z(y))$, and relative probability $P(m^2)$ of the even resonance KK modes on the brane generated by a K-field with $b = 2$, $d = -2$, $\lambda = -0.6$, and $f_R|_{y=0} = 0.106$

$$-\frac{3}{2}A'' = \phi'^2 \geq 0, \tag{49}$$

which results in a lumplike solution of the wrap factor.

For the $f(R)$-brane, we have Eq. (18). Fixing the solution of the warp factor, if we expect some huge differences of the potentials $W(z(y))$ between $f(R)$-brane and GR-brane, the value of $\frac{f_R'}{f_R}$ or $\frac{f_R''}{f_R}$ must be large enough at some locations to form deep wells or high barriers. Considering that $f(R)$ should be continuous and satisfy $f_R > 0$, the most promising approach may be that f_R approaches zero at some locations.

Because both $f(R)$ and f_R are even functions of the extra dimension y and, generally, $f(R)$ should be a constant at infinity, we need only consider the situation that f_R approaches a very small value at one or two locations (more locations make no difference).

From Eq. (48) we know that ΔW is also an even function. We assume ΔW has two infinitely deep potential wells (the same analysis is appropriate for barriers) symmetrically situated at $y = \pm y_0$. For simplicity, we consider the case that these locations are minima of $f_R(y)$. So, we have $f_R' = 0$ and $f_R'' \geq 0$ at these locations. We plug $f_R' = 0$ into Eq. (18) at $y = \pm y_0$, then we have $-3A'' f_R \geq f_R''$.

Obviously, the first two terms of ΔW in Eq. (48) vanish at $y = \pm y_0$. Since $-3A'' f_R \geq f_R''$, the ranges of ΔW at $y = \pm y_0$ are both $0 < \Delta W = \frac{1}{2}\frac{a^2 f_R''}{f_R} < \frac{-3a^2 A''}{2} = a^2\phi'^2$, which is finite. So, it is hard for the potential $W(z(y))$ to have deep potential wells or barriers around $y = \pm y_0$ even if $f_R|_{y=\pm y_0}$ approach zero. The same analysis also applies to the situation that $W(z(y))$ is supposed to have one infinite potential well at $y = 0$.

As for the Bloch brane, although the solutions of the warp factor and scalar fields are not the same as the previous model, the restrictions on the warp factor and $f(R)$ are similar (see Eq. (37)). So, we also have the same conclusion.

Finally, we discuss why the third $f(R)$-brane model has a strong influence on the potential $W(z(y))$. Taking $f(R) = R$ in Eq. (47), we find that although λ is a small parameter, it can lead to a non-monotonic warp factor from $y = 0$ to $y = +\infty$. Therefore, the shape of the potential $W(z(y))$ of the $f(R)$-brane becomes complicated. Furthermore, obviously when $\lambda < 0$, it is possible that the function $f_R(y)$ approaches zero at some locations. In Fig. 14a, we see that when $f_R(y)$ gets close to zero, there are two high barriers at the corresponding locations (Fig. 14b) but the GR-brane never has such a kind of feature.

5 Conclusion and discussion

In this work we discussed gravitational resonances in various $f(R)$-brane models. For the branes generated by one or two canonical scalar fields, we found that the $f(R)$-brane and GR-brane are similar from the view point of gravitational resonance. Then we investigated the brane generated by a single K-field. For a positive λ (or a negative λ with the warp factor varying monotonically), the structure of gravitational resonance of $f(R)$-brane does not deviate much from the GR-brane. This result is similar to the case with the standard kinetic term. When the parameter λ is negative and the wrap factor is non-monotonic, the structure of the gravitational resonance of the $f(R)$-brane can seriously deviate from the GR-brane. The degree of deviation relies mostly on the value of the parameter $f_R|_{y=0}$.

Finally, we analyzed why the $f(R)$-brane and GR-brane have similar gravitational resonance structures when the background scalar field is canonical. Our conclusion is that the similarity is due to the limitations $f_R > 0$ and Eq. (18), which constrain strictly the potential $W(z(y))$ of the $f(R)$-brane on the whole space. For the cases with more scalar fields (such as the Bloch brane), one may draw the same conclusion, as long as all kinetic terms of the scalar fields are canonical. However, for the brane generated by a K-field, the function $f_R(y)$ could approach zero at some locations. Therefore, the effective potential $W(z(y))$ in the $f(R)$-brane model may have plentiful structures and result in a series of gravitational resonances, which is very different from the GR-brane model.

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