

A systematic study of finite BRST-BV transformations within W - X formulation of the standard and the $Sp(2)$ -extended field-antifield formalism

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Abstract Finite BRST-BV transformations are studied systematically within the W - X formulation of the standard and the $Sp(2)$ -extended field-antifield formalism. The finite BRST-BV transformations are introduced by formulating a new version of the Lie equations. The corresponding finite change of the gauge-fixing master action X and the corresponding Ward identity are derived.

1 Introduction

In recent papers [1–6], finite BRST transformations have been studied systematically both in the Hamiltonian and Lagrangian formalism in their standard and $Sp(2)$ -extended versions [7–18]. The so-called W - X formulation [19–27] is known as the most symmetric form of the Lagrangian field-antifield formalism. Dynamical gauge-generating master action W serves as a deformation to the original action of the theory. On the other hand, the gauge-fixing master action X serves just as to eliminate the antifield variables. It is remarkable that these complementary master actions W and X do satisfy a set of quantum master equations transposed to each other.

In the present paper we study systematically finite BRST-BV transformations within the W - X formulation both in the standard and $Sp(2)$ -extended field-antifield formalism. We introduce these transformations by formulating the respective Lie equations. Among other things, we derive in this way the effective change in the gauge-fixing master action X , as induced by the finite BRST-BV transformation defined.

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2 W - X formulation to the standard field-antifield formalism

Let z^A be the complete set of the variables necessary within the standard field-antifield formalism

$$z^A = \{\Phi^\alpha; \Phi_\alpha^*\}, \quad (2.1)$$

whose Grassmann parities are

$$\varepsilon(z^A) = \{\varepsilon_\alpha; \varepsilon_\alpha + 1\}. \quad (2.2)$$

We denote the respective z^A -derivatives as

$$\partial_A = \{\partial_\alpha; \partial_\alpha^*\}. \quad (2.3)$$

Let Z be the partition function

$$Z = \int Dz D\lambda \exp \left\{ \frac{i}{\hbar} [W + X] \right\}, \quad (2.4)$$

where λ^α are Lagrange multipliers for gauge-fixing with the Grassmann parity

$$\varepsilon(\lambda^\alpha) = \varepsilon_\alpha + 1. \quad (2.5)$$

In the partition function (2.4), the dynamical gauge-generating master action W and the gauge-fixing master action X are defined so as to satisfy the respective quantum master equations,

$$\left(\Delta \exp \left\{ \frac{i}{\hbar} W \right\} \right) = 0 \Leftrightarrow \frac{1}{2}(W, W) = i\hbar(\Delta W), \quad (2.6)$$

$$\left(\Delta \exp \left\{ \frac{i}{\hbar} X \right\} \right) = 0 \Leftrightarrow \frac{1}{2}(X, X) = i\hbar(\Delta X). \quad (2.7)$$

In the above quantum master equations (2.6) and (2.7), Δ and $(,)$ are the standard nilpotent odd Laplacian

$$\Delta = \partial_\alpha \partial_\alpha^* (-1)^{\varepsilon_\alpha}, \quad (2.8)$$

and the standard antibracket

$$(f, g) = (-1)^{\varepsilon_f} [[\Delta, f], g] = f \overleftarrow{\partial}_\alpha \overrightarrow{\partial}_*^\alpha g - (f \leftrightarrow g)(-1)^{(\varepsilon_f+1)(\varepsilon_g+1)}, \tag{2.9}$$

respectively. Equations (2.8) and (2.9) tell us that the anti-canonical pairs $(\Phi^\alpha; \Phi_\alpha^*)$ serve as Darboux coordinates on the flat field–antifield phase space with measure density $\rho = 1$ and no odd scalar curvature $\nu_\rho = 0$.

At $\hbar = 0$, $\Phi_\alpha^* = 0$, the W -action coincides with the original action of the theory. As to the X -action, it can be chosen in the form related to the gauge-fixing fermion $\Psi(\Phi)$,

$$X = (\Phi_\alpha^* - \Psi(\Phi) \overleftarrow{\partial}_\alpha) \lambda^\alpha = \Phi_\alpha^* \lambda^\alpha - \Psi(\Phi) \overleftarrow{d}, \tag{2.10}$$

where

$$\overleftarrow{d} = \overleftarrow{\partial}_\alpha \lambda^\alpha \tag{2.11}$$

is a nilpotent fermionic differential that acts from the right.

In the integrand of the path integral (2.4), consider now the following infinitesimal BRST-BV transformation:

$$\begin{aligned} \delta z^A &= -\mu(Y, z^A) - \frac{\hbar}{i}(\mu, z^A) \\ &= -\frac{\hbar}{i}y^{-1}(y\mu, z^A), \end{aligned} \tag{2.12}$$

where we have defined for later convenience

$$Y := X - W, \quad y := \exp \left\{ \frac{i}{\hbar} Y \right\}, \tag{2.13}$$

and where $\mu(z)$ is an infinitesimal fermionic function with $\varepsilon(\mu) = 1$.

