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Scale-invariant power spectra from a Weyl-invariant scalar–tensor theory

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Abstract We obtain scale-invariant scalar and tensor power spectra from a Weyl-invariant scalar–tensor theory in de Sitter spacetime. This implies that the Weyl invariance guarantees the implementation of the scale invariance of the power spectrum in de Sitter spacetime. We establish a deep connection between the Weyl invariance of the action and the scale invariance of the power spectrum in de Sitter spacetime.

1 Introduction

The conformal gravity $C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma}$ of being invariant under the Weyl transformation of $g_{\mu\nu} \to \Omega^2(x)\tilde{g}_{\mu\nu}$ has its own interests in quantum gravity and cosmology [1]. Its appearance, related to the trace anomaly, was established in [2,3]. Stelle [4] was first to introduce the quadratic curvature gravity of $a(R_{\mu\nu}^2 - R^2/3) + bR^2$ to improve the perturbatively renormalizable property of Einstein gravity in Minkowski spacetime. For the case of $ab \neq 0$, the renormalizability was achieved but the unitarity was violated. This means that even though the a-term improves the ultraviolet divergence, it induces simultaneously ghost excitations which spoil the unitarity. This issue has not been resolved completely until now in Minkowski spacetime.

A purely conformal gravity implication to cosmological perturbation was first studied in [5], indicating that there exists a difference between conformal and Einstein gravities in their perturbed equations in de Sitter (dS) spacetime. Later, one of the authors has computed an observable of tensor power spectrum [6,7], which is scale invariant during dS inflation. In Einstein–Weyl gravity, the role of the Weyl-squared term was extensively studied in dS spacetime [8–11].

On the other hand, the Lee–Wick scalar theory [12–14], has an action is given by

$$S_{\rm LW} = -\frac{1}{2} \int {\rm d}^4 x \sqrt{-g} \left[(\partial\phi)^2 + \frac{1}{M^2}(\Box\phi)^2 \right], \qquad (1)$$

where the scalar has a mass dimension 1 and $S_{\rm LW}$ is not invariant under the Weyl transformations of $g_{\mu\nu} \to \Omega^2(x)\tilde{g}_{\mu\nu}$ and $\phi \to \tilde{\phi}/\Omega$. It has provided a scale-invariant scalar spectrum when one requires $M^2 = 2H^2$ in dS spacetime [15]. Also, a fourth-order scalar theory with nonminimal derivative coupling could induce a scale-invariant scalar spectrum by requiring that the nonminimal derivative coupling constant be $\xi = 2/3$ in dS spacetime [16].

Hence, it is quite interesting to find a proper scalar theory which may give us a scale-invariant scalar power spectrum without introducing any artificial adjustments. This might be a desired Weyl-invariant scalar theory. As is well known, the simplest example of a Weyl-invariant theory is a massless vector theory described by $F^{\mu\nu} F_{\mu\nu}/4g^2$ and the massless Dirac equation is also Weyl covariant. In order to obtain scale-invariant scalar and tensor spectra, one has to combine this would-be scalar theory with conformal gravity, leading to a Weyl-invariant scalar–tensor theory.

In this work, we propose two candidates for the Weyl-invariant scalar–tensor theory. We will show that the Weyl invariance guarantees the implementation of the scale-invariant power spectra in dS spacetime, which are independent of the wave number k. This work establishes a deep connection between the Weyl invariance of the action and scale invariance of the power spectrum in dS spacetime clearly.

2 Weyl-invariant scalar–tensor theory

In this work we wish to explore a deep connection between the Weyl invariance of the action and the scale invariance

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of the power spectrum in dS spacetime. In two dimensions the second-order scalar operator $\Box_2$ is covariant under the Weyl transformation with $\Omega = e^\sigma$ in the sense that $\Box_2 \to e^{-2\sigma} \tilde{\Box}_2$. However, this is not true in other dimensions. For example, one finds that $\Box \to e^{-2\sigma}[\tilde{\Box} + 2(\tilde{\nabla}^\mu \sigma)\tilde{\nabla}_\mu]$ in four dimensions. Accordingly, the fourth-order scalar operator $\Box^2$ is not Weyl covariant. In order to make it Weyl covariant, we need to introduce additional terms like $2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}\nabla^\mu R \nabla_\mu$ [17,18].

Let us first consider a Weyl-invariant scalar–tensor theory whose action is given by fourth-order scalar theory and conformal gravity,

$$S_{ST1} = \frac{1}{2}\int d^4x \sqrt{-g}\Big[-(\Box\phi)^2 + 2\Big(R_{\mu\nu} - \frac{R}{3}g_{\mu\nu}\Big)\partial^\mu\phi\partial^\nu\phi - \frac{\alpha}{2}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}\Big] \quad (2)$$

with α a dimensionless coupling constant. The appearance of the second term is necessary to have the Weyl-invariant scalar theory. Here the conformal gravity of the Weyl-squared term is given by

$$C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}(\equiv C^2) = 2\Big(R^{\mu\nu}R_{\mu\nu} - \frac{1}{3}R^2\Big) + \Big(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2\Big) \quad (3)$$

with the Weyl tensor

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}\Big(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}\Big) + \frac{1}{6}R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \quad (4)$$

Alternatively, the action (2) can be rewritten as

$$S_{ST2} = \frac{1}{2}\int d^4x \sqrt{-g}\Big[-(\Box\phi)^2 + 2\Big(G_{\mu\nu} + \frac{R}{6}g_{\mu\nu}\Big)\partial^\mu\phi\partial^\nu\phi - \frac{\alpha}{2}C^2\Big], \quad (5)$$

where $G_{\mu\nu}\partial^\mu\phi\partial^\mu\phi$ denotes the nonminimal derivative coupling term [19] which may render slow-roll inflation even for a steep potential [20,21]. Here $G_{\mu\nu} = R_{\mu\nu} - (R/2)g_{\mu\nu}$ is the Einstein tensor. This expression shows clearly why (2) differs from the fourth-order scalar theory with nonminimal derivative coupling model [16].

