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Superradiance and instability of small rotating charged AdS black holes in all dimensions

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Abstract Rotating small AdS black holes exhibit the superradiant instability to low-frequency scalar perturbations, which is amenable to a complete analytic description in four dimensions. In this paper, we extend this description to all higher dimensions, focusing on slowly rotating charged AdS black holes with a single angular momentum. We divide the spacetime of these black holes into the near-horizon and far regions and find solutions to the scalar wave equation in each of these regions. Next, we perform the matching of these solutions in the overlap between the regions, by employing the idea that the orbital quantum number ℓ can be thought of as an approximate integer. Thus, we obtain the complete low-frequency solution that allows us to calculate the complex frequency spectrum of quasinormal modes, whose imaginary part is determined by a small damping parameter. Finally, we find a remarkably instructive expression for the damping parameter, which appears to be a complex quantity in general. We show that the real part of the damping parameter can be used to give a *universal* analytic description of the superradiant instability for slowly rotating charged AdS black holes in all spacetime dimensions.

1 Introduction

Superradiance is a property of rotating black holes by which waves of certain frequencies are amplified when scattering by the black holes. The amplification effect occurs due to the extraction of rotational energy from the black holes. This is a wave analog of the hypothetical Penrose process [1] wherein the extraction of rotational energy occurs by orbiting and breaking up a particle in the ergoregion of the black holes. Zel'dovich apparently first pointed out that a rotating black hole may possess the superradiant property just as an axially symmetric body rotating with an angular velocity Ω in a resonant cavity, where oscillation modes with the fre-

quency ω and the azimuthal number m, fulfilling the condition $\omega < m\Omega$, undergo amplification [2,3]. The superradiant property was independently discussed by Misner [4] as well, who demonstrated that scalar waves scattered by a Kerr black hole become amplified if the wave frequency $\omega < m\Omega_H$, where Ω_H is the angular velocity of the black hole. Developing Zel'dovich's heuristic idea on the possibility of exponential amplification of waves, when surrounding a rotating black hole by a semitransparent mirror, Press and Teukolsky [5] proposed a "black hole bomb" mechanism as an intriguing probe of the strong superradiant instability in the black hole–mirror system.

In the spirit of these fruitful ideas, Starobinsky [6] developed a quantitative theory of superradiance for scalar waves in the Kerr metric. Similar considerations for electromagnetic and gravitational waves were given in [7]. Ever since, superradiance and the black hole bomb mechanism have been the subject of many investigations. The authors of Refs. [8,9] have shown that rotating black holes are stable against all modes of massless bosonic perturbations, whereas it appeared to be opposite for massive bosonic perturbations [10–12] (see also [13] and references therein). It is the nonvanishing mass of a bosonic field that provides a natural mirror around the black hole, thereby resulting in the superradiant instability of bound state modes of the bosonic field. In addition, it has recently been shown that the instability time scale for these bound state modes may become orders of magnitude shorter in many cases of physical interest [14–19] and thus giving rise to potentially observable effects of the superradiant instability. In this regard, the case of ultralight axions appearing in the "axiverse" scenario [20] of string theory compactifications is particularly intriguing. It appears that for axions in a certain mass range, the time scale of the superradiant instability becomes significantly short, creating gaps in the mass–spin spectrum of astrophysical black holes [20].

Yet another physically more realistic setup for the black hole instability can occur in spacetimes with natural reflec-
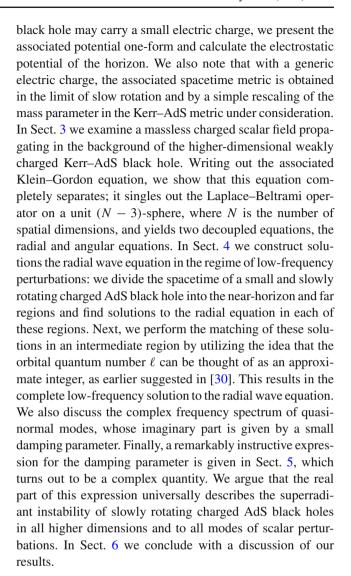
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tive boundaries. For example, the "confining-box" behavior of asymptotically anti-de Sitter (AdS) or Gödel spacetimes can serve as a resonant cavity between the rotating black holes and spatial infinity. This in turn would result in the instability of superradiant modes of massless bosonic perturbations. As shown in [21], small Kerr–AdS black holes do indeed exhibit the superradiant instability to scalar perturbations (see also [22–25] for related work). Using the arguments of Refs. [21,26], a quantitative description of the superradiant instability of small rotating charged AdS black holes in five dimensions was given in [27]. Meanwhile, the unstable superradiant modes of scalar perturbations, which occur for rotating black holes in the Gödel universe of five-dimensional minimal ungauged supergravity, were studied in [28] by using numerical calculations. More recently, the superradiant effects of a massive and charged scalar field for small Reissner–Nordström–AdS black holes were analyzed in all spacetime dimensions, by employing analytical and numerical techniques [29]. This analysis has shown that in odd spacetime dimensions the matching procedure of [27] fails for some values of the orbital quantum number ℓ, thereby entailing the apparent absence of the instability for these modes. This occurs because the near-horizon and far regions solutions to the Klein–Gordon equation have different functional dependence (involving the logarithmic term) that makes their matching impossible. Solving numerically the Klein–Gordon equation for this special case, where the analytical method is no longer valid, the authors of [29] have shown that the superradiant instability does exist in this case as well.

In spite of the subtleties with the matching procedure in odd spacetime dimensions, one can still successfully use an analytical approach to give a complete description of the superradiant instability in higher dimensions. In fact, this was demonstrated in [30] by focusing on the black hole bomb model for scalar perturbations in five dimensions. The key ingredients of this description are based on a theoretical assumption that the orbital quantum number ℓ is not exactly, but nearly integer. This allows one to perform the matching of the near-horizon and far regions solutions in their overlap region, avoiding the solution with a logarithmic term that excludes the matching procedure. The resulting frequency spectrum of the bound state modes, which is calculated by taking the corresponding limits as ℓ approaches even or odd integers, shows that all ℓ modes undergo the superradiant instability.

