

Master actions for massive spin-3 particles in $D = 2 + 1$

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Abstract We present here a relationship between massive self-dual models for spin-3 particles in $D = 2 + 1$ via the master action procedure. Starting with a first-order model (in the derivatives) $S_{SD(1)}$ we have constructed a master action which interpolates between a sequence of four self-dual models $S_{SD(i)}$ where $i = 1, 2, 3, 4$. By analyzing the particle content of the mixing terms, we give additional arguments that explain why it is apparently impossible to jump from the fourth-order model to a higher-order model. We have also analyzed similarities and differences between the fourth-order K -term in the spin-2 case and the analogous fourth-order term in the spin-3 context.

1 Introduction

Higher spin massive particles are present in the spectrum of string theories. Although we have not observed higher spin elementary particles in nature yet, resonant states have been detected; see [1] as a recent example of spin-3 resonance observation. In general, Lagrangians for massive particles in $D = 4$ have the same form in arbitrary dimensions, like e.g. the Maxwell-Proca (spin-1) and Fierz-Pauli (spin-2) theories and even higher spin theories. Since in $D = 3$ one can build up such Lagrangians, see [2,3] for the spin-1 and spin-2 cases, respectively, via soldering of opposite helicity states, one may claim that the Lagrangians which represent helicity eigenstates in $D = 3$ are the basic building blocks for bosonic massive higher spin particle Lagrangians. Thus, the study of such parity breaking theories (self-dual models) may have some interesting connection with massive higher spin particles in arbitrary dimensions. Here we investigate the simplest case after the spin-2 one, i.e., the massive spin-3 states in $D = 3$. Based on our previous experience with

massive spin-2 gauge theories [4,5], we are going to analyze it by means of the master action technique.

Recently, we have addressed [6] this subject through another dualization procedure, the Noether Gauge Embedding, NGE, which is based on the existence of a local symmetry in the highest derivative term of the self-dual model which is not present in the lower derivative terms. In complete analogy with the spin-2 case we have shown that starting with the first-order non-gauge invariant self-dual model [7] it is possible to obtain the second- [8], third- [9] and fourth-order self-dual models, where the last one has all the auxiliary fields needed to correctly describe only one helicity $+3$ or -3 particle. In [6] we have faced the problem of missing gauge symmetries which are required in order to proceed with the technique and go beyond the fourth-order self-dual model, which might be naively expected since looking at the spin-1 and spin-2 examples, one can see that there are two and four self-dual descriptions for the singlets, respectively, indicating that there might be some $2s$ rule for the highest order of the spin- s self-dual model, where s is the spin. In fact the authors in [10] have proposed fifth- and sixth-order equations of motion, for a massive spin-3 particle, however, without considering auxiliary fields, where, however, the fifth-order model contains ghosts.

Here, we come back to this point. We find an alternative explanation of why it is apparently impossible to complete the chain of $2s$ models in the spin-3 case, and give a demonstration that the classical equivalence between first-, second-, third- and fourth-order spin-3 models obtained in [6] holds also at the quantum level.

In the master action approach a fundamental ingredient consists of finding appropriate mixing terms between the dual fields, which cannot have by themselves any particle content when diagonalized. In other words, the equations of motion of the mixing term lead to no propagating mode whatsoever, otherwise the particle content of the two models connected via master action would not be the same. So, here we organize the paper first providing a discussion of the particle content

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of mixing terms for spins 2 and 3. We then propose a master action that interpolates between the first three spin-3 self-dual models, and obtain their dual maps. As a last step we show that is possible to construct a new master action using only totally symmetric fields, which interpolates between the third and the fourth-order self-dual models. In the last section final remarks on the particle content of the fourth-order term clarify the difficulties in going beyond the fourth-order self-dual model.

2 Trivial Lagrangians

2.1 Chain of self-dual models

In order to capture the essential features of the master action [11] approach used here in $D = 2 + 1$ to jump from a k th-order self-dual model $\mathcal{L}_{SD}^{(k)}$ to $\mathcal{L}_{SD}^{(k+1)}$, we have found it instructive to write down a toy model version of $\mathcal{L}_{SD}^{(k)}$ in a symbolic notation suppressing all Lorentz indices and complicated details as regards the different operators appearing in the Lagrangian densities, namely,

$$\mathcal{L}_{SD}^{(k)}[A] = m^2 b_k \left[(-1)^k A \hat{\partial}^k A + A \hat{\partial}^{k-1} A \right]. \tag{1}$$

For convenience we define a dimensionless derivative $\hat{\partial} = \partial/m$, and $b_k = (-1)^{k(k+1)/2}$ such that it satisfies

$$b_{k+1} = (-1)^{k+1} b_k. \tag{2}$$

The first-order theory $\mathcal{L}_{SD}^{(1)}[A]$ may represent the self-dual model of [12] in the spin-1 case or the self-dual models of [7, 13] in the spin-2 and spin-3 cases, respectively, while $\mathcal{L}_{SD}^{(2)}$ represents the spin-1 Maxwell–Chern–Simons (MCS) model of [14] as well as the second-order models of spin-2 and spin-3 defined in [8, 15], respectively. The Lagrangian density $\mathcal{L}_{SD}^{(3)}$ stands for both the linearized topologically massive gravity of [14] and the third-order spin-3 model of [9]. Finally $\mathcal{L}_{SD}^{(4)}$ may represent either the linearized version of the higher derivative topologically massive gravity of [5, 16] or the spin-3 fourth-order model whose main equations of motion are given in [10] and the whole action in [6]. More precisely, the A -field in (1) stands for a rank- s tensor field and in the spin- s case while $\mathcal{L}_{SD}^{(k)}$ corresponds only to the two terms of the self-dual model which are quadratic in the rank- s field.