Noting that the Weyl-squared term is covariant ($C^2 = e^{-4\sigma}\tilde{C}^2$) under the Weyl transformation of $g_{\mu\nu} \to e^{2\sigma(x)}\tilde{g}_{\mu\nu}$ [22], the first-two terms of (2) can be expressed as

$$\frac{1}{2}\int d^4x\sqrt{-g}\phi\Delta_4\phi. \quad (6)$$

Here the Weyl operator Δ_4 takes the form

$$\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}\nabla^\mu R\nabla_\mu, \quad (7)$$

which is obviously Weyl covariant ($\Delta_4 = e^{-4\sigma}\tilde{\Delta}_4$) under the Weyl transformation. It is clear that the action (2) is Weyl invariant provided that the scalar field is dimensionless, when one takes into account $\sqrt{-g} \to e^{4\sigma}\sqrt{-\tilde{g}}$. Hence, the action (2) is regarded as a promising Weyl-invariant scalar–tensor theory, compared to the Lee–Wick scalar theory (1) and the fourth-order scalar theory with nonminimal derivative coupling.

Now, we derive the Einstein equation from (2):

$$-\alpha B_{\mu\nu} = T_{\mu\nu}, \quad (8)$$

where the Bach tensor is defined by

$$B_{\mu\nu} = 2R^{\rho\sigma}\Big(R_{\mu\rho\nu\sigma} - \frac{1}{4}R_{\rho\sigma}g_{\mu\nu}\Big) - \frac{2}{3}R\Big(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}\Big) + \nabla^2 R_{\mu\nu} - \frac{1}{6}\nabla^2 Rg_{\mu\nu} - \frac{1}{3}\nabla_\mu\nabla_\nu R. \quad (9)$$

Here $T_{\mu\nu}$ is the total energy-momentum tensor derived from the first three terms of (2), which takes the form [17]

$$T_{\mu\nu} = -2\nabla_{(\mu}\phi\nabla_{\nu)}\Box\phi + 2\nabla^\rho(\nabla_\rho\phi\nabla_\mu\nabla_\nu\phi)$$
$$- \frac{2}{3}\nabla_\mu\nabla_\nu(\nabla_\rho\phi\nabla^\rho\phi)$$
$$+ \frac{2}{3}R_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi - 4R_{\rho(\mu}\nabla_{\nu)}\phi\nabla^\rho\phi + \frac{2}{3}R\nabla_\mu\phi\nabla_\nu\phi$$
$$+ \frac{1}{6}g_{\mu\nu}\Big\{ -3(\Box\phi)^2 + \Box(\nabla_\rho\phi\nabla^\rho\phi) + 2(3R^{\rho\sigma} - Rg^{\rho\sigma})\nabla_\rho\phi\nabla_\sigma\phi \Big\}. \quad (10)$$

On the other hand, its scalar equation is given by

$$\Delta_4\phi = 0. \quad (11)$$

For a maximally symmetric spacetime with $\bar{R} = \text{const}$, one finds

$$\bar{R}_{\mu\nu\rho\sigma} = \frac{\bar{R}}{12}\Big(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}\Big), \quad \bar{R}_{\mu\nu} = \frac{\bar{R}}{4}\bar{g}_{\mu\nu}. \quad (12)$$

In this case, the Bach tensor is always zero ($\bar{B}_{\mu\nu} = 0$). Hence, choosing $\bar{\phi} = \text{const}$ ($\bar{T}_{\mu\nu} = 0$), we have solutions of dS ($\bar{R} > 0$), Minkowski ($\bar{R} = 0$), and anti de Sitter ($\bar{R} < 0$) spacetime. In this work, we concentrate on the dS solution, as regards cosmological implications, whose curvature quantities are given by

$$\bar{R}_{\mu\nu\rho\sigma} = H^2(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}),$$
$$\bar{R}_{\mu\nu} = 3H^2\bar{g}_{\mu\nu}, \quad \bar{R} = 12H^2 \tag{13}$$

with $H = $ const.

Now, let us choose the dS background explicitly by choosing a conformal time η,

$$ds^2_{\mathrm{dS}} = \bar{g}_{\mu\nu}dx^\mu dx^\nu = a(\eta)^2\Big[-d\eta^2 + d\mathbf{x}\cdot d\mathbf{x}\Big], \tag{14}$$

where the conformal scale factor is

$$a(\eta) = -\frac{1}{H\eta}, \tag{15}$$

while the cosmic scale factor is given by $a(t) = e^{Ht}$ in a flat Friedmann–Robertson–Walker (FRW) background; $ds^2_{\mathrm{FRW}} = -dt^2 + a^2(t)d\mathbf{x}\cdot d\mathbf{x}$. We note that the dS solution is not distinctive since any maximally symmetric spacetime can be a solution to Einstein and scalar equations. This redundancy of the solutions is a feature of Weyl-invariant scalar–tensor theory (2). The dS SO(1,4)-invariant distance between two spacetime points x^μ and x'^μ is defined by

$$\mathcal{Z}(x,x') = 1 - \frac{-(\eta-\eta')^2 + |\mathbf{x}-\mathbf{x}'|^2}{4\eta\eta'}$$
$$= 1 - \frac{(x-x')^2}{4\eta\eta'} \tag{16}$$

since $\mathcal{Z}(x,x')$ has the ten symmetries which leave the metric of dS spacetime invariant. Here $(x-x')^2$ is the Lorentz-invariant flat spacetime distance.