The Jacobian of the infinitesimal BRST-BV transformation (2.12) has the form

$$\ln J = (-1)^{\varepsilon_A} (\partial_A \delta z^A) = (Y, \mu) + 2(\Delta Y)\mu + 2\frac{\hbar}{i}(\Delta\mu). \tag{2.14}$$

The complete action in the partition function (2.4) transforms as

$$\delta[W + X] = [-(W, W) + (X, X)]\mu + \frac{\hbar}{i}(W + X, \mu). \tag{2.15}$$

Due to the quantum master equations (2.6) and (2.7), we then have from Eqs. (2.14) and (2.15) that

$$\frac{i}{\hbar} \delta[W + X] + \ln J = 2(\sigma(X)\mu), \tag{2.16}$$

where $\sigma(X)$ is a quantum BRST generator

$$(\sigma(X)f) = (X, f) + \frac{\hbar}{i}(\Delta f). \tag{2.17}$$

Equation (2.16) tells us that the BRST transformation (2.12) induces the following variation:

$$\delta X = 2\frac{\hbar}{i}(\sigma(X)\mu) \tag{2.18}$$

to the X -action in the integrand of the path integral (2.4). We conclude that the partition function (2.4) and the quantum master equation (2.7) for X are both stable under the infinitesimal variation (2.18).

Next let t be a bosonic parameter. It is natural to define a one-parameter subgroup $t \mapsto \bar{z}^A(t)$ of finite BRST-BV transformations by the Lie equation¹

$$\frac{d\bar{z}^A}{dt} = \overline{\mathcal{H}}^A, \quad \bar{z}^A|_{t=0} = z^A. \tag{2.19}$$

where

$$\begin{aligned} \mathcal{H} &= \mathcal{H}^A \partial_A = -\mu \operatorname{ad}(Y) - \frac{\hbar}{i} \operatorname{ad}(\mu) \\ &= -\frac{\hbar}{i}y^{-1} \operatorname{ad}(y\mu), \quad \operatorname{ad}(f)g := (f, g), \end{aligned} \tag{2.20}$$

is the corresponding vector field with components

$$\mathcal{H}^A := -\mu(Y, z^A) - \frac{\hbar}{i}(\mu, z^A) = -\frac{\hbar}{i}y^{-1}(y\mu, z^A). \tag{2.21}$$

Note that $\mu(z)$ is now an arbitrary finite fermionic function. In other words, the Lie equation (2.19) is

$$\frac{d\bar{z}^A}{dt} = (\overline{\mathcal{H}}\bar{z}^A), \quad \overline{\mathcal{H}} := \overline{\mathcal{H}}^A \underline{\partial}_A \tag{2.22}$$

with solution

$$\bar{z}^A = \left(e^{t\mathcal{H}} z^A \right). \tag{2.23}$$

Recall that the antibracket for any fermion $F = y\mu$ with itself is zero: $(F, F) = 0$. This fact yields a conservation law

$$\begin{aligned} \frac{d(y\mu)}{dt} &= \frac{d\bar{z}^A}{dt} \underline{\partial}_A (\overline{y\mu}) = -\frac{\hbar}{i}y^{-1} \overline{(y\mu, z^A)} \underline{\partial}_A (\overline{y\mu}) \\ &= -\frac{\hbar}{i}y^{-1} \overline{(y\mu, y\mu)} = 0, \end{aligned} \tag{2.24}$$

so that the following invariance property holds:

$$\overline{y\mu} = y\mu. \tag{2.25}$$

The Jacobian of these transformations satisfies the following equation:

$$\begin{aligned} \frac{d \ln J}{dt} &= \overline{\operatorname{div} \mathcal{H}}, \quad \operatorname{div} \mathcal{H} := (-1)^{\varepsilon_A} \partial_A \mathcal{H}^A \\ &= (Y, \mu) + 2(\Delta Y)\mu + 2\frac{\hbar}{i}(\Delta\mu). \end{aligned} \tag{2.26}$$

¹ For an arbitrary function $f = f(z)$, we use the shorthand notation $\overline{f} = f(\bar{z}) = (e^{t\mathcal{H}} f)$ for the corresponding function with shifted arguments.

The transformed complete action satisfies the equation

$$\begin{aligned} \frac{d}{dt} [\overline{W} + \overline{X}] &= \frac{d\overline{z}^A}{dt} \partial_A [\overline{W} + \overline{X}] \\ &= \left[-\overline{\mu}(Y, z^A) - \frac{\hbar}{i} \overline{(\mu, z^A)} \right] \partial_A [\overline{W} + \overline{X}] \\ &= \left[-\overline{(W, W)} + \overline{(X, X)} \right] \overline{\mu} + \frac{\hbar}{i} \overline{(W + X, \mu)}. \end{aligned} \tag{2.27}$$

Due to the transformed master equations (2.6) and (2.7), it follows that

$$\frac{d}{dt} \left[\overline{W} + \overline{X} + \frac{\hbar}{i} \ln J \right] = \frac{\hbar}{i} \overline{a}, \tag{2.28}$$

where we have defined for later convenience

$$a := 2(\sigma(X)\mu). \tag{2.29}$$

By integrating Eq. (2.28) within $0 \leq t \leq 1$, we get

$$\overline{W} + \overline{X} + \frac{\hbar}{i} \ln J = W + X + \frac{\hbar}{i} A, \tag{2.30}$$

where we have defined the average

$$A := \int_0^1 dt \overline{a} = \int_0^1 dt (e^{t\mathcal{H}} a) = (E(\mathcal{H})a). \tag{2.31}$$