The basic idea of the master action approach is to add to $\mathcal{L}_{SD}^{(k)}$ a “mixing term” between the A -field and the dual B -field and define a master model:

$$\mathcal{L}_M[A, B] = m^2 b_k \left[(-1)^k A \hat{\partial}^k A + A \hat{\partial}^{k-1} A - (-1)^k (A - B) \hat{\partial}^k (A - B) \right]. \tag{3}$$

The mixing term is essentially the highest derivative term of $\mathcal{L}_{SD}^{(k)}$. After the trivial shift $B \rightarrow \tilde{B} + A$, which produces a trivial Jacobian in the path integral, we have

$$\mathcal{L}_M = \mathcal{L}_{SD}^{(k)}[A] + m^2 b_k (-1)^{k+1} \tilde{B} \hat{\partial}^k \tilde{B}. \tag{4}$$

On the other hand, we can rewrite (3) neglecting total derivatives:

$$\mathcal{L}_M[A, B] = m^2 b_k \left\{ B \hat{\partial}^{k+1} B - (-1)^k B \hat{\partial}^k B + \left[A - (-1)^k \hat{\partial} B \right] \hat{\partial}^{k-1} \left[A - (-1)^k \hat{\partial} B \right] \right\}. \tag{5}$$

After the shift $A \rightarrow \tilde{A} + (-1)^k \hat{\partial} B$ and use of (2) we have

$$\mathcal{L}_M = \mathcal{L}_{SD}^{(k+1)}[B] + m^2 b_k \tilde{A} \hat{\partial}^{k-1} \tilde{A}. \tag{6}$$

Therefore if both $\tilde{B} \hat{\partial}^k \tilde{B}$ and $\tilde{A} \hat{\partial}^{k-1} \tilde{A}$ have no particle content, it is clear that (4) and (6) establish the physical equivalence (duality) of $\mathcal{L}_{SD}^{(k)}$ and $\mathcal{L}_{SD}^{(k+1)}$. It amounts to assume that $\mathcal{L}_{SD}^{(k)}$ in (1) is made out of two terms without physical content (trivial). In the next subsection we review the triviality of some spin-2 Lagrangian densities as an introduction to the spin-3 case.

2.2 Spin-2 and spin-3

As a warm up let us briefly review the very known terms that can be used as mixing terms in the spin-2 context. First we have a Chern–Simons-like term of first order in the derivatives:

$$S_{CS2}^{(1)} = \frac{m}{2} \int d^3x \epsilon^{\mu\nu\alpha} f_{\mu\beta} \partial_\nu f_\alpha^\beta, \tag{7}$$

where $f_{\mu\nu}$ is a non-symmetric tensor. Similar to the spin-1 case one can show that the general solution to the equations of motion is also pure gauge, with $f_{\mu\nu} = \partial_\mu \xi_\nu$.

We can also use the linearized version of the Einstein–Hilbert term, which is second order in the derivatives. We write it in the following way:

$$S_{EH} = -\frac{1}{2} \int d^3x f_{(\mu\nu)} E^{\mu\alpha} E^{\nu\beta} f_{(\alpha\beta)}, \tag{8}$$

where $E^{\mu\nu} = \epsilon^{\mu\nu\rho} \partial_\rho$. From the equations of motion with respect to $f_{(\mu\nu)}$ ¹ we have the second-order differential equations $E^{\mu\alpha} E^{\nu\beta} f_{(\alpha\beta)} = 0$. By applying twice the Levi-Civita symbol in the equations of motion, we have

$$\epsilon_{\mu\gamma\rho} \epsilon_{\nu\lambda\xi} E^{\mu\alpha} E^{\nu\beta} f_{(\alpha\beta)} = R_{\gamma\rho\lambda\xi}^L(f) = 0. \tag{9}$$

Here $R_{\gamma\rho\lambda\xi}^L(f)$ stands for the linearized Riemann tensor. The general solution of (9) (see [14]) is also pure gauge, $f_{(\mu\nu)} = \partial_\mu \Phi_\nu + \partial_\nu \Phi_\mu$.

¹ Here $f_{(\mu\nu)} = (f_{\mu\nu} + f_{\nu\mu})/2$.

Finally, besides (7) and (8) one has yet a third option, which is the third-order topological Chern–Simons term whose linearized version is

$$S_{CS2}^{(3)} = \frac{1}{2m} \int d^3x f^{(\mu\nu)} \square \theta_{\mu\alpha} E_{\nu\beta} f^{(\alpha\beta)}. \tag{10}$$

Here we have introduced the transverse projection operator $\theta_{\mu\nu} = (\eta_{\mu\nu} - \partial_\mu \partial_\nu / \square)$. In [14] the authors demonstrate through the helicity decomposition method that this term has no particle content. So, it is possible to use it as a mixing term.

For spin-3 the first candidate for a mixing term is the first-order Chern–Simons-like term, which was first introduced in [7]:

$$S_{CS3}^{(1)} = \frac{m}{2} \int d^3x \epsilon_{\mu\nu\alpha} \omega^{\mu(\beta\gamma)} \partial^\nu \omega^{\alpha(\beta\gamma)}, \tag{11}$$

where $\omega_{\mu(\beta\gamma)}$, is symmetric and traceless i.e. $\omega_{\mu(\beta\gamma)} = \omega_{\mu(\gamma\beta)}$ and $\eta^{\beta\gamma} \omega_{\mu(\beta\gamma)} = 0$ with $\omega_\gamma \equiv \eta^{\mu\beta} \omega_{\mu(\beta\gamma)}$. For the same reasons as mentioned before it is not difficult to convince oneself that the general solution of the equation of motion derived from (11) is pure gauge, $\omega_{\mu(\beta\gamma)} = \partial_\mu \Lambda_{(\beta\gamma)}$ where $\eta^{\beta\gamma} \Lambda_{(\beta\gamma)} = 0$.

