

Coupled dark energy: a dynamical analysis with complex scalar field

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Received: 3 November 2015 / Accepted: 11 January 2016

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Abstract The dynamical analysis for coupled dark energy with dark matter is presented, where a complex scalar field is taken into account and it is considered in the presence of a barotropic fluid. We consider three dark-energy candidates: quintessence, phantom, and tachyon. The critical points are found and their stabilities analyzed, leading to the three cosmological eras (radiation, matter, and dark energy), for a generic potential. The results presented here extend the previous analyses found in the literature.

1 Introduction

Observations of Type IA Supernova indicate that the universe undergoes an accelerated expansion [1, 2], which is dominant at present times ($\sim 68\%$) [3]. Despite of ordinary matter, the remaining 27% is an unknown form of matter that interacts in principle only gravitationally, known as dark matter. The nature of the dark sector is still mysterious and it is one of the biggest challenges in the modern cosmology. The simplest dark-energy candidate is the cosmological constant, whose equation of state $w_\Lambda = p_\Lambda/\rho_\Lambda = -1$ is in agreement with the Planck results [3]. This attempt, however, suffers from the so-called cosmological constant problem, a huge discrepancy of 120 orders of magnitude between the theoretical prediction and the observed data.

Among a wide range of alternatives, a scalar field is a viable candidate to be used. Its usage includes the canonical scalar field, called “quintessence” [4–8], and the scalar field with the opposite-sign in the kinetic term, known as “phantom” [9, 10]. Beyond the real scalar field case, a complex quintessence was also used in [11] to account for the acceleration of the universe. The $U(1)$ symmetry associated with this complex scalar leads to a more sophisticated structure for the dark sector, and unless the standard model of particle physics is a very special case in the nature, there is no reason

(apart from simplicity) not to consider a richer physics of the dark sector.¹

Another possibility of noncanonical scalar field is the tachyon and it comes from string theory. In the bosonic string, its ground state is the tachyon field, whereas in supersymmetric string theory a real tachyon is present in non-BPS Dp-branes, while the complex tachyon appears in a brane-anti-brane system [13]. The tachyon potential has a minimum [14, 15] and at this minimum the tachyon field behaves like a pressureless gas [16]. As soon as tachyon condensation in string theory had been proposed, the tachyon was also regarded as a dark-energy candidate [17–19].

Still regarding the dynamical dark energy, there exists a possibility of interaction between dark energy and dark matter [20, 21], since their densities are comparable and, depending on the coupling used, the interaction can alleviate the coincidence problem [22, 23]. This approach was applied to phantom and tachyon as well, in Refs. [24–29].

When a scalar field is in the presence of a barotropic fluid (with equation of state $w_m = p_m/\rho_m$) the relevant evolution equations can be converted into an autonomous system and the asymptotic states of the cosmological models can be analyzed. Such approach was taken for uncoupled dark energy (quintessence, the tachyon field, and the phantom field for instance [30–34]) and coupled dark energy [21, 28, 35–41]. Since the complex scalar field and the coupled dark energy are generalizations of the real field and the uncoupled case, respectively, we aim to study both possibilities together, in the light of the linear dynamical systems theory. Thus we investigate in this paper the critical points that come from the evolution equations for the complex scalar field (quintessence, phantom, and tachyon), considering the possibility of interaction between the two components of the dark sector. The dynamical equations are derived and the critical point are analyzed, to see what kind of universe might

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¹ A current example of a vector field that perhaps interacts with dark matter is the so-called “dark photon” (see [12] for a quick review).

emerge. As we have said, this is a natural extension of the previous work [21, 28, 37, 39]. As a result, we have found similar fixed points, for all the three scalar field cases, although their stabilities are affected in some cases. The critical points can be compared with previous analyses, and the general behavior can be seen.

The rest of the paper is organized as follows. In Sect. 2 we present the basics of the interacting dark energy and the dynamical analysis theory, where we show the couplings we have used. In Sects. 3 and 4 we use the dynamical system theory for the canonical (and phantom) and the tachyon field, respectively. Within the respective section we show the critical points with their stabilities, and also the viable sequence of cosmological eras (radiation–matter–dark energy). Section 5 is reserved for our conclusions. We use Planck units ($\hbar = c = 1 = M_{pl} = 1$) throughout the text.