Here E is the function

$$E(x) = \int_0^1 dt e^{tx} = \frac{\exp\{x\} - 1}{x}. \tag{2.32}$$

Equation (2.30) shows the finite effective change in X induced by the finite transformation $z^i \rightarrow \overline{z}^i$ in the partition function (2.4). Now consider the left-hand side $\overline{\mathcal{Y}}$ of the transformed quantum master equation (2.7), where

$$\mathcal{Y} := \frac{1}{2}(X, X) + \frac{\hbar}{i}(\Delta X). \tag{2.33}$$

We have the following Cauchy initial value problem:

$$\frac{d\overline{\mathcal{Y}}}{dt} = (\overline{\mathcal{H}} \overline{\mathcal{Y}}) \quad \wedge \quad \overline{\mathcal{Y}}|_{t=0} = 0 \quad \Rightarrow \quad \overline{\mathcal{Y}} \equiv 0 \tag{2.34}$$

for arbitrary t .

Thereby, we have confirmed that the quantum master equation (2.7) is stable under the finite BRST-BV transformation generated by Eq. (2.19). Of course, the general expression (2.4) itself is stable under the same transformation, as well.

At this point we would like to investigate the quantum master equation

$$\left(\Delta \exp \left\{ \frac{i}{\hbar} X' \right\} \right) = 0, \tag{2.35}$$

where we have denoted the new gauge-fixing master action,

$$X' = X + \frac{\hbar}{i} A. \tag{2.36}$$

Equation (2.35) is equivalent to

$$(\sigma(X) \exp\{A\}) = 0 \quad \Leftrightarrow \quad \frac{\hbar}{2i}(A, A) + (\sigma(X)A) = 0. \tag{2.37}$$

The exponential $\exp\{A\}$ rewrites in the form

$$\begin{aligned} \exp\{A\} &= e^{(E(\mathcal{H})a)} = (\exp\{\mathcal{H} + a\}1) \\ &= (\exp\{H + 2[\sigma(X), \mu]\}1), \end{aligned} \tag{2.38}$$

where we have defined the first-order operator

$$\begin{aligned} H &:= \frac{\hbar}{i} y \operatorname{ad}(y^{-1}\mu) = -\mu \operatorname{ad}(Y) + \frac{\hbar}{i} \operatorname{ad}(\mu) \\ &= \mathcal{H} + 2\frac{\hbar}{i} \operatorname{ad}(\mu), \end{aligned} \tag{2.39}$$

and we used the formula

$$[\sigma(X), f] = (\sigma(X)f) + (-1)^{\varepsilon_f} \frac{\hbar}{i} \operatorname{ad}(f) \tag{2.40}$$

for a function f . Hence Eqs. (2.35)/(2.37) are equivalent to

$$[\sigma(X), \exp\{H + 2[\sigma(X), \mu]\}]1 = 0. \tag{2.41}$$

In general, it looks as if Eqs. (2.35)/(2.37)/(2.41) serve as a condition for finite field-dependent parameter $\mu(z)$. This equation is certainly satisfied with arbitrary infinitesimal $\mu(z) \rightarrow 0$, to the first order in that. We do not know if the same situation holds for arbitrary finite $\mu(z)$, as the Eqs. (2.35)/(2.37)/(2.41) are rather complicated in the general case.

Also, there is a potential obstacle that the dynamical master action W actually enters that equation. Thus, the finite parameter $\mu(z)$ being restricted in its field-dependence, that circumstance would be a crucial specific feature of the W - X formulation.

One can proceed from a solution A to the quantum master equation (2.37). If we ignore Eqs. (2.29) and (2.31), then the quantum master equation (2.37) knows nothing about the parameter μ . Moreover, A serves as an external source in the left-hand side of Eq. (2.31). The right-hand side of Eq. (2.31) knows about the parameter μ via its explicit appearance in Eqs. (2.20) and (2.31). Thereby, the aspects related to the quantum master equation (2.37) by itself, and to the parameter μ , are separated naturally. From this point of view, it sounds not so plausible that Eq. (2.31) could allow for a finite arbitrary parameter $\mu(z)$. If one rescales the parameters in Eqs. (2.37) and (2.31),

$$\mu \rightarrow \varepsilon \mu, \tag{2.42}$$

with ε being a boson parameter, and then expands μ and A in formal power series,

$$\mu = \mu_0 + \varepsilon \mu_1 + \dots, \tag{2.43}$$

$$A = A_0 + \varepsilon A_1 + \dots, \tag{2.44}$$

one gets to the first order in ε

$$0 = A_0, \tag{2.45}$$

$$2(\sigma(X)\mu_0) = A_1, \tag{2.46}$$

$$(\sigma(X)A_1) = 0, \tag{2.47}$$

so that μ_0 remains arbitrary to that order. However, to the second order in ε , one has