We can also take the usual Singh–Hagen second-order massless spin-3 term [17–19] as a mixing term; it comes into the game as an analog of the spin-2 Einstein–Hilbert term. Just as in the case of spin 2, one can check that it is possible to write it in terms of partially symmetric tensors or in terms of totally symmetric tensors:

$$\begin{aligned} S_{SH} &= \frac{1}{2} \int d^3x \xi_{\mu(\beta\gamma)} \Omega^{\mu(\beta\gamma)}(\xi) \\ &= \frac{1}{2} \int d^3x \phi_{\mu\nu\alpha} G^{\mu\nu\alpha}(\phi), \end{aligned} \tag{12}$$

where

$$\begin{aligned} \Omega_{\mu(\beta\gamma)}(\xi) &= 3(\xi_{\beta(\mu\gamma)} + \xi_{\gamma(\mu\beta)} - \xi_{\mu(\beta\gamma)}) \\ &\quad - 2\eta_{\beta\gamma} \xi_\mu, \xi_{\mu(\beta\gamma)} = E_\mu^\lambda \omega_{\lambda(\beta\gamma)}. \end{aligned} \tag{13}$$

On the right hand site of (12) we have used the spin-3 ‘‘Einstein tensor’’ given in terms of the totally symmetric field $\phi_{\mu\beta\gamma}$ and defined in [9,20] as

$$G^{\mu\beta\gamma}(\phi) \equiv R^{\mu\nu\lambda} - \frac{1}{2} \eta^{(\mu\nu} R^{\lambda)}, \tag{14}$$

where the ‘‘Ricci’’ tensor is given by $R^{\mu\nu\lambda} = \square \phi^{\mu\nu\lambda} - \partial_\alpha \partial^{(\mu} \phi^{\alpha\nu\lambda)} + \partial^{(\mu} \partial^\nu \phi^{\lambda)}$ and its trace $R^\lambda = \eta_{\mu\nu} R^{\mu\nu\lambda}$.

Finally, the equivalence between the two notations in (12) is possible thanks to the general decomposition below:

$$\begin{aligned} \omega_{\mu(\beta\gamma)} &= \frac{1}{\sqrt{3}} \left[\phi_{\mu\beta\gamma} + \frac{1}{4} (\eta_{\lambda\beta} \phi_{\gamma\lambda} + \eta_{\lambda\gamma} \phi_{\beta\lambda}) - \frac{1}{2} \eta_{\beta\gamma} \phi_\lambda \right] \\ &\quad + \left(\epsilon_{\mu\nu\beta} \chi_\nu^\gamma + \epsilon_{\mu\nu\gamma} \chi_\nu^\beta \right), \end{aligned} \tag{15}$$

where $\chi_{\mu\nu}(x) = \chi_{\nu\mu}(x)$ and $\eta^{\mu\nu} \chi_{\mu\nu} = \chi = 0$. The numerical factors in (15) are obtained in such a way that our results fit the results of [9]. Besides, one can verify that both sides of (15) have the same number of independent components in $D = 2 + 1$.

From the equations of motion derived from (12) with respect to $\phi_{\mu\beta\gamma}$ we conclude that the Einstein tensor (14) vanishes:

$$G_{\mu\nu\lambda}(\phi) = 0, \tag{16}$$

which implies that the ‘‘Ricci’’ tensor is null, $R_{\mu\nu\lambda} = 0$. In [9] the authors demonstrate that, as in the spin-2 case, in $D = 2 + 1$ the curvature Riemann tensor for spin-3,

$$\begin{aligned} R_{\mu\nu\lambda\alpha\beta\gamma}(\phi) &\equiv \partial_\alpha \partial_\beta \partial_\gamma \phi_{\mu\nu\lambda} - \partial_\mu \partial_\beta \partial_\gamma \phi_{\alpha\nu\lambda} - \partial_\alpha \partial_\beta \partial_\lambda \phi_{\mu\nu\gamma} \\ &\quad + \partial_\mu \partial_\beta \partial_\lambda \phi_{\alpha\nu\gamma} - \partial_\alpha \partial_\nu \partial_\gamma \phi_{\mu\beta\lambda} + \partial_\mu \partial_\nu \partial_\gamma \phi_{\alpha\beta\lambda} \\ &\quad + \partial_\alpha \partial_\nu \partial_\lambda \phi_{\mu\beta\gamma} - \partial_\mu \partial_\nu \partial_\lambda \phi_{\alpha\beta\gamma}, \end{aligned} \tag{17}$$

can be expressed in terms of the Ricci tensor, since the Weyl tensor vanishes in $D = 2 + 1$; thus $R_{\mu\nu\lambda\alpha\beta\gamma} = 0$ follows from $R_{\mu\nu\lambda} = 0$. The general solution for the null curvature Riemann tensor is pure gauge, $\phi_{\mu\nu\lambda} = \partial_{(\mu} \tilde{\Lambda}_{\nu\lambda)}$ where $\tilde{\Lambda} = \eta^{\mu\nu} \tilde{\Lambda}_{\mu\nu} = 0$. Note that $\delta_\xi S_{SH} = 0$ where $\delta_\xi \omega_{\mu(\beta\gamma)} = \partial_\mu \xi_{(\beta\gamma)}$ with $\eta^{\beta\gamma} \xi_{(\beta\gamma)} = 0$. Then the action (12) has no particle content in $D = 2 + 1$.