The main aim of this paper is to continue the spirit of [30] and give a complete analytic description of the superradiant instability in all higher dimensions, focusing on the case of slowly rotating charged AdS black holes with a single angular momentum. In Sect. 2 we begin with the spacetime metric for a higher-dimensional Kerr–AdS black hole and discuss some of its basic properties. Assuming that the

black hole may carry a small electric charge, we present the associated potential one-form and calculate the electrostatic potential of the horizon. We also note that with a generic electric charge, the associated spacetime metric is obtained in the limit of slow rotation and by a simple rescaling of the mass parameter in the Kerr–AdS metric under consideration. In Sect. 3 we examine a massless charged scalar field propagating in the background of the higher-dimensional weakly charged Kerr–AdS black hole. Writing out the associated Klein–Gordon equation, we show that this equation completely separates; it singles out the Laplace–Beltrami operator on a unit $(N - 3)$-sphere, where N is the number of spatial dimensions, and yields two decoupled equations, the radial and angular equations. In Sect. 4 we construct solutions the radial wave equation in the regime of low-frequency perturbations: we divide the spacetime of a small and slowly rotating charged AdS black hole into the near-horizon and far regions and find solutions to the radial equation in each of these regions. Next, we perform the matching of these solutions in an intermediate region by utilizing the idea that the orbital quantum number ℓ can be thought of as an approximate integer, as earlier suggested in [30]. This results in the complete low-frequency solution to the radial wave equation. We also discuss the complex frequency spectrum of quasinormal modes, whose imaginary part is given by a small damping parameter. Finally, a remarkably instructive expression for the damping parameter is given in Sect. 5, which turns out to be a complex quantity. We argue that the real part of this expression universally describes the superradiant instability of slowly rotating charged AdS black holes in all higher dimensions and to all modes of scalar perturbations. In Sect. 6 we conclude with a discussion of our results.

2 The spacetime metric

The spacetime metric for general multiply rotating AdS black holes in all higher dimensions was found in [31]. We will focus on the higher-dimensional AdS black hole with a single angular momentum (the Kerr–AdS black hole), thus avoiding complications introduced by multiple rotations. The metric for this black hole is given by

$$ds^2 = -\frac{\Delta_r}{\Sigma} \left(dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right)^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2$$
$$+ \frac{\Delta_\theta \sin^2 \theta}{\Sigma} \left(a\, dt - \frac{r^2 + a^2}{\Xi} d\phi \right)^2 + r^2 \cos^2 \theta\, d\Omega_{N-3}^2,$$

(1)

where N is the number of spatial dimensions ($N \geq 3$) and

$$d\Omega_{N-3}^2 = d\chi_1^2 + \sin^2 \chi_1 (d\chi_2^2 + \sin^2 \chi_2 (\dots d\chi_{N-3}^2 \dots)),$$

(2)

is the metric on a unit $(N-3)$-sphere. The metric functions are given by

$$\Delta_r = \left(r^2 + a^2\right)\left(1 + \frac{r^2}{l^2}\right) - m\,r^{4-N}, \quad \Sigma = r^2 + a^2\cos^2\theta,$$

$$\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta, \quad \Xi = 1 - \frac{a^2}{l^2}. \tag{3}$$

Here l denotes the curvature radius of the AdS space, which is given by the negative cosmological constant, $\Lambda = -l^{-2}N(N-1)/2$. The parameters m and a are, respectively, determined by the mass and angular momentum of the black hole, as given in [32–35]. Meanwhile, for the determinant of metric (1) we have

$$\sqrt{-g} = \frac{\Sigma \sin\theta}{\Xi}\sqrt{\gamma}\,r^{N-3}\cos^{N-3}\theta, \tag{4}$$

where γ is the determinant of metric (2).

Next, we need to know the angular velocity and the electrostatic potential of the black hole, assuming that it may possess a small electric charge. They can be obtained by using the time-translational and rotational isometries of metric (1), described by two commuting Killing vectors $\xi_{(t)} = \partial_t$ and $\xi_{(\phi)} = \partial_\phi$. Defining a family of locally nonrotating observers on orbits with constant r and θ, for which the velocity vector u^μ obeys the condition $u \cdot \xi_{(\phi)} = 0$, we find that the coordinate angular velocity of these observers at the horizon $r \to r_+$ ($\Delta_r = 0$) is given by

$$\Omega_H = \frac{a\,\Xi}{r_+^2 + a^2}. \tag{5}$$

This in turn gives the angular velocity of the black hole. With this angular velocity, the co-rotating Killing vector $\chi = \xi_{(t)} + \Omega_H\,\xi_{(\phi)}$ becomes tangent to the null surface $\Delta_r = 0$, thus correctly describing the isometry of the horizon (see [35] for details).

Meanwhile, the potential one-form for a small electric charge of the Kerr–AdS black hole is determined by the difference between the timelike Killing isometries of metric (1) and those of its reference (the vanishing mass parameter, $m = 0$) background, as shown in [34,35]. Consequently, we have

$$A = -\frac{Q\,r^{4-N}}{(N-2)\,\Sigma}\left(\mathrm{d}t - \frac{a\sin^2\theta}{\Xi}\,\mathrm{d}\phi\right), \tag{6}$$

where the parameter Q is given by the electric charge of the black hole through Gauss' law. The associated contravariant components are given by

$$A^0 = \frac{Q\,r^{4-N}}{(N-2)\,\Sigma}\,\frac{r^2 + a^2}{\Delta_r}, \quad A^3 = \frac{Q\,r^{4-N}}{(N-2)\,\Sigma}\,\frac{a\,\Xi}{\Delta_r}, \tag{7}$$

which, by Eq. (6), yield

$$A_\mu A^\mu = -\left(\frac{Q}{N-2}\right)^2\frac{r^{2(4-N)}}{\Sigma\,\Delta_r}. \tag{8}$$

With these expressions in mind, it is now straightforward to show that the electrostatic potential of the horizon, defined as $\Phi_H = -A \cdot \chi$, is given by

$$\Phi_H = \frac{Q}{(N-2)}\,\frac{r_+^{4-N}}{r_+^2 + a^2}. \tag{9}$$

We also note that by a rescaling of the mass parameter in metric (1),

$$m \to m - q^2/r^{N-2}, \tag{10}$$

one can introduce a generic electric charge into the black hole spacetime. Then, for $N = 3$ we have the familiar Kerr–Newman–AdS metric in four dimensions, in which case $m = 2M$ and the potential one-form has the form as given in (6). However, in higher dimensions the system of Einstein–Maxwell equations becomes consistent only in the limit of slow rotation [36]. Thus, with Eqs. (6) and (10) and keeping only linear in a terms, the metric in (1) describes a slowly rotating higher-dimensional AdS black hole with an arbitrary amount of the electric charge. The horizon of such a black hole is governed by the equation $\Delta_r = 0$ in (3), after performing the rescaling in accordance with (10) and dropping the a^2 term. Thus, we have the equation

$$r^{2(N-2)}\left(1 + \frac{r^2}{l^2}\right) - mr^{N-2} + q^2 = 0, \tag{11}$$

where the parameter q is related to the electric charge of the black hole by the relation

$$q^2 = \frac{8\pi G Q^2}{(N-2)(N-1)}. \tag{12}$$

We note that in the limit of slow rotation, the parameter Q coincides with electric charge of the AdS black hole [35,36].

3 Scalar field

We now consider a massless scalar field Φ propagating in the background of the Kerr–AdS black hole, given by spacetime metric (1). Assuming that the black hole may also have a small electric charge, it is straightforward to show that the Klein–Gordon equation

$$(\nabla_\mu - ieA_\mu)(\nabla^\mu - ieA^\mu)\Phi = 0, \tag{13}$$

where ∇_μ is a covariant derivative operator and e is the charge of the scalar field, can be written out in the form

$$\frac{1}{\Sigma\,r^{N-3}}\frac{\partial}{\partial r}\left(\Delta_r\,r^{N-3}\frac{\partial\Phi}{\partial r}\right) + \frac{1}{\Sigma\sin\theta\cos^{N-3}\theta}\frac{\partial}{\partial\theta}$$
$$\times\left(\Delta_\theta\sin\theta\cos^{N-3}\theta\frac{\partial\Phi}{\partial\theta}\right) + g^{ab}\frac{\partial^2\Phi}{\partial x^a\partial x^a} - 2ieA^a\frac{\partial\Phi}{\partial x^a}$$
$$-e^2A_aA^a\Phi + \frac{1}{r^2\cos^2\theta}\triangle_{(N-3)}\Phi = 0. \tag{14}$$

Here we have introduced the Laplace–Beltrami operator $\triangle_{(N-3)}$ on a unit $(N-3)$-sphere,

$$\triangle_{(N-3)}\Phi = \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{\gamma}\,\gamma^{\alpha\beta}\frac{\partial\Phi}{\partial x^\beta}\right), \tag{15}$$

decomposing the indices as $\mu = \{r, \theta, a, \alpha\}$, where $a = 0, 3 \equiv t,\ \phi$ and $\alpha = 1, \ldots, N-3$.

Next, to separate variables in Eq. (14) we assume the ansatz in the form

$$\Phi = e^{-i\omega t + im\phi}\,R(r)\,S(\theta)\,Y_j(\Omega), \tag{16}$$

where m is the "magnetic" quantum number that takes integer values being associated with rotation in the ϕ-direction. In the following, we will focus on the case of positive frequency ($\omega > 0$) and positive m. The hyperspherical harmonics $Y_j(\Omega)$ are eigenfunctions of the Laplace–Beltrami operator (see e.g. [37]). The corresponding eigenvalues are given by

$$\triangle_{(N-3)}Y_j(\Omega) = -j(j+N-4)Y_j(\Omega). \tag{17}$$

With this in mind, it is not difficult to show that separation ansatz (16) yields two decoupled ordinary differential equations; the angular equation

$$\frac{1}{\sin\theta\cos^{N-3}\theta}\frac{d}{d\theta}\left(\Delta_\theta\sin\theta\cos^{N-3}\theta\frac{dS}{d\theta}\right)$$
$$+\left[\lambda - \frac{1}{\Delta_\theta}\left(\frac{m\Xi}{\sin\theta} - a\omega\sin\theta\right)^2 - \frac{j(j+N-4)}{\cos^2\theta}\right]S = 0 \tag{18}$$

and the radial equation

$$\frac{\Delta_r}{r^{N-3}}\frac{d}{dr}\left(\Delta_r r^{N-3}\frac{dR}{dr}\right) + U(r)\,R = 0, \tag{19}$$

where

$$U(r) = -\Delta_r\left[\lambda + \frac{j(j+N-4)a^2}{r^2}\right] + (r^2+a^2)^2$$
$$\times\left(\omega - \frac{am\Xi}{r^2+a^2} - \frac{eQ}{N-2}\frac{r^{4-N}}{r^2+a^2}\right)^2. \tag{20}$$

It should be noted that the angular equation (18), subject to the regular boundary conditions at $\theta = 0$ and $\theta = \pi/2$, yields a well-defined eigenvalue problem for the separation constant $\lambda = \lambda_\ell(\omega)$, where ℓ is an integer which can interpreted as being an "orbital" quantum number. The associated eigenfunctions are AdS modified higher-dimensional spheroidal harmonics $S(\theta) = S_{\ell mj}(\theta|a\omega)$. In general, the eigenvalues are determined numerically, but in some special cases they can also be calculated analytically by employing a series expansion in $a\omega$ and demanding its convergence, as shown in [38]. Similarly, assuming that $a/l \ll 1$ and focusing on

low frequencies one can show that

$$\lambda = \ell(\ell + N - 2) + \mathcal{O}\left(a^2\omega^2, a^2/l^2\right), \tag{21}$$

where ℓ is constrained by the condition $\ell \geq m + j$. This condition follows from the requirement that the series expansion is convergent in the limit $a\omega \to 0$, i.e. terminates within finite terms (see e.g. [38]).