$$2(\sigma(X)\mu_1) + (\mathcal{H}(\mu_0)\sigma(X)\mu_0) = A_2, \tag{2.48}$$

$$\frac{\hbar}{2i}(A_1, A_1) + (\sigma(X)A_2) = 0, \tag{2.49}$$

so that μ_1 remains arbitrary to that order, while (2.49) restricts μ_0 ,

$$(\sigma(X)H(\mu_0)\sigma(X)\mu_0) = 0, \tag{2.50}$$

with $H(\mu_0)$ being the operator (2.39) as taken at $\mu = \mu_0$. To the third order in ε , μ_2 remains arbitrary, while μ_1 is restricted to satisfy the condition

$$\begin{aligned} &(\sigma(X)H(\mu_0)\sigma(X)\mu_1) + (\sigma(X)H(\mu_1)\sigma(X)\mu_0) \\ &+ \left(\frac{1}{3}\sigma(X)(H(\mu_0) + 2[\sigma(X), \mu_0])^2\sigma(X)\mu_0\right) = 0. \end{aligned} \tag{2.51}$$

The same situation holds to higher orders in ε : to each subsequent order, the respective coefficient in μ remains arbitrary, while the preceding coefficient in μ becomes restricted. Of course, it looks rather difficult to estimate on being such a strange procedure “convergent” to infinite order in ε .

It may look a bit strange that the operator H from Eq. (2.39) appears in Eqs. (2.50) and (2.51) while \mathcal{H} from Eq. (2.20) enters Eq. (2.31). In fact, one could, in principle, proceed directly from Eq. (2.41) formulated via the operator H from the very beginning. Then one could use Eq. (2.41), together with the properties

$$(\sigma(X)1) = 0, \quad (H1) = 0, \tag{2.52}$$

to derive Eqs. (2.50) and (2.51). Also, notice that there is an implication,

$$(\sigma(X))^2 = 0 \Rightarrow (\sigma(X) \mathcal{O} \sigma(X))^2 = 0, \tag{2.53}$$

with \mathcal{O} being any operator.

Finally we present a simple general argument, based on the integration by parts, that the partition function (2.4) is independent of finite arbitrariness in a solution to the gauge-fixing master action X ,

$$\exp\left\{\frac{i}{\hbar}X'\right\} = \left(\exp\{[\Delta, \mu]\} \exp\left\{\frac{i}{\hbar}X\right\}\right) \tag{2.54}$$

$$= \exp\left\{\frac{i}{\hbar}X\right\} + \left(\Delta \mu E([\Delta, \mu]) \exp\left\{\frac{i}{\hbar}X\right\}\right), \tag{2.55}$$

where μ is any finite fermionic operator and the function $E(x)$ is defined in Eq. (2.32). By substituting Eq. (2.55) into

Eq. (2.4) with X' standing for X , and then integrating by parts with Eq. (2.6) taken into account, one observes that the second term in the right-hand side in Eq. (2.55) does not contribute to the integral (2.4). Thereby, the integral (2.4) with X' standing for X reduces to the case of the initial X standing for itself. Thus, the partition function is independent of a particular representative of the class (2.54).

3 Ward identities in the standard W - X formulation

Let J_A be external sources to the variables z^A ; then the integral (2.4) generalizes to the generating functional,

$$Z[J] = \int Dz D\lambda \exp\left\{\frac{i}{\hbar}[W + X + J_A z^A]\right\}. \tag{3.1}$$

Arbitrary variation δz^A yields the equations of motion,

$$\langle \partial_B(W + X) \rangle_J + J_B(-1)^{\varepsilon_B} = 0, \tag{3.2}$$

where $\langle \dots \rangle_J$ is the source-dependent mean value

$$\langle \dots \rangle_J = \frac{1}{Z[J]} \int Dz D\lambda(\dots) \exp\left\{\frac{i}{\hbar}[W + X + J_A z^A]\right\}. \tag{3.3}$$

It follows from Eq. (3.2) that

$$\langle J_A \omega^{AB} \partial_B(W + X) \rangle_J + J_A \omega^{AB} J_B(-1)^{\varepsilon_B} = 0, \tag{3.4}$$

where

$$\omega^{AB} = (z^A, z^B) = \text{const} \tag{3.5}$$

is the fundamental invertible antibracket. In Eq. (3.1), the BRST-BV variation (2.12) yields

$$\langle J_A \omega^{AB} \partial_B Y \rangle_J = 0 \tag{3.6}$$

due to Eq. (2.16) for $\mu = \text{const}$.

It follows then from Eqs. (3.4) and (3.6) that

$$\langle J_A \omega^{AB} \partial_B W \rangle_J = -\frac{1}{2} J_A \omega^{AB} J_B(-1)^{\varepsilon_B}. \tag{3.7}$$

Thus we have eliminated the average (2.31) of the gauge-fixing master action X from the new Ward identity (3.7). The price is that we have got the non-homogeneity quadratic in the external sources J in the right-hand side in Eq. (3.7).

Finally, at the level of finite BRST-BV transformations, Eq. (2.30) yields

$$\left\langle \exp\left\{\frac{i}{\hbar}J_A(\bar{z}^A - z^A) + A\right\}\right\rangle_J = 1. \tag{3.8}$$

However, it is impossible to eliminate the average (2.31) of the gauge-fixing master action X from (3.8).