Besides the first two terms (11) and (12) introduced before, one can also use as a mixing term the third-order spin-3 Chern–Simons term, which can be written in terms of a partially symmetric tensor or a totally symmetric tensor:

$$\begin{aligned} S_{CS3}^{(3)} &= \frac{1}{2m} \int d^3x \Omega_{\mu(\beta\gamma)}(\xi) E_\lambda^\mu \Omega^{\lambda(\beta\gamma)}(\xi) \\ &= \frac{3}{2m} \int d^3x \phi_{\mu\nu\lambda} E_\gamma^\mu G^{\gamma\nu\lambda}(\phi) \\ &= \frac{1}{2m} \int d^3x C_{\mu\beta\gamma}(\phi) G^{\mu\beta\gamma}(\phi). \end{aligned} \tag{18}$$

We have used the symmetrized curl defined in [9], given by

$$C_{\mu\beta\gamma}(\phi) \equiv E_\mu^\nu \phi_{\nu\beta\gamma} + E_\beta^\nu \phi_{\nu\beta\mu} + E_\gamma^\nu \phi_{\nu\beta\mu} \tag{19}$$

and $G^{\mu\beta\gamma}$ is given in (14). The authors of [10] have suggested that the more natural analog of the Einstein tensor for spin-3 should be a rank-3 third-order tensor instead of (14). This alternative is particularly useful for us, since defining $\mathcal{G}_{\mu\nu\rho} \equiv E_\mu^\alpha E_\nu^\beta E_\rho^\gamma \phi_{\alpha\beta\gamma}$ the action (18) becomes

$$\begin{aligned} S_{CS3}^{(3)} &= \frac{1}{2m} \int d^3x \phi_{\mu\nu\lambda} \mathcal{G}^{\mu\nu\lambda} \\ &= \frac{1}{2m} \int d^3x \phi_{\mu\nu\lambda} E^{\mu\alpha} E^{\nu\beta} E^{\lambda\gamma} \phi_{\alpha\beta\gamma}, \end{aligned} \tag{20}$$

and this makes evident the gauge symmetry $\delta \phi_{\alpha\beta\gamma} = \partial_{(\alpha} \Lambda_{\beta\gamma)}$ with $\Lambda_{\beta\gamma} = \Lambda_{\gamma\beta}$ an arbitrary symmetric parameter—compare (8) with (20). Finally by taking the equations of motion from (20) $\mathcal{G}_{\mu\nu\lambda} = 0$, we have the pure gauge solution

$\phi_{\mu\nu\lambda} = \partial_{(\mu}\Lambda_{\nu\lambda)}$. One can notice that like in the spin-2 case, once we have

$$\epsilon_{\gamma\mu\nu}\epsilon_{\delta\lambda\alpha}\epsilon_{\rho\beta\gamma}\mathcal{G}^{\gamma\delta\rho} = 0, \tag{21}$$

this results in $R_{\mu\nu\lambda\alpha\beta\gamma} = 0$. Then, again according to the theorem demonstrated in [9], one can verify that the third-order Chern–Simons term (20) has no particle content, so it can be used as a mixing term in order to construct a master action just like (11) and (12).

3 First-, second- and third-order spin-3 self-dual models

The master action is constructed from the first-order self-dual model proposed in [7] which is the spin 3 analog of the spin-2 and spin-1 self-dual models of [12, 13], respectively:

$$S_{SD(1)}[\omega, A] = \int d^3x \left[\frac{m}{2}\epsilon^{\mu\nu\alpha}\omega_{\mu(\beta\gamma)}\partial_\nu\omega_\alpha^{(\beta\gamma)} + \frac{m^2}{6}(\omega_\mu\omega^\mu - \omega_{\mu(\beta\gamma)}\omega^{\beta(\mu\gamma)}) + m^2\omega_\mu A^\mu \right] + S_1[A], \tag{22}$$

where

$$S_1[A] = \int d^3x \left[-9m\epsilon^{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha - 9m^2A_\mu A^\mu - 12(\partial_\mu A^\mu)^2 \right], \tag{23}$$

is the required auxiliary action such that (22) describes only one massive spin-3 particle. The Fierz–Pauli conditions can be obtained from the equations of motion derived from (22), demonstrations can be found in [6, 7].

By adding mixing terms without particle content we aim to construct a master action from (22). We can use the first-order Chern–Simons term (11), the usual massless second-order spin-3 term (12) and the third-order Chern–Simons term (18) as mixing terms. However, as we have observed in [6] when we get the third-order self-dual model the whole action can be described by totally symmetric tensors through the decomposition (15), so, first we are going to construct a master action interpolating between the first three self-dual models, and then as a last step an action interpolating between the third-order self-dual model and the fourth-order self-dual model, both of them in terms of totally symmetric tensors. The first master action is suggested as follows:

$$S_M = \int \left[\frac{m}{2}\omega \cdot d\omega + \frac{m^2}{6}(\omega^2) - \frac{m}{2}(\omega - g) \cdot d(\omega - g) + \frac{1}{2}(h - g) \cdot d\Omega(h - g) \right] + m^2 \int d^3x \omega_\mu A^\mu + S_1[A], \tag{24}$$

where $g_{\mu(\beta\gamma)}$ and $h_{\mu(\beta\gamma)}$ are new auxiliary fields. Here, we use the same shorthand notation adopted in [4], where

$$\int (\omega^2) \equiv \int d^3x (\omega_\mu\omega^\mu - \omega_{\mu(\beta\gamma)}\omega^{\beta(\mu\gamma)}), \tag{25}$$

$$\int \omega \cdot d\omega \equiv \int d^3x \epsilon^{\mu\nu\alpha}\omega_{\mu(\beta\gamma)}\partial_\nu\omega_\alpha^{(\beta\gamma)}, \tag{26}$$

$$\int \omega \cdot d\Omega(\omega) \equiv \int d^3x \xi_{\mu(\beta\gamma)}(\omega)\Omega^{\mu(\beta\gamma)}[\xi(\omega)]. \tag{27}$$