2 Interacting dark energy and dynamical analysis

We consider that dark energy is described by a scalar field with energy density ρ_ϕ and pressure p_ϕ , and with an equation of state given by $w_\phi = p_\phi/\rho_\phi$. We assume that the scalar field is coupled with dark matter, in such a way that total energy-momentum is still conserved. In the flat Friedmann–Robertson–Walker background with a scale factor a , the continuity equations for both components and for radiation are

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -\mathcal{Q}, \quad (1)$$

$$\dot{\rho}_m + 3H\rho_m = \mathcal{Q}, \quad (2)$$

$$\dot{\rho}_r + 4H\rho_r = 0, \quad (3)$$

respectively, where $H = \dot{a}/a$ is the Hubble rate, \mathcal{Q} is the coupling and the dot is a derivative with respect to the cosmic time t . The indices m and r stand for matter and radiation, respectively. The case of $\mathcal{Q} > 0$ corresponds to dark-energy transformation into dark matter, while $\mathcal{Q} < 0$ is the transformation in the opposite direction. In principle, the coupling can depend on several variables $\mathcal{Q} = \mathcal{Q}(\rho_m, \rho_\phi, \dot{\phi}, H, t, \dots)$, so we assume for the canonical scalar (quintessence) and for the phantom field the coupling is $\mathcal{Q} = Q\rho_m\dot{\phi}$ [20, 21], while for the tachyon field the coupling is $\mathcal{Q} = Q\rho_m\rho_\phi\dot{\phi}/H$ [39], where Q is a positive constant. The case with negative Q is similar and we will not consider it here, because the minus sign of the case $\mathcal{Q} < 0$ can be absorbed into the $\dot{\phi}$, instead of considering $Q < 0$.

To deal with the dynamics of the system, we will define dimensionless variables. The new variables are going to characterize a system of differential equations in the form

$$X' = f[X], \quad (4)$$

where X is a column vector of dimensionless variables and the prime is the derivative with respect to $\log a$, where we set the present scale factor a_0 to be one. The critical points X_c are the ones that satisfy $X' = 0$. In order to study stability of the fixed points, we consider linear perturbations U around them, thus $X = X_c + U$. At the critical point the perturbations U satisfy the following equation:

$$U' = \mathcal{J}U, \quad (5)$$

where \mathcal{J} is the Jacobian matrix. The eigenvalues of \mathcal{J} determine if the critical points are stable (if all eigenvalues are negative), unstable (if all eigenvalues are positive) or saddle points (if at least one eigenvalue is positive and the others are negative, or vice versa).

3 Quintessence and phantom dynamics

The complex scalar field Φ can be written as $\Phi = \phi e^{i\theta}$, where ϕ is the absolute value of the scalar field and θ is a phase. Both canonical and phantom fields are described by the Lagrangian

$$\mathcal{L} = -\sqrt{-g} \left(\frac{\epsilon}{2} \partial^\mu \Phi \partial_\mu \Phi + V(|\Phi|) \right), \quad (6)$$

where $V(|\Phi|)$ is the potential for the complex scalar and we consider it depends only on the absolute value of the scalar field $\phi \equiv |\Phi|$. We have $\epsilon = +1$ for the canonical field (quintessence) and $\epsilon = -1$ for the phantom field. For a homogeneous field $\phi \equiv \phi(t)$ and $\theta \equiv \theta(t)$, in an expanding universe with Friedmann–Robertson–Walker metric with scale factor $a \equiv a(t)$, the equations of motion are

$$\epsilon \ddot{\phi} + 3\epsilon H \dot{\phi} + V'(\phi) - \epsilon \phi \dot{\theta}^2 = 0, \quad (7)$$

$$\epsilon \ddot{\theta} + \left(3H + \frac{2\dot{\phi}}{\phi} \right) \dot{\theta} = 0, \quad (8)$$

where the prime denotes the derivative with respect to ϕ . In the uncoupled case Eq. (8) gives rise to the effective potential in (7), which is $\frac{d}{d\phi} \left(\frac{\omega^2}{2a^6} \frac{1}{\phi^2} + V(\phi) \right)$ [11], where ω is an integration constant interpreted as angular velocity. The first term in the brackets drives ϕ away from zero and the factor a^{-6} may make the term decreases very fast, provided that ϕ does not decrease faster than $a^{-3/2}$.

We assume the interaction between the scalar field with dark matter through the coupling $Q\rho_m\dot{\phi}$ and it enters in the right-hand side of Eq. (7).

In the presence of matter and radiation, the Friedmann equations for the canonical (phantom) field are

$$H^2 = \frac{1}{3} \left(\frac{\epsilon}{2} \dot{\phi}^2 + \frac{\epsilon}{2} \phi^2 \dot{\theta}^2 + V(\phi) + \rho_m + \rho_r \right), \quad (9)$$

$$\dot{H} = -\frac{1}{2} \left(\epsilon \dot{\phi}^2 + \epsilon \phi^2 \dot{\theta}^2 + \rho_m + \frac{4}{3} \rho_r \right), \tag{10}$$

and the equation of state becomes

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 + \phi^2 \dot{\theta}^2 - 2\epsilon V(\phi)}{\dot{\phi}^2 + \phi^2 \dot{\theta}^2 + 2\epsilon V(\phi)}. \tag{11}$$

We are now ready to proceed with the dynamical analysis of the system.

3.1 Autonomous system

The dimensionless variables are defined as

$$x_1 \equiv \frac{\dot{\phi}}{\sqrt{6}H}, \quad x_2 \equiv \frac{\phi \dot{\theta}}{\sqrt{6}H}, \quad x_3 \equiv \frac{\sqrt{6}}{\phi}, \quad y \equiv \frac{\sqrt{V(\phi)}}{\sqrt{3}H}, \tag{12}$$

$$z \equiv \frac{\sqrt{\rho_r}}{\sqrt{3}H}, \quad \lambda \equiv -\frac{V'}{V}, \quad \Gamma \equiv \frac{VV''}{V'^2}. \tag{12}$$

The dark-energy density parameter is written in terms of these new variables as

$$\Omega_\phi \equiv \frac{\rho_\phi}{3H^2} = \epsilon x_1^2 + \epsilon x_2^2 + y^2, \tag{13}$$

so that Eq. (9) can be written as

$$\Omega_\phi + \Omega_m + \Omega_r = 1, \tag{14}$$

where the matter and radiation density parameters are defined by $\Omega_i = \rho_i/(3H^2)$, with $i = m, r$. From Eqs. (13) and (14) we see that x_1, x_2 , and y are restricted in the phase plane by the relation

$$0 \leq \epsilon x_1^2 + \epsilon x_2^2 + y^2 \leq 1, \tag{15}$$

due to $0 \leq \Omega_\phi \leq 1$. Notice that if $y = 0$ the restriction (15) forbids the possibility of phantom field ($\epsilon = -1$) because for this case $\Omega_\phi < 0$.

The equation of state w_ϕ becomes

$$w_\phi = \frac{\epsilon x_1^2 + \epsilon x_2^2 - y^2}{\epsilon x_1^2 + \epsilon x_2^2 + y^2}, \tag{16}$$

which is a trivial extension of the real scalar field case. The total effective equation of state is

$$w_{\text{eff}} = \frac{p_\phi + p_r}{\rho_\phi + \rho_m + \rho_r} = \epsilon x_1^2 + \epsilon x_2^2 - y^2 + \frac{z^2}{3}, \tag{17}$$

with an accelerated expansion for $w_{\text{eff}} < -1/3$. The dynamical system for the variables x_1, x_2, x_3, y, z and λ are

$$\frac{dx_1}{dN} = -3x_1 + x_2^2 x_3 + \frac{\sqrt{6}}{2} \epsilon y^2 \lambda - \frac{\sqrt{6}}{2} \epsilon Q (1 - x_1^2 - x_2^2 - y^2 - z^2) - x_1 H^{-1} \frac{dH}{dN}, \tag{18}$$

$$\frac{dx_2}{dN} = -3x_2 - x_1 x_2 x_3 - x_2 H^{-1} \frac{dH}{dN}, \tag{19}$$

$$\frac{dx_3}{dN} = -x_1 x_3^2, \tag{20}$$

$$\frac{dy}{dN} = -\frac{\sqrt{6}}{2} x_1 y \lambda - y H^{-1} \frac{dH}{dN}, \tag{21}$$

$$\frac{dz}{dN} = -2z - z H^{-1} \frac{dH}{dN}, \tag{22}$$

$$\frac{d\lambda}{dN} = -\sqrt{6} \lambda^2 x_1 (\Gamma - 1), \tag{23}$$

where

$$H^{-1} \frac{dH}{dN} = -\frac{3}{2} (1 + \epsilon x_1^2 + \epsilon x_2^2 - y^2) - \frac{z^2}{2}. \tag{24}$$

3.2 Critical points

The fixed points of the system are obtained by setting $dx_1/dN = 0, dx_2/dN = 0, dx_3/dN = 0, dy/dN = 0, dz/dN = 0$ and $d\lambda/dN = 0$ in Eq. (18)–(23). When $\Gamma = 1, \lambda$ is constant the potential is $V(\phi) = V_0 e^{-\lambda\phi}$ [30,31].² The fixed points are shown in Table 1. Notice that x_3 and y cannot be negative and recall that $\Omega_r = z^2$. Some of the fixed points do not exist for the phantom field because for those cases Ω_ϕ is negative.

The eigenvalues of the Jacobian matrix were found for each fixed point in Table 1. The results are shown in Table 2.

The eigenvalues μ_{4e} and μ_{5e} are

$$\mu_{4e,5e} = -\frac{3(\lambda + 2Q)}{4(\lambda + Q)} \times \left(1 \pm \sqrt{1 + \frac{8[3 - \epsilon\lambda(\lambda + Q)][3\epsilon + 2Q(\lambda + Q)]}{3(\lambda + 2Q)^2}} \right). \tag{25}$$

At the first sight, one might think that the linear analysis would not give a complete description of the stability, because all fixed points, but (f), have at least one eigenvalue equals zero. However, as pointed out in [42], fixed points that have at least one positive and one negative eigenvalue are always unstable, and methods such as center manifold [42] should be used to analyze the stability of the critical points that can be stable [(g) and (h)]. Even so, for almost all fixed points, but (a) and (c), $x_3 = 0$, which means $\phi \rightarrow \infty$. However,

² The equation for λ is also equal zero when $x_1 = 0$ or $\lambda = 0$, so that λ should not necessarily be constant, for the fixed points with this value of x_1 . However, for the case of dynamical λ , the correspondent eigenvalue is equal zero, indicating that the fixed points is not hyperbolic.

Table 1 Critical points ($x_1, x_2, x_3, y,$ and z) of the Eq. (18) for the quintessence and phantom field

Point	Existence	x_1	x_2	x_3	y	z	w_ϕ	Ω_ϕ	w_{eff}
(a)	$Q = 0$	0	0	Any	0	0	–	0	0
(b)	$\epsilon = +1$	$\frac{-\sqrt{6}Q}{3}$	0	0	0	0	1	$\frac{2Q^2}{3}$	$\frac{2Q^2}{3}$
(c)	Any	0	0	Any	0	1	–	0	$\frac{1}{3}$
(d)	$\epsilon = +1$	$\frac{-1}{\sqrt{6}Q}$	0	0	0	$\sqrt{1 - \frac{1}{2Q^2}}$	1	$\frac{1}{6Q^2}$	$\frac{1}{3}$
(e)	$\epsilon = +1$	$\frac{2\sqrt{6}}{3\lambda}$	0	0	$\frac{2\sqrt{3}}{3\lambda}$	$\sqrt{1 - \frac{4}{\lambda^2}}$	$\frac{1}{3}$	$\frac{4}{\lambda^2}$	$\frac{1}{3}$
(f)	$\epsilon = +1$	Any	$\sqrt{1 - x_1^2}$	0	0	0	1	1	1
(g)	Any	$\frac{\sqrt{6}}{2(\lambda+Q)}$	0	0	$\sqrt{\frac{2Q(Q+\lambda)+3\epsilon}{2(\lambda+Q)^2}}$	0	$\frac{-Q(Q+\lambda)}{Q(Q+\lambda)+3\epsilon}$	$\frac{Q(Q+\lambda)+3\epsilon}{(\lