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Unified dark energy and dust dark matter dual to quadratic purely kinetic K-essence

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Abstract We consider a modified gravity plus single scalar-field model, where the scalar Lagrangian couples symmetrically both to the standard Riemannian volume-form (spacetime integration measure density) given by the square root of the determinant of the Riemannian metric, as well as to another non-Riemannian volume-form in terms of an auxiliary maximal-rank antisymmetric tensor gauge field. As shown in a previous paper, the pertinent scalar-field dynamics provides an exact unified description of both dark energy via dynamical generation of a cosmological constant, and dark matter as a "dust" fluid with geodesic flow as a result of a hidden Noether symmetry. Here we extend the discussion by considering a non-trivial modification of the purely gravitational action in the form of $f(R) = R - \alpha R^2$ generalized gravity. Upon deriving the corresponding "Einstein-frame" effective action of the latter modified gravity-scalar-field theory we find explicit duality (in the sense of weak versus strong coupling) between the original model of unified dynamical dark energy and dust fluid dark matter, on one hand, and a specific quadratic purely kinetic "k-essence" gravity–matter model with special dependence of its coupling constants on only two independent parameters, on the other hand. The canonical Hamiltonian treatment and Wheeler–DeWitt quantization of the dual purely kinetic "k-essence" gravity–matter model is also briefly discussed.

1 Introduction

The unified description of dark energy and dark matter as a manifestation of a single entity of matter has been an important challenge in cosmology in the last decade or so (for extensive reviews of dark energy see [1–3], and for reviews of dark matter see [4–6]).

Originally, a unified treatment of dark energy and dark matter was proposed in the "Chaplygin gas" models [7–10]. Another trend aimed at unifying dark energy and dark matter is based on the class of "k-essence" models [11–14], in particular, on the so-called "purely kinetic k-essence" models [15] (for further developments, see [16–20]), which successfully avoid difficulties inherent in the generalized Chaplygin gas models related to the non-negligible sound speed.

Also, recently a lot of interest has been attracted by the so-called "mimetic" dark matter model proposed in [21,22]. The latter employs a special covariant isolation of the conformal degree of freedom in Einstein gravity, whose dynamics mimics cold dark matter as a pressureless "dust". Further generalizations and extensions of "mimetic" gravity are studied in Refs. [23,24].

Models of explicitly coupled dark matter and dark energy described in terms of two different scalar fields were proposed in Ref. [25].

In this paper we study a class of generalized models of gravity interacting with a single scalar field employing the method of non-Riemannian volume-forms on the pertinent spacetime manifold, i.e., generally covariant integration measure densities independent of the standard Riemannian one given in terms of the square root of the determinant of the metric [26–29]. (For further developments, see Ref. [30].) In this general class of models, also called "two-measure gravity theories", the non-Riemannian volume-forms are defined in terms of auxiliary maximal-rank antisymmetric tensor gauge fields ("measure gauge fields").

The introduction of the two integration measures (one standard Riemannian and the other a non-Riemannian one) opens the possibility to obtain both dark energy and dark matter from a single scalar field dynamics, as already observed in Ref. [31]. Subsequently, in Ref. [32] we have gone further and have discovered the fundamental reason that a class of mod-

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els generalizing those studied in [31] describes a unification of dark matter and dark energy as an exact sum of two separate contributions in the pertinent energy-momentum tensor (see also the review in Sect. 2 below). This is because of:

1. The appearance of an arbitrary integration constant from a dynamical constraint on the scalar Lagrangian as a result of the equations of motion for the auxiliary "measure" gauge field. This integration constant is identified as a dynamically generated cosmological constant which provides the dark energy component.
2. The existence of a hidden Noether symmetry of the non-Riemannian-measure-modified scalar Lagrangian implying a conserved Noether current, which produces a dark matter component as a "dust" fluid flowing along geodesics.

This behavior is totally independent of the specific form of the scalar field Lagrangian, as long as the scalar field couples in a symmetric way to both of the measures. The latter also ensures that the hidden "dust" Noether symmetry holds. In addition, the fact that the dynamically generated cosmological constant arises as an arbitrary integration constant makes the observed vacuum energy density totally decoupled from the parameters of the initial scalar Lagrangian.

In the present paper our main object of study is a nontrivial extension of the modified gravity plus single scalar-field model considered in [32], describing unification of dark matter and dark energy. Namely, we now modify the purely gravitational part of the action by introducing an extended $f(R)$-gravity action with $f(R) = R - \alpha R^2$ within the first-order Palatini formalism.[1]

This new non-Riemannian-measure-modified gravity-scalar-field theory has some very interesting features. On one hand, the hidden "dust" Noether symmetry of the scalar field Lagrangian remains intact and so does the picture of unified description through the pertinent energy-momentum tensor of both the dynamical dark energy (via the dynamical generation of the cosmological constant) and the "dust" dark matter fluid, i.e., they appear as an exact sum of two separate contributions to the corresponding energy density.

On the other hand, upon performing a transition to the effective "Einstein frame" and upon the appropriate scalar-field redefinition we arrive at a dual theory of a specific quadratic purely kinetic "k-essence" form. In the dual purely kinetic "k-essence" formulation the original "dust" Noether symmetry is replaced by a simple shift symmetry of the transformed scalar field.

It is essential to put stress on the following important properties of the dual theory:

(a) All three constant coefficients in the quadratic purely kinetic "k-essence" action are given in terms of only two independent parameters: α (the R^2 coupling constant) and an arbitrary integration constant M produced by a dynamical constraint resulting from the equation of motion of the "measure" gauge field in the original theory.
(b) Let us emphasize that in the present context applying the notion of "duality" is justified in the usual sense of weak coupling in the original theory versus strong coupling in the dual theory. Indeed, as we will see below (Eq. (39)) all coupling constants in the dual quadratic purely kinetic "k-essence" theory are functions of $1/\alpha$.
(c) In the limit $\alpha \to 0$ (strong coupling limit in the dual "k-essence" theory) the physical quantities—energy density, pressure etc. have well-defined smooth limiting values in spite of the singularities in the coefficients of the "k-essence" action. The latter limiting values coincide with the corresponding values in the original gravity–matter theory with a standard Einstein–Hilbert gravity action (where $\alpha = 0$). Thus, the established duality reveals an important feature of the approximate description of unified dark energy and dark matter via the quadratic purely kinetic "k-essence" model (Eq. 40 below)—it becomes an exact unification of both dark species in the strong coupling limit $\alpha \to 0$.
(d) In the limit $\alpha \to 0$ the dual theory parameter M precisely coincides with the dynamically generated effective cosmological constant in the original theory [32]. Thus, in this sense we can say that the dual purely kinetic "k-essence" theory is (partially) dynamically generated.

The above list of properties epitomizes the new features in our treatment of the purely quadratic "k-essence" theory w.r.t. to the previous treatment in [15].

Now, an important remark is in order. In a number of papers dealing with "dust-like" dark matter one uses the so-called "Lagrangian multiplier gravity" formalism [34,35]. We would like to point out that this formalism is in fact a special particular case of the above mentioned more general and powerful approach based on non-Riemannian spacetime volume-forms (being used also in the present paper), which appeared earlier and which has a profound impact in any (field theory) models with general coordinate reparametrization invariance, such as general relativity and its extensions [26–32,36,37], strings and (higher-dimensional) membranes [38], and supergravity [39].

Indeed, dynamical constraints like the one on the scalar-field Lagrangian in Eq. (8) below, which routinely appear in all instances of applying the non-Riemannian volume-

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[1] Let us recall that $R + R^2$ gravity within the second-order formalism (which was also the first inflationary model) was originally proposed in Ref. [33].

form method in gravity–matter theories, resembles at first sight analogous constraints on scalar-field Lagrangians in the "Lagrangian multiplier gravity" [34,35]. However, employing the concept of non-Riemannian volume-forms in the form of dual field strengths of auxiliary maximal-rank tensor gauge fields ("measure gauge fields") as in Eq. (3) below, instead of bare Lagrange multiplier fields, has certain essential advantages:

1. Dynamical constraints in "two-measure" gravity–matter theories result from the equations of motion of the auxiliary "measure" gauge fields and, thus, they always involve an arbitrary integration constant like M in Eq. (8) below, as opposed to picking some a priori fixed constant within the "Lagrange multiplier gravity" formalism. Depending on the specific gravity–matter theories with non-Riemannian volume-forms under consideration, the pertinent arbitrary integration constant acquires the meaning of a dynamically generated cosmological constant like the integration constant M below (cf. Ref. [26,39]).

2. Employing the canonical Hamiltonian formalism for Dirac-constrained systems we find that the auxiliary "measure" gauge fields are in fact almost pure gauge degrees of freedom except for the above mentioned arbitrary integration constants which are identified with the conserved Dirac-constrained canonical momenta conjugated to the "magnetic" components of the "measure" gauge fields (see Appendix A in Ref. [37]).

3. Upon applying the non-Riemannian volume-form formalism to minimal $N = 1$ supergravity the dynamically generated cosmological constant triggers spontaneous supersymmetry breaking and mass generation for the gravitino (supersymmetric Brout–Englert–Higgs effect) [39]. Applying the same formalism to anti-de Sitter supergravity allows one to produce simultaneously a very large physical gravitino mass and a very small positive observable cosmological constant [39] in accordance with modern cosmological scenarios for the slowly expanding universe of the present epoch [1–3].

4. Employing two independent non-Riemannian volume-forms in generalized gravity-gauge+scalar-field models [36], thanks to the appearance of several arbitrary integration constants through the equations of motion w.r.t. the "measure" gauge fields, we obtain a remarkable effective scalar potential with two infinitely large flat regions (one for large negative and another one for large positive values of the scalar field φ) with vastly different scales appropriate for a unified description of both the early and the late universe's evolution. An interesting feature is the existence of a stable initial phase of non-singular universe creation preceding the inflationary phase—a stable "emergent universe" without "Big-Bang" [36].

The plan of the paper is as follows. In Sect. 2, for self-consistency of the exposition, we briefly review [32], namely, the basic aspects of the non-Riemannian volume-form approach leading to dynamical generation of a cosmological constant (dynamical dark energy) and revealing a hidden Noether symmetry allowing for a "dust" fluid representation of dark matter, such that both dark species appear as a sum of two separate contributions to the energy-momentum tensor. Section 3 contains the main result—establishing duality (in the standard sense of weak versus strong coupling) between the original quadratic $f(R)$-gravity plus a modified-measure scalar-field model, whose matter part yields an exact unified description of dynamical dark energy and dust fluid dark matter, on one hand, and a specific quadratic purely kinetic "k-essence" gravity–matter model, on the other hand. In Sect. 4 we analyze the above models within the canonical Dirac-constrained Hamiltonian formalism and derive the corresponding quantum Wheeler–DeWitt equations in the cosmological Friedmann–Lemaitre–Robertson–Walker framework.

2 Gravity–matter theory with a non-Riemannian volume-form in the scalar-field action: exact unification of dark energy and dust fluid dark matter

2.1 Non-Riemannian volume-form formalism

We start by considering the following non-conventional gravity-scalar-field action—a particular case of the general class of the so called "two-measure" gravity–matter theories [27–29] (for simplicity we use units with the Newton constant $G_N = 1/16\pi$):

$$S = S_{\text{grav}}[g_{\mu\nu}, \Gamma^\lambda_{\mu\nu}] + \int \mathrm{d}^4x \left(\sqrt{-g} + \Phi(B)\right) L(\varphi, X). \quad (1)$$

The notations used here and below are as follows:

- The first term in (1) is the purely gravitational action in the first-order (Palatini) formalism, where the Riemannian metric $g_{\mu\nu}$ and the affine connection $\Gamma^\lambda_{\mu\nu}$ are a priori independent variables. In the previous paper [32] we have studied (1) with the simplest choice of S_{grav}:

$$S = \int \mathrm{d}^4x \sqrt{-g}\, R + \int \mathrm{d}^4x \left(\sqrt{-g} + \Phi(B)\right) L(\varphi, X), \quad (2)$$

where R denotes the scalar curvature—in this case the first-order (Palatini) formalism is equivalent to the ordinary second-order (metric) formalism.

- The second term in (1)—the scalar-field action—is constructed in terms of two mutually independent volume-forms:

  (a) $\sqrt{-g} \equiv \sqrt{-\det \|g_{\mu\nu}\|}$ is the standard Riemannian integration measure density (spacetime volume-form);

  (b) $\Phi(B)$ denotes an alternative non-Riemannian generally covariant integration measure density defining an alternative non-Riemannian volume-form:

$$\Phi(B) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu B_{\nu\kappa\lambda}, \tag{3}$$

  where $B_{\mu\nu\lambda}$ is an auxiliary maximal-rank antisymmetric tensor gauge field independent of the Riemannian metric. $B_{\mu\nu\lambda}$ (3) is called a "measure gauge field".

- $L(\varphi, X)$ is the general coordinate invariant Lagrangian of a single scalar field $\varphi(x)$, which can be of an arbitrary generic "k-essence" form [11–14]:

$$L(\varphi, X) = \sum_{n=1}^{N} A_n(\varphi)X^n - V(\varphi), \tag{4}$$

$$X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi,$$

  i.e., a nonlinear (in general) function of the scalar kinetic term X.

In this section we will concentrate on the scalar-field action—the second term in (1). First, due to general coordinate invariance we have covariant conservation of the pertinent energy-momentum tensor:

$$T_{\mu\nu} = g_{\mu\nu}L(\varphi, X) + \left(1 + \frac{\Phi(B)}{\sqrt{-g}}\right)\frac{\partial L}{\partial X}\partial_\mu\varphi\,\partial_\nu\varphi, \tag{5}$$

$$\nabla^\nu T_{\mu\nu} = 0.$$

It follows from the equation of motion w.r.t. φ that

$$\frac{\partial L}{\partial \varphi} + (\Phi(B)$$
$$+\sqrt{-g})^{-1}\partial_\mu\left[(\Phi(B) + \sqrt{-g})\,g^{\mu\nu}\partial_\nu\varphi\frac{\partial L}{\partial X}\right] = 0. \tag{6}$$

Further, variation of the action (1) w.r.t. the "measure" gauge field $B_{\mu\nu\lambda}$ reads

$$\partial_\mu L(\varphi, X) = 0, \tag{7}$$

i.e., the $B_{\mu\nu\lambda}$-equations of motion yield the following dynamical constraint on the scalar-field Lagrangian:

$$L(\varphi, X) = -2M = \text{const}, \tag{8}$$

where M is arbitrary integration constant. The factor 2 in front of M is for later convenience; moreover, we will take $M > 0$ in view of its interpretation as a dynamically generated cosmological constant [see Eqs. (10), (16), and (19) below].

It is important to stress that the scalar-field dynamics is determined entirely by the first-order differential equation—the dynamical constraint Eq. (8), which, in the simplest case of (4) ($L(\varphi, X) = X - V(\varphi)$) to be considered henceforth for simplicity, implies

$$X - V(\varphi) = -2M \longrightarrow X = V(\varphi) - 2M. \tag{9}$$

The standard second-order differential equation (6) is in fact a consequence of (8) together with the energy-momentum conservation $\nabla^\mu T_{\mu\nu} = 0$ with

$$T_{\mu\nu} = -2Mg_{\mu\nu} + \left(1 + \frac{\Phi(B)}{\sqrt{-g}}\right)\partial_\mu\varphi\partial_\nu\varphi. \tag{10}$$

The physical meaning of the "measure" gauge field $B_{\mu\nu\lambda}$ (3) as well as the meaning of the integration constant M are most straightforwardly seen within the canonical Hamiltonian treatment of (the scalar-field part of) (1)—this is systematically derived in Sect. 2 of Ref. [32]. Namely, using short-hand notations for the components of $B_{\mu\nu\lambda}$ (3):

$$\Phi(B) = \partial_\mu\mathcal{B}^\mu = \dot{\mathcal{B}} + \partial_i\mathcal{B}^i, \quad \mathcal{B}^\mu \equiv \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}B_{\nu\kappa\lambda}, \tag{11}$$

$$\mathcal{B} \equiv \mathcal{B}^0 = \frac{1}{3!}\varepsilon^{mkl}B_{mkl}, \quad \mathcal{B}^i \equiv -\frac{1}{2}\varepsilon^{ikl}B_{0kl}, \tag{12}$$

we obtain for the canonically conjugated momenta $\pi_\mathcal{B}, \pi_{\mathcal{B}i}$ the set of Dirac first-class constraints:

$$\pi_{\mathcal{B}i} = 0, \quad \partial_i\pi_\mathcal{B} = 0 \longrightarrow \pi_\mathcal{B} = \text{const} \equiv -2M, \tag{13}$$

where we have $\pi_\mathcal{B} = L(\varphi, X)$ straightforwardly obtainable from the explicit form of the action (1) taking into account the representation of $\Phi(B)$ in (11). The above Dirac constraints imply that all components of the "measure" gauge field $B_{\mu\nu\lambda}$ (12) are pure gauge (non-propagating) degrees of freedom. The last relation in (13) is the canonical Hamiltonian analog of Eq. (8) within the Lagrangian formalism—in other words, the integration constant M in the $B_{\mu\nu\lambda}$-equations of motion is (modulo a trivial numerical factor) a conserved Dirac-constrained canonical momentum $\pi_\mathcal{B}$ conjugated to the "magnetic" component $\mathcal{B}$ of the "measure"-gauge field $B_{\mu\nu\lambda}$ (12).

For more details as regards the canonical Hamiltonian treatment of general gravity–matter theories with (several independent) non-Riemannian volume-forms we refer to [37].

2.2 "Dust" fluid conservation laws

In Ref. [32] we have shown that the scalar-field action in (1) possesses a hidden Noether symmetry, namely (1) is invariant (up to a total derivative) under the following nonlinear symmetry transformations:

$$\delta_\epsilon \varphi = \epsilon \sqrt{X}, \quad \delta_\epsilon g_{\mu\nu} = 0,$$
$$\delta_\epsilon B_{\mu\nu\lambda} = -\epsilon \frac{1}{2\sqrt{X}} \varepsilon_{\mu\nu\lambda\kappa} g^{\kappa\rho} \partial_\rho \varphi \left(\Phi(B) + \sqrt{-g} \right). \tag{14}$$

Then the standard Noether procedure yields the conserved current:

$$\nabla_\mu J^\mu = 0, \quad J^\mu \equiv \left(1 + \frac{\Phi(B)}{\sqrt{-g}} \right) \sqrt{2X} g^{\mu\nu} \partial_\nu \varphi \frac{\partial L}{\partial X}. \tag{15}$$

Let us stress at this point that the existence of the hidden symmetry (14) of the action (1) does not depend on the specific form of the scalar-field Lagrangian (4). The only requirement is that the kinetic term X must be positive.

$T_{\mu\nu}$ (5) and J^μ (15) can be cast into the a relativistic hydrodynamical form (taking into account (8)):

$$T_{\mu\nu} = -2M g_{\mu\nu} + \rho_0 u_\mu u_\nu, \quad J^\mu = \rho_0 u^\mu, \tag{16}$$

where

$$\rho_0 \equiv \left(1 + \frac{\Phi(B)}{\sqrt{-g}} \right) 2X \frac{\partial L}{\partial X}, \tag{17}$$

$$u_\mu \equiv \frac{\partial_\mu \varphi}{\sqrt{2X}} \quad (\text{note } u^\mu u_\mu = -1). \tag{18}$$

For the pressure p and energy density ρ we have accordingly

$$p = -2M = \text{const}, \tag{19}$$

$$\rho = \rho_0 - p = \left(1 + \frac{\Phi(B)}{\sqrt{-g}} \right) 2X \frac{\partial L}{\partial X} + 2M, \tag{20}$$

where the integration constant M appears as dynamically generated cosmological constant.

Let us note that constancy (19) of the pressure $p = -2M$ in (16) together with the covariant conservation of $T_{\mu\nu}$:

$$\nabla^\nu T_{\mu\nu} = \nabla^\nu \left(\rho_0 u_\mu u_\nu \right)$$
$$= u_\mu \nabla^\nu \left(\rho_0 u_\nu \right) + \rho_0 \left(u_\nu \nabla^\nu u_\mu \right) = 0, \tag{21}$$

upon projecting (21) along the "velocity" vector u_μ and orthogonally w.r.t. the latter by $\Pi^{\mu\lambda} = g^{\mu\lambda} + u^\mu u^\lambda$), immediately implies both the covariant conservation of J^μ (15):

$$\nabla_\mu \left(\rho_0 u^\mu \right) = 0, \tag{22}$$

and the geodesic flow equation:

$$u_\nu \nabla^\nu u_\mu = 0. \tag{23}$$

As discussed in [32] the energy-momentum tensor (16) consists of two parts with the following interpretation according to the standard Λ-CDM model [40–42] [using the notations $p = p_{DM} + p_{DE}$ and $\rho = \rho_{DM} + \rho_{DE}$ in (19)–(20)]:

- A dark energy part given by the first cosmological constant term in $T_{\mu\nu}$ (16), which arises due to the dynamical constraint on the scalar-field Lagrangian (8) with $p_{DE} = -2M$, $\rho_{DE} = 2M$;

- A dark matter part given by the second term in (16) with $p_{DM} = 0$, $\rho_{DM} = \rho_0$ [ρ_0 as in (17)], which in fact describes a dust fluid. According to the general definitions, see e.g. [43], $\rho_{DM} = \rho_0$ (17) and $\rho_{DE} = 2M$ are the rest-mass and internal fluid energy densities, so that the Noether conservation law (15) describes dust dark matter "particle number" conservation.

3 Quadratic gravity interacting with a dark energy-dark matter unifying scalar field: dual to purely kinetic "K-Essence"

3.1 Derivation of the dual kinetic pure "K-Essence" theory

Let us now consider a modification of the gravitational part of the gravity-scalar-field action (2) as follows:

$$S = \int d^4 x \sqrt{-g} \left(R(g, \Gamma) - \alpha R^2(g, \Gamma) \right)$$
$$+ \int d^4 x \left(\sqrt{-g} + \Phi(B) \right) L(\varphi, X), \tag{24}$$

where we have introduced $f(R) = R - \alpha R^2$ extended gravity action in the first-order Palatini formalism:

$$R(g, \Gamma) = g^{\mu\nu} R_{\mu\nu}(\Gamma), \tag{25}$$

i.e., with an a priori independent metric $g_{\mu\nu}$ and affine connection $\Gamma^\mu_{\nu\lambda}$.

Clearly, since the scalar-field action—the second term in (24)—remains the same as in the original action (2), all results in Sect. 2.2 remain valid.[2] In other words, the modified gravity-scalar-field action (24) possesses a "hidden" Noether symmetry (14) producing a "dust" fluid energy density conserved current (22) and we have the interpretation of φ as describing simultaneously dark energy and dust dark matter with geodesic dust fluid flow (23).

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[2] Adding the bare cosmological constant term $-2\Lambda_0 \sqrt{-g}$ to the gravity action in (24) is irrelevant, since this is equivalent to a constant shift of the scalar Lagrangian $L(\varphi, X) \to L(\varphi, X) - 2\Lambda_0$ [recall that the non-Riemannian measure density $\Phi(B)$ is a total derivative (3)], which in turn amounts on-shell to a trivial redefinition of the arbitrary integration constant M ($M \to M + \Lambda_0$) in Eq. (8).

The gravitational equations of motion resulting from (24) are, however, not of the standard Einstein form. The equations of motion w.r.t. $g^{\mu\nu}$ read

$$R_{\mu\nu}(\Gamma) = \frac{1}{2f'_R}\left[T_{\mu\nu} + f(R)g_{\mu\nu}\right], \qquad (26)$$

$$f(R) = R(g,\Gamma) - \alpha R^2(g,\Gamma), \quad f'_R = 1 - 2\alpha R(g,\Gamma), \qquad (27)$$

with $T_{\mu\nu}$ the same as in (10). The trace of Eqs. (26) yields

$$R(g,\Gamma) = -\frac{1}{2}T, \quad T = g^{\mu\nu}T_{\mu\nu}, \qquad (28)$$

$$T = -8M - \left(1 + \frac{\Phi(B)}{\sqrt{-g}}\right)2\left(V(\varphi) - 2M\right). \qquad (29)$$

Variation of (24) w.r.t. $\Gamma^{\mu}_{\nu\lambda}$ yields

$$\int d^4x\,\sqrt{-g}\,g^{\mu\nu}f'_R\left(\nabla_\kappa\delta\Gamma^\kappa_{\mu\nu} - \nabla_\mu\delta\Gamma^\kappa_{\kappa\nu}\right) = 0, \qquad (30)$$

which shows, following the analogous derivation in Ref. [27], that $\Gamma^{\mu}_{\nu\lambda}$ becomes a Levi-Civita connection:

$$\Gamma^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda}(\overline{g}) = \frac{1}{2}\overline{g}^{\mu\kappa}\left(\partial_\nu\overline{g}_{\lambda\kappa} + \partial_\lambda\overline{g}_{\nu\kappa} - \partial_\kappa\overline{g}_{\nu\lambda}\right), \qquad (31)$$

w.r.t. to the Weyl-rescaled metric $\overline{g}_{\mu\nu}$:

$$\overline{g}_{\mu\nu} = f'_R\, g_{\mu\nu}. \qquad (32)$$

Before going over to the physical "Einstein frame" it is useful to perform the following φ-field redefinition:

$$\varphi \to \widetilde{\varphi} = \int \frac{d\varphi}{\sqrt{(V(\varphi) - 2M)}}, \qquad (33)$$

$$X \to \widetilde{X} = -\frac{1}{2}\overline{g}^{\mu\nu}\partial_\mu\widetilde{\varphi}\partial_\nu\widetilde{\varphi} = \frac{1}{f'_R}, \qquad (34)$$

where the latter relation follows from the Lagrangian dynamical constraint (9) together with (32).

Now, using Eqs. (27)–(29), which together with (34) imply:

$$\frac{1}{\widetilde{X}} = 1 - \alpha\left[8M + \left(1 + \frac{\Phi(B)}{\sqrt{-g}}\right)2\left(V(\varphi) - 2M\right)\right], \qquad (35)$$

as well as Eqs. (31)–(32) and (34) we can rewrite all equations of motion resulting from (24), in particular the quadratic $f(R)$-gravity Eqs. (26), in terms of the new metric $\overline{g}_{\mu\nu}$ (32) and the new scalar field $\widetilde{\varphi}$ (33) in the standard form of Einstein gravity equations:

$$\overline{R}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{R} = \frac{1}{2}\overline{T}_{\mu\nu} \qquad (36)$$

with the following notations:

- Here $\overline{R}_{\mu\nu}$ and $\overline{R}$ are the standard Ricci tensor and scalar curvature of the Einstein-frame metric (32).
- The Einstein-frame energy-momentum tensor

$$\overline{T}_{\mu\nu} = \overline{g}_{\mu\nu}\overline{L}_{\text{eff}} - 2\frac{\partial\overline{L}_{\text{eff}}}{\partial\overline{g}^{\mu\nu}} \qquad (37)$$

is given in terms of the following effective $\widetilde{\varphi}$-scalar field Lagrangian of a specific quadratic purely kinetic "k-essence" form:

$$\overline{L}_{\text{eff}}(\widetilde{X}) = A_2\widetilde{X}^2 - A_1\widetilde{X} + A_0 \qquad (38)$$

$$A_2 \equiv \frac{1}{4\alpha} - 2M, \quad A_1 \equiv \frac{1}{2\alpha}, \quad A_0 \equiv \frac{1}{4\alpha}. \qquad (39)$$

Let us stress that the three constant coefficients in (38) depend only on two independent parameters (α, M), the second one being a dynamically generated integration constant in the original theory (24).

Thus, we have established a duality between the modified-measure gravity-scalar-field theory (24) within the original $g_{\mu\nu}$ frame and the special quadratic purely kinetic "k-essence" theory within the conformally rescaled $\overline{g}_{\mu\nu}$ frame (Einstein frame):

$$S_{\text{k-ess}} = \int d^4\sqrt{-\overline{g}}\left[\overline{R} + \left(\frac{1}{4\alpha} - 2M\right)\widetilde{X}^2 - \frac{1}{2\alpha}\widetilde{X} + \frac{1}{4\alpha}\right]. \qquad (40)$$

with a matter Lagrangian (38)–(39).

The Einstein-frame effective energy-momentum tensor (37) in the perfect fluid representation reads [taking into account (38)–(39)]

$$\overline{T}_{\mu\nu} = \overline{g}_{\mu\nu}\widetilde{p} + \widetilde{u}_\mu\widetilde{u}_\nu\left(\widetilde{\rho} + \widetilde{p}\right), \qquad (41)$$

$$\widetilde{p} = \left(\frac{1}{4\alpha} - 2M\right)\widetilde{X}^2 - \frac{1}{2\alpha}\widetilde{X} + \frac{1}{4\alpha}, \qquad (42)$$

$$\widetilde{\rho} = 3\left(\frac{1}{4\alpha} - 2M\right)\widetilde{X}^2 - \frac{1}{2\alpha}\widetilde{X} - \frac{1}{4\alpha}, \qquad (43)$$

$$\widetilde{u}_\mu \equiv \frac{\partial_\mu\widetilde{\varphi}}{\sqrt{2\widetilde{X}}}, \quad \overline{g}^{\mu\nu}\widetilde{u}_\mu\widetilde{u}_\nu = -1.$$

Because of the obvious Noether symmetry of (40) under constant shift of $\widetilde{\varphi}$:

$$\widetilde{\varphi} \to \widetilde{\varphi} + \text{const} \qquad (44)$$

the corresponding Noether conservation law is identical to the $\widetilde{\varphi}$-equations of motion:

$$\overline{\nabla}_\mu\left(\overline{g}^{\mu\nu}\partial_\nu\widetilde{\varphi}\frac{\partial\widetilde{L}_{\text{eff}}}{\partial\widetilde{X}}\right) = 0, \qquad (45)$$

where $\overline{\nabla}_\mu$ indicates a covariant derivative w.r.t. Levi-Civita connection in the $\overline{g}_{\mu\nu}$ (Einstein) frame.

Introducing now the standard thermodynamical notions of enthalpy per unit particle $\widetilde{h}$ and particle number density $\widetilde{n}$ within the above Einstein-frame system, where $\widetilde{\rho} + \widetilde{p} = \widetilde{h}\widetilde{n}$, and taking into account the expressions (42)–(43), the Noether current conservation law (45) and the covariant conservation $\overline{\nabla}^\nu \widetilde{T}_{\mu\nu} = 0$ of (41) can be written, respectively, as

$$\overline{\nabla}_\mu \left(\widetilde{n}\widetilde{u}^\mu\right) = 0, \quad \widetilde{u}^\nu \overline{\nabla}_\nu \widetilde{u}^\mu + \widetilde{\Pi}_\mu^\nu \partial_\nu \ln \widetilde{h} = 0, \tag{46}$$

where

$$\widetilde{h} = \sqrt{2\widetilde{X}}, \quad \widetilde{\Pi}^{\mu\nu} = \overline{g}^{\mu\nu} + \widetilde{u}^\mu \widetilde{u}^\nu,$$
$$\widetilde{n} = \sqrt{2\widetilde{X}}\frac{\partial \widetilde{L}_{\text{eff}}}{\partial \widetilde{X}} = \sqrt{2\widetilde{X}}\left[\frac{1}{2\alpha}\left(\widetilde{X} - 1\right) - 4M\widetilde{X}\right]. \tag{47}$$

Comparing Eqs. (46) in the Einstein frame with the corresponding relations in the original $g_{\mu\nu}$-frame (22)–(23) we conclude that:

- Conservation of the dust dark matter energy density current (22) in the $g_{\mu\nu}$ frame is dual to the conservation of the particle number density current within the Einstein frame—first Eq. (46), which in fact is the standard $\widetilde{\varphi}$-equation of motion resulting from the action (40). That is, the "hidden" nonlinear Noether symmetry of (24) is dual to the shift symmetry (44) of (40).
- In the original $g_{\mu\nu}$ frame the dust dark matter flows along the geodesics (23), whereas in the dual Einstein frame the dual purely kinetic "k-essence" fluid does not any more flow along the geodesics [second Eq. (46)].

The purely kinetic "k-essence" theory (40) apart from the trivial vacuums $\widetilde{\varphi} = \text{const}$ possesses in addition a non-trivial "kinetic vacuum" solution $\widetilde{X}_{\text{vac}}$ of the equations of motion (45):

$$\left.\frac{\partial \widetilde{L}_{\text{eff}}}{\partial \widetilde{X}}\right|_{\widetilde{X}_{\text{vac}}} = 0 \quad \longrightarrow \quad \widetilde{X}_{\text{vac}} = \frac{1}{1 - 8\alpha M}, \tag{48}$$

implying the dark energy property [cf. (41)]:

$$(\widetilde{\rho} + \widetilde{p})\big|_{\widetilde{X}_{\text{vac}}} = 0, \quad \widetilde{\rho}\big|_{\widetilde{X}_{\text{vac}}} = \frac{2M}{1 - 8\alpha M} \equiv 2\Lambda_{\text{eff}} \tag{49}$$

with effective cosmological constant:

$$\Lambda_{\text{eff}} = \frac{M}{1 - 8\alpha M}. \tag{50}$$

The explicit form of (48) reads

$$\overline{g}^{\mu\nu}\partial_\mu\widetilde{\varphi}_{\text{vac}}\partial_\nu\widetilde{\varphi}_{\text{vac}} + \frac{2}{1 - 8\alpha M} = 0. \tag{51}$$

It has the form of the standard Hamilton–Jacobi equation for the action of a massive relativistic point-particle moving in a $\overline{g}_{\mu\nu}$-background with mass squared:

$$m_0^2 \equiv \frac{2}{1 - 8\alpha M}. \tag{52}$$

In other words $\widetilde{\varphi}_{\text{vac}}(x) = m_0 T$, where T is the proper-time for the particle to reach the spacetime point x from some reference point $x_{(0)}$.

3.2 FLRW reduction of the dual kinetic pure "K-Essence" theory

Let us now consider a reduction of the dual quadratic purely kinetic "k-essence" gravity-scalar-field model (40) for the Friedman–Lemaitre–Robertson–Walker (FLRW) class of metrics:

$$ds^2 = -N^2(t)dt^2 + a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{53}$$

Then the action (40) acquires the form (using again shorthand notations (39); in what follows we will take the spatial FLRW curvature $K = 0$ for simplicity):

$$S = \int dt \left[-\frac{a\dot{a}^2}{N} + \frac{N}{4}a^3\left(\frac{1}{\alpha} - \frac{1}{\alpha}\frac{\dot{\phi}^2}{N^2}\right.\right.$$
$$\left.\left. + \left(\frac{1}{4\alpha} - 2M\right)\frac{\dot{\phi}^4}{N^4}\right)\right], \tag{54}$$

with $\dot{\phi} \equiv \frac{d\widetilde{\varphi}}{dt}$. The $\phi \equiv \widetilde{\varphi}$-equation of motion from (54) yields (using henceforth the gauge $N = 1$):

$$\frac{dp_\phi}{dt} = 0 \quad \longrightarrow \quad p_\phi = a^3\left[-\frac{1}{2\alpha}\dot{\phi} + \left(\frac{1}{4\alpha} - 2M\right)\dot{\phi}^3\right], \tag{55}$$

where p_ϕ is the constant conserved canonically conjugated momentum of $\phi \equiv \widetilde{\varphi}$.

The equation of motion w.r.t. a resulting from (54)—the Friedmann equation—reads

$$\dot{a}^2 = \frac{1}{6}a^2\rho(p_\phi/a^3), \tag{56}$$

with $\rho(p_\phi/a^3)$ denoting the energy density as a function of $\frac{p_\phi}{a^3}$:

$$\rho(p_\phi/a^3) = -A_0 - \frac{1}{2}A_1\dot{\phi}^2(p_\phi/a^3) + \frac{3}{4}A_2\dot{\phi}^4(p_\phi/a^3)$$
$$= \frac{1}{8\alpha}\dot{\phi}^2(p_\phi/a^3) + \frac{3}{4}\frac{p_\phi}{a^3}\dot{\phi}(p_\phi/a^3) - \frac{1}{4\alpha}, \tag{57}$$

where in the second line of (57) relation (55) was used. Here $\dot{\phi}(p_\phi/a^3) \equiv y$ is one of the roots of the cubic equation (55):

$$y^3 - \frac{2}{1-8\alpha M}y - \frac{4\alpha}{1-8\alpha M}\frac{p_\phi}{a^3} = 0, \tag{58}$$

which explicitly read

$$y_1 = \tilde{A} + \tilde{B}, \quad y_{2,3} = -\frac{1}{2}\left(\tilde{A}+\tilde{B}\right) \pm i\frac{\sqrt{3}}{2}\left(\tilde{A}-\tilde{B}\right), \tag{59}$$

where

$$\tilde{A}, \tilde{B} \equiv \frac{2\alpha}{1-8\alpha M}\frac{p_\phi}{a^3}$$
$$\pm \sqrt{\left(\frac{2\alpha}{1-8\alpha M}\frac{p_\phi}{a^3}\right)^2 - \left(\frac{2}{3(1-8\alpha M)}\right)^3}. \tag{60}$$

Similarly, for the pressure we have

$$p(p_\phi/a^3) = A_0 - \frac{1}{2}A_1\dot{\phi}^2(p_\phi/a^3) + \frac{1}{4}A_2\dot{\phi}^4(p_\phi/a^3)$$
$$= -\frac{1}{8\alpha}\dot{\phi}^2(p_\phi/a^3) + \frac{p_\phi}{4a^3}\dot{\phi}(p_\phi/a^3) + \frac{1}{4\alpha}, \tag{61}$$

where again in the second line the cubic equation (58) has been used.

The Friedmann equation (56) can be solved approximately for small and large values of a using expressions (59)–(60):

$$\dot{\phi}(p_\phi/a^3) \simeq \left(\frac{2\alpha p_\phi}{1-8\alpha M}\right)^{1/3} a^{-1} \quad \text{for } a \to 0, \tag{62}$$

$$\dot{\phi}(p_\phi/a^3) \simeq \sqrt{\frac{2}{1-8\alpha M}} + \alpha\frac{p_\phi}{a^3} \quad \text{for } a \to \infty. \tag{63}$$

Accordingly, we have for the energy density (57):

$$\rho(p_\phi/a^3) \simeq \frac{3}{4}\left(\frac{4\alpha}{1-8\alpha M}\right)^{1/3} p_\phi^{4/3}a^{-4} \quad \text{for } a \to 0, \tag{64}$$

i.e. radiation domination for small a, and [using the notations (50) and (52)]:

$$\rho(p_\phi/a^3) \simeq \frac{2M}{1-8\alpha M} + \sqrt{\frac{2}{1-8\alpha M}}\frac{p_\phi}{a^3} \tag{65}$$

$$= 2\Lambda_{\text{eff}} + m_0\frac{p_\phi}{a^3} \quad \text{for } a \to \infty, \tag{66}$$

i.e., dark energy domination plus a subleading "dust" dark matter contribution for large a.

An important property of the above derivation of the duality between the original $g_{\mu\nu}$-frame quadratic $f(R)$-gravity plus non-Riemannian-modified-measure scalar-field action (24), on one hand, and the Einstein-frame special quadratic purely kinetic "k-essence" theory (40), on the other hand, is that there exists a smooth limit $\alpha \to 0$ of the energy density (57) and pressure (61) in the latter theory in spite of the singularity at the strong coupling limit $\alpha \to 0$ in all kinetic "k-essence" coefficients (38)–(39). The corresponding limiting values at $\alpha = 0$ of the energy density and pressure of the dual purely kinetic "k-essence" theory (40) are those of the original theory (24) with the standard Einstein gravity action ($\alpha = 0$), i.e., the theory (2) [32]. In particular, in the limit $\alpha \to 0$ we get precisely the expression for the energy density being an exact sum of dark energy and dust dark matter contributions produced by (2) [32].

Indeed, from (55) and (59)–(60) we obtain for $\alpha \to 0$:

$$\dot{\phi}(p_\phi/a^3) \simeq \sqrt{2} + \alpha\left(4\sqrt{2}M + \frac{p_\phi}{a^3}\right)$$
$$+ \alpha^2\left[24\sqrt{2}M - \frac{3\sqrt{2}}{4}\left(\frac{p_\phi}{a^3}\right)^2\right] + \text{O}(\alpha^3), \tag{67}$$

wherefrom (57) and (61) yield for small α:

$$\rho(p_\phi/a^3) \simeq 2M + \sqrt{2}\frac{p_\phi}{a^3}$$
$$+ \alpha\left[16M^2 + 4\sqrt{2}M\frac{p_\phi}{a^3} + \frac{1}{2}\left(\frac{p_\phi}{a^3}\right)^2\right] + \text{O}(\alpha^2), \tag{68}$$

$$p(p_\phi/a^3) \simeq -2M - \alpha\left[16M^2 - \frac{1}{2}\left(\frac{p_\phi}{a^3}\right)^2\right] + \text{O}(\alpha^2). \tag{69}$$

The expression (68) resembles the large-a asymptotics (65) for the energy density, however, unlike the latter Eq. (68) is valid for generic values of the FLRW factor a [and small values of α—the R^2 coupling constant in the original theory (24)].

The cosmological implications of general purely kinetic "k-essence" models have been previously studied extensively in Refs. [15–20]. In particular, an important strong inequality (Eqs. (26)–(27) in [15]) involving the parameters of a generic quadratic purely kinetic "k-essence" theory was derived from the requirement that the onset of dark matter behavior must occur before the epoch of equal matter and radiation, and also that in the present stage of evolution the dark energy component must exceed twice the dark matter component in the pertinent energy density, The counterpart of the above Scherrer inequality in the present settings becomes the requirement:

$$\alpha M \ll 1, \tag{70}$$

which completely conforms to the result (68).

From Eqs. (57)–(61) we obtain for the squared sound speed:

$$c_s^2 = \frac{\partial p}{\partial \rho} = \frac{1}{3}\left[1 - \frac{1}{\sqrt{1 + 3\alpha\left[(1 - 8\alpha M)\rho - 2M\right]}}\right] \quad (71)$$

with ρ as in (57)–(58). Obviously $c_s^2 \to 0$ for $\alpha \to 0$, which conforms to the limiting values (68)–(69). Also, c_s^2 (71) is well defined (positive) and reasonably small for $\rho > 2M(1 - 8\alpha M)^{-1}$, which is obviously fulfilled for small α according to (68).

The new aspects in our current treatment w.r.t. Refs. [15–20] are as follows:

- We have derived an explicit duality between the original $g_{\mu\nu}$-frame quadratic $f(R)$-gravity plus non-Riemannian-modified-measure scalar-field action (24), on one hand, and the Einstein-frame special quadratic purely kinetic "k-essence" theory (40).
- Unlike the general purely kinetic "k-essence" treatment [15–20] all three coefficients in our kinetic "k-essence" action (38)–(39) explicitly depend on only two independent parameters (α, M), the latter being a dynamically generated integration constant.
- In spite of the singularity of all "k-essence" coefficients A_1, A_2, A_3 (39) for small α [the coupling constant of the R^2 in the original theory (24)], the energy density and pressure in the dual purely kinetic "k-essence" theory have smooth limit for $\alpha \to 0$, whereby their limiting values (68) and (69) exactly coincide with the corresponding values in the original theory (24) with $\alpha = 0$, i.e., (2) [32].
- The established duality between (24) and (40) elucidates the origin of the unified description of dark energy and dark matter within the approach based on "purely kinetic k-essence" models [15].

4 Canonical Hamiltonian formalism and Wheeler–DeWitt equation

For the canonical Hamiltonian formalism applied to generic generalized gravity–matter models with one or more non-Riemannian spacetime volume-forms we refer to [37]. In particular, for the theory (2) the canonical Hamiltonian analysis was discussed in detail in Sect. 2 of Ref. [32].

Here we will consider specifically the Hamiltonian treatment and quantization of the action (54)—reduction of the purely kinetic "k-essence" model (40) for the Friedman–Lemaitre–Robertson–Walker (FLRW) class of metrics (53).

From the explicit form of the FLRW action (54) we deduce the canonically conjugated momenta p_a, π_N and p_ϕ w.r.t. a, N and $\phi \equiv \widetilde{\varphi}$:

$$p_a = -\frac{2a\dot{a}}{N}, \quad \pi_N = 0,$$
$$p_\phi = a^3\left[-\frac{1}{2\alpha}\frac{\dot{\phi}}{N} + \left(\frac{1}{4\alpha} - 2M\right)\frac{\dot{\phi}^3}{N^3}\right], \quad (72)$$

where the second relation for π_N is a primary Dirac first-class constraint. The total canonical Hamiltonian becomes

$$\mathcal{H}_{\text{total}} = N\left[-\frac{p_a^2}{24a} + a^3\rho(p_\phi/a^3)\right] \quad (73)$$

with $\rho(p_\phi/a^3)$ as in (57), thus $\mathcal{H}_{\text{total}}$ by itself is a secondary Dirac first-class constraint with N playing the role ot its Lagrange multiplier. Since $\phi \equiv \widetilde{\varphi}$ is a cyclic variable its canonically conjugated momentum p_ϕ is conserved.

Quantization according to the Dirac-constrained Hamiltonian formalism proceeds by imposing the quantized operator version of the Hamiltonian constraint—the expression in the square brackets in (73)—on the quantum wave function $\Psi(a, p_\phi)$—the Wheeler–DeWitt equation. The ordering ambiguity in the quantized version of the first term there is resolved by changing variables:

$$a \to \widetilde{a} = \frac{4}{\sqrt{3}}a^{3/2}, \quad (74)$$

and taking the special operator ordering:

$$\frac{p_a^2}{24a} \to \frac{1}{\sqrt{12a}}\widehat{p}_a\frac{1}{\sqrt{12a}}\widehat{p}_a = -\frac{1}{2}\frac{\partial^2}{\partial\widetilde{a}^2}. \quad (75)$$

Therefore, the Wheeler–DeWitt equation acquires the form of Schrödinger equation for zero energy eigenvalue:

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\widetilde{a}^2} + \mathcal{V}_{\text{eff}}(\widetilde{a}, p_\phi)\right]\Psi(\widetilde{a}, p_\phi) = 0 \quad (76)$$

with effective potential:

$$\mathcal{V}_{\text{eff}}(\widetilde{a}, p_\phi) = -a^3\rho(p_\phi/a^3) \quad (77)$$

[$\rho(p_\phi/a^3)$ as in (57), and a and $\widetilde{a}$ related as in (74)]. Explicitly:

$$\mathcal{V}_{\text{eff}}(\widetilde{a}, p_\phi) = -\frac{3\widetilde{a}^2}{16}\left[\frac{1}{8\alpha}f^2(\widetilde{a}, p_\phi) + \frac{4p_\phi}{\widetilde{a}^2}f(\widetilde{a}, p_\phi) - \frac{1}{4\alpha}\right], \quad (78)$$

where $f(\widetilde{a}, p_\phi) \equiv y$ is a root of the cubic equation [cf. Eqs. (55), (58)–(59)]:

$$y^3 - \frac{2}{1 - 8\alpha M}y - \frac{64\alpha p_\phi}{3(1 - 8\alpha M)}\widetilde{a}^{-2} = 0. \quad (79)$$

Using the asymptotics (64)–(65) we obtain for the small $\widetilde{a}$ and large $\widetilde{a}$ of $\mathcal{V}_{\text{eff}}(\widetilde{a}, p_\phi)$:

$$
\mathcal{V}_{\text{eff}}(\widetilde{a}, p_\phi) \simeq -\left(\frac{9\alpha p_\phi^4}{1 - 8\alpha M}\right)^{1/3} \widetilde{a}^{-2/3}
$$
$$
-\left(\frac{2\alpha p_\phi}{1 - 8\alpha M}\right)^{2/3} \frac{1}{8\alpha} \tag{80}
$$

for small $\widetilde{a}$, and

$$
\mathcal{V}_{\text{eff}}(\widetilde{a}, p_\phi) \simeq -\frac{3M}{8(1 - 8\alpha M)}\widetilde{a}^2 - \sqrt{\frac{2}{1 - 8\alpha M}}\, p_\phi \tag{81}
$$

for large $\widetilde{a}$. Using (79) we find that $\mathcal{V}_{\text{eff}}(\widetilde{a}, p_\phi)$ (78) is negative for all $\widetilde{a} \in (0, \infty)$ and diverges to $-\infty$ at both ends of the interval according to (80)–(81).

According to Scherrer's inequality (70) α must be very small. Thus, substituting (68) into (77) and accounting for the change of variables (74) renders the Wheeler–DeWitt effective potential into the following form:

$$
\mathcal{V}_{\text{eff}}(\widetilde{a}, p_\phi) \simeq -\frac{3M}{8}(1 + 8\alpha M)\widetilde{a}^2
$$
$$
-\alpha \frac{8p_\phi^2}{3}\widetilde{a}^{-2} - \sqrt{2}p_\phi(1 + 4\alpha M), \tag{82}
$$

valid for any $\widetilde{a} \in (0, \infty)$. In other words, for small α (82) is a sum of inverted harmonic oscillator potential with negative frequency squared:

$$
\omega^2 = -\frac{3M}{4}(1 + 8\alpha M) = -\frac{3}{4}\Lambda_{\text{eff}}, \tag{83}
$$

where Λ_{eff} is the effective cosmological constant (50) for small α, plus an inverse square potential and with an "energy" eigenvalue $\mathcal{E} = \sqrt{2}p_\phi(1 + 4\alpha M)$.

Thus, in the limit $\alpha \to 0$ the Wheeler–DeWitt equation (76) becomes the Schrödinger equation for an inverted harmonic oscillator with negative frequency squared (83) and with "energy" eigenvalue $\mathcal{E} = \sqrt{2}p_\phi$:

$$
\left[-\frac{1}{2}\frac{\partial^2}{\partial\widetilde{a}^2} - \frac{3M}{8}\widetilde{a}^2 - \sqrt{2}p_\phi\right]\Psi(\widetilde{a}, p_\phi) = 0. \tag{84}
$$

The inverted harmonic oscillator was extensively studied in Ref. [44] (for a more recent account and further references, see [45]). In particular, the inverted oscillator was applied in [46] to the study of the quantum mechanical dynamics of the scalar field in the so called "new inflationary" scenario. Since the energy eigenvalue spectrum of the inverted harmonic oscillator is continuous ($\mathcal{E} \in (-\infty, +\infty)$) and the corresponding energy eigenfunctions are not square-integrable, its application in the context of cosmology [46] required the

employment of wave-packets w.r.t. $\mathcal{E}$ instead of energy eigenfunctions.

Similarly, in the present case the value of p_ϕ—the conserved canonical momentum of the kinetic "k-essence" scalar field $\phi \equiv \widetilde{\varphi}$ in (54)—is the analog of energy eigenvalue $\mathcal{E}$ in the Wheeler–DeWitt Schrödinger-like equation (84) (modulo the factor $\sqrt{2}$). Therefore, now in the strong coupling limit $\alpha \to 0$ the "k-essence" field $\phi \equiv \widetilde{\varphi}$ plays the role of a Wheeler–DeWitt time $\tau \equiv \frac{1}{\sqrt{2}}\phi$ and Eq. (84) acquires the form of a "time-dependent" Schrödinger-like equation for the inverted harmonic oscillator upon Fourier transforming the "energy"-eigenvalue Eq. (84):

$$
i\frac{d}{d\tau}\Psi(\widetilde{a}, \tau) = \left[-\frac{1}{2}\frac{\partial^2}{\partial\widetilde{a}^2} - \frac{3M}{8}\widetilde{a}^2\right]\Psi(\widetilde{a}, \tau), \tag{85}
$$

$$
\Psi(\widetilde{a}, \tau) = \int_{-\infty}^{\infty} dp_\phi \frac{1}{\sqrt{2\pi}}e^{-i\sqrt{2}p_\phi\tau}\Psi(\widetilde{a}, p_\phi). \tag{86}
$$

Following [46] the appropriate normalized to unity [on the semiaxis $\widetilde{a} \in (0, \infty)$] wave-packet solution of the "time-dependent" Wheeler–DeWitt equation (85) is of the form [using the notation ω (83) for $\alpha = 0$]:

$$
\Psi(\widetilde{a}, \tau) = \left(\frac{2\omega}{\pi}\sin(2b)\right)^{1/4}(\cos(b - i\omega\tau))^{-1/2}
$$
$$
\times \exp\left\{-\frac{1}{2}\widetilde{a}^2\omega\tan(b - i\omega\tau)\right\}, \tag{87}
$$

where b is an integration constant describing the width of the wave packet. Accordingly, the average value of the FLRW scale factor $a = \frac{\sqrt{3}}{4}\widetilde{a}^{2/3}$ [cf. (74)–(75)] is given by

$$
\langle\widetilde{a}\rangle \equiv \int_0^{\infty} d\widetilde{a}\,\widetilde{a}|\Psi(\widetilde{a}, \tau)|^2 = \left[\frac{\cos(2b) + \cosh(2\omega\tau)}{\pi\omega\sin(2b)}\right]^{1/2}, \tag{88}
$$

exhibiting no singularity ($\langle\widetilde{a}\rangle \to 0$) at any "time" τ.

5 Conclusions

In the present paper we have discussed in some detail the main properties of a generalized model of gravity interacting with a single scalar field, where we have employed the method of non-Riemannian spacetime volume-forms (alternative generally covariant integration measure densities) constructed in terms of auxiliary maximal-rank tensor gauge fields ("measure" gauge fields).

In the preceding paper [32] we have shown that the non-Riemannian-measure-modified scalar-field action (2) yields a simple unified description of dark energy and dust dark matter. Namely, the corresponding energy density arises as an exact sum of a dark energy component in the form of a dynamically generated cosmological constant appearing

as an arbitrary integration constant in the solution of the "measure" gauge field equations, and a dark matter component produced by a hidden Noether symmetry (not affecting the gravity part) giving rise to a Noether conserved current, which identifies the scalar-field dynamics as a dust fluid motion along the geodesics.

Here we extended the above treatment by coupling the non-Riemannian-measure-modified scalar-field dynamics to quadratic $f(R)$ gravity. We have found an explicit duality in the usual sense of "weak versus strong coupling" between the original non-standard gravity-scalar-field model providing an exact unified description of dynamical dark energy and dust fluid dark matter in the matter sector, on one hand, and a quadratic purely kinetic "k-essence" gravity–matter model, on the other hand. The latter dual theory arises as the "Einstein-frame" theory of its original counterpart. It is special in a sense that the couplings in the dual quadratic kinetic "k-essence" action are given in terms of only two parameters (α, M)—the R^2-coupling constant in the original action (1) and a dynamically generated integration constant M upon solving the equations of motion for the auxiliary "measure" gauge field in (1). Moreover, in spite of the divergence of the "k-essence" coupling constants when $\alpha \to 0$ (strong coupling limit), both the "k-essence" energy density and the "k-essence" pressure have smooth limits at $\alpha = 0$ with the limiting values coinciding with their respective values in (2)—the weak coupling limit of the original theory (1), leading to an explicit unified description of dark energy and dust dark matter as an exact sum of two separate contributions to the total energy density.

The established duality in the present paper explains the ability of the purely kinetic "k-essence" models [15] to provide approximately a unified description of dark energy and dark matter and reveals that this unified description becomes exact in the strong coupling limit of a special type of quadratic purely kinetic "k-essence" theory.

Finally, we have used the standard Dirac approach to constrained Hamiltonian systems for a canonical Hamiltonian treatment of the non-Riemannian-measure-modified gravity-scalar-field theory, specifically for the reduction of the latter in the case of FLRW class of cosmological spacetime metrics. In the limit of vanishing R^2 coupling the associated Wheeler–DeWitt equation acquires the form of a Schrödinger-like equation with the effective potential of an inverted harmonic oscillator. The quantum average value of the FLRW scale factor does not exhibit any singularities in its time evolution.

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