In order to interpolate between the dual models, obtaining dual maps let us introduce a source term $j_{\mu(\beta\gamma)}$ and define the generating functional:

$$W_M[j] = \int \mathcal{D}\omega \mathcal{D}g \mathcal{D}h \mathcal{D}A \exp i \times \left(S_M + \int d^3x j_{\mu(\beta\gamma)}\omega^{\mu(\beta\gamma)} \right). \tag{28}$$

The first thing we note is that in order to recover the first-order self-dual model we just need to make the shifts $h \rightarrow h + g$ and $g \rightarrow g + \omega$ in (24). Then we get the mixing terms decoupled. Since they have no particle content, see (11) and (12), we end up with the content of the first-order self-dual model (22). So by the derivatives with respect to the sources we find the following identity:

$$\langle \omega_{\mu_1(\beta_1\gamma_1)}(x_1) \dots \omega_{\mu_N(\beta_N\gamma_N)}(x_N) \rangle_M = \langle \omega_{\mu_1(\beta_1\gamma_1)}(x_1) \dots \omega_{\mu_N(\beta_N\gamma_N)}(x_N) \rangle_{SD(1)}. \tag{29}$$

On the other hand making only the shift $h \rightarrow h + g$ and then functionally integrating over h we have

$$S_M = \int d^3x \omega_{\mu(\beta\gamma)}\tilde{g}^{\mu(\beta\gamma)} + \int \left(\frac{m^2}{6}(\omega^2) - \frac{m}{2}g \cdot dg \right) + S_1[A], \tag{30}$$

where

$$\tilde{g}^{\mu(\beta\gamma)} = m\epsilon^{\mu\nu\alpha}\partial_\nu g_\alpha^{(\beta\gamma)} + \frac{m^2}{2}f^{\mu(\beta\gamma)}(A) + j^{\mu(\beta\gamma)}, \tag{31}$$

with

$$f^{\mu(\beta\gamma)}(A) = \eta^{\beta\mu}A^\gamma + \eta^{\gamma\mu}A^\beta - \frac{2}{3}\eta^{\beta\gamma}A^\mu. \tag{32}$$

In (30) we have a quadratic term and a linear term in ω . This suggests an integration over ω in such a way that we obtain an action for g .

Due to the absence of particle content of terms like (11) we have a second-order self-dual action:

$$S_{SD(2)} = \int \left[-\frac{1}{2}g \cdot d\Omega(g) - \frac{m}{2}g \cdot dg - \frac{m}{2}f(A) \cdot dg + j_{\mu(\beta\gamma)}F^{\mu(\beta\gamma)}(g, A) + \mathcal{O}(j^2) \right] + S_2[A], \tag{33}$$

where $\mathcal{O}(j^2)$ stands for quadratic terms in the source and

$$F^{\mu(\beta\gamma)}(g, A) = \frac{\Omega^{\mu(\beta\gamma)}[\xi(g)]}{m} + f^{\mu(\beta\gamma)}(A). \tag{34}$$

The action (33) is exactly the one for the second-order self-dual model given in [8], except for the source term it automatically includes, the new action for the auxiliary field A_μ given by

$$S_2[A] = \int d^3x \left[-9m \epsilon^{\mu\nu\alpha} A_\mu \partial_\nu A_\alpha - \frac{32m^2}{3} A_\mu A^\mu - 12(\partial_\mu A^\mu)^2 \right]. \tag{35}$$

Now, by the derivatives with respect to the source in (24) and (33) we have the correlation functions duality:

$$\langle \omega_{\mu_1(\beta_1\gamma_1)}(x_1) \dots \omega_{\mu_N(\beta_N\gamma_N)}(x_N) \rangle_{SD1} = \langle F_{\mu_1(\beta_1\gamma_1)}(x_1) \dots F_{\mu_N(\beta_N\gamma_N)}(x_N) \rangle_{SD2} + C.T, \tag{36}$$

where $C.T$ stands for contact terms. The relation (36) gives us the dual map at classical and quantum level:

$$\omega^{\mu(\beta\gamma)} \longleftrightarrow F^{\mu(\beta\gamma)}(g, A). \tag{37}$$

Moreover, one can easily demonstrate that the interaction term between the spin-3 field and the vector field is the same as the one obtained previously through the NGE procedure in [6] i.e., $-\frac{m}{2} f(A) \cdot dg = 2m\xi_\mu(g)A^\mu$. The equations of motion of the first-order self-dual model $S_{SD(1)}$ can be dually mapped into the equations of motion for the second-order self-dual model $S_{SD(2)}$ via (37), as we have demonstrated in [6].

Considering again the master action written in (24), without shifting $h \rightarrow h + g$, instead of (33), the master action would be

$$S_M = \int \left[-\frac{1}{2}g \cdot d\Omega(g) - \frac{m}{2}g \cdot dg + \frac{1}{2}(h - g) \cdot d\Omega(h - g) - \frac{m}{2}f(A) \cdot dg - j_{\mu(\beta\gamma)}F^{\mu(\beta\gamma)}(g, A) + \mathcal{O}(j^2) \right] + S_2[A]. \tag{38}$$

It is straightforward to show that

$$\int h \cdot d\Omega(g) = \int g \cdot d\Omega(h). \tag{39}$$

So, we can rewrite (38) in such a way that

$$S_M = \int \left[-\frac{m}{2}(g + C) \cdot d(g + C) + \frac{m}{2}C \cdot dC + \frac{1}{2}h \cdot d\Omega(h) - j_{\mu(\beta\gamma)}F^{\mu(\beta\gamma)}(g, A) + \mathcal{O}(j^2) \right] + S_2[A], \tag{40}$$

where we have defined

$$C = \frac{\Omega(h)}{m} - \frac{\Omega(j)}{m^2} + f(A). \tag{41}$$

The shifts $g \rightarrow g - C$ and $g_{\mu(\beta\gamma)} \rightarrow 3(j_{\beta(\mu\gamma)} + j_{\gamma(\mu\beta)} - j_{\mu(\beta\gamma)}) - 2\eta_{\beta\gamma}j_\mu$ in (40) will completely decouple g from C and j . Then we can integrate over g . Substituting back C we have the third-order self-dual action of [9]:

$$S_{SD(3)} = \int \left[\frac{1}{2}h \cdot d\Omega(h) + \frac{1}{2m}\Omega(h) \cdot d\Omega(h) + f(A) \cdot d\Omega(h) + j_{\mu(\beta\gamma)}H^{\mu(\beta\gamma)}(h, A) + \mathcal{O}(j^2) \right] + S_3[A], \tag{42}$$

where $H^{\mu(\beta\gamma)}(h, A)$ gives the dual map:

$$\omega^{\mu(\beta\gamma)} \longleftrightarrow H^{\mu(\beta\gamma)} \tag{43}$$

where

$$H^{\mu(\beta\gamma)} \equiv -\frac{1}{m}\Omega^{\mu(\beta\gamma)} \left[\frac{\Omega[\xi(h)]}{m} + f(A) \right] + f^{\mu(\beta\gamma)}(A). \tag{44}$$

Again, the auxiliary action is automatically redefined and given in agreement with [9] by

$$S_3[A] = \int d^3x \left[-\frac{32m}{3} \epsilon^{\mu\nu\alpha} A_\mu \partial_\nu A_\alpha - \frac{32m^2}{3} A_\mu A^\mu - 12(\partial_\mu A^\mu)^2 \right]. \tag{45}$$

By the derivatives with respect to the source term in (24) and (42) we have the equivalence of the correlation functions:

$$\langle \omega_{\mu_1(\beta_1\gamma_1)}(x_1) \dots \omega_{\mu_N(\beta_N\gamma_N)}(x_N) \rangle_{SD1} = \langle H_{\mu_1(\beta_1\gamma_1)}(x_1) \dots H_{\mu_N(\beta_N\gamma_N)}(x_N) \rangle_{SD3} + C.T. \tag{46}$$

In the next section we are going to perform the interpolation between the third-order self-dual model and the new fourth-order self-dual model.

4 Master action in terms of totally symmetric fields

From now on, we propose a new master action only in terms of totally symmetric fields. It can be constructed from the third-order self-dual model obtained before, by means of the decomposition (15) in (42), with $\omega_{\mu(\beta\gamma)}$ replaced by $h_{\mu(\beta\gamma)}$,

$$S_{SD(3)}[\phi, A] = \int d^3x \left[-\frac{1}{2}\phi_{\mu\beta\gamma}G^{\mu\beta\gamma}(\phi) - \frac{1}{2m}C_{\mu\beta\gamma}(\phi)G^{\mu\beta\gamma}(\phi) \right]$$

$$-\frac{4}{3\sqrt{3}}\tilde{A}_{\mu\beta\gamma}G^{\mu\beta\gamma}(\phi) + \tilde{j}_{\mu\beta\gamma}G^{\mu\beta\gamma}(\phi) \Big] + S_3[A]. \tag{47}$$

For simplicity we have considered only the totally symmetric source term, $\tilde{j}_{\mu\beta\gamma}$, which will give the correlation functions of the totally symmetric fields $\phi_{\mu\beta\gamma}$. We have also defined the totally symmetric combination for the spin-1 field:

$$\tilde{A}_{\mu\nu\lambda} \equiv A_\mu\eta_{\nu\lambda} + A_\nu\eta_{\mu\lambda} + A_\lambda\eta_{\nu\mu}. \tag{48}$$

It is useful for the next step to notice that the first two terms in (47) are self-adjoint, i.e; $\phi_{\mu\nu\lambda}G^{\mu\nu\lambda}(\psi) = \psi_{\mu\nu\lambda}G^{\mu\nu\lambda}(\phi)$ and $\phi_{\mu\nu\lambda}C^{\mu\nu\lambda}(\psi) = \psi_{\mu\nu\lambda}C^{\mu\nu\lambda}(\phi)$ hold inside space-time integrals.

Omitting the indices for simplicity, since all quantities are totally symmetric third rank tensors, we propose the following master action:

$$S_M = \int d^3x \left[-\frac{1}{2}\phi G(\phi) - \frac{1}{2m}C(\phi)G(\phi) + \frac{1}{2m}C(\phi - \psi)G(\phi - \psi) - \frac{4}{3\sqrt{3}}\tilde{A}G(\phi) + \tilde{j}G(\phi) \right] + S_3[A]. \tag{49}$$

We have used the third-order Chern–Simons term as the mixing term to interpolate between $SD(3)$ and $SD(4)$. The field $\psi_{\mu\nu\lambda}$ corresponds to a new totally symmetric field. It is trivial to observe that with the shift $\psi \rightarrow \psi + \phi$ we have the correspondence $S_M \Leftrightarrow S_{SD(3)}$. Using the property $C(\phi)G(\sigma) = C(\sigma)G(\phi)$ one can rewrite S_M as

$$S_M = \int \left[-\frac{1}{2}(\phi - \sigma)G(\phi - \sigma) + \frac{1}{2}\sigma G(\sigma) + \frac{1}{2m}C(\psi)G(\psi) \right] + S_3[A], \tag{50}$$

where σ is defined by

$$\sigma = -\frac{C(\psi)}{m} - \frac{4}{3\sqrt{3}}\tilde{A} + \tilde{j}, \tag{51}$$

making the shift $\phi \rightarrow \phi + \sigma$, as the second-order term of the kind (12) has no particle content we end up after a gaussian integration on ϕ with

$$S_M = \int \left[\frac{1}{2}\sigma G(\sigma) + \frac{1}{2m}C(\psi)G(\psi) \right] + S_3[A], \tag{52}$$

substituting back (51) in (52) we have, after manipulations, the complete spin-3 fourth-order self-dual model that we have found in [6]:

$$S_{SD(4)} = \int d^3x \left[\frac{1}{2m}C(\psi)G(\psi) + \frac{1}{2m^2}C(\psi)G[C(\psi)] + \frac{4}{3\sqrt{3}m}C(\psi)G(\tilde{A}) - \frac{1}{m}C(\psi)G(\tilde{j}) \right] + S_4[A, \tilde{j}]. \tag{53}$$

Now the auxiliary action has gained a new second-order term in the derivatives, which, combined with $(\partial_\mu A^\mu)^2$, is precisely the Maxwell term, written in terms of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as

$$S_4[A] = -\frac{32}{3} \int d^3x \left[-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + m\epsilon^{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha + m^2A_\mu A^\mu - \tilde{j}_{\mu\beta\gamma}G^{\mu\beta\gamma}(\tilde{A}) \right]. \tag{54}$$

By the derivatives with respect to the totally symmetric source in (47) and in the fourth-order self-dual model (53), we have the equivalence between the correlation functions:

$$\langle G_{\mu_1\beta_1\gamma_1}(\phi) \dots G_{\mu_N\beta_N\gamma_N}(\phi) \rangle_{SD3} = \left\langle G_{\mu_1\beta_1\gamma_1} \left[-\frac{4C(\psi)}{3\sqrt{3}m} - \tilde{A} \right] \dots \times G_{\mu_N\beta_N\gamma_N} \left[-\frac{4C(\psi)}{3\sqrt{3}m} - \tilde{A} \right] \right\rangle_{SD4} + C.T., \tag{55}$$

which implies

$$G_{\mu\beta\gamma} \left[\phi + \frac{4C(\psi)}{3\sqrt{3}m} + \tilde{A} \right] = 0 \tag{56}$$

whose general solution is pure gauge:

$$\phi_{\mu\beta\gamma} = \partial_{(\mu}\tilde{\Lambda}_{\beta\gamma)} - \frac{4C_{\mu\beta\gamma}(\psi)}{3\sqrt{3}m} + \tilde{A}_{\mu\beta\gamma} \tag{57}$$

with $\tilde{\Lambda}_{\beta\gamma}$ symmetric and traceless. So we have obtained the local dual map for the totally symmetric field $\phi_{\mu\beta\gamma}$.

5 Final remarks and conclusion

In our recent work [6] we have obtained the fourth-order self-dual model (53) via NGE. In [6] we have faced the problem of not being able to find any new gauge symmetry in the fourth-order term of (53), which would take us to a fifth-order self-dual model, and perhaps to a sixth-order self-dual model. The authors of [10] have proposed their equations of motion but not their complete actions. One could also think that, since for the spin-1 case we have two self-dual models, and for the spin-2 four self-dual models, maybe there is a rule of the type $2s$ where s is the spin.

As an alternative method in obtaining those complete models we have tried the master action approach. As we have seen, a fundamental point when dealing with master actions consists of introducing mixing terms without particle content.

Once we can prove that the fourth-order self-dual model has no particle content it would be possible to go beyond it, but unfortunately that is not the case, as we are going to analyze in the following. The fourth-order term is given by

$$\begin{aligned} & \frac{1}{2m^2} \int d^3x C(\psi)G[C(\psi)] \\ &= \frac{9}{2m^2} \int d^3x \psi^{\mu\nu\lambda} \square \theta_\mu^\alpha E_\nu^\beta E_\lambda^\gamma \psi_{\alpha\beta\gamma}. \end{aligned} \tag{58}$$