4 W - X formulation to the $Sp(2)$ -symmetric field-antifield formalism

Let z^A be the complete set of the variables necessary to the W - X formulation of the $Sp(2)$ -symmetric field-antifield formalism [15, 17, 18]

$$z^A = \{\Phi^\alpha, \pi^{\alpha a}; \Phi_{\alpha a}^*, \Phi_\alpha^{**}\} \tag{4.1}$$

whose Grassmann parities are

$$\varepsilon(z^A) = \{\varepsilon_\alpha, \varepsilon_\alpha + 1; \varepsilon_\alpha + 1, \varepsilon_\alpha\}. \tag{4.2}$$

We denote the respective z^A derivatives as

$$\partial_A = \{\partial_\alpha, \partial_{\alpha a}; \partial_{*}^{\alpha a}, \partial_{**}^\alpha\}. \tag{4.3}$$

Let Z be the partition function:

$$Z = \int Dz D\lambda \exp \left\{ \frac{i}{\hbar} [W + X] \right\}, \tag{4.4}$$

where λ^α are Lagrange multipliers for gauge-fixing with Grassmann parities,

$$\varepsilon(\lambda^\alpha) = \varepsilon_\alpha. \tag{4.5}$$

In the partition function (4.4), the dynamical gauge-generating master action W and the gauge-fixing master action X is defined to satisfy the respective quantum master equation

$$\begin{aligned} \left(\Delta_+^a \exp \left\{ \frac{i}{\hbar} W \right\} \right) &= 0 \\ \Leftrightarrow \frac{1}{2} (W, W)^a + (V^a W) &= i\hbar (\Delta^a W), \end{aligned} \tag{4.6}$$

$$\begin{aligned} \left(\Delta_-^a \exp \left\{ \frac{i}{\hbar} X \right\} \right) &= 0 \\ \Leftrightarrow \frac{1}{2} (X, X)^a - (V^a X) &= i\hbar (\Delta^a X). \end{aligned} \tag{4.7}$$

In the above quantum master equations (4.6) and (4.7), the $\Delta^a, (,)^a, V^a$, and Δ_\pm^a are the $Sp(2)$ -vector-valued odd Laplacian

$$\Delta^a = \partial_\alpha \partial_{*}^{\alpha a} (-1)^{\varepsilon_\alpha} + \varepsilon^{ab} \partial_{\alpha b} \partial_{**}^\alpha (-1)^{\varepsilon_\alpha + 1}, \tag{4.8}$$

the antibracket

$$\begin{aligned} (f, g)^a &= (-1)^{\varepsilon_f} [[\Delta^a, f], g] \\ &= f \left[\overleftarrow{\partial}_\alpha \overrightarrow{\partial}_*^{\alpha a} + \varepsilon^{ab} \overleftarrow{\partial}_{\alpha b} \overrightarrow{\partial}_{**}^\alpha \right] g \\ &\quad - (f \leftrightarrow g) (-1)^{(\varepsilon_f + 1)(\varepsilon_g + 1)}, \end{aligned} \tag{4.9}$$

and the special vector field

$$V^a = V^{Aa} \partial_A = \varepsilon^{ab} \Phi_{\alpha b}^* \partial_{**}^\alpha, \quad (V^a z^A) = V^{Aa}, \tag{4.10}$$

and

$$\Delta_\pm^a := \Delta^a \pm \frac{i}{\hbar} V^a, \tag{4.11}$$

respectively.

For the W -action, one should require that W is independent of $\pi^{\alpha a}$,

$$(\Phi_\alpha^{**}, W) = 0. \tag{4.12}$$

As to the X -action, it can be chosen in the form related to the gauge-fixing boson $F(\Phi)$,

$$\begin{aligned} X &= \Phi_{\alpha a}^* \pi^{\alpha a} + (\Phi_\alpha^{**} - F \overleftarrow{\partial}_\alpha) \lambda^\alpha \\ &\quad + \frac{1}{2} F \overleftarrow{\partial}_\alpha \pi^{\alpha a} \overleftarrow{\partial}_\beta \pi^{\beta b} \varepsilon_{ba} \\ &= \Phi_{\alpha a}^* \pi^{\alpha a} + \Phi_\alpha^{**} \lambda^\alpha + \frac{1}{2} F \overleftarrow{d}^a \overleftarrow{d}^b \varepsilon_{ba}, \end{aligned} \tag{4.13}$$

where

$$\overleftarrow{d}^a = \overleftarrow{\partial}_\alpha \pi^{\alpha a} - \overleftarrow{\partial}_{ab} \lambda^\alpha \varepsilon^{ba} \tag{4.14}$$

is a $Sp(2)$ -vector-valued fermionic differential that acts from the right.

In the integrand of the path integral (4.4), consider now the following infinitesimal BRST transformation:

$$\begin{aligned} \delta z^A &= -\mu_a (Y, z^A)^a - \frac{\hbar}{i} (\mu_a, z^A)^a + 2\mu_a V^{Aa} \\ &= -\frac{\hbar}{i} y^{-1} (y \mu_a, z^A)^a + 2\mu_a V^{Aa}, \end{aligned} \tag{4.15}$$

where we have defined for later convenience

$$Y := X - W, \quad y := \exp \left\{ \frac{i}{\hbar} Y \right\}, \tag{4.16}$$

and where $\mu_a = \mu_a(z)$ is an infinitesimal $Sp(2)$ co-vector-valued fermionic function. Its Jacobian has the form

$$\begin{aligned} \ln J &= (-1)^{\varepsilon_A} (\partial_A \delta z^A) = (Y, \mu_a)^a + 2(\Delta^a Y) \mu_a \\ &\quad + 2\frac{\hbar}{i} (\Delta_-^a \mu_a). \end{aligned} \tag{4.17}$$

The complete action in the partition function (4.4) transforms as

$$\begin{aligned} \delta[W + X] &= [-(W, W)^a + (X, X)^a] \mu_a \\ &\quad + \frac{\hbar}{i} (W + X, \mu_a)^a - 2[V^a (W + X)] \mu_a \\ &= 2i\hbar (\Delta^a Y) \mu_a + \frac{\hbar}{i} (W + X, \mu_a)^a. \end{aligned} \tag{4.18}$$

It follows from Eqs. (4.17) and (4.18) that

$$\frac{i}{\hbar} \delta[W + X] + \ln J = 2(\sigma_-^a(X) \mu_a), \tag{4.19}$$

where

$$(\sigma_-^a(X) f) = (X, f)^a + \frac{\hbar}{i} (\Delta_-^a f) \tag{4.20}$$

is the $Sp(2)$ -vector-valued quantum BRST generator.

Equation (4.19) tells us that the BRST transformation (4.15) induces the following variation:

$$\delta X = 2\frac{\hbar}{i} (\sigma_-^a(X) \mu_a). \tag{4.21}$$

to the X -action in the integrand of the path integral (4.4).

We conclude that the partition function (4.4) and the quantum master equation (4.7) for X are both stable under the infinitesimal variation (4.21).

Next let t be a bosonic parameter. It is natural to define a one-parameter subgroup $t \mapsto \bar{z}^A(t)$ of finite BRST transformations by the Lie equation

$$\frac{d\bar{z}^A}{dt} = \overline{\mathcal{H}}^A, \quad \bar{z}^A|_{t=0} = z^A, \tag{4.22}$$

where

$$\begin{aligned} \mathcal{H} &= \mathcal{H}^A \partial_A = -\mu_a \text{ad}^a(Y) - \frac{\hbar}{i} \text{ad}^a(\mu_a) + 2\mu_a V^a \\ &= -\frac{\hbar}{i} y^{-1} \text{ad}^a(y\mu_a) + 2\mu_a V^a \end{aligned} \tag{4.23}$$

is the corresponding vector field with components

$$\begin{aligned} \mathcal{H}^A &:= -\mu_a (Y, z^A)^a - \frac{\hbar}{i} (\mu_a, z^A)^a + 2\mu_a V^{Aa} \\ &= -\frac{\hbar}{i} y^{-1} (y\mu_a, z^A)^a + 2\mu_a V^{Aa}. \end{aligned} \tag{4.24}$$

This equation implies the $Sp(2)$ -vector-valued counterpart to the Eq. (2.24),

$$\begin{aligned} \left(\frac{d}{dt} - 2\overline{\mu}_a \overline{V}^a \right) (\overline{y\mu}_b) &= \left(\frac{d\bar{z}^A}{dt} - 2\overline{\mu}_a \overline{V}^{Aa} \right) \partial_A (\overline{y\mu}_b) \\ &= -\frac{\hbar}{i} \overline{y}^{-1} \overline{(y\mu_a, z^A)^a} \partial_A (\overline{y\mu}_b) = -\frac{\hbar}{i} \overline{y}^{-1} \overline{(y\mu_a, y\mu_b)^a}, \end{aligned} \tag{4.25}$$

which cannot be completely integrated explicitly to yield a counterpart to the conservation law (2.25).

The Jacobian of the transformation (4.22) satisfies the equation

$$\begin{aligned} \frac{d \ln J}{dt} &= \overline{\text{div}} \mathcal{H}, \quad \text{div} \mathcal{H} := (-1)^{\varepsilon_A} \partial_A \mathcal{H}^A \\ &= (Y, \mu_a)^a + 2(\Delta Y^a) \mu_a + 2\frac{\hbar}{i} (\Delta_-^a \mu_a). \end{aligned} \tag{4.26}$$

The complete action in Eq. (4.4) satisfies the equation

$$\begin{aligned} \frac{d}{dt} [\overline{W} + \overline{X}] &= \frac{d\bar{z}^A}{dt} \partial_A [\overline{W} + \overline{X}] \\ &= \left[-\overline{\mu}_a \overline{(Y, z^A)^a} - \frac{\hbar}{i} \overline{(\mu_a, z^A)^a} + 2\overline{\mu}_a \overline{V}^{Aa} \right] \partial_A [\overline{W} + \overline{X}] \\ &= \left[-\overline{(W, W)^a} + \overline{(X, X)^a} \right] \overline{\mu}_a + \frac{\hbar}{i} \overline{(W + X, \mu_a)^a} \\ &\quad - 2[\overline{V}^a (\overline{W} + \overline{X})] \overline{\mu}_a. \end{aligned} \tag{4.27}$$

It follows from Eqs. (4.26) and (4.27) that

$$\frac{d}{dt} \left[\frac{i}{\hbar} (\overline{W} + \overline{X}) + \ln J \right] = \overline{a}, \tag{4.28}$$

where we have defined for later convenience

$$a := 2(\sigma_-^a(X)\mu_a). \tag{4.29}$$

By integrating within $0 \leq t \leq 1$, we get from Eq. (4.28)

$$\overline{W} + \overline{X} + \frac{\hbar}{i} \ln J = W + X + \frac{\hbar}{i} A, \tag{4.30}$$

where we have defined the average

$$A := \int_0^1 dt \bar{a} = \int_0^1 dt (e^{t\mathcal{H}} a) = E(\mathcal{H})a. \tag{4.31}$$

Equation (4.30) shows the finite effective change in X induced by the finite transformation $z^A \rightarrow \bar{z}^A$ in Eq. (4.4).

Now consider the left-hand side $\overline{\mathcal{Y}}^a$ of the transformed quantum master equation (4.7), where

$$\mathcal{Y}^a := \frac{1}{2}(X, X)^a + \frac{\hbar}{i} (\Delta_-^a X). \tag{4.32}$$

We have the following Cauchy initial value problem:

$$\frac{d\overline{\mathcal{Y}}^a}{dt} = (\overline{\mathcal{H}} \overline{\mathcal{Y}}^a) \wedge \overline{\mathcal{Y}}^a|_{t=0} = 0 \Rightarrow \overline{\mathcal{Y}}^a \equiv 0 \tag{4.33}$$

for arbitrary t .

Thereby, we have confirmed that the quantum master equation (4.7) is stable under the finite BRST-BV transformation generated by Eq. (4.22). Of course, the general expression (4.4) itself is stable under the same transformations, as well.

The $Sp(2)$ -extended quantum master equation

$$\left(\Delta_-^a \exp \left\{ \frac{i}{\hbar} X' \right\} \right) = 0 \tag{4.34}$$

for the new gauge-fixing master action,

$$X' = X + \frac{\hbar}{i} A, \tag{4.35}$$

must be interpreted similarly to what we have explained in Sect. 2 with respect to Eq. (2.35).

For instance, the $Sp(2)$ -vector-valued counterpart to Eq. (2.37) reads

$$(\sigma_-^a(X) \exp\{A\}) = 0 \Leftrightarrow \frac{\hbar}{2i} (A, A)^a + (\sigma_-^a(X)A) = 0. \tag{4.36}$$

Finally, the respective $Sp(2)$ symmetric counterpart to the Eqs. (2.54) and (2.55) reads

$$\exp \left\{ \frac{i}{\hbar} X' \right\} = \left(\exp \left\{ \frac{1}{2} \varepsilon_{ab} [\Delta_-^b, [\Delta_-^a, \nu]] \right\} \exp \left\{ \frac{i}{\hbar} X \right\} \right) \tag{4.37}$$

$$\begin{aligned} &= \exp \left\{ \frac{i}{\hbar} X \right\} + \left(\frac{1}{2} \varepsilon_{ab} \Delta_-^b \Delta_-^a \nu E \left(\frac{1}{2} \varepsilon_{cd} [\Delta_-^d, [\Delta_-^c, \nu]] \right) \right) \\ &\quad \times \exp \left\{ \frac{i}{\hbar} X \right\}, \end{aligned} \tag{4.38}$$

with ν being any finite bosonic operator.

5 Ward identities in the $Sp(2)$ -extended W - X formulation

Let J_A be external sources to the variables z^A ; then the integral (4.4) generalizes to the generating functional

$$Z[J] = \int Dz D\lambda \exp \left\{ \frac{i}{\hbar} [W + X + J_A z^A] \right\}. \tag{5.1}$$

An arbitrary variation δz^A yields the equations of motion, $\langle \partial_B (W + X) \rangle_J + J_B (-1)^{\varepsilon_B} = 0,$ (5.2)

where $\langle \dots \rangle_J$ is the source-dependent mean-value $\langle \dots \rangle_J = \frac{1}{Z[J]} \int Dz D\lambda (\dots) \exp \left\{ \frac{i}{\hbar} [W + X + J_A z^A] \right\}.$ (5.3)

It follows from Eq. (5.2) that $\langle J_A \omega^{ABa} \partial_B (W + X) \rangle_J + J_A \omega^{ABa} J_B (-1)^{\varepsilon_B} = 0,$ (5.4)

where $\omega^{ABa} = (z^A, z^B)^a = \text{const}$ (5.5)

is the fundamental $Sp(2)$ antibracket. In Eq. (5.1), the BRST-BV variation (4.15) yields

$$\left\langle J_A [\omega^{ABa} \partial_B Y - 2V^{Aa} (-1)^{\varepsilon_A}] \right\rangle_J = 0, \tag{5.6}$$

due to Eq. (4.19) for $\mu_a = \text{const}$. It follows then from Eqs. (5.4) and (5.6) that

$$\left\langle J_A [\omega^{ABa} \partial_B W + V^{Aa} (-1)^{\varepsilon_A}] \right\rangle_J = -\frac{1}{2} J_A \omega^{ABa} J_B (-1)^{\varepsilon_B}. \tag{5.7}$$

Thus we have eliminated the average (4.31) of the gauge-fixing master action X from the new Ward identity (5.7).

The price is that we have got the non-homogeneity quadratic in the external sources J in the right-hand side in Eq. (5.7).

Finally, at the level of finite BRST-BV transformations, the relation (4.30) yields

$$\left\langle \exp \left\{ \frac{i}{\hbar} J_A (\bar{z}^A - z^A) + A \right\} \right\rangle_J = 1. \tag{5.8}$$

However, it is impossible to eliminate the average (4.31) of the gauge-fixing master action X from Eq. (5.8).

6 Conclusions

Notice that, on one hand (and in contrast to the original $Sp(2)$ -formulation [15, 17, 18]), in the $Sp(2)$ -symmetric W - X formulation, the anticanonical dynamical activity of the variables $\{\pi^{\alpha a}, \Phi_{\alpha}^{**}\}$ [22], as represented by the second term in Eq. (4.8) and in the square bracket of Eq. (4.9), is of crucial

importance to satisfy the quantum master equation (4.7) with the ansatz (4.13) for X .

On the other hand, $\pi^{\alpha a}$ and Φ_{α}^{**} are kept as dynamically passive (antibracket-commuting) variables in the W -action. Thus, one may realize what the price is of the coexistence of the $Sp(2)$ -symmetry and the complementary W - X duality of the quantum master equations (4.6) and (4.7).

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Appendix A: Algebra of the σ -operators

In this appendix we present the general formal algebra of the σ -operators, both in the standard and the $Sp(2)$ case.

In the standard case we introduce the σ -operator

$$\begin{aligned} \sigma(F) &:= \frac{\hbar}{i} \exp \left\{ -\frac{i}{\hbar} F \right\} \Delta \exp \left\{ \frac{i}{\hbar} F \right\} \\ &= \frac{\hbar}{i} \Delta + \text{ad}(F) + \left(\Delta F + \frac{i}{2\hbar} (F, F) \right) \end{aligned} \tag{A.1}$$

for any bosonic functional F . It inherits the nilpotency

$$\Delta^2 = 0 \Rightarrow (\sigma(F))^2 = 0. \tag{A.2}$$

Then straightforward calculation gives the following results for the commutator of $\sigma(W)$ and $\sigma(X)$:

$$[\sigma(W), \sigma(X)] = \text{ad}(C), \tag{A.3}$$

where

$$\begin{aligned} C &:= \frac{\hbar}{i} (\Delta(W + X)) + (W, X) = -\frac{1}{2} (Y, Y) \\ &= \frac{4\hbar}{i} \left(\sigma \left(\frac{W + X}{2} \right) 1 \right), \end{aligned} \tag{A.4}$$

and where the quantum master equations for W and X are used.

In the $Sp(2)$ case the set of operators $\sigma^a(F)$, $\sigma_{\pm}^a(F)$ for any bosonic functional F is introduced

$$\begin{aligned} \sigma^a(F) &:= \frac{\hbar}{i} \exp\left\{-\frac{i}{\hbar}F\right\} \Delta^a \exp\left\{\frac{i}{\hbar}F\right\} \\ &= \frac{\hbar}{i} \Delta^a + \text{ad}^a(F) + \left(\Delta^a F + \frac{i}{2\hbar}(F, F)^a\right), \end{aligned} \tag{A.5}$$

$$\begin{aligned} \sigma_{\pm}^a(F) &:= \frac{\hbar}{i} \exp\left\{-\frac{i}{\hbar}F\right\} \Delta_{\pm}^a \exp\left\{\frac{i}{\hbar}F\right\} \\ &= \frac{\hbar}{i} \Delta_{\pm}^a + \text{ad}^a(F) + \left(\Delta_{\pm}^a F + \frac{i}{2\hbar}(F, F)^a\right). \end{aligned} \tag{A.6}$$

The $Sp(2)$ nilpotency reads

$$\begin{aligned} [\Delta^a, \Delta^b] &= 0, \quad [\Delta^a, V^b] = 0, \quad [V^a, V^b] = 0, \\ \Rightarrow [\sigma^a(F), \sigma^b(F)] &= 0, \quad [\sigma_{\pm}^a(F), \sigma_{\pm}^b(F)] = 0. \end{aligned} \tag{A.7}$$

Taking into account the quantum master equations for W and X from Eqs. (A.5) and (A.6) it follows that

$$[\sigma_+^a(W), \sigma_-^b(X)] = \text{ad}^a(C^b), \tag{A.8}$$

where

$$\begin{aligned} C^b &:= \frac{\hbar}{i}(\Delta_-^b W) + \frac{\hbar}{i}(\Delta_+^b X) + (W, X)^b \\ &= -\frac{1}{2}(Y, Y)^b + 2(V^b Y) \\ &= \frac{4\hbar}{i} \left(\sigma^b \left(\frac{W + X}{2} \right) 1 \right). \end{aligned} \tag{A.9}$$

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