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Perturbative quantization of Yang–Mills theory with classical double as gauge algebra

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Abstract Perturbative quantization of Yang–Mills theory with a gauge algebra given by the classical double of a semisimple Lie algebra is considered. The classical double of a real Lie algebra is a nonsemisimple real Lie algebra that admits a nonpositive definite invariant metric, the indefiniteness of the metric suggesting an apparent lack of unitarity. It is shown that the theory is UV divergent at one loop and that there are no radiative corrections at higher loops. One-loop UV divergences are removed through renormalization of the coupling constant, thus introducing a renormalization scale. The terms in the classical action that would spoil unitarity are proved to be cohomologically trivial with respect to the Slavnov–Taylor operator that controls gauge invariance for the quantum theory. Hence they do not contribute gauge invariant radiative corrections to the quantum effective action and the theory is unitary.

1 Introduction

Nonreductive metric Lie algebras are Lie algebras that (i) cannot be written as a direct product of semisimple and Abelian Lie algebras but (ii) admit a metric, where by a metric is meant a nondegenerate symmetric bilinear form that is invariant under the adjoint action. Here we will be concerned with perturbative quantization of Yang–Mills theory for a particular class of such algebras, known as classical doubles. These algebras describe the gauge symmetries in a variety of problems, including three-dimensional gravity [1,2], asymptotically flat solutions to the Einstein equations in three and four dimensions [3–5], string actions in doubled space [6], or $1/N_{color}$ expansions for baryons in QCD [7].

Many WZW models [8–14] based on nonreductive Lie algebras, though of a different type, have a simpler structure than the WZW models based on semisimple Lie algebras. This suggests considering Yang–Mills theories in four

dimensions and investigate if the simplifications introduced in two dimensions by going nonreductive carry through to four dimensions. The problem was addressed in Ref. [15] for a class of nonreductive algebras called double extensions. One-loop radiative corrections for certain models were computed and it was argued that, if renormalizability is assumed, there would not be higher-loop corrections. An apparent lack of unitarity was found.

The classical double, we denote it as $\mathfrak{g}_\ltimes$, of any real Lie algebra $\mathfrak{g}$ is a Lie algebra of dimension twice the dimension of $\mathfrak{g}$ that admits a metric. This metric determines a Yang–Mills Lagrangian for the field that results from gauging the algebra. For $\mathfrak{g}$ simple, the self-antiself dual instantons of the $\mathfrak{g}_\ltimes$ Yang–Mills theory in four-dimensional Euclidean space have been studied elsewhere [16]. Every $\mathfrak{g}_\ltimes$ instanton has an embedded $\mathfrak{g}$ instanton with the same instanton number and twice the number of collective coordinates. This doubling of degrees of freedom and the simpler structure of classical doubles as compared to double extensions suggest considering perturbative quantization of Yang–Mills theory with gauge algebra $\mathfrak{g}_\ltimes$.

The theory is shown to have UV divergences at one loop but no radiative corrections at higher loops. As in the classical case, the physical degrees of freedom of the quantum theory are doubled with respect to the $\mathfrak{g}$ theory. In particular, the first and only coefficient (since there are no radiative corrections beyond one loop) of the beta function is twice that of the $\mathfrak{g}$ theory. To disentangle truly gauge invariant one-loop corrections from those due to gauge fixing, the Slavnov–Taylor operator for the quantum theory is used. The term in the classical action that would spoil unitarity is cohomologically trivial with respect to the Slavnov–Taylor operator, so in the quantum effective action it can be put to zero through a field redefinition and poses no problem for unitarity.

The manuscript is organized as follows. Section 2 contains a brief reminder of classical doubles $\mathfrak{g}_\ltimes$ and their Lie groups $G_\ltimes$. Classical Yang–Mills theory with gauge

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group $G_\ltimes$ is formulated in Sect. 3, and the path integral generating the theory's Green functions is derived. With the mind set in avoiding miscounting the theory's degrees of freedom, special consideration is given to gauge fixing, and three derivations for the gauge fixing terms in Landau gauge are presented. Section 3 also discusses the emergence of the classical theory as a limit of Yang–Mills theory with gauge algebra the direct product $\mathfrak{g} \times \mathfrak{g}$. Section 4 contains our perturbative analysis, with the calculation of one-loop 1PI radiative corrections and the proof that there are no higher-loop corrections. The one-loop divergences are removed in Sect. 5 by adding a gauge invariant counterterm consistent with unitarity, so the one-loop contribution to the quantum effective action is positive definite. We conclude in Sect. 6.

2 Semidirect products of Lie algebras and their groups

We start by recalling the definitions of classical double of a Lie algebra and its Lie group. Consider an arbitrary Lie algebra $\mathfrak{g}$ of dimension n with basis $\{T_a\}$ and commutation relations $[T_a, T_b] = f_{ab}{}^c T_c$. As a vector space, $\mathfrak{g}$ has a dual vector space $\mathfrak{g}^*$. Take on $\mathfrak{g}^*$ the canonical dual basis $\{Z^a\}$, given by $Z^a(T_b) = \delta^a{}_b$. The classical double of $\mathfrak{g}$, denoted by $\mathfrak{g}_\ltimes$, is the Lie algebra of dimension $2n$ with basis $\{T_a, Z^b\}$ and commutators

$$[T_a, T_b] = f_{ab}{}^c T_c, \quad [T_a, Z^b] = -f_{ac}{}^b Z^c, [Z^a, Z^b] = 0. \tag{2.1}$$

It is trivial to check that these commutators satisfy the Jacobi identity, so indeed they define a Lie bracket. In a more mathematical language [14], the algebra $\mathfrak{g}_\ltimes$ is the semidirect product $\mathfrak{g} \ltimes \mathfrak{g}^*$ obtained by the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$.

It is clear from the commutators (2.1) that $\mathfrak{g}_\ltimes$ is not semisimple, so its Killing form is degenerate and cannot be used as a metric. For convenience we recall that a metric on a Lie algebra is a symmetric, nondegenerate, bilinear form Ω such that

$$\Omega(A, [B, C]) = \Omega([A, B], C) \tag{2.2}$$

for all A, B, C in the algebra. The relevance of condition (2.2) is that it implies invariance under the adjoint action of the algebra's Lie group,

$$\Omega(e^{-C} A e^C, e^{-C} B e^C) = \Omega(A, B). \tag{2.3}$$

It is straightforward to check that, for $\{T_a, Z^b\}$ above, the bilinear symmetric form

$$\Omega = \begin{array}{c} \\ T_a \\ Z^a \end{array} \begin{array}{c} T_b \quad\; Z^b \\ \begin{pmatrix} \omega_{ab} & \delta_a^b \\ \delta_a^b & 0 \end{pmatrix} \end{array}. \tag{2.4}$$

is nondegenerate and solves Eq. (2.2), hence is a metric on $\mathfrak{g}_\ltimes$. Here $\omega_{ab} = \omega(T_a, T_b)$ is an arbitrary invariant, symmetric, bilinear form on $\mathfrak{g}$ that may be degenerate. It is worth remarking that $\mathfrak{g}$ is arbitrary and need not be metric itself.

Denote by $G_\ltimes$, G, and N the Lie groups obtained by exponentiation of the algebras $\mathfrak{g}_\ltimes$, $\mathfrak{g}$, and $\mathfrak{g}^*$. Every element h of G and every n of N can be uniquely written as $h = e^T$ and $n = e^Z$, for some T in $\mathfrak{g}$ and some Z in $\mathfrak{g}^*$. In turn, every element g of $G_\ltimes$ can be uniquely written as $g = hn$. The product $g_3 = g_1 g_2$ of two elements $g_1 = h_1 n_1$ and $g_2 = h_2 n_2$ of $G_\ltimes$ is given by $g_3 = h_3 n_3$, with $h_3 = h_1 h_2$ and $n_3 = (h_2^{-1} n_1 h_2) n_2$. It is clear that g_3 is in $G_\ltimes$ since the Campbell–Hausdorff formula and the commutators (2.1) imply that h_3 is in G and n_3 is in N. It is very easy to see that the group $G_\ltimes$ is the semidirect product of G with the normal Abelian subgroup N. Since N is isomorphic to $\mathbf{R}^n$, the group $G_\ltimes$ is noncompact.

So far no restriction has been placed on $\mathfrak{g}$. Assume now that it is semisimple. In this case, $\mathfrak{g}_\ltimes$ can be viewed as a limit of the direct product of $\mathfrak{g}$ with itself [16]. To see this, take ω_{ab} in Eq. (2.4) proportional to the Killing form of $\mathfrak{g}$. Since ω_{ab} is nondegenerate, indices in the structure constants $f_{ab}{}^c$ can be raised and lowered using ω_{ab} and its inverse ω^{ab}, defined by $\omega^{ab}\omega_{bc} = \delta^a{}_c$. This gives completely antisymmetric structure constants,

$$f_{abc} = f_{ab}{}^d \omega_{dc}, \qquad f_{abc} = -f_{bac} = f_{bca}.$$

Perform in $\mathfrak{g}^*$ the change of generators $Z^a \to Z_a = \omega_{ab} Z^b$. In the basis $\{T_a, Z_b\}$ the Lie bracket (2.1) reads

$$[T_a, T_b] = f_{ab}{}^c T_c, \quad [T_a, Z_b] = f_{ab}{}^c Z_c, \quad [Z_a, Z_b] = 0 \tag{2.5}$$

and the metric Ω is recast as

$$\Omega = \begin{array}{c} \\ T_a \\ Z_a \end{array} \begin{array}{c} T_b \quad\; Z_b \\ \begin{pmatrix} \omega_{ab} & \omega_{ab} \\ \omega_{ab} & 0 \end{pmatrix} \end{array}. \tag{2.6}$$

Consider now the commutators

$$[T_a, T_b] = f_{ab}{}^c T_c, \quad [T_a, Z_b] = f_{ab}{}^c Z_c,$$
$$[Z_a, Z_b] = t^2 f_{ab}{}^c T_c, \tag{2.7}$$

where t is an arbitrary real parameter. It is trivial to show that they satisfy the Jacobi identity for all t. Hence they define a Lie algebra, call it $\mathfrak{g}_\ltimes^t$, that reduces to $\mathfrak{g}_\ltimes$ in the limit $t \to 0$. Furthermore, $\mathfrak{g}_\ltimes^t$ admits the invariant metric

$$\Omega^t = \begin{array}{c} \\ T_a \\ Z_a \end{array} \begin{pmatrix} \overset{T_b}{\omega_{ab}} & \overset{Z_b}{\omega_{ab}} \\ \omega_{ab} & t^2\omega_{ab} \end{pmatrix}. \tag{2.8}$$

The change of basis

$$X_a^\pm = \frac{1}{2}\left(T_a \pm \frac{1}{t} Z_a\right), \tag{2.9}$$

in $\mathfrak{g}_\ltimes^t$ transforms the Lie bracket (2.7) in

$$[X_a^\pm, X_b^\pm] = f_{ab}{}^c X_c^\pm, \quad [X_a^+, X_b^-] = 0 \tag{2.10}$$

and gives for the metric Ω^t in Eq. (2.8) the block diagonal form

$$\Omega^t = \begin{array}{c} \\ X_a^+ \\ X_a^- \end{array} \begin{pmatrix} \overset{X_b^+}{\frac{1}{2}\left(1+\frac{1}{t}\right)\omega_{ab}} & \overset{X_b^-}{0} \\ 0 & \frac{1}{2}\left(1-\frac{1}{t}\right)\omega_{ab} \end{pmatrix}. \tag{2.11}$$

It is clear from Eqs. (2.10) and (2.11) that $\mathfrak{g}_\ltimes^t$ is the direct product $\mathfrak{g} \times \mathfrak{g}$, so its Lie group $G_\ltimes^t$ is the direct product $G \times G$.

3 The gauge fixed classical action

We assume from now on that $\mathfrak{g}$ is semisimple and work in the basis $\{T_a, Z_b\}$ of $\mathfrak{g}_\ltimes$. The commutation relations are as in Eq. (2.5) and the metric Ω as in (2.6). The Lie groups of $\mathfrak{g}$ and $\mathfrak{g}_\ltimes$ will be denoted by G and $G_\ltimes$. All quantities taking values in $\mathfrak{g}_\ltimes$ will be written in boldface, whereas their T_a and Z_a components will be labeled with subscripts T and Z. For the gauge field A_μ and its field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, we have

$$A_\mu = A_{T\mu}^a T_a + A_{Z\mu}^a Z_a,$$
$$F_{\mu\nu} = F_{T\mu\nu}^a T_a + F_{Z\mu\nu}^a Z_a.$$

The expressions of $F_{T\mu\nu}^a$ and $F_{Z\mu\nu}^a$ in terms of $A_{T\mu}^a$ and $A_{Z\mu}^a$ follow from the commutators (2.5),

$$F_{T\mu\nu}^a = \partial_\mu A_{T\nu}^a - \partial_\nu A_{T\mu}^a + f_{bc}{}^a A_{T\mu}^b A_{Z\nu}^c,$$
$$F_{Z\mu\nu}^a = \partial_\mu A_{Z\nu}^a - \partial_\nu A_{Z\mu}^a + f_{bc}{}^a \left(A_{T\mu}^b A_{Z\nu}^a - A_{T\nu}^b A_{Z\nu}^a\right).$$

The action of the covariant derivative $D_\mu = \partial_\mu + [A_\mu, \ ]$ on any tensor $\Phi = \Phi_T^a T_a + \Phi_Z^a Z_a$ has components

$$\left(D_\mu \Phi\right)_T^a = \partial_\mu \Phi_T^a + f_{bc}{}^a A_{T\mu}^b \Phi_T^c,$$
$$\left(D_\mu \Phi\right)_Z^a = \partial_\mu \Phi_Z^a + f_{bc}{}^a A_{T\mu}^b \Phi_Z^c + f_{bc}{}^a A_{Z\mu}^b \Phi_T^c.$$

On the right hand side of the first equation, one recognizes the covariant derivative for the algebra $\mathfrak{g}$, which we denote by D_μ,

$$D_c^{a\mu} := \delta_c^a \partial_\mu + f_{bc}{}^a A_{T\mu}^b. \tag{3.1}$$

Finite gauge transformations read

$$A_\mu \to A_\mu' = g^{-1}\partial_\mu g + g^{-1} A_\mu g,$$
$$F_{\mu\nu} \to F_{\mu\nu}' = g^{-1} F_{\mu\nu} g, \tag{3.2}$$

with g an arbitrary group element in $G_\ltimes$,

$$g(x) = \exp[\theta_T^a(x) T_a] \exp[\theta_Z^a(x) Z_a].$$

Infinitesimally these transformations take the form

$$\delta A_{T\mu}^a = D_\mu \theta_T^a, \tag{3.3}$$
$$\delta A_{Z\mu}^a = D_\mu \theta_Z^a + f_{bc}{}^a A_{Z\mu}^b \theta_T^c \tag{3.4}$$

for the gauge field, and

$$\delta F_{T\mu\nu}^a = f_{bc}{}^a F_{T\mu\nu}^b \theta_T^c, \tag{3.5}$$
$$\delta F_{Z\mu\nu}^a = f_{bc}{}^a \left(F_{Z\mu\nu}^b \theta_T^c + F_{T\mu\nu}^b \theta_Z^c\right) \tag{3.6}$$

for the field strength. The invariance condition (2.3) for the metric Ω and the transformation law (3.2) for the field strength $F_{\mu\nu}$ imply that the Lagrangian density

$$\mathcal{L}_{YM} = \frac{1}{4g^2} \Omega\left(F_{\mu\nu}, F^{\mu\nu}\right)$$
$$= \frac{1}{4g^2} \omega_{ab}\left(F_{T\mu\nu}^a F_T^{b\mu\nu} + 2 F_{T\mu\nu}^a F_Z^{b\mu\nu}\right), \tag{3.7}$$

where g is a coupling constant, is gauge invariant. We are interested in perturbatively quantizing Yang–Mills theory with Lagrangian $\mathcal{L}_{YM}$. To obtain a path integral that generates the theory's Green functions, we next fix the gauge. We do this in three different ways.

Gauge fixing I. Introduce a ghost field c, an antighost field $\bar{c}$ and a Lagrange multiplier field b,

$$c = c_T^a T_a + c_Z^a Z_a,$$
$$\bar{c} = \bar{c}_T^a T_a + \bar{c}_Z^a Z_a,$$
$$b = b_T^a T_a + b_Z^a Z_a.$$

Use the infinitesimal form of the gauge transformations to define a BRS operator s by its action on the fields,

$$sA_\mu = D_\mu c, \quad sc = -cc, \quad s\bar{c} = b, \quad sb = 0. \tag{3.8}$$

The operator s commutes with ∂_μ, A_μ and b, and anticommutes with c and $\bar{c}$. The BRS transformations for the T and Z components of the fields can be obtained either from the definition of s in Eq. (3.8) and the commutations relations (2.5), or directly from Eqs. (3.3) and (3.4). They read

$$sA_{T\mu} = D_\mu c_T^a, \quad sc_T^a = -\frac{1}{2} f_{bc}{}^a c_T^b c_T^c,$$
$$s\bar{c}_T^a = b_T^a, \qquad sb_T^a = 0, \tag{3.9}$$

and

$$sA^a_{Z\mu} = D_\mu c^a_Z + f_{bc}{}^a A^b_{Z\mu} c^c_T,$$

$$sc^a_Z = -f_{bc}{}^a c^b_T c^c_Z, \quad s\bar{c}^a_Z = b^a_Z, \quad sb^a_Z = 0. \tag{3.10}$$

Using Eqs. (3.8), or their equivalent Eqs. (3.9) and (3.10), it is very easy to check that $s^2 = 0$. Note that the BRS transformations (3.9) are the same as for the semisimple gauge algebra $\mathfrak{g}$. In Lorenz gauge, the gauge fixing Lagrangian is the BRS variation

$$\mathcal{L}_{\text{GF}} = -\frac{1}{g^2} \, s \, \Omega\left(\bar{c}, \, \frac{\alpha}{2} \, b + \partial A\right),$$

where α is the gauge parameter and a contraction of the space-time indices in ∂_μ and A_μ is understood. Expanding in terms of field components, $\mathcal{L}_{\text{GF}}$ becomes

$$\begin{aligned}
\mathcal{L}_{\text{GF}} = \frac{1}{g^2} \, \omega_{ab} \Big[&-\frac{\alpha}{2} \, (b^a_T b^b_T + 2 \, b^a_T b^b_Z) - b^a_T \, (\partial A^b_T) \\
&- b^a_T \, (\partial A^b_Z) - b^a_Z \, (\partial A^b_T) \\
&+ \bar{c}^a_T \, (\partial D c^b_T) + \bar{c}^a_T \, (\partial D c^b_Z) \\
&+ \bar{c}^a_T \, \partial \, (f_{cd}{}^b A^c_Z c^d_T) + \bar{c}^a_Z \, (\partial D c^b_T) \Big],
\end{aligned} \tag{3.11}$$

with D_μ the $\mathfrak{g}$ covariant derivative in Eq. (3.1). The gauge fixed Lagrangian is the sum

$$\mathcal{L}_{\ltimes} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{GF}}. \tag{3.12}$$

Introduce external sources $\{J_\Phi\} = \{J^\mu, \bar{\xi}, \zeta, B\}$ for the fields $\{\Phi\} = \{A_\mu, c, \bar{c}, b\}$, with

$$J_\Phi = J^a_{\Phi T} T_a + J^a_{\Phi Z} Z_a.$$

The path integral that generates the theory's Green functions is given by

$$\begin{aligned}
Z[J, \bar{\xi}, \zeta, B] = \int [\mathrm{d}A] \, [\mathrm{d}c] \, [\mathrm{d}\bar{c}] \, [\mathrm{d}b] \\
\exp\left[-\int \mathrm{d}^4 x \, \mathcal{L}_{\ltimes} + S_{\text{ext}} \right],
\end{aligned} \tag{3.13}$$

where $[\mathrm{d}\Phi] = [\mathrm{d}\Phi_T] \, [\mathrm{d}\Phi_Z]$ for every field Φ, and S_{ext} is the source term

$$\begin{aligned}
S_{\text{ext}} = \frac{1}{g^2} \int \mathrm{d}^4 x \, \big[\, \Omega(J, A) + \Omega(\bar{\xi}, c) + \Omega(\bar{c}, \zeta) \\
+ \Omega(B, b) \, \big].
\end{aligned} \tag{3.14}$$

The external sources are coupled to fields through the Lie algebra metric Ω, so that

$$\Omega(J_\Phi, \Phi) = \omega_{ab} \, \big(J^a_{\Phi T} \Phi^b_T + J^a_{\Phi T} \Phi^b_Z + J^a_{\Phi Z} \Phi^b_T \big)$$

for every field Φ and its source J_Φ. The Green functions are obtained by functionally differentiating with respect to the external sources. For example, $\langle A^a_{T\mu}(x) A^b_{Z\nu}(y) \rangle$ is given by

$$\omega^{ac} \, \frac{\delta}{\delta J^{c\mu}_Z(x)} \, \omega^{bd} \left[\frac{\delta}{\delta J^{d\nu}_T(y)} - \frac{\delta}{\delta J^{d\nu}_Z(y)} \right] \ln Z[J_\phi] \Bigg|_{J_\phi = 0}.$$

Gauge fixing II. Equivalently, the path integral (3.13) can be derived as follows. In the naive path integral

$$\int [\mathrm{d}A] \, \exp\left[-\int \mathrm{d}^4 x \, \mathcal{L}_{\text{YM}} \right], \tag{3.15}$$

to avoid integrating over gauge equivalent degrees of freedom, replace

$$\int [\mathrm{d}A] \, \rightarrow \, \int [\mathrm{d}A] \, \delta(\partial A - f) \, \Delta_{\partial A}, \tag{3.16}$$

where the Dirac delta imposes the Lorenz gauge fixing condition $\partial A = f$ and

$$\Delta_{\partial A} = \det\left[\frac{\delta}{\delta \theta(y)} \, \partial(A + \delta A)(x) \right]_{\partial A = 0}$$

is the corresponding Faddeev–Popov determinant. Proceed now as usual:

(i) Average over f with Gaussian type weight. That is, introduce in the measure

$$\int [\mathrm{d}f] \exp\left[-\frac{1}{2\alpha g^2} \int \mathrm{d}^4 x \, \Omega(f, f) \right].$$

(ii) Exponentiate $\delta(\partial A - f)$ by means of an auxiliary field b,

$$\delta(\partial A - f) = \int [\mathrm{d}b] \, \exp\left[\frac{i}{g^2} \int \Omega(b, \partial A - f) \right].$$

(iii) Write the determinant

$$\Delta_{\partial A} = \det \delta^{(4)}(x - y) \begin{pmatrix} \partial^\mu D_c{}^a_\mu(x) & 0 \\ f_{bc}{}^a \partial^\mu A^b_{Z\mu}(x) & \partial^\mu D_c{}^a_\mu(x) \end{pmatrix}$$

as a path integral over Grassmann fields c^a_1, c^a_2, and $\bar{c}_{1a}$, $\bar{c}_{2a}$,

$$\begin{aligned}
\Delta_{\partial A} = \int [\mathrm{d}\bar{c}_1] \, [\mathrm{d}\bar{c}_2] \, [\mathrm{d}c_1] \, [\mathrm{d}c_2] \, \exp \\
\times \Big[-\frac{1}{g^2} \int \mathrm{d}^4 x \, \big(\bar{c}_{1a} \partial D c^a_1 + \bar{c}_{2a} f^a{}_{cb} \\
\times \partial_\mu (A^{c\mu}_T c^b_1) + \bar{c}_{2a} \partial D c^a_2 \big) \Big].
\end{aligned}$$

Use next (i)–(iii) in Eq. (3.16), integrate over f, make the change[1]

$$c_T^a = c_1^a, \quad c_Z^a = c_2^a$$
$$\bar{c}_T^a = \omega^{ab} \bar{c}_{2b}, \quad \bar{c}_Z^a = \omega^{ab} (\bar{c}_{1b} - \bar{c}_{2b}), \tag{3.17}$$

and replace b with ib. This gives for the path integral the expression in Eq. (3.13).

Gauge fixing III. The observation at the end of Sect. 2 concerning deformations of $\mathfrak{g}_{\ltimes}$ suggests that classical $G_{\ltimes}$ Yang–Mills theory can be regarded as a limit of $G \times G$ Yang–Mills theory. To see this, consider two copies of a Yang–Mills theory, both with gauge group the semisimple Lie group G. Label the copies with the subscripts + and −, so that $A_{\pm\mu}^a$, $F_{\pm\mu\nu}^a$, and $g_{\pm}$ denote their gauge fields, field strengths and coupling constants. Fix in both copies a Lorenz gauge by introducing ghost, antighost, and auxiliary fields $c_{\pm}^a$, $\bar{c}_{\pm}^a$, and $b_{\pm}^a$, and a BRS operators s given by

$$sA_{\pm\mu}^a = D_{\pm\mu} c_{\pm}^a, \quad sc_{\pm}^a = -\frac{1}{2} f_{bc}{}^a c_{\pm}^b c_{\pm}^c,$$
$$s\bar{c}_{\pm}^a = b_{\pm}^a, \quad sb_{\pm}^a = 0.$$

The corresponding gauge fixed Lagrangians read

$$\mathcal{L}_{\pm} = \frac{1}{4g_{\pm}^2} \omega_{ab} F_{\pm\mu\nu}^a F_{\pm}^{b\,\mu\nu}$$
$$- \frac{1}{g_{\pm}^2} \omega_{ab} s \left[\bar{c}_{\pm}^a \left(\frac{\alpha_{\pm}}{2} b_{\pm}^b + \partial A_{\pm}^b \right) \right].$$

The sum

$$\mathcal{L}_{+-} = \mathcal{L}_{+} + \mathcal{L}_{-}$$

describes two theories not interacting with each other. Write $g_{\pm}$ in terms of a coupling constant g and a parameter t as

$$\frac{1}{g_{\pm}^2} = \frac{1}{2g^2} \left(1 \pm \frac{1}{t} \right),$$

and introduce fields Φ_T^a and Φ_Z^a given by

$$\Phi_T^a = \frac{1}{2} (\Phi_+^a + \Phi_-^a), \quad \Phi_Z^a = \frac{1}{2t} (\Phi_+^a - \Phi_-^a)$$

for $\Phi_{\pm}^a = A_{\pm\mu}^a$, $c_{\pm}^a$, $\bar{c}_{\pm}^a$, $b_{\pm}^a$, Take now $\alpha_+ = \alpha_- = \alpha$ and send $t \to 0$. In this limit, the + and − sectors couple and the Lagrangian $\mathcal{L}_{+-}$ becomes the Lagrangian $\mathcal{L}_{\ltimes}$ of the $G_{\ltimes}$ theory given by Eqs. (3.7), (3.11) and (3.12).

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[1] A more precise notation for the T and Z components of a $\mathfrak{g}_{\ltimes}$-valued field is $\Phi = \Phi^{T_a} T_a + \Phi^{Z_a} Z_a$. Not to load the writing, we have used instead $\Phi_T^a := \Phi^{T_a}$ and $\Phi_Z^a := \Phi^{Z_a}$. Using Ω and Ω^{-1} to lower and raise indices, one has

$$\Phi_{T_a} = \omega_{ab} (\Phi_T^b + \Phi_Z^b), \qquad \Phi_{Z_a} = \omega_{ab} \Phi_T^b$$
$$\Phi_T^a = \omega^{ab} \phi_{Z_b}, \qquad \Phi_Z^a = \omega^{ab} (\Phi_{T_b} - \Phi_{Z_b}).$$

Hence $(\bar{c}_{1a}, \bar{c}_{2a})$ in Eq. (3.17) is nothing but $(\bar{c}_{T_a}, \bar{c}_{Z_a})$.

4 Radiative corrections

To explicitly calculate radiative corrections, we take in this section $\mathfrak{g}$ to be $\mathfrak{su}(N)$. The group G is then SU(N) and for $G_{\ltimes}$ we use the notation SU(N)$_{\ltimes}$.

In the conventions of Sect. 2, in which a group element h of SU(N) is written as e^T, the elements T of $\mathfrak{su}(N)$ in the defining (fundamental) representation are traceless anti-hermitean matrices. We normalize the structure constants $f_{ab}{}^c$ of $\mathfrak{su}(N)$ by requiring $f_{ca}{}^d f_{db}{}^c = N\delta_{ab}$. This gives for the Killing form $k_{ab} = N\delta_{ab}$ and amounts to taking $\text{tr}[T_{(R)a} T_{(R)b}] = C_2 \delta_{ab}$ in a representation R, with $C_2 = N$ in the adjoint representation and $C_2 = -1/2$ in the defining representation. For ω_{ab} in Eq. (2.6), we take

$$\omega_{ab} = \delta_{ab}, \tag{4.1}$$

so indices in $f_{ab}{}^c$ are lowered and raised with δ_{ab} and δ^{ab}.

Feynman rules in Lorenz gauge. Introduce external sources K^{μ} and H for the nonlinear BRS transforms sA_{μ} and sc,

$$K^{\mu} = K_T^{a\mu} T_a + K_Z^{a\mu} Z_a,$$
$$H^{\mu} = H_T^a T_a + H_Z^a Z_a.$$

The path integral (3.13) becomes

$$Z[J, \bar{\xi}, \zeta, B; K, H] = \int [dA] [dc] [d\bar{c}] [db]$$
$$\times \exp \left[-\int d^4x \, \mathcal{L}_{\ltimes} + S_{\text{ext}} + S_{\text{KH}} \right], \tag{4.2}$$

where S_{KH} is given by

$$S_{\text{KH}} = \frac{1}{g^2} \int d^4x \left[\Omega(K, sA) - \Omega(H, sc) \right] \tag{4.3}$$

and $\Omega(K, A)$ and $\Omega(H, sc)$ read

$$\Omega(K, sA) = \omega_{ab} \left(K_T^{a\mu} sA_{T\mu}^b + K_T^{a\mu} sA_{Z\mu}^b + K_Z^{a\mu} sA_{T\mu}^b \right),$$
$$\Omega(H, sc) = \omega_{ab} \left(H_T^a sc_T^b + H_T^a sc_Z^b + H_Z^a sc_T^b \right).$$

The Feynman rules of the theory follow from $Z[\cdots]$ in (4.2) and are collected in Figs. 1, 2 and 3, where solid lines denote gauge propagators and dashed lines ghost propagators. The expressions of $D_{\mu\nu}^{ab}(p)$ and $\Delta^{ab}(p)$ in Fig. 1 are

$$D_{\mu\nu}^{ab}(p) = g^2 \frac{\delta^{ab}}{p^2} \left[\delta_{\mu\nu} + (\alpha - 1) \frac{p_{\mu} p_{\nu}}{p^2} \right] \tag{4.4}$$

and

$$\Delta^{ab}(p) = -g^2 \frac{\delta^{ab}}{p^2}. \tag{4.5}$$

Fig. 1 Free propagators for SU(N)$_\ltimes$ Yang–Mills in Lorenz gauge

Fig. 2 Vertices for SU(N)$_\ltimes$ Yang–Mills theory in Lorenz gauge

In turn, the gauge vertices $V_{\mu\nu\rho}^{abc}(p,q,k)$ and $W_{\mu\nu\rho\sigma}^{abcd}$ in Fig. 2 read

$$V_{\mu\nu\rho}^{abc}(p,q,k) = if^{abc}\left[(q-k)_\mu\,\delta_{\nu\rho} + (k-p)_\nu\,\delta_{\rho\mu} \right.$$
$$\left. +(p-q)_\rho\,\delta_{\mu\nu}\right]$$

and

$$W_{\mu\nu\rho\sigma}^{abcd} = -\left[f^{abe}\,f^{cde}\left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right)\right.$$
$$+f^{ace}\,f^{dbe}\left(\delta_{\mu\sigma}\delta_{\rho\nu} - \delta_{\mu\nu}\delta_{\rho\sigma}\right)$$
$$\left. +f^{ade}\,f^{bce}\left(\delta_{\mu\nu}\delta_{\sigma\rho} - \delta_{\mu\rho}\delta_{\sigma\nu}\right)\right].$$

We remark that the free propagators do not have TT components, and that their TZ component is equal to their ZT component. This will play an important role in the analysis below.

Consider for comparison conventional SU(N) Yang–Mills theory in Lorenz gauge. Its gauge fixed Lagrangian is recovered from $\mathcal{L}_\ltimes$ by setting all the Z components equal to

zero. To avoid confusion, we reserve the subscripts T and Z for the field components of the SU(N)$_\ltimes$ theory, and use A_μ^a, b^a, $\bar{c}^a$, c^a, and K_μ^a, H^a without subscripts for the fields and the nonlinear BRS sources of the SU(N) theory. The gauge field and ghost free propagators of the SU(N) theory are given by $D_{\mu\nu}^{ab}(p)$ and $\Delta^{ab}(p)$ in Eqs. (4.4) and (4.5), which are equal to the TZ and ZT free propagators of the SU(N)$_\ltimes$ theory. The Feynman rules for the vertices A^3, A^4, $\bar{c}Ac$, KAc, and Hcc of the SU(N) theory are as in Figs. 2 and 3.

We now proceed to compute radiative corrections. To regulate whatever UV divergences may occur, we will use dimensional regularization with $D = 4 - 2\epsilon$, so from now on all diagrams and Green functions should be understood as dimensionally regularized. Since dimensional regularization manifestly preserves BRS invariance, the dimensionally regularized Green functions will solve the functional identities associated to BRS invariance.

Fig. 3 External vertices for SU(N)$_\ltimes$ Yang–Mills theory in Lorenz gauge

$$= -\frac{1}{g^2} f^{abc} \delta_{\mu\nu}$$

$$= \frac{1}{g^2} f^{abc}$$
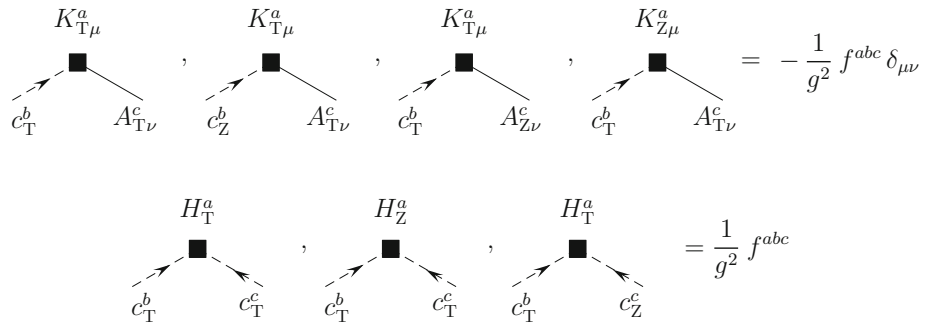
One-loop radiative corrections. The only 1PI one-loop diagrams that occur in perturbation theory have all their external legs of type T. To prove this, note first that the vertices of the theory, see Figs. 2 and 3, have either no or one leg of type Z. Assume now that there is a 1PI one-loop diagram with an external Z leg, and call U_1 to the vertex to which the leg is attached. All the other legs of U_1 will be of type T. To close a loop, two of these T legs must be internal. Since there are no TT propagators, each internal T leg must propagate into type Z. Each one of the resulting Z legs will in turn be attached to a different vertex. Call these vertices U_2 and U_3. From U_2 and U_3 only T legs will come out. One may go on and introduce new vertices, but the loop will never close since there are no TT propagators to join two T legs. Hence 1PI one-loop diagrams have all their external legs of type T. The only nonzero 1PI Green functions at one loop are then $\langle \Psi_{1T}(p_1) \ldots \Psi_{nT}(p_n) \rangle_{SU(N)_\ltimes}$, where Ψ_{iT} stands for any of the fields $A_{T\mu}^a$, $\bar{c}_T^a$, c_T^a or the sources $K_{T\mu}^a$, H_T^c.

Consider for example the two-point 1PI function $\langle A_{T\mu}^a A_{T\nu}^b \rangle_{SU(N)_\ltimes}$. At one loop, it receives contributions from the diagrams in Fig. 4, where the number under each diagram is the diagram's symmetry factor. We have drawn in Fig. 5 the one-loop diagrams that contribute to the 1PI function $\langle A_\mu^a A_\nu^b \rangle_{SU(N)}$ of SU(N) Yang–Mills theory. Since the propagators and vertices in both sets of diagrams are the same, we conclude that

$$\langle A_{T\mu}^a A_{T\nu}^b \rangle_{SU(N)_\ltimes} = 2 \langle A_\mu^a A_\nu^b \rangle_{SU(N)}. \tag{4.6}$$

Writing only the divergent part as $\epsilon \to 0$, this gives

$$\langle A_{T\mu}^a(-p) A_{T\nu}^b(p) \rangle_{SU(N)_\ltimes} = \left(\frac{13}{3} - \alpha \right) C_\epsilon \, \Pi_{\mu\nu}^{ab}(p) + O(\epsilon^0),$$

where C_ϵ is the constant

$$C_\epsilon = -\frac{C_2}{16\pi^2 \epsilon}. \tag{4.7}$$

Equation (4.6) can be extended to all 1PI functions as follows. Consider a 1PI one-loop diagram in the SU(N)$_\ltimes$ theory. Since all its external legs are of type T and there are no TT propagators, all its vertices have one internal leg of type Z. Label clockwise the vertices in the loop as $U_1, \ldots, U_n$. The Z leg coming out of vertex U_1 must be connected to an internal T
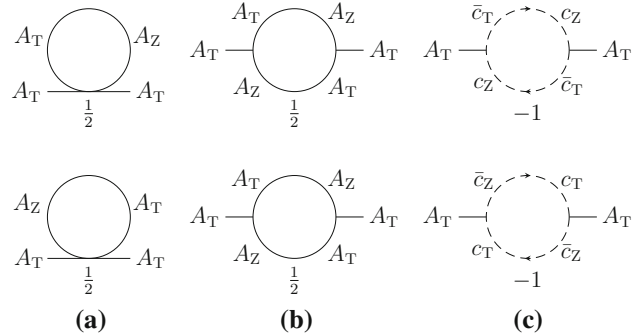

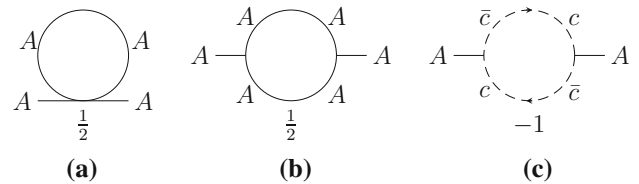**Fig. 4** One-loop corrections to $\langle A_T A_T \rangle_{SU(N)_\ltimes}$

Fig. 5 One-loop corrections to $\langle AA \rangle_{SU(N)}$

leg of a neighboring vertex, say U_2, through a ZT propagator. In turn U_2 is connected to U_3 through another ZT propagator, and so on, until the loop is closed, with U_n connecting to U_1 via a ZT propagator. For every such diagram, there is a diagram with the same vertices and the only difference that now the internal Z leg from U_1 connects with an internal T leg in U_n through a ZT line, rather than with U_2. This implies that U_n connects with U_{n-1} through a ZT line, and so on until the loop is closed with a ZT propagator from U_2 with U_1. These two diagrams give the same contribution to the 1PI function, which in turn is equal to the contribution of the equivalent 1PI diagram in the SU(N) theory. Hence we have

$$\langle \Phi_{1,T}(p_1) \ldots \Phi_{n,T}(p_n) \rangle_{SU(N)_\ltimes} = 2 \langle \Phi_1(p_1) \ldots \Phi_n(p_n) \rangle_{SU(N)}.$$

This reduces the calculation of the one-loop 1PI Green functions in the SU(N)$_\ltimes$ theory to that in the SU(N) theory. The left column In Table 1 collects all the one-loop 1PI Green functions in SU(N)$_\ltimes$ Yang–Mills theory that are UV divergent, whereas the column in the center lists their UV diver-

Table 1 UV divergent 1PI Green functions in SU(N)$_\ltimes$ theory and their counterterms

1PI UV divergent Green function	Contribution from $\bar\Gamma_1^\epsilon$	Contribution from $\bar\Gamma_1^{ct}$
$\langle A_{T\mu}^a(-p)A_{T\nu}^b(p)\rangle$	$\left(\frac{13}{3}-\alpha\right)C_\epsilon\,\Pi_{\mu\nu}^{ab}(p)$	$-\left(c_1+2c_2\right)\Pi_{\mu\nu}^{ab}(p)$
$\langle A_{T\mu}^a(p)A_{T\nu}^b(q)A_{T\rho}^c(k)\rangle$	$\left(\frac{17}{6}-\frac{3}{2}\alpha\right)C_\epsilon\,V_{\mu\nu\rho}^{abc}(p,q,k)$	$-\left(c_1+3c_2\right)V_{\mu\nu\rho}^{abc}(p,q,k)$
$\langle A_{T\mu}^a(p)A_{T\nu}^b(q)A_{T\rho}^c(k)A_{T\sigma}^d(r)\rangle$	$\left(\frac{4}{3}-2\alpha\right)C_\epsilon\,W_{\mu\nu\rho\sigma}^{abcd}$	$-\left(c_1+4c_2\right)W_{\mu\nu\rho\sigma}^{abcd}$
$\langle \bar c_T^a(-p)c_T^b(p)\rangle$	$\frac{1}{2}\left(3-\alpha\right)C_\epsilon\delta^{ab}p^2$	$(c_2-c_3)\,\delta^{ab}p^2$
$\langle \bar c_T^a(p)A_{T\mu}^b(q)c^c(k)\rangle$	$i\alpha C_\epsilon\,f^{abc}\,p_\mu$	$ic_3 f^{abc}\,p_\mu$
$\langle K_{T\mu}^a(-p)\,c_T^b(p)\rangle$	$\frac{1}{2}\left(3-\alpha\right)C_\epsilon\,\delta^{ab}\,ip_\mu$	$(c_2-c_3)\,\delta^{ab}\,ip_\mu$
$\langle K_{T\mu}^a(p)\,A_{T\nu}^b(q)\,c_T^c(k)\rangle$	$-\alpha C_\epsilon\,f^{abc}\delta_{\mu\nu}$	$-c_3 f^{abc}\delta_{\mu\nu}$
$\langle H_T^a(p)\,c_T^b(q)\,c_T^c(k)\rangle$	$-\alpha C_\epsilon\,f^{abc}$	$-c_3\,f^{abc}$

gent contributions as computed in dimensional regularization. The column on the right will be discussed in Sect. 5.

Vanishing of 1PI radiative corrections beyond one loop. Any 1PI n-loop diagram can be obtained by joining two external legs in a 1PI $(n–1)$-loop diagram. In our case, since 1PI one-loop diagrams have all their external legs of type T and there are no TT propagators, it is impossible to have two- and higher-loop 1PI diagrams.

We end this section by noting that, again because 1PI Green functions have all their external legs of type T and to these it is only possible to attach free TZ propagators, the only on-shell Green functions that receive radiative corrections are those having all their external legs of type Z.

5 The BRS identity, renormalization, and unitarity

The effective action that generates the 1PI Green functions of G$_\ltimes$ Yang–Mills theory is obtained by writing

$$Z[J,\bar\xi,\xi,B;K,H]=\exp\left(-W[J,\bar\xi,\xi,B;K,H]\right)$$

and performing a Legendre transformation on $W[J,\bar\xi,\xi,B;K,H]$ as follows. Introduce Legendre fields $\{\tilde A,\tilde c,\bar{\tilde c},\tilde b\}$ for the sources $\{J,\bar\xi,\xi,B\}$ through the functional derivatives

$$\tilde A_{T\mu}^a(x)=-\omega^{ab}\frac{\delta W}{\delta J_Z^{b\mu}(x)},$$

$$\tilde A_{Z\mu}^a(x)=\omega^{ab}\left[\frac{\delta W}{\delta J_Z^{b\mu}(x)}-\frac{\delta W}{\delta J_T^{b\mu}(x)}\right],$$

$$\tilde b_T^a(x)=-\omega^{ab}\frac{\delta W}{\delta B_Z^b(x)},$$

$$\tilde b_Z^a(x)=\omega^{ab}\left[\frac{\delta W}{\delta B_Z^b(x)}-\frac{\delta W}{\delta B_T^b(x)}\right],$$

$$\tilde c_T^a(x)=-\omega^{ab}\frac{\delta W}{\delta\bar\xi_Z^b(x)},$$

$$\tilde c_Z^a(x)=\omega^{ab}\left[\frac{\delta W}{\delta\bar\xi_Z^b(x)}-\frac{\delta W}{\delta\bar\xi_T^b(x)}\right],$$

$$\bar{\tilde c}_T^a(x)=\omega^{ab}\frac{\delta W}{\delta\xi_Z^b(x)},$$

$$-\bar{\tilde c}_Z^a(x)=\omega^{ab}\left[\frac{\delta W}{\delta\xi_Z^b(x)}-\frac{\delta W}{\delta\xi_T^b(x)}\right].$$

Solve these equations for $\{J,\bar\xi,\xi,B\}$ in terms of fields $\{\tilde A,\tilde c,\bar{\tilde c},\tilde b\}$, and use the solutions to construct the effective action functional

$$\begin{aligned}\Gamma[\tilde A,\tilde c,\bar{\tilde c},\tilde B;K,H]&=W[J,\bar\xi,\xi,B;K,H]\\&+\int d^4x\,\omega_{ab}\left(J_T^a\tilde A_T^b+J_T^a\tilde A_Z^b+J_Z^a\tilde A_T^b+\bar\xi_T^a\tilde c_T^b+\bar\xi_T^a\tilde c_Z^b\right.\\&+\bar\xi_Z^a\tilde c_T^b+B_T^a\tilde b_T^b+B_T^a\tilde b_Z^b+B_Z^a\tilde b_T^b\\&\left.+\bar{\tilde c}_T^a\xi_T^b+\bar{\tilde c}_T^a\xi_Z^b+\bar{\tilde c}_Z^a\xi_T^b\right).\end{aligned}$$

The very same methods as for Yang–Mills theory with semisimple gauge group show that Γ has the form

$$\begin{aligned}\Gamma=\bar\Gamma-\int d^4x\,\omega_{ab}\\\times\left[\frac{\alpha}{2}\left(\tilde b_T^a\tilde b_T^b+2\tilde b_T^a\tilde b_Z^b\right)+\tilde b_T^a\partial\tilde A_T^b+\tilde b_T^a\partial\tilde A_Z^b+\tilde b_Z^a\partial\tilde A_T^b\right],\end{aligned}\tag{5.1}$$

where the functional

$$\bar\Gamma=\bar\Gamma[\tilde A_\mu,\tilde c,G_\mu,H]$$

depends on K_μ and $\bar{\tilde c}$ through the combination

$$G_\mu=K_\mu+\partial_\mu\bar{\tilde c}$$

and satisfies the BRS identity

$$\begin{aligned}\int d^4x\,\omega^{ab}&\left[\frac{\delta\bar\Gamma}{\delta\tilde A_T^a}\frac{\delta\bar\Gamma}{\delta G_Z^b}+\frac{\delta\bar\Gamma}{\delta\tilde A_Z^a}\left(\frac{\delta\bar\Gamma}{\delta G_T^b}-\frac{\delta\bar\Gamma}{\delta G_Z^b}\right)\right.\\&\left.-\frac{\delta\bar\Gamma}{\delta\tilde c_T^a}\frac{\delta\bar\Gamma}{\delta H_Z^b}-\frac{\delta\bar\Gamma}{\delta\tilde c_Z^a}\left(\frac{\delta\bar\Gamma}{\delta H_T^b}-\frac{\delta\bar\Gamma}{\delta H_Z^b}\right)\right]=0.\end{aligned}\tag{5.2}$$

The analysis in Sect. 4 implies that $\bar{\Gamma}$ is the sum

$$\bar{\Gamma} = \bar{\Gamma}_0 + \hbar\, \bar{\Gamma}_1 \tag{5.3}$$

of a tree-level contribution

$$\bar{\Gamma}_0 = \frac{1}{g^2} \int d^4x\, \omega_{ab} \left[\frac{1}{4}\, \tilde{F}_T^a \tilde{F}_T^b + \frac{1}{2}\, \tilde{F}_T^a \tilde{F}_Z^b - G_T^a \tilde{D} \tilde{c}_T^b \right.$$
$$- G_T^a \left(\tilde{D} \tilde{c}_Z^b + f_{cd}{}^b\, \tilde{A}_Z^c \tilde{c}_T^c \right)$$
$$- G_Z^a \tilde{D} \tilde{c}_T^b - \frac{1}{2}\, f_{cd}{}^b\, H_T^a \tilde{c}_T^c \tilde{c}_T^d$$
$$\left. - f_{cd}{}^b\, H_T^a \tilde{c}_T^c \tilde{c}_Z^d - \frac{1}{2}\, f_{cd}{}^b\, H_Z^a \tilde{c}_T^c \tilde{c}_T^d \right] \tag{5.4}$$

and a one-loop contribution $\bar{\Gamma}_1$. The term $\bar{\Gamma}_0$ satisfies the BRS identity (5.2). Substituting Eq. (5.3) in Eq. (5.2), it follows that $\bar{\Gamma}_1$ must satisfy

$$\Delta \bar{\Gamma}_1 = 0,$$

where Δ is the Slavnov–Taylor operator

$$\Delta = \int d^4x\, \omega^{ab} \left[\frac{\delta\bar{\Gamma}_0}{\delta\tilde{A}_T^a} \frac{\delta}{\delta G_Z^b} + \frac{\delta\bar{\Gamma}_0}{\delta G_Z^a} \frac{\delta}{\delta\tilde{A}_T^b} \right.$$
$$+ \frac{\delta\bar{\Gamma}_0}{\delta\tilde{A}_Z^a} \left(\frac{\delta}{\delta G_T^b} - \frac{\delta}{\delta G_Z^b} \right) + \left(\frac{\delta\bar{\Gamma}_0}{\delta G_T^a} - \frac{\delta\bar{\Gamma}_0}{\delta G_Z^a} \right) \frac{\delta}{\delta\tilde{A}_Z^b}$$
$$- \frac{\delta\bar{\Gamma}_0}{\delta\tilde{c}_T^a} \frac{\delta}{\delta\tilde{H}_Z^b} - \frac{\delta\bar{\Gamma}_0}{\delta\tilde{H}_Z^a} \frac{\delta}{\delta\tilde{c}_T^b}$$
$$\left. - \frac{\delta\bar{\Gamma}_0}{\delta\tilde{c}_Z^a} \left(\frac{\delta}{\delta H_T^b} - \frac{\delta}{\delta H_Z^b} \right) - \left(\frac{\delta\bar{\Gamma}_0}{\delta H_T^a} - \frac{\delta\bar{\Gamma}_0}{\delta H_Z^a} \right) \frac{\delta}{\delta\tilde{c}_Z^b} \right]. \tag{5.5}$$

The very same arguments as for the semisimple case show that Δ is nilpotent, $\Delta^2 = 0$. The explicit expressions for the action of Δ on $(\tilde{A}_{T\mu}^a, \tilde{A}_{Z\mu}^a)$, $(\tilde{c}_T^a, \tilde{c}_Z^a)$ $(G_{T\mu}^a, G_{Z\mu}^a)$ and (H_T^a, H_Z^a) are given in the appendix. The operator Δ is the quantum analog of the BRS operator and controls gauge invariance for the quantum theory. The only gauge invariant radiative corrections are those which are cohomologically nontrivial with respect to Δ. That is, those that cannot be written as ΔX for any X. Cohomologically trivial corrections are of the form ΔX, originate in gauge fixing and do not contribute to on-shell amplitudes.

We can add to $\bar{\Gamma}_1$ any functional $\bar{\Gamma}_1^{ct}$ such that $\Delta\bar{\Gamma}_1^{ct} = 0$. If $\bar{\Gamma}_1^{ct}$ subtracts the UV divergences in $\bar{\Gamma}_1$, the sum $\bar{\Gamma}_1' = \bar{\Gamma}_1 + \bar{\Gamma}_1^{ct}$ will be finite and still satisfy $\Delta\bar{\Gamma}_1' = 0$, thus can be taken as the one-loop contribution to the quantum effective action. Since the UV divergences in the theory are local, we are interested in the solution of equation $\Delta\bar{\Gamma}_1^{ct} = 0$ over the space of local integrated functionals of mass dimension four

and ghost number zero.[2] The most general solution over this space has the form

$$\bar{\Gamma}_1^{ct} = c_1 S_{TT} + \Delta X, \tag{5.6}$$

where c_1 is an arbitrary real coefficient, S_{TT} is the G Yang–Mills classical action

$$S_{TT} = \frac{1}{4} \int d^4x\, \omega_{ab}\, \tilde{F}_T^a \tilde{F}_T^b \tag{5.7}$$

and X is any local integrated functional of mass dimension three and ghost number -1. Note that the cohomologically nontrivial part of the solution (5.6) does not have a term

$$S_{TZ} = \frac{1}{2} \int d^4x\, \omega_{ab}\, \tilde{F}_T^a \tilde{F}_Z^b. \tag{5.8}$$

This is so since S_{TZ} can be written as ΔY, with Y given by

$$Y = \int d^4x\, \omega_{ab} \left(G_T^a \tilde{A}_Z^b + H_T^a \tilde{c}_Z^b \right). \tag{5.9}$$

We observe here an important difference between the cohomologies of the BRS operator s and the Slavnov–Taylor operator Δ over the space of local integrated functionals of mass dimension four and ghost number zero. While S_{TT} and S_{TZ} are both nontrivial with respect to s, only S_{TT} is nontrivial with respect to Δ.

Recall now that the classical action is the sum of the terms S_{TZ} and S_{TZ}. The first one of them is positive definite and the second one is not. This would seem to point to a loss of unitarity. This, however, is only apparent since, being cohomologically trivial with respect to Δ, S_{TZ} does not carry gauge invariant radiative corrections in the quantum effective action.

In Sect. 4 we have used dimensional regularization to compute the one-loop contribution, call it $\bar{\Gamma}_1^{dreg}$, to the quantum effective action for $SU(N)_\ltimes$. It consists of a divergent part $\bar{\Gamma}_1^\epsilon$ as $\epsilon \to 0$, formed by the terms listed in the left and center columns in Table 1, and a finite part $\bar{\Gamma}_1^{fin}$,

$$\bar{\Gamma}_1^{dreg} = \bar{\Gamma}_1^\epsilon + \bar{\Gamma}_1^{fin}.$$

Since dimensional regularization is BRS invariant, both $\bar{\Gamma}_1^\epsilon$ and $\bar{\Gamma}_1^{fin}$ satisfy $\Delta\bar{\Gamma}_1^\epsilon = \Delta\bar{\Gamma}_1^{fin} = 0$. To remove the UV divergences, we take $\bar{\Gamma}_1^{ct}$ as in Eq. (5.6), with X given by

$$X = \int d^4x\, \omega_{ab} \left(c_2\, G_T^a \tilde{A}_T^b + c_3\, H_T^a \tilde{c}_T^b \right) \tag{5.10}$$

and c_2 and c_3 real coefficients. The counterterm $\bar{\Gamma}_1^{ct}$ then produces the contributions listed in the right column of Table 1. We choose c_1, c_2 and c_3 so that $\bar{\Gamma}_1^\epsilon + \bar{\Gamma}_1^{ct} = 0$. This corresponds to a minimal subtraction scheme and defines a finite renormalized effective action

$$\bar{\Gamma} = \lim_{\epsilon \to 0} \left(\bar{\Gamma}_0 + \bar{\Gamma}_1^{fin} \right).$$

---

[2] Both the BRS operator and the Slavnov–Taylor operator have mass dimension one and ghost number 1.

Using the results in Table 1, we have

$$c_1 = \frac{22}{3}\, C_\epsilon \quad c_2 = -\frac{\alpha+3}{2}\, C_\epsilon\,, \quad c_3 = -\alpha\, C_\epsilon\,, \qquad (5.11)$$

with C_ϵ as in Eq. (4.7). These values for c_1, c_2 and c_2 are twice those for SU(N) Yang–Mills theory. This implies in particular that the first coefficient of the beta function for $SU(N)_\ltimes$ is $-22/3$, rather than the usual $-11/3$.

Multiplicative renormalization. The subtraction performed by the counterterm $\bar\Gamma_1^{ct}$ in Eqs. (5.6) and (5.10) is equivalent to multiplicative renormalization. To see this, recall that in multiplicative renormalization, the fields and the coupling constant in the tree-level action $\bar\Gamma_0$ in Eq. (5.4) are regarded as bare fields $\{A_{0\mu}, c_0, G_{0\mu}, H_0\}$ and bare coupling constant g_0. Renormalized quantities are then introduced through the equations

$$\begin{aligned}
A_{0T\mu}^a &= Z_A^T\, \tilde A_{T\mu}^a\,, & A_{0Z\mu}^a &= Z_A^Z\, \tilde A_{Z\mu}^a\,, \\
c_{0T}^a &= Z_c^T\, \tilde c_T^a\,, & c_{0Z}^a &= Z_c^Z\, \tilde c_Z^a\,, \\
G_{0T}^a &= Z_G^T\, G_T^a\,, & G_{0Z}^a &= Z_G^Z\, G_{Z\mu}^a\,, \\
H_{0T}^a &= Z_G^T\, H_T^a\,, & H_{0Z}^a &= Z_G^Z\, H_Z^a
\end{aligned}$$

and

$$g_0 = Z_g\, g\,.$$

Writing every renormalization constant as $Z = 1 + \delta Z$, with δZ first-order in perturbation theory, the action $\bar\Gamma_0[\Psi_0, g_0]$ is recast as

$$\begin{aligned}
\bar\Gamma_0[\tilde A_0, \tilde c_0, G_0, H_0, g_0] &= \bar\Gamma_0[\tilde A, \tilde c, G, H, g] \\
&\quad + \delta\bar\Gamma_0[\tilde A, \tilde c, G, H, g]\,,
\end{aligned}$$

where the counterterm $\delta\bar\Gamma_0[\cdots]$ collects all contributions of order one,

$$\begin{aligned}
\delta\bar\Gamma_0[\tilde A, \tilde c, G, H, g] &= \frac{1}{g^2}\int \mathrm{d}^4x\, \omega_{ab} \\
&\times \left[\frac{1}{2}\left(2\,\delta Z_A^T - 2\,\delta Z_g\right)\tilde A_T^{a\mu}\left(\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)\tilde A_T^{b\nu} + \cdots\right].
\end{aligned}$$

The requirement that $\delta\bar\Gamma_0[\tilde\Psi, g]$ must cancel the UV divergences fixes

$$\delta Z_g = -\frac{g^2 c_1}{2}\,, \qquad \delta Z_c^T + \delta Z_G^T = -g^2(c_1 + c_2 + c_3)\,,$$
$$\delta Z_A^T = g^2 c_2\,, \qquad \delta Z_H^T + 2\,\delta Z_c^T = g^2(c_3 - c_1)\,,$$

where c_1, c_2, and c_3 are the coefficients in the right column in Table 1. Since there are no one-loop 1PI diagrams with Z external legs, there are no conditions for δZ_Ψ^Z. The question then arises as to what is the meaning of the Z terms in $\delta\bar\Gamma_0[\tilde\Psi, g]$. Our analysis above indicates that they should be cohomologically trivial with respect to the Slavnov–Taylor operator. This is indeed the case. A long but straightforward

calculation shows that, for

$$\delta Z_A^Z = \delta Z_A^T\,, \quad \delta Z_c^Z = \delta Z_c^T\,, \quad \delta Z_G^Z = \delta Z_G^T\,, \quad \delta Z_H^Z = \delta Z_H^T\,,$$

the functional $\delta\bar\Gamma_0[\tilde A, \tilde c, G, H, g]$ can be written as

$$\delta\bar\Gamma_0[\tilde A, \tilde c, G, H, g] = c_1 S_{TT} + \Delta(X + U)\,, \qquad (5.12)$$

where X is given by Eq. (5.10) and U has the form

$$\begin{aligned}
U = \int \mathrm{d}^4x\, \omega_{ab} &\Big[(c_1 + c_2)\, G_T^a \tilde A_Z^b - (c_1 + 2c_2 + c_3)\, H_T^a \tilde c_Z^b \\
&+ c_2\, G_Z^a \tilde A_T^b - c_3\, H_Z^a \tilde c_T^b\Big].
\end{aligned}$$

The difference between the counterterms $\delta\bar\Gamma_0$ above and $\bar\Gamma_1^{ct}$ in Eqs. (5.6) and (5.10) is ΔU, which is cohomologically trivial.

All in all, the only gauge invariant radiative corrections are those in c_1, which account for a renormalization of the coupling constant. This introduces a renormalization scale in the quantum effective action and the quantum theory is asymptotically free.

6 Discussion

The pattern observed for the gauge invariant degrees of freedom in the quantum theory resembles very much that for the self-antiself dual instantons of the classical theory [16]. In the classical case, the number of collective coordinates of the $G_\ltimes$ instantons is twice that of the embedded G instantons, yet $\omega_{ab} F_{T\mu\nu}^a F_Z^{b\mu\nu}$ does not contribute to the instanton number. Now the gauge invariant radiative corrections are doubled and $\omega_{ab} F_{T\mu\nu}^a F_Z^{b\mu\nu}$ is cohomologically trivial with respect to the Slavnov–Taylor operator.

Our discussion may have some implications for Yang–Mills theories with more general nonreductive real metric Lie algebras. There is a structure theorem [17] that states that all real metric Lie algebras are direct products of Abelian algebras, simple real Lie algebras, and double extensions $\mathfrak{d}(\mathfrak{h}, \mathfrak{g})$ of a real metric Lie algebra $\mathfrak{h}$ by an algebra $\mathfrak{g}$.[3] The double extension $\mathfrak{d}(\mathfrak{h}, \mathfrak{g})$ is obtained [14,17] by forming the classical double $\mathfrak{g}_\ltimes$ and then by acting with $\mathfrak{g}$ on $\mathfrak{h}$ via antisymmetric derivations. Incidentally we mention that the classical double $\mathfrak{g}_\ltimes$ can be viewed as the double extension of the trivial algebra by $\mathfrak{g}$.

According to the theorem, since $\mathfrak{h}$ must be metric, three possibilities must be considered for $\mathfrak{h}$ in forming double extensions $\mathfrak{d}(\mathfrak{h}, \mathfrak{g})$. The first one is that $\mathfrak{h}$ is a simple real Lie algebra. In this case [14], the algebra of antisymmetric derivations of $\mathfrak{h}$ is $\mathfrak{h}$ itself and the double extension $\mathfrak{d}(\mathfrak{h}, \mathfrak{h})$ is isomorphic to the direct product $\mathfrak{h} \times \mathfrak{h}_\ltimes$. The resulting

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[3] The theorem goes further and specifies the nature of $\mathfrak{g}$ in the double extension.

Yang–Mills theory then separates into two Yang–Mills theories, not interacting with each other, one with gauge group H and one with group $H_\ltimes$. The second possibility is that $\mathfrak{h}$ is Abelian, of dimension m. Being Abelian, any nondegenerate, symmetric bilinear form on $\mathfrak{h}$ is a metric, and it can always be brought to a diagonal form with all the entries in the diagonal equal to $+1$ and -1. If the number of occurrences of $+1$ is p, and the number of occurrences of -1 is q, the algebra of antisymmetric derivations of $\mathfrak{h}$ is any subalgebra of $\mathfrak{so}(p, q)$ [14]. Many of the nonsemisimple WZW models considered in the literature [8–13] and their four-dimensional Yang–Mills analogs [15] fall into this class. In this instance unitarity remains an open problem. We think that a thorough analysis of the corresponding Slavnov–Taylor operator should shed some light on the problem. The third possibility for $\mathfrak{h}$ is that it is a double extension, which takes us back to the starting point.

Appendix

The action of the Slavnov–Taylor operator (5.5) on the fields $(\tilde{A}^a_{\text{T}\mu}, \tilde{A}^a_{\text{Z}\mu})$ and $(\tilde{c}^a_\text{T}, \tilde{c}^a_\text{Z})$ is given by

$$\Delta \tilde{A}^a_{\text{T}\mu} = -\tilde{D}_\mu \tilde{c}^a_\text{T},$$
$$\Delta \tilde{A}^a_{\text{Z}\mu} = -\left(\tilde{D}_\mu \tilde{c}^a_\text{Z} + f_{bc}{}^a \tilde{A}^b_{\text{Z}\mu} \tilde{c}^c_\text{T}\right),$$
$$\Delta \tilde{c}^a_\text{T} = \frac{1}{2} f_{bc}{}^a \tilde{c}^b_\text{T} \tilde{c}^c_\text{T},$$
$$\Delta \tilde{c}^a_\text{Z} = f_{bc}{}^a \tilde{c}^b_\text{T} \tilde{c}^c_\text{Z},$$

Modulo an irrelevant overall sign, these expressions generalize the classical BRS operator s in Eqs. (3.9) and (3.10). The action on $(G^a_{\text{T}\mu}, G^a_{\text{Z}\mu})$ and $(H^a_\text{T}, H^a_\text{Z})$ is in turn

$$\Delta G^a_{\text{T}\mu} = -\left(\tilde{D}^\rho \tilde{F}^a_{\text{T}\rho\mu} + f^{abc} G^b_{\text{T}\mu} \tilde{c}^c_\text{T}\right),$$
$$\Delta G^a_{\text{Z}\mu} = -\left(\tilde{D}^\rho \tilde{F}^a_{\text{Z}\rho\mu} + f_{bc}{}^a \tilde{A}^{b\rho}_\text{Z} \tilde{F}^c_{\text{T}\rho\mu}\right)$$
$$\qquad + f_{bc}{}^a \left(G^b_{\text{Z}\mu} \tilde{c}^c_\text{T} + G^b_{\text{T}\mu} \tilde{c}^c_\text{Z}\right),$$

$$\Delta H^a_\text{T} = \tilde{D}_\mu G^{a\mu}_\text{T} - f_{bc}{}^a H^b_\text{T} \tilde{c}^c_\text{T},$$
$$\Delta H^a_\text{Z} = \tilde{D}_\mu G^{a\mu}_\text{Z} - f_{bc}{}^a G^{b\mu}_\text{T} \tilde{A}^{c\mu}_\text{Z} - f_{bc}{}^a \left(H^b_\text{Z} \tilde{c}^c_\text{T} + H^b_\text{T} \tilde{c}^c_\text{Z}\right).$$

With this, it is matter of algebra to check that (i) the action of Δ on $\bar{\Gamma}^\text{ct}_1$ in Eqs. (5.6) and (5.10) produces the terms in the right column of Table 1, (ii) that S_{TZ} in Eq. (5.8) equals ΔY, with Y given by Eq. (5.9), and (iii) that $\delta \bar{\Gamma}_0[\tilde{A}, \tilde{c}, G, H, g]$ can be written as in Eq. (5.12).

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