In order to verify the particle content of this term we start with a lower-order version of this theory with the help of an auxiliary totally symmetric field $h_{\mu\nu\lambda}$, which is introduced in the following way:

$$\begin{aligned} S[\psi, h] = & \frac{9}{m^2} \int d^3x \left[h_{\mu\nu\lambda} G^{\mu\nu\lambda}(\psi) \right. \\ & \left. - \frac{1}{2} (h_{\mu\nu\lambda} h^{\mu\nu\lambda} - 3\alpha h_\mu h^\mu) \right], \end{aligned} \tag{59}$$

where $h_\mu = \eta^{\nu\lambda} h_{\mu\nu\lambda}$. Notice that, by Gaussian integrating over $h_{\mu\nu\lambda}$ in (59), we have a fourth-order term. In order to reproduce (58) we set $\alpha = 7/8$. So, if and only if we have this value for α we have a second-order version of (58). This reminds us of the spin-2 case (new massive gravity [23]) where the fourth-order K -term ($R_{\mu\nu}R^{\mu\nu} - 3R^2/8$) can be obtained via a gaussian integral over a symmetric auxiliary field $f_{\mu\nu}$ coupled to the spin-2 Einstein tensor:

$$\begin{aligned} S[g_{\mu\nu}, f_{\mu\nu}] = & \int d^3x \sqrt{-g} \\ & \times \left[f_{\mu\nu} G^{\mu\nu} - \frac{m^2}{2} (f_{\mu\nu} f^{\mu\nu} - f^2) \right]. \end{aligned} \tag{60}$$

There is subtle difference now, the f^2 term is of the Fierz–Pauli type while the h^2 term of (59) does not fit in the usual spin-3 mass term ($\alpha = 1$). Instead of integrating over $h_{\mu\nu\lambda}$, if we take the equations of motion for $\psi_{\mu\nu\lambda}$ in (58), we have $G_{\mu\nu\lambda}(h) = 0$. We already have seen that this immediately implies that the Ricci tensor vanishes, which in its turn implies that the Riemann tensor vanishes. Thus, we have the general solution $h_{\mu\nu\lambda} = \partial_{(\mu} \tilde{\Lambda}_{\nu\lambda)}$ with $\eta^{\nu\lambda} \tilde{\Lambda}_{\nu\lambda} = 0$. Substituting back this result in the non-derivative term of (59), we have the rank-2 traceless theory below:

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} (h_{\mu\nu\lambda} h^{\mu\nu\lambda} - 3\alpha h_\mu h^\mu) \\ = & \frac{3}{2} \left[\tilde{\Lambda}_{\mu\nu} \square \tilde{\Lambda}^{\mu\nu} + a (\partial^\mu \tilde{\Lambda}_{\mu\nu})^2 \right], \end{aligned} \tag{61}$$

where we have redefined the tensors in order to get rid of the overall factor $9/m^2$ and defined $a = 4\alpha - 2$. In the specific case we are interested in, i.e., $\alpha = 7/8$ we have $a = 3/2$. At this special point the theory (61) becomes invariant under the local traceless scalar symmetry:

$$\delta_\Phi \tilde{\Lambda}_{\mu\nu} = \partial_\mu \partial_\nu \Phi - \frac{\eta_{\mu\nu}}{3} \square \Phi. \tag{62}$$

The equations of motion of (61) are given by

$$\square \tilde{\Lambda}_{\mu\nu} = \frac{3}{4} (\partial_\mu A_\nu + \partial_\nu A_\mu) - \frac{1}{2} \eta_{\mu\nu} \partial_\rho A^\rho, \tag{63}$$

where we have defined the vector field

$$A^\mu \equiv \partial_\rho \tilde{\Lambda}^{\rho\mu}. \tag{64}$$

From the equations of motion (63) it is easy to deduce the Maxwell equations:

$$\square A^\mu - \partial^\mu (\partial \cdot A) = 0. \tag{65}$$

Due to the scalar symmetry (62) one may fix the Lorenz gauge $\partial_\mu \partial_\nu \tilde{\Lambda}^{\mu\nu} = \partial \cdot A = 0$. Apparently we have a massless spin-1 theory. However, although the Lorenz gauge still has residual gauge invariance under harmonic functions $\square \Phi = 0$ as in the Maxwell theory, such transformations do not shift the vector field since $\delta_\Phi A_\mu = 2 \square \partial_\mu \Phi / 3 = 0$. Consequently, we are left with $D - 1 = 2$ degrees of freedom instead of $D - 2 = 1$ as we expect for the 3D Maxwell theory. The extra degree of freedom is a ghost. As a double check one can verify that there is a double massless pole in the spin-1 sector of the propagator of the $\tilde{\Lambda}$ -theory.

This is another similarity with the spin-2 case where, however, the fourth-order, K term [23], is fully equivalent to the Maxwell theory; see [24]. So, in the spin-2 case we have a physical massless spin-1 particle instead of a ghost. The difference comes from the non-derivative nature of the Weyl symmetry [25], which induces a $U(1)$ change in the vector field $\delta_w h_{\mu\nu} = \eta_{\mu\nu} \phi \rightarrow \delta_w A_\mu = \partial_\mu \phi$ even for a harmonic function ϕ . This allows us to get rid of the “would be” ghost field present in the vector field.

Anyway, in both spin-2 and spin-3 cases the nontrivial particle content of the fourth-order term invalidates its use as a mixing term in the master action approach which avoids the transition to a possible fifth-order dual theory. So the dualization procedure stops at the fourth-order in both cases.

In the case of the usual spin-3 mass term $\alpha = 1$, ($a = 2$), the traceless model (61) becomes exactly the W-TDIFF model in $D = 3$, see [21], which has no particle content in $D = 3$. This allows us [22] to look for a spin-3 analog of spin-2 NMG of sixth-order in the derivatives.

As a final comment, we notice that in [6] we have deduced the higher-order spin-3 self-dual models from the first-order one of [7] via gauge embedding without any proof of a spectrum equivalence, which is now clear in the master action approach used here.

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