To proceed with the radial equation (19), it is useful to introduce a new radial function, defined by

$$Y = \left[r^{N-3}(r^2+a^2)\right]^{1/2}R, \tag{22}$$

and a new radial coordinate r_*, given by

$$\frac{dr_*}{dr} = \frac{r^2+a^2}{\Delta_r}. \tag{23}$$

Using these relations in Eq. (19), we transform it to the Schrödinger form

$$\frac{d^2Y}{dr_*^2} + V(r)Y = 0, \tag{24}$$

where the effective potential is given by

$$V(r) = \frac{U(r)}{(r^2+a^2)^2} - \frac{1}{f}\frac{d^2f}{dr_*^2}. \tag{25}$$

Here $U(r)$ is the same as that given in Eq. (20) and $f = [r^{N-3}(r^2+a^2)]^{1/2}$.

Next, we need to impose boundary conditions on the propagating scalar waves at spatial infinity and at the horizon. Recalling that the AdS spacetime yields a natural reflective boundary at spatial infinity due to its confining-box behavior, it is tempting to impose the vanishing field boundary condition,

$$\Phi \to 0,\quad r \to \infty. \tag{26}$$

Meanwhile, from the physical point of view, it is clear that at the horizon one must impose an ingoing wave boundary condition. From Eq. (25) it follows that at the horizon, $r \to r_+$, the effective potential reduces to

$$V(r_+) = (\omega - m\Omega_H - e\Phi_H)^2. \tag{27}$$

This in turn yields the asymptotic solution that represents a purely ingoing wave at the horizon,

$$\Phi = e^{-i\omega t + im\phi}\,e^{-i(\omega - \omega_p)r_*}S(\theta)Y_j(\Omega), \tag{28}$$

where ω_p is the threshold frequency, given by

$$\omega_p = m\Omega_H + e\Phi_H. \tag{29}$$

It follows that for the frequency range

$$0 < \omega < \omega_p, \tag{30}$$

the phase velocity of the wave, $v_{ph} = \omega/(\omega_p - \omega)$, is in the opposite direction with respect to the group velocity,

$v_{gr} = -1$. This fact signifies the appearance of superradiance, resulting in the energy outflow from the black hole.

It is clear that the above boundary conditions at spatial infinity and at the horizon render the frequency spectrum of bound state modes *quasinormal*, with complex frequencies. If the imaginary part of a characteristic frequency is positive, exponential growth of the associated mode amplitude occurs, as follows from decomposition (16). In this case, the system would eventually develop instability. In the following, we will describe this phenomenon for low-frequency modes in which case such a description appears to be amenable to analytic consideration.

4 Low-frequency solutions

We will now construct solutions to radial wave equation (19), focusing on low-frequency perturbations, i.e. in the limit when the frequency of the typical perturbation is much less than the inverse of the horizon scale, $\omega \ll 1/r_+$. It is not difficult to show that even in this case, one cannot find the general solution to Eq. (19) by employing the methods which are known in the theory of ordinary differential equations. On the other hand, this can be done by dividing the spacetime into the near-horizon region, $r - r_+ \ll 1/\omega$, and the far-horizon region, $r - r_+ \gg r_+$, with the subsequent approximating Eq. (19) for each of these regions. Clearly, the solution of the resulting equation in the far-region which is valid only for large r, might also correspond to small $\omega(r - r_+)$ for sufficiently small frequencies, i.e. provided that $\omega r_+ \ll \omega(r - r_+) \ll 1$. Meanwhile, for $\omega \to 0$, the near-horizon region solution tends to cover whole spacetime. Altogether, one can conclude that for sufficiently small frequencies there must exist a region, given by $r_+ \ll r - r_+ \ll 1/\omega$, where the near-horizon solution overlaps the far-region solution. Then matching of these solutions in the overlapping region will yield the complete low-frequency solution to Eq. (19). Below, we show that this is indeed the case, by presenting the explicit form of these solutions and performing their appropriate asymptotic analysis. We note that a similar approach was first developed by Starobinsky [6] in the theory of superradiance for a Kerr black hole in four dimensions. To avoid potential complications, we will here focus on a small AdS black hole in the regime of slow rotation, i.e. when $r_+ \ll l$ and $a \ll r_+$. In the slow rotation regime, we will also consider an arbitrary electric charge for the black hole, using Eq. (10) in metric (1) and dropping all terms higher than linear order in rotation parameter a. With these assumptions in mind, we find solutions to the radial equation (19), first in the near-horizon region and then in the far-region of the spacetime.

Near-horizon solution: In the near-horizon region, we have $r - r_+ \ll 1/\omega$ and, for a slowly rotating AdS black hole of small size, Eq. (19) is approximated by

$$(N-2)^2 \Delta_x \frac{d}{dx}\left(\Delta_x \frac{dR}{dx}\right) + \left[x_+^{\frac{2(N-1)}{N-2}} \left(\omega - \omega_p\right)^2 \right.$$
$$\left. - \ell(\ell + N - 2)\Delta_x \right] R = 0, \tag{31}$$

where we have defined a new metric function Δ_x, given by

$$\Delta_x = \Delta_r r^{2(N-3)} = (x - x_+)(x - x_-), \tag{32}$$

and a new coordinate $x = r^{N-2}$. Consequently, the quantities x_+ and x_- correspond to the radii of outer and inner horizons, which are determined by Eq. (11), with $r_+ \ll l$. In obtaining Eq. (31) we have also used Eq. (21), dropping small correction terms in it. Next, it is not difficult to show that by defining a new dimensionless coordinate,

$$z = \frac{x - x_+}{x - x_-}, \tag{33}$$

and performing a few simple manipulations, Eq. (31) can be transformed into the form

$$(1-z)\frac{d^2 R}{dz^2} + (1-z)\frac{dR}{dz} + \left[\frac{1-z}{z}\Omega^2 - \frac{\ell}{N-2} \right.$$
$$\left. \times \left(1 + \frac{\ell}{N-2}\right)\frac{1}{1-z} \right] R = 0, \tag{34}$$

where

$$\Omega = \frac{x_+^{\frac{N-1}{N-2}}}{N-2} \frac{\omega - \omega_p}{x_+ - x_-}. \tag{35}$$