At this stage, it seems appropriate to comment on the other Weyl-invariant scalar–tensor theory of massive conformal gravity [23,24],

$$S_{\mathrm{ST3}} = -\frac{1}{2}\int d^4x\sqrt{-g}\left[(\partial\phi)^2 + \frac{1}{6}\phi^2 R + \frac{\alpha}{2}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}\right], \tag{17}$$

where its Weyl invariance can be achieved up to surface terms by requiring both $\phi \to \tilde{\phi}e^{-\sigma}$ and $g_{\mu\nu} \to e^{2\sigma}\tilde{g}_{\mu\nu}$. Hence, we wish to point out a difference between ST1 [17,18] and ST3: the scaling dimension of ϕ in (2) is zero, while the scaling dimension of ϕ in (17) is -1 (or mass dimension 1) as ϕ in kinetic term of $\phi\square\phi$ does have. Also, ϕ in (2) is Weyl invariant ($\phi \to \tilde{\phi}$), whereas ϕ in (17) transforms as $\phi \to \tilde{\phi}e^{-\sigma}$. Furthermore, since (17) provides a conformal scalar propagation in dS spacetime, it is not a promising candidate for our purpose.

Adapting the action (17) to find the background solution, one finds the Einstein and scalar equations,

$$-\alpha B_{\mu\nu} = T^\phi_{\mu\nu}, \tag{18}$$
$$\left(\square - \frac{R}{6}\right)\phi = 0, \tag{19}$$

where

$$T^\phi_{\mu\nu} = \nabla_\mu\phi\nabla_\mu\phi - \frac{1}{2}(\nabla\phi)^2 g_{\mu\nu}$$
$$+ \frac{1}{6}G_{\mu\nu}\phi^2 - \frac{1}{6}\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\square\right)\phi^2. \tag{20}$$

Its trace is zero when using (19). For $\bar{\phi} = $ const, the Minkoswksi spacetime of $\bar{R} = 0(\bar{G}_{\mu\nu} = 0)$ is only a solution. In the case of $\bar{\phi} = 0$, any maximally symmetric spacetime is a solution and, thus, the dS solution is not distinctive.

Finally, if one wishes really to obtain a dS solution, one has to insert a term of $R - 2\Lambda(\Lambda = 3H^2)$ into the action (2) which breaks the Weyl invariance manifestly. This leads to a fourth-order scalar theory coupled to Einstein–Weyl gravity, where one could not obtain a scale-invariant tensor spectrum.

3 Perturbed equations on de Sitter spacetime

In order to derive the perturbed equation (linearized equation) around the dS spacetime, we introduce a perturbed scalar φ as

$$\phi = \bar{\phi} + \varphi. \tag{21}$$

For a metric perturbation, we choose the Newtonian gauge [25] of $B = E = 0$ and $\bar{E}_i = 0$, leading to $10 - 4 = 6$ degrees of freedom (DOF). In this case, the cosmologically perturbed metric can be simplified to

$$ds^2 = a(\eta)^2\Big[-(1+2\Psi)d\eta^2 + 2\Psi_i d\eta dx^i$$
$$+ \Big\{(1+2\Phi)\delta_{ij} + h_{ij}\Big\}dx^i dx^j\Big] \tag{22}$$

with the transverse vector $\partial_i\Psi^i = 0$ and transverse-traceless tensor $\partial_i h^{ij} = h = 0$. It is worth to note that choosing the SO(3)-perturbed metric (22) contrasts with the covariant approach to the cosmological conformal gravity [5].

In order to get the cosmological perturbed equations, one should first obtain the bilinear action and then vary it to yield the linearized equations. According to the previous work [9], we expand the Weyl-invariant scalar–tensor action (2) up to quadratic order in the perturbations of φ, Ψ, Φ, Ψ_i, and h_{ij} around the dS background. Then the bilinear action is composed of four terms, thus:

$$\delta S_{\mathrm{ST1}} = \delta S_S + \delta S_{\mathrm{CG}}^{(S)} + \delta S_{\mathrm{CG}}^{(V)} + \delta S_{\mathrm{CG}}^{(T)}, \tag{23}$$

where

$$\delta S_{\text{S}} = \frac{1}{2} \int \text{d}^4x \, \varphi \bar{\Delta}_4 \varphi, \tag{24}$$

$$\delta S_{\text{CG}}^{(\text{S})} = \frac{\alpha}{3} \int \text{d}^4x \left[\nabla^2(\Psi - \Phi) \right]^2, \tag{25}$$

$$\delta S_{\text{CG}}^{(\text{V})} = \frac{\alpha}{4} \int \text{d}^4x \left(\partial_i \Psi'_j \partial^i \Psi'^j - \nabla^2 \Psi_i \nabla^2 \Psi^i \right), \tag{26}$$

$$\delta S_{\text{CG}}^{(\text{T})} = \frac{\alpha}{8} \int \text{d}^4x \left(h''_{ij} h''^{ij} - 2\partial_k h'_{ij} \partial^k h'^{ij} + \nabla^2 h_{ij} \nabla^2 h^{ij} \right). \tag{27}$$

Here $'$ (a prime) denotes differentiation with respect to the conformal time η. Note that all bilinear actions are independent of the conformal scale factor $a(\eta)$, showing that the Weyl invariance persists in the bilinear action.

From (24), we obtain the fourth-order perturbed scalar equation

$$\bar{\Delta}_4 \varphi = -\Box(-\Box + 2H^2)\varphi = 0, \tag{28}$$

which shows a second factorization of $\bar{\Delta}_4$ into two second-order operators in dS spacetime (and in fact any conformally flat spacetime). Here $\Box = -\text{d}^2/\text{d}\eta^2 + \nabla^2$ with $\nabla^2 = \partial_i^2$ is the Laplacian operator. Varying (26) and (27) with respect to Ψ^i and h^{ij} leads to linearized equations of motion for vector and tensor perturbations,

$$\nabla^2 \Box \Psi_i = 0, \tag{29}$$

$$\Box^2 h_{ij} = 0. \tag{30}$$

It is emphasized again that (28)–(30) are independent of $a^2(\eta)$ of the expanding dS background in the Weyl-invariant scalar–tensor theory.

Finally, we would like to mention the two scalars Φ and Ψ. Two scalar equations are given by $\nabla^2 \Psi = \nabla^2 \Phi = 0$, which imply that they are obviously non-propagating modes in the dS background. Hereafter, we will not consider these irrelevant metric scalars. This means that the Weyl-invariant theory (2) describes 7 DOF (1 scalar+ 2 of vector +4 of tensor modes), where the last becomes four because h_{ij} satisfies a fourth-order equation.

4 Primordial power spectra

The power spectrum is usually given by the two-point correlation function which could be computed when one chooses the vacuum state $|0\rangle$. It is defined by

$$\langle 0|\mathcal{F}(\eta, \mathbf{x})\mathcal{F}(\eta, \mathbf{x}')|0\rangle = \int \text{d}^3\mathbf{k} \frac{\mathcal{P}_\mathcal{F}(\eta, k)}{4\pi k^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \tag{31}$$

where $\mathcal{F}$ denotes a scalar, vector or tensor, and $k = |\mathbf{k}|$ is the wave number. For simplicity, we may use the zero-point

correlation function to define the power spectrum by [26]

$$\langle 0|\mathcal{F}(\eta, 0)\mathcal{F}(\eta, 0)|0\rangle = \int \frac{\text{d}k}{k} \mathcal{P}_\mathcal{F}(\eta, k). \tag{32}$$

In general, fluctuations are created on all length scales with wave number k. Cosmologically relevant fluctuations start their lives inside the Hubble radius which defines the subhorizon: $k \gg aH$ ($z = -k\eta \gg 1$). On the other hand, the comoving Hubble radius $(aH)^{-1}$ shrinks during inflation while the comoving wave number k is constant. Therefore, eventually all fluctuations exit the comoving Hubble radius, which defines the superhorizon: $k \ll aH$ ($z = -k\eta \ll 1$). One may compute the two-point function by taking the Bunch–Davies vacuum $|0\rangle$. In the case of dS inflation, we choose the subhorizon limit of $z \to \infty$ to define the Bunch–Davies vacuum, while we choose the superhorizon limit of $z \to 0$ to get a definite form of the power spectrum which stays alive after decaying.

4.1 Scalar power spectrum

There are two ways to obtain the scalar power spectrum: One is to find the inverse Weyl operator $\bar{\Delta}_4^{-1}$ and Fourier transforming it leads to the scalar power spectrum. The other is to compute the power spectrum (32) directly by using the quantization scheme of the non-degenerate Pais–Uhlenbeck (PU) oscillator. We briefly describe both computation schemes.

The inverse Weyl operator is given by [17,18]

$$\bar{\Delta}_4^{-1}[\mathcal{Z}(x, x')] = \frac{1}{2H^2} \left[\frac{1}{-\Box} - \frac{1}{-\Box + 2H^2} \right]$$
$$= \frac{1}{2H^2}[G_{\text{mmc}}[\mathcal{Z}(x, x')] - G_{\text{mcc}}[\mathcal{Z}(x, x')]], \tag{33}$$

where the propagators of massless minimally coupled (mmc) scalar [27] and massless conformally coupled (mcc) scalar [28] in dS spacetime are given by

$$G_{\text{mmc}}[\mathcal{Z}(x, x')] = \frac{H^2}{(4\pi)^2} \left[\frac{1}{1 - \mathcal{Z}} - 2\ln(1 - \mathcal{Z}) + c_0 \right],$$

$$G_{\text{mcc}}[\mathcal{Z}(x, x')] = \frac{H^2}{(4\pi)^2} \frac{1}{1 - \mathcal{Z}}, \tag{34}$$

where the former is the dS-invariant renormalized two-point function (on the space of non-constant modes), while the latter is the conformally coupled scalar two-point function on dS spacetime. As opposed to Ref. [29], the inverse Weyl operator (33) is dS invariant because it is a function of $1 - \mathcal{Z}$. Substituting (34) into (33), the propagator takes the form

$$\bar{\Delta}_4^{-1}[\mathcal{Z}(x, x')] = \frac{1}{16\pi^2} \left(-\ln[1 - \mathcal{Z}(x, x')] + \frac{c_0}{2} \right), \tag{35}$$

which is a pure logarithm up to an additive constant c_0 and is a dS-invariant two-point function. The scalar power spectrum is defined by Fourier transforming the propagator at equal times $\eta = \eta'$ as

$$\mathcal{P}_\varphi = \frac{1}{(2\pi)^3} \int d^3\mathbf{r} \, 4\pi k^3 \bar{\Delta}_4^{-1} [\mathcal{Z}(\eta, \mathbf{x}; \eta, \mathbf{x}')] e^{-i\mathbf{k}\cdot\mathbf{r}}, \quad \mathbf{r} = \mathbf{x} - \mathbf{x}' \tag{36}$$

$$= \frac{1}{(2\pi)^3} \frac{k^3}{4\pi} \int d^3\mathbf{r} \left(-\ln\left[\frac{r^2}{4\eta^2}\right] + \frac{c_0}{2} \right) e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{37}$$

$$= -\frac{k^2}{8\pi^3} \int_0^\infty dr \left\{ r \sin[kr] \ln[r^2] \right\} \tag{38}$$

$$= \frac{1}{8\pi^2}, \tag{39}$$

where we have used the Cesàro-summation method in deriving (38) to (39) [16,30].

On the other hand, Eq. (28) implies two second-order equations for the mmc and mcc scalars,

$$\Box \varphi^{\text{mmc}} = 0, \tag{40}$$

$$(\Box - 2H^2)\varphi^{\text{mcc}} = 0. \tag{41}$$

Expanding $\varphi^{\text{mmc,mcc}}$ in terms of Fourier modes $\phi_\mathbf{k}^{\text{mmc,mcc}}(\eta)$,

$$\varphi^{\text{mmc,mcc}}(\eta, \mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3\mathbf{k} \, \phi_\mathbf{k}^{\text{mmc,mcc}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{42}$$

With $z = -k\eta$, Eqs. (40) and (41) become

$$\left(\frac{d^2}{dz^2} - \frac{2}{z} \frac{d}{dz} + 1 \right) \phi_\mathbf{k}^{\text{mmc}} = 0, \tag{43}$$

$$\left(\frac{d^2}{dz^2} - \frac{2}{z} \frac{d}{dz} + 1 + \frac{2}{z^2} \right) \phi_\mathbf{k}^{\text{mcc}} = 0. \tag{44}$$

The solutions to (43) and (44) are given by

$$\phi_\mathbf{k}^{\text{mmc}} = c_{\text{mmc}}(i + z)e^{iz}, \tag{45}$$

$$\phi_\mathbf{k}^{\text{mcc}} = c_{\text{mcc}} i z e^{iz}, \tag{46}$$

where c_{mmc} and c_{mcc} are constants to be determined. Then the field operator $\hat{\varphi}$ can be expanded in the Fourier modes as

$$\hat{\varphi}(\eta, \mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3\mathbf{k} \left[\left(\hat{a}_1(\mathbf{k}) \phi_\mathbf{k}^{\text{mmc}}(\eta) \right. \right.$$
$$\left. \left. + \hat{a}_2(\mathbf{k}) \phi_\mathbf{k}^{\text{mcc}}(\eta) \right) e^{i\mathbf{k}\cdot\mathbf{x}} + \text{h.c.} \right], \tag{47}$$

where the two commutation relations take the forms

$$[\hat{a}_i(\mathbf{k}), \hat{a}_j^\dagger(\mathbf{k}')] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \delta^3(\mathbf{k} - \mathbf{k}'). \tag{48}$$

It is noted that two mode operators $(\hat{a}_1(\mathbf{k}), \hat{a}_2(\mathbf{k}))$ are necessary to take into account fourth-order theory quantum mechanically as the Pais–Uhlenbeck fourth-order oscillator has been shown in Ref. [31]. In addition, two Wronskian conditions are found to be

$$\left[\phi_\mathbf{k}^{\text{mmc}} \left\{ \left(\phi_\mathbf{k}^{*\text{mmc}}(\eta) \right)''' + +2k^2 \left(\phi_\mathbf{k}^{*\text{mmc}}(\eta) \right)' \right. \right.$$
$$\left. - 2aHk^2 \phi_\mathbf{k}^{*\text{mmc}}(\eta) \right\}$$
$$- \phi_\mathbf{k}^{\text{mcc}} \left\{ \left(\phi_\mathbf{k}^{*\text{mcc}}(\eta) \right)''' + 2k^2 \left(\phi_\mathbf{k}^{*\text{mmc}}(\eta) \right)' \right.$$
$$\left. \left. - 2aHk^2 \phi_\mathbf{k}^{*\text{mcc}}(\eta) \right\} \right] - c.c. = i,$$

$$\left[\left(\phi_\mathbf{k}^{\text{mmc}} \right)' \left\{ \left(\phi_\mathbf{k}^{*\text{mmc}}(\eta) \right)'' + 2aH \left(\phi_\mathbf{k}^{*\text{mmc}}(\eta) \right)' \right\} \right.$$
$$- \left(\phi_\mathbf{k}^{\text{mcc}} \right)' \left\{ \left(\phi_\mathbf{k}^{*\text{mcc}}(\eta) \right)'' \right.$$
$$\left. \left. + 2aH \left(\phi_\mathbf{k}^{*\text{mcc}}(\eta) \right)' \right\} \right] - c.c. = -i, \tag{49}$$

which will be used to fix c_{mmc} and c_{mcc} as

$$\phi_\mathbf{k}^{\text{mmc}} = \frac{1}{\sqrt{2^2 k^3}} (i + z) e^{iz}, \quad \phi_\mathbf{k}^{\text{mcc}} = \frac{1}{\sqrt{2^2 k^3}} i z e^{iz}. \tag{50}$$

On the other hand, the power spectrum [26] of the scalar is defined by

$$\langle 0 | \hat{\varphi}(\eta, 0) \hat{\varphi}(\eta, 0) | 0 \rangle = \int \frac{dk}{k} \mathcal{P}_\varphi(\eta, k). \tag{51}$$

Considering the Bunch–Davies vacuum state imposed by $\hat{a}_\mathbf{k}|0\rangle = 0$ and $\hat{b}_\mathbf{k}|0\rangle = 0$, (51) is computed as

$$\mathcal{P}_\varphi(\eta, k) = \frac{k^3}{2\pi^2} \left(\left| \phi_\mathbf{k}^{\text{mmc}} \right|^2 - \left| \phi_\mathbf{k}^{\text{mcc}} \right|^2 \right) \tag{52}$$

$$= \frac{1}{8\pi^2} \left[(1 + z^2) - z^2 \right] \tag{53}$$

$$= \frac{1}{8\pi^2}. \tag{54}$$

Importantly, the minus sign $(-)$ in (52) appears because the unusual commutation relation $(\hat{a}_2(\mathbf{k}), \hat{a}_2^\dagger(\mathbf{k}'))$ for the ghost state was used. There is a cancelation between z^2 and $-z^2$ thanks to its ghost-like contribution.

Finally, the conformally coupled scalar equation (19) from (17) leads to the linearized equation around the dS spacetime, thus:

$$(\Box - 2H^2)\varphi^{\text{mcc}} = 0, \tag{55}$$

which is the same equation as in (41). Its normalized solution takes the form

$$\tilde{\phi}_\mathbf{k}^{\text{mcc}} = \frac{H}{\sqrt{2k^3}} i z e^{iz}. \tag{56}$$

In this case, we obviously have a scale-variant scalar power spectrum,

$$\mathcal{P}_{\text{mcc}} = \frac{k^3}{2\pi^2}\left|\tilde{\phi}_{\mathbf{k}}^{\text{mcc}}\right|^2 = \left(\frac{k}{2\pi a}\right)^2, \tag{57}$$

which depends on the wave number k.

4.2 Vector power spectrum

Let us consider Eq. (29) for a vector perturbation and then expand Ψ_i in plane waves with the linearly polarized states

$$\Psi_i(\eta, \mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3\mathbf{k} \sum_{s=1,2} p_i^s(\mathbf{k}) \Psi_{\mathbf{k}}^s(\eta) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{58}$$

where $p_i^{1/2}$ are linear polarization vectors with $p_i^{1/2} p^{1/2,i} = 1$. Also, $\Psi_{\mathbf{k}}^s$ denote linearly polarized vector modes. Plugging (58) into Eq. (29), one finds the equation

$$\left[\frac{d^2}{d\eta^2} + k^2\right] \Psi_{\mathbf{k}}^s(\eta) = 0. \tag{59}$$

Introducing $z = -k\eta$, Eq. (59) takes a simple form:

$$\left[\frac{d^2}{dz^2} + 1\right] \Psi_{\mathbf{k}}^s(z) = 0, \tag{60}$$

whose solution is given by

$$\Psi_{\mathbf{k}}^s(z) \sim e^{\pm iz}. \tag{61}$$

Here a positive frequency solution is given by e^{iz}.

Now, let us calculate vector power spectrum. For this purpose, we define a commutation relation for the vector. In the bilinear action (26), the momentum for the field Ψ_j is found to be

$$\pi_\Psi^j = -\frac{\alpha}{2}\nabla^2 \Psi'^j. \tag{62}$$

Note that one observes an unusual factor of the Laplacian ∇^2 which reflects that the vector Ψ_i is not a canonically well-defined vector because it originates from the fourth-order conformal gravity. The quantization is implemented by imposing the commutation relation

$$[\hat{\Psi}_j(\eta, \mathbf{x}), \hat{\pi}_\Psi^j(\eta, \mathbf{x}')] = 2i\delta(\mathbf{x} - \mathbf{x}'), \tag{63}$$

with $\hbar = 1$. Then the operator $\hat{\Psi}_j$ can be expanded in Fourier modes as

$$\hat{\Psi}_j(\eta, \mathbf{x})$$
$$= \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3\mathbf{k} \sum_{s=1,2} \left(p_j^s(\mathbf{k})\hat{a}_{\mathbf{k}}^s \Psi_{\mathbf{k}}^s(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + \text{h.c.}\right) \tag{64}$$

and the operator $\hat{\pi}_\Psi^j = \frac{\alpha k^2}{2}\hat{\Psi}'^j$ can be obtained from (64). Plugging (64) and $\hat{\pi}_\Psi^j$ into (63), we find the commutation relation and Wronskian condition for the normalization as

$$\left[\hat{a}_{\mathbf{k}}^s, \hat{a}_{\mathbf{k}'}^{s'\dagger}\right] = \delta^{ss'}\delta^3(\mathbf{k} - \mathbf{k}'), \tag{65}$$

$$\Psi_{\mathbf{k}}^s\left(\frac{\alpha k^2}{2}\right)(\Psi_{\mathbf{k}}^{*s})' - \text{c.c.} = i \rightarrow \Psi_{\mathbf{k}}^s \frac{d\Psi_{\mathbf{k}}^{*s}}{dz} - \text{c.c.}$$
$$= -\frac{2i}{\alpha k^3}. \tag{66}$$

We choose the positive frequency mode normalized by the Wronskian condition,

$$\Psi_{\mathbf{k}}^s(z) = \sqrt{\frac{1}{\alpha k^3}}e^{iz}, \tag{67}$$

as the solution to (60). On the other hand, the vector power spectrum is defined by

$$\langle 0|\hat{\Psi}_j(\eta, 0)\hat{\Psi}^j(\eta, 0)|0\rangle = \int \frac{dk}{k}\mathcal{P}_\Psi(\eta, k), \tag{68}$$

where we take the Bunch–Davies vacuum $|0\rangle$ by imposing $\hat{a}_{\mathbf{k}}^s|0\rangle = 0$. The vector power spectrum $\mathcal{P}_\Psi$ leads to

$$\mathcal{P}_\Psi \equiv \sum_{s=1,2} \frac{k^3}{2\pi^2}\left|\Psi_{\mathbf{k}}^s\right|^2. \tag{69}$$

Plugging (67) into (69), we find a scale-invariant power spectrum for a vector perturbation,

$$\mathcal{P}_\Psi = \frac{1}{\pi^2\alpha^2}. \tag{70}$$

4.3 Tensor power spectrum

Now, let us take Eq. (30) to compute the tensor power spectrum. In this case, the metric tensor h_{ij} can be expanded in Fourier modes:

$$h_{ij}(\eta, \mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3\mathbf{k} \sum_{s=+,\times} p_{ij}^s(\mathbf{k})h_{\mathbf{k}}^s(\eta) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{71}$$

where p_{ij}^s are linear polarization tensors with $p_{ij}^s p^{s,ij} = 1$. Also, $h_{\mathbf{k}}^s(\eta)$ represent linearly polarized tensor modes. Plugging (71) into (30) leads to the fourth-order differential equation

$$(h_{\mathbf{k}}^s)'''' + 2k^2(h_{\mathbf{k}}^s)'' + k^4 h_{\mathbf{k}}^s = 0, \tag{72}$$

which is further rewritten as a factorized form

$$\left[\frac{d^2}{d\eta^2} + k^2\right]^2 h_{\mathbf{k}}^s(\eta) = 0. \tag{73}$$

Introducing $z = -k\eta$, Eq. (73) can be rewritten as a degenerate fourth-order equation,

$$\left[\frac{d^2}{dz^2} + 1\right]^2 h_{\mathbf{k}}^s(z) = 0. \tag{74}$$

This is the same equation as for a degenerate Pais–Uhlenbeck (PU) oscillator [31] and its solution is given by

$$h_{\mathbf{k}}^s(z) = \frac{N}{2k^2}\left[(a_2^s + a_1^s z)e^{iz} + c.c.\right] \tag{75}$$

with N the normalization constant. After quantization, a_2^s and a_1^s are promoted to operators $\hat{a}_2^s(\mathbf{k})$ and $\hat{a}_1^s(\mathbf{k})$ ($h_{\mathbf{k}}^s \to \hat{h}_{\mathbf{k}}^s$). The presence of z in $(\cdots)$ reflects clearly that $h_{\mathbf{k}}^s(z)$ is a solution to the degenerate Eq. (74). Together with $N = \sqrt{2/\alpha}$, the canonical quantization could be accomplished by introducing commutation relations between $\hat{a}_i^s(\mathbf{k})$ and $\hat{a}_j^{s'\dagger}(\mathbf{k}')$, thus [32]:

$$[\hat{a}_i^s(\mathbf{k}), \hat{a}_j^{s'\dagger}(\mathbf{k}')] = 2k\delta^{ss'}\begin{pmatrix} 0 & -i \\ i & 1 \end{pmatrix}\delta^3(\mathbf{k} - \mathbf{k}'). \tag{76}$$

On the other hand, the tensor power spectrum is defined by

$$\langle 0|\hat{h}_{ij}(\eta, 0)\hat{h}^{ij}(\eta, 0)|0\rangle = \int \frac{dk}{k}\mathcal{P}_h(\eta, k). \tag{77}$$

Here we choose the Bunch–Davies vacuum $|0\rangle$ by imposing $\hat{a}_i^s(\mathbf{k})|0\rangle = 0$ for $i = 1, 2$. Using the definition

$$\mathcal{P}_h \equiv \sum_{s,s'=+,\times} \mathcal{P}_h^{ss'} \tag{78}$$

and substituting $\hat{h}_{ij}(\eta, 0)$ together with (76) into (77), then one finds that $\mathcal{P}_h^{ss'}$ is given by

$$\begin{aligned}
\mathcal{P}_h^{ss'} = \frac{k}{4\pi^2\alpha^2}\int d^3\mathbf{k}'&\left[\frac{1}{k'^2}p_{ij}^s(\mathbf{k})p^{ijs'}(\mathbf{k}')\right.\\
&\times\left\langle 0\left|\left(\left[\hat{a}_2^s(\mathbf{k}), \hat{a}_2^{s'\dagger}(\mathbf{k}')\right] + z\left[\hat{a}_2^s(\mathbf{k}), \hat{a}_1^{s'\dagger}(\mathbf{k}')\right]\right.\right.\right.\\
&\left.\left.\left.+ z\left[\hat{a}_1^s(\mathbf{k}), \hat{a}_2^{s'\dagger}(\mathbf{k}')\right] + z^2\left[\hat{a}_1^s(\mathbf{k}), \hat{a}_1^{s'\dagger}(\mathbf{k}')\right]\right)\right|0\right\rangle\right]
\end{aligned} \tag{79}$$

$$= \frac{1}{2\pi^2\alpha^2}p_{ij}^s p^{ijs'}\delta^{ss'}[1 - iz + iz + 0 \times z^2]. \tag{80}$$

In obtaining (80), we used the commutation relations of (76) which reflect the quantum nature of Weyl-invariant tensor theory like a degenerate PU oscillator. A cancelation between iz and $-iz$ occurs, showing that this is slightly different from that between z^2 and $-z^2$ in (53) for the Weyl-invariant scalar theory of a non-degenerate PU oscillator. Finally, from (78)

and (80), we obtain a scale-invariant tensor power spectrum,

$$\mathcal{P}_h = \frac{1}{\pi^2\alpha^2}, \tag{81}$$

which is of the same form as the vector power spectrum (70).

5 Discussions

First of all, we have emphasized a deep connection between the Weyl invariance of the action (fourth-order theory) and the scale invariance of the power spectrum in dS spacetime.

In deriving the power spectra, we have used two different quantization schemes for fourth-order theory: a non-degenerate PU oscillator was employed for quantizing the Weyl-invariant scalar theory, while a degenerate PU oscillator could be used to quantize the Weyl-invariant tensor theory of conformal gravity.

However, we have found the same ambiguity of the power spectra in dS spacetime; the scalar power spectrum takes the form $\to 1/2(2\pi)^2$ instead of the conventional spectrum $(H/2\pi)^2[1 + (k/aH)^2]$ for a second-order scalar theory of the massless scalar and the tensor power spectra, given by $1/\pi^2\alpha^2$ instead of $2(H/\pi M_P)^2[1 + (k/aH)^2]$ for a second-order tensor theory (Einstein gravity) of massless gravitons. Here H^2 was missing and there is no way to restore it in this approach. If one had used the Krein space quantization, which is the generalization of the Hilbert space, to quantize a massless scalar in dS space [33], its power spectrum would have led to $(H/2\pi)^2$, which is also scale invariant as a result of elimination of the scale-dependent term of $(k/2\pi a)^2$. However, this method to derive a scale-invariant scalar spectrum is an ad hoc approach because it has dealt with a second-order scalar theory.

Now, we ask whether our model (2) is just a toy model for providing scale-invariant power spectra of scalar and tensor fields or really has an application to the early stage of the universe (inflation). We remind the reader that power spectra have been computed based on the dS spacetime (eternal inflation). However, a slow-roll inflation is in order for a quasi-dS spacetime with a graceful exit. In the slow-roll inflation, the scale-dependence of the power spectra appears when fluctuations of scalar and tensor exit the comoving Hubble radius $[1/(aH)]$ even for choosing the superhorizon limit of $z = k/aH \to 0$ [26]. It seems difficult to compute power spectra of the scalar and tensor cases when one takes slow-roll inflation. Hence, our model (2) is suggested to be a toy model for providing scale-invariant power spectra of the scalar and tensor fields in dS inflation, which are independent of the scale $z(k)$ in the whole range of z.

At this stage, we would like to comment on two different Weyl-invariant theories. The action S_{ST1} in Eq. (2) gives

us a scale-invariant scalar spectrum of $1/2(2\pi)^2$, while the action S_{ST3} in Eq. (17) provides a scale-variant scalar spectrum of $(k/2\pi a)^2$. The former scalar is dimensionless (Weyl invariant), whereas the latter has dimension 1 (Weyl variant).

Finally, we should mention the ghost issues (negative-norm state) because Weyl-invariant scalar–tensor theory is a fourth-order scalar–tensor theory. In general, a fourth-order scalar theory implies two second-order theories with opposite signs in the diagonalized commutation relations (48), while a degenerate fourth-order tensor theory implies two second-order tensor theories with the non-diagonalized commutation relations (76). In the Weyl-invariant scalar theory, there is a cancelation between z^2 (positive-norm state from mmc) and $-z^2$ (negative-norm state from mcc scalar) in the power spectrum (53). This reflects the quantization of the non-degenerate PU oscillator. On the other hand, in the Weyl-invariant tensor theory, a cancelation between iz and $-iz$ in the power spectrum (80) occurs as in the quantization scheme of degenerate PU oscillator. Consequently, there are no negative-norm states in the scalar and tensor power spectra in dS spacetime. This may indicate that the Weyl invariance forbids the ghost states of the power spectra in dS spacetime. As counter examples, one did not obtain a positive scalar (tensor) power spectrum from the nonminimal derivative coupling with fourth-order term [16] (Einstein–Weyl gravity [10,11]), which are not obviously Weyl invariant.

References

1. P.D. Mannheim, Prog. Part. Nucl. Phys. **56**, 340 (2006). doi:10.1016/j.ppnp.2005.08.001. arXiv:astro-ph/0505266
2. N.D. Birrell, P.C.W. Davies, *Quantum Field in Curved Space* (Cambridge UP, Cambridge, 1982)
3. R.J. Riegert, Phys. Rev. Lett. **53**, 315 (1984)
4. K.S. Stelle, Phys. Rev. D **16**, 953 (1977)
5. P.D. Mannheim, Phys. Rev. D **85**, 124008 (2012). arXiv:1109.4119 [gr-qc]
6. Y.S. Myung, T. Moon, arXiv:1407.0441 [gr-qc]
7. Y.S. Myung, T. Moon, Mod. Phys. Lett. A **30**(32), 1550172 (2015). doi:10.1142/S0217732315501722. arXiv:1501.01749 [gr-qc]
8. T. Clunan, M. Sasaki, Class. Quant. Grav. **27**, 165014 (2010). arXiv:0907.3868 [hep-th]
9. N. Deruelle, M. Sasaki, Y. Sendouda, A. Youssef, JCAP **1103**, 040 (2011). arXiv:1012.5202 [gr-qc]
10. N. Deruelle, M. Sasaki, Y. Sendouda, A. Youssef, JHEP **1209**, 009 (2012). arXiv:1202.3131 [gr-qc]
11. Y.S. Myung, T. Moon, JCAP **1408**, 061 (2014). arXiv:1406.4367 [gr-qc]
12. T.D. Lee, G.C. Wick, Nucl. Phys. B **9**, 209 (1969)
13. T.D. Lee, G.C. Wick, Phys. Rev. D **2**, 1033 (1970)
14. Y.F. Cai, T.T. Qiu, R. Brandenberger, X.M. Zhang. Phys. Rev. D **80**, 023511 (2009). arXiv:0810.4677 [hep-th]
15. Y.S. Myung, T. Moon, JCAP **1502**(02), 044 (2015). arXiv:1412.7263 [gr-qc]
16. Y.S. Myung, T. Moon, arXiv:1502.03881 [gr-qc]
17. E. Mottola, Acta Phys. Polon. B **41**, 2031 (2010). arXiv:1008.5006 [gr-qc]
18. I. Antoniadis, P.O. Mazur, E. Mottola, JCAP **1209**, 024 (2012). arXiv:1103.4164 [gr-qc]
19. S.V. Sushkov, Phys. Rev. D **80**, 103505 (2009). arXiv:0910.0980 [gr-qc]
20. C. Germani, A. Kehagias, Phys. Rev. Lett. **105**, 011302 (2010). arXiv:1003.2635 [hep-ph]
21. C. Germani, Y. Watanabe, JCAP **1107**, 031 (2011) [Addendum-ibid. **1107**, A01 (2011)]. arXiv:1106.0502 [astro-ph.CO]
22. R.J. Riegert, Phys. Lett. B **134**, 56 (1984)
23. F.F. Faria, Adv. High Energy Phys. **2014**, 520259 (2014). arXiv:1312.5553 [gr-qc]
24. Y.S. Myung, Phys. Lett. B **730**, 130 (2014). arXiv:1401.1890 [gr-qc]
25. V.F. Mukhanov, H.A. Feldman, R.H. Brandenberger, Phys. Rep. **215**, 203 (1992)
26. D. Baumann, arXiv:0907.5424 [hep-th]
27. J. Bros, H. Epstein, U. Moschella, Lett. Math. Phys. **93**, 203 (2010). arXiv:1003.1396 [hep-th]
28. A. Higuchi, Y.C. Lee, Class. Quant. Grav. **26**, 135019 (2009). arXiv:0903.3881 [gr-qc]
29. P.R. Anderson, E. Mottola, Phys. Rev. D **89**(10), 104039 (2014). arXiv:1310.1963 [gr-qc]
30. A. Youssef, Phys. Lett. B **718**, 1095 (2013). arXiv:1203.3171 [gr-qc]
31. P.D. Mannheim, A. Davidson, Phys. Rev. A **71**, 042110 (2005). arXiv:hep-th/0408104
32. Y.W. Kim, Y.S. Myung, Y.J. Park, Phys. Rev. D **88**, 085032 (2013). arXiv:1307.6932
33. M. Mohsenzadeh, M.R. Tanhayi, E. Yusofi, Eur. Phys. J. C **74**, 2920 (2014). arXiv:1306.6722 [hep-th]