This is a hypergeometric type differential equation which can be solved by the substitution

$$R(z) = z^{i\Omega}(1-z)^{1+\frac{\ell}{N-2}} F(z), \tag{36}$$

where the function $F(z) = F(\alpha, \beta, \gamma, z)$ is given by the standard hypergeometric equation

$$z(1-z)\frac{d^2 F}{dz^2} + \left[\gamma - (\alpha + \beta + 1)z\right]\frac{dF}{dz} - \alpha\beta F = 0 \tag{37}$$

with

$$\alpha = 1 + \frac{\ell}{N-2} + 2i\Omega, \quad \beta = 1 + \frac{\ell}{N-2}, \quad \gamma = 1 + 2i\Omega. \tag{38}$$

We are interested in the solution which at the horizon ($z \to 0$) obeys the ingoing wave boundary condition in (28). It is not difficult to show that the desired solution has the form

$$R(z) = A_{(+)}^{in} z^{-i\Omega}(1-z)^{1+\frac{\ell}{N-2}}$$
$$\times F\left(1 + \frac{\ell}{N-2}, 1 + \frac{\ell}{N-2} - 2i\Omega, 1 - 2i\Omega, z\right), \tag{39}$$

where $A^{in}_{(+)}$ is a constant. In the following we need to match this solution to the far-region solution of Eq. (19), in an intermediate region of their overlap. This in turn requires us to know the behavior of this solution at large r. However, it turns out that the matching procedure fails for some values of ℓ (in odd spacetime dimensions) regardless of whether the black hole is static or rotating. For the static black hole, this subtlety was first noted in [29], emphasizing the inevitability of a numerical analysis in the case under consideration. Remarkably, one can avoid the failure of the matching procedure by assuming that ℓ is nearly integer, which can always be thought of as a pure mathematical trick by introducing a small deviation from its exact value. In the rotating case, such an assumption could be argued physically as well, if one takes into account the correction terms in Eq. (21). However, we are dealing with the regime of slow rotation (ignoring a^2 and higher-order terms) therefore, we will henceforth employ a formal mathematical trick, assuming that ℓ is an approximate integer, which approaches the exact value when the small deviation from it vanishes.

Using now the standard formula (see e.g. [39]) that relates the hypergeometric functions of the arguments z and $1 - z$, we find that the large r ($z \to 1$) limit of solution (39) is given by

$$
R \simeq A^{in}_{(+)} \Gamma(1 - 2i\Omega)
$$
$$
\times \left[\frac{\Gamma\left(-1 - \frac{2\ell}{N-2}\right) (x_+ - x_-)^{1+\frac{\ell}{N-2}}}{\Gamma\left(-\frac{\ell}{N-2}\right) \Gamma\left(-\frac{\ell}{N-2} - 2i\Omega\right)} r^{2-N-\ell} \right.
$$
$$
\left. + \frac{\Gamma\left(1 + \frac{2\ell}{N-2}\right) (x_+ - x_-)^{-\frac{\ell}{N-2}}}{\Gamma\left(1 + \frac{\ell}{N-2}\right) \Gamma\left(1 + \frac{\ell}{N-2} - 2i\Omega\right)} r^{\ell} \right]. \tag{40}
$$

We recall once again that the number ℓ in this expression is supposed to be nearly integer since the quotient of gamma functions $\Gamma(-1 - \frac{2\ell}{N-2})/\Gamma(-\frac{\ell}{N-2})$ appearing in the first line requires a special care for $N \geq 4$, yielding in some cases divergent results for an exact integer ℓ.

Far-region solution: In the far-horizon region, $r - r_+ \gg r_+$, and for the black hole under consideration it follows that we can approximate Eq. (19) by its purely AdS limit. Thus, we have the equation

$$
\left(1 + \frac{r^2}{l^2}\right) \frac{d^2 R}{dr^2} + \left[\frac{N-1}{r} + (N+1)\frac{r}{l^2}\right] \frac{dR}{dr}
$$
$$
+ \left[\frac{\omega^2}{1 + \frac{r^2}{l^2}} - \frac{\ell(\ell + N - 2)}{r^2}\right] R = 0, \tag{41}
$$

where we again assume that ℓ is nearly integer. Next, we define a new radial coordinate y, given by

$$
y = \left(1 + \frac{r^2}{l^2}\right), \tag{42}
$$

in terms of which Eq. (41) takes the form

$$
y(1 - y)\frac{d^2 R}{dy^2} + \left[1 - \left(1 + \frac{N}{2}\right)y\right]\frac{dR}{dy}
$$
$$
- \frac{1}{4}\left[\frac{\omega^2 l^2}{y} - \frac{\ell(\ell + N - 2)}{y - 1}\right] R = 0. \tag{43}
$$

The solution to this equation we are interested in must be regular at the origin of the AdS space, $r \to 0$, and satisfy the vanishing field condition at spatial infinity [see Eq. (26)]. It is straightforward to verify that the substitution

$$
R = y^{\frac{\omega l}{2}} (1 - y)^{\frac{\ell}{2}} F(y), \tag{44}
$$

where the hypergeometric function $F(y)$ is given by Eq. (37), with the associated parameters

$$
\alpha = \frac{N}{2} + \frac{\ell}{2} + \frac{\omega l}{2}, \quad \beta = \frac{\ell}{2} + \frac{\omega l}{2}, \quad \gamma = 1 + \omega l, \tag{45}
$$

results in the desired solution. Finally, we find that

$$
R(y) = A_\infty\, y^{-\frac{\ell}{2} - \frac{N}{2}} (1 - y)^{\frac{\ell}{2}} F\left(\frac{N}{2} + \frac{\ell}{2} + \frac{\omega l}{2}, \right.
$$
$$
\left. \frac{N}{2} + \frac{\ell}{2} - \frac{\omega l}{2}, 1 + \frac{N}{2}, 1/y\right), \tag{46}
$$

where A_∞ is a constant. We also need to know the behavior of this solution at small distances, which becomes manifest when expressing it in terms of the hypergeometric functions of the argument $1 - y$. After some algebra, for $y \to 1$ we find the expansion

$$
R(r) \simeq A_\infty (-1)^{\ell/2} \Gamma\left(1 + \frac{N}{2}\right)
$$
$$
\times \left[\frac{\Gamma\left(\frac{N}{2} + \ell - 1\right) l^{N+\ell-2}\, r^{2-N-\ell}}{\Gamma\left(\frac{N}{2} + \frac{\ell}{2} + \frac{\omega l}{2}\right) \Gamma\left(\frac{N}{2} + \frac{\ell}{2} - \frac{\omega l}{2}\right)} \right.
$$
$$
\left. + \frac{\Gamma\left(1 - \ell - \frac{N}{2}\right) l^{-\ell}\, r^{\ell}}{\Gamma\left(1 - \frac{\ell}{2} + \frac{\omega l}{2}\right) \Gamma\left(1 - \frac{\ell}{2} - \frac{\omega l}{2}\right)} \right], \tag{47}
$$

which specifies the small r behavior of solution (46). In order that this solution be finite at the origin of the AdS space, $r = 0$, we must set the "quantization" condition

$$
\frac{N}{2} + \frac{\ell}{2} - \frac{\omega l}{2} = -n, \tag{48}
$$

which follows from the pole structure of the gamma function, $\Gamma(\frac{N}{2} + \frac{\ell}{2} - \frac{\omega l}{2}) = \infty$. Here n is a non-negative integer which can be thought of as a "principal" quantum number. Clearly, this condition governs the discrete frequency spectrum for scalar perturbations in the higher-dimensional AdS spacetime, resulting in the remarkably simple formula

$$
\omega_n = \frac{2n + \ell + N}{l}. \tag{49}
$$

This is in agreement with the result given in [29]. Recall that we consider only the positive frequency spectrum. Mean-

while, the presence of a black hole would render the decay of bound state modes by tunneling the waves through the potential barrier into the horizon. In this case, as mentioned above, the frequency spectrum becomes quasinormal, given by the complex frequencies

$$\omega = \omega_n + i\delta, \tag{50}$$

where δ is a small damping parameter that "measures" the decay of bound state modes due to the presence of the black hole.

Overlap region: Comparing now Eq. (47) with that given in (40), it is not difficult to see that there exists an overlapping region, $r_+ \ll r - r_+ \ll 1/\omega$, of validity for the near-horizon and far-region solutions. Thus, with relations (49) and (50) in mind, one can perform the matching of these solutions in the overlapping region. This allows us to find the damping parameter by iteration and, to first order, it is given by

$$
\delta = 2i\, \frac{(-1)^n}{n!}\, \frac{(x_+ - x_-)^{1+\frac{2\ell}{N-2}}}{l^{N+2\ell-1}}\, \frac{\Gamma\left(1+\frac{\ell}{N-2}\right)}{\Gamma\left(1+\frac{2\ell}{N-2}\right)}
$$
$$
\times \frac{\Gamma\left(N+\ell+n\right)}{\Gamma\left(\frac{N}{2}+n+1\right)\Gamma\left(\frac{N}{2}+\ell-1\right)} \times \frac{\Gamma\left(-1-\frac{2\ell}{N-2}\right)}{\Gamma\left(-\frac{\ell}{N-2}\right)}
$$
$$
\times \frac{\Gamma\left(1+\frac{\ell}{N-2}-2i\Omega\right)}{\Gamma\left(-\frac{\ell}{N-2}-2i\Omega\right)}\, \frac{\Gamma\left(1-\ell-\frac{N}{2}\right)}{\Gamma\left(1-\ell-\frac{N}{2}-n\right)}. \tag{51}
$$

Here the quantity Ω has the same form as given in Eq. (35), where one must now put $\omega = \omega_n$.

5 Instability

As mentioned above, when the damping parameter is positive, a characteristic field mode would exponentially grow its amplitude, resulting in the instability of the system. Therefore, to proceed with Eq. (51), we need to establish its sign for the cases of interest. We begin by simplifying the quotients of gamma functions, appearing in the second line of this expression. Using the standard functional relation for gamma functions $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$ and performing straightforward calculations, we find that

$$
\frac{\Gamma\left(-1-\frac{2\ell}{N-2}\right)}{\Gamma\left(-\frac{\ell}{N-2}\right)} = -\frac{1}{2\cos\left(\frac{\pi\ell}{N-2}\right)}\, \frac{\Gamma\left(1+\frac{\ell}{N-2}\right)}{\Gamma\left(2+\frac{2\ell}{N-2}\right)}, \tag{52}
$$

$$
\frac{\Gamma\left(1+\frac{\ell}{N-2}-2i\Omega\right)}{\Gamma\left(-\frac{\ell}{N-2}-2i\Omega\right)} = -\frac{1}{\pi}\left|\Gamma\left(1+\frac{\ell}{N-2}-2i\Omega\right)\right|^2
$$
$$
\times \left[\sin\left(\frac{\pi\ell}{N-2}\right)\cosh(2\pi\Omega) + i\cos\left(\frac{\pi\ell}{N-2}\right)\sinh(2\pi\Omega)\right], \tag{53}
$$

$$
\frac{\Gamma\left(1-\ell-\frac{N}{2}\right)}{\Gamma\left(1-\ell-\frac{N}{2}-n\right)} = (-1)^n \prod_{k=1}^{n}\left(\frac{N}{2}+\ell+k-1\right). \tag{54}
$$

In obtaining Eq. (54) we have also used the relation $\Gamma(z+k) = (z)_k \Gamma(z)$, where $(z)_k$ is the Pochhammer factorial, given by

$$
(z)_k = z(z+1)\cdots(z+k-1) = \prod_{i=1}^{k}(z+i-1) \tag{55}
$$

Substituting now these relations in (51), we find that the expression for the damping parameter can be put in the following remarkable form:

$$
\delta = \frac{(x_+ - x_-)^{1+\frac{2\ell}{N-2}}}{n!\, l^{N+2\ell-1}}\, \frac{\Gamma^2\left(1+\frac{\ell}{N-2}\right)}{\Gamma\left(1+\frac{2\ell}{N-2}\right)\Gamma\left(2+\frac{2\ell}{N-2}\right)}
$$
$$
\times \frac{\left|\Gamma\left(1+\frac{\ell}{N-2}-2i\Omega\right)\right|^2}{\pi\cos\left(\frac{\pi\ell}{N-2}\right)}
$$
$$
\times \left[i\sin\left(\frac{\pi\ell}{N-2}\right)\cosh(2\pi\Omega) - \cos\left(\frac{\pi\ell}{N-2}\right)\sinh(2\pi\Omega)\right]
$$
$$
\times \frac{\Gamma\left(N+\ell+n\right)}{\Gamma\left(\frac{N}{2}+n+1\right)\Gamma\left(\frac{N}{2}+\ell-1\right)}
$$
$$
\times \prod_{k=1}^{n}\left(\frac{N}{2}+\ell+k-1\right). \tag{56}
$$

We note that this is a complex quantity, whose real part describes the damping of modes, while the imaginary part gives the frequency shift of modes with respect to the AdS spectrum. It is easy to see that the overall sign of the real part is entirely determined by the sign of the quantity Ω and it is positive for $\Omega < 0$, i.e. in the superradiant regime. Meanwhile, no such sign-changing occurs for the imaginary part that behaves as being not sensitive to the superradiance.

In order to give further insight into Eq. (56), let us now assume that the orbital quantum number ℓ approaches a non-negative integer in the limit $\epsilon \to 0$ and consider the following exhaustive cases: (i) $\frac{\ell}{N-2} = p + \epsilon$, where p is a non-negative integer.[1] In this case, the imaginary part of

---

[1] Note that to retain the original form of Eq. (56), for the sake of uniformity, we have not explicitly used p therein, when discussing the above cases.

Table 1 The damping parameter of fundamental modes for a Reissner–Nordström–AdS black hole with $r_+ = 0.01$ in five and seven dimensions ($D = N + 1$); the scalar field charge $e = 12$

q/q_e	$\delta_{\frac{\ell}{N-2}=(p+1/2)+\epsilon},\ N=4,\ \ell=1,\ \epsilon \to 10^{-8}$	$\delta_{\frac{\ell}{N-2}=(p+1/2)+\epsilon},\ N=6,\ \ell=2,\ \epsilon \to 10^{-15}$
0.1	$-1.453 \times 10^{-9} - 0.370 \times 10^{-8}\,i/\epsilon$	$-1.509 \times 10^{-16} - 0.431 \times 10^{-15}\,i/\epsilon$
0.3	$-6.342 \times 10^{-10} - 0.311 \times 10^{-8}\,i/\epsilon$	$-1.013 \times 10^{-16} - 0.363 \times 10^{-15}\,i/\epsilon$
0.5	$+5.444 \times 10^{-11} - 0.210 \times 10^{-8}\,i/\epsilon$	$-5.275 \times 10^{-17} - 0.246 \times 10^{-15}\,i/\epsilon$
0.7	$+4.299 \times 10^{-10} - 0.098 \times 10^{-8}\,i/\epsilon$	$-1.497 \times 10^{-17} - 0.114 \times 10^{-15}\,i/\epsilon$
0.9	$+3.174 \times 10^{-10} - 0.020 \times 10^{-8}\,i/\epsilon$	$+2.209 \times 10^{-18} - 0.011 \times 10^{-15}\,i/\epsilon$

Table 2 The damping parameter of fundamental modes for a Kerr–AdS black hole with $r_+ = 0.01$ in five and seven dimensions ($D = N + 1$); the magnetic quantum number $m = 1$

a/r_+	$\delta_{\frac{\ell}{N-2}=(p+1/2)+\epsilon},\ N=4,\ \ell=1,\ \epsilon \to 10^{-8}$	$\delta_{\frac{\ell}{N-2}=(p+1/2)+\epsilon},\ N=6,\ \ell=2,\ \epsilon \to 10^{-15}$
0.1	$1.853 \times 10^{-9} - 0.378 \times 10^{-8}\,i/\epsilon$	$4.318 \times 10^{-17} - 0.437 \times 10^{-15}\,i/\epsilon$
0.2	$5.639 \times 10^{-9} - 0.408 \times 10^{-8}\,i/\epsilon$	$2.597 \times 10^{-16} - 0.443 \times 10^{-15}\,i/\epsilon$
0.3	$9.657 \times 10^{-9} - 0.468 \times 10^{-8}\,i/\epsilon$	$4.790 \times 10^{-16} - 0.458 \times 10^{-15}\,i/\epsilon$
0.33	$1.093 \times 10^{-8} - 0.493 \times 10^{-8}\,i/\epsilon$	$5.457 \times 10^{-16} - 0.465 \times 10^{-15}\,i/\epsilon$

(56) vanishes and the remaining real part correctly describes the instability of the associated modes in the superradiant regime, $\Omega < 0$. Clearly, this choice also encompasses the case of superradiant instability for rotating AdS black holes in four-dimensional spacetime ($N = 3$), earlier considered in [21]; (ii) $\frac{\ell}{N-2} \neq (p + 1/2) + \epsilon$, in which case, it is not difficult to see that the damping parameter in (56) remains complex, describing both the frequency shift and the superradiant instability of the associated modes, by its imaginary and real parts, respectively; (iii) $\frac{\ell}{N-2} = (p+1/2)+\epsilon$, which requires a special care and may occur only in odd spacetime dimensions. Substituting this in Eq. (56) and taking the limit $\epsilon \to 0$, we find that

$$
\delta = -\frac{(x_+ - x_-)^{1+\frac{2\ell}{N-2}}}{\pi\, n!\, l^{N+2\ell-1}}
$$
$$
\times \frac{\Gamma^2\left(1 + \frac{\ell}{N-2}\right)\left|\Gamma\left(1 + \frac{\ell}{N-2} - 2i\Omega\right)\right|^2}{\Gamma\left(1 + \frac{2\ell}{N-2}\right)\Gamma\left(2 + \frac{2\ell}{N-2}\right)}
$$
$$
\times \left[\sinh(2\pi\Omega) + \frac{i}{\pi\epsilon}\cosh(2\pi\Omega)\right]
$$
$$
\times \frac{\Gamma(N + \ell + n)}{\Gamma\left(\frac{N}{2} + n + 1\right)\Gamma\left(\frac{N}{2} + \ell - 1\right)}
$$
$$
\times \prod_{k=1}^{n}\left(\frac{N}{2} + \ell + k - 1\right). \tag{57}
$$

Again, we see that the real part of this expression becomes positive in the supperradiant regime, $\Omega < 0$, thus resulting in exponential growth of the mode amplitudes. Meanwhile, its imaginary part contains $1/\epsilon$ type divergence in the limit $\epsilon \to$

0. However, it should be remembered that we are dealing with small-size black holes: The horizon radius r_+ is small enough, fitting the range of validity of the low-frequency solution under consideration, $r_+ \ll r - r_+ \ll 1/\omega$. With this in mind, one can argue that the ratio $\frac{(x_+ - x_-)^{1+\frac{2\ell}{N-2}}}{\epsilon}$ showing up in the imaginary part of (57), to high accuracy, can be driven to a small finite quantity even for the lowest mode, $p = 0$. This expectation is also confirmed by a numerical analysis of expression (57). The results of numerical calculations are presented in Tables 1 and 2.

In Table 1 we give the numerical results for a small Reissner–Nordström–AdS black hole in five and seven spacetime dimensions. We normalize all quantities in terms of the AdS curvature scale l and take $l = 1$, for certainty. From Eq. (11), we find that the extreme charge of the black hole $q_e = r_+^2$. In both cases, the calculations are performed for $r_+ = 0.01$, $e = 12$ and for the lowest modes ($n = 0$, $p = 0$). Accordingly, we take $\ell = 1$ in five dimensions and $\ell = 2$ in seven dimensions. We see that for sufficiently large values of the electric charge, the superradiant instability appears in both cases when the real part of the damping parameter becomes positive. We also see that the imaginary part of this parameter can be thought of as representing a small frequency shift in the spectrum by choosing $\epsilon \to 10^{-8}$ in five dimensions and $\epsilon \to 10^{-15}$ in seven dimensions. Table 2 presents the results of similar numerical analysis of the damping parameter for a slowly rotating Kerr–AdS black hole. We have the superradiant instability to all modes under consideration and again, a small frequency shift in the spectrum; for the $\ell = 1$ mode as $\epsilon \to 10^{-8}$ and for the $\ell = 2$ mode as $\epsilon \to 10^{-15}$. Altogether, the above numeri-

cal analysis shows that the idea of an approximate integer ℓ works well and is sufficient to give an analytical description of the superradiant instability in odd spacetime dimensions for all ℓ, approaching the integer, as given in (57).

Finally, we conclude that the real part of Eq. (56) can be regarded as giving a *universal* description of the superradiant instability for small and slowly rotating charged AdS black holes in all spacetime dimensions. The analysis of expression (56) shows that the instability time scale, $\tau = 1/\delta$, significantly grows as the number of dimensions increases.

6 Conclusion

The superradiant instability of small rotating black holes in four-dimensional spacetimes has been thoroughly investigated in the literature. Remarkably, for low-frequency scalar perturbations this phenomenon appears to be amenable to a complete quantitative description for both black hole–mirror systems and AdS black holes. The basic idea of this description traces back to a pioneering work of Starobinsky [6], where the complete low-frequency solution to the Klein–Gordon equation was found by matching together the near-horizon and far regions solutions.

However, it appears that there exist some subtleties with the matching procedure in higher dimensions, where it fails to be valid for certain modes of scalar perturbations. This makes the use of numerical integration inevitable [29], thereby creating a gap in the complete analytic description of the superradiant instability in higher-dimensional spacetimes. In this paper, we have filled this gap, extending the complete analytic description of the black hole superradiant instability to all higher dimensions. As an instructive model of crucial importance, we have elaborated on the case of a small rotating charged AdS black hole, in the regime of slow rotation and with a single angular momentum.

First, we have demonstrated that Klein–Gordon equation for a massless charged scalar field, propagating in the background of a higher-dimensional weakly charged Kerr–AdS black hole completely separates. Then, focusing on the spacetime of a slowly rotating charged AdS black hole, we have found solutions to the radial wave equation in the near-horizon and far regions of this spacetime. Utilizing the idea of our previous work [30], which relies on the assumption that the orbital quantum number ℓ can be considered as an approximate integer, we have performed the matching of these solutions in the overlap region. Thus, we have obtained the complete low-frequency solution to the radial wave equation, resulting in the complex frequency spectrum for quasinormal scalar modes. The small damping parameter appearing in the imaginary part of this spectrum measures the decay of bound state modes.

Finally, we have calculated the damping parameter, ending up with a manifestly instructive expression which appeared to be a complex quantity. Performing a detailed analysis of this expression for all pertinent cases, as the orbital quantum number ℓ approaches a non-negative integer, we have concluded that its real part correctly describes the negative damping of modes in the regime of superradiance, i.e. the superradiant instability of the higher-dimensional AdS black holes under consideration.

Note added While completing this paper, the work of [40] appeared where the superradiant instability of slowly rotating (uncharged) AdS black holes in higher dimensions is also discussed, thoroughly utilizing the key idea of [30] on the "nearly" integer orbital quantum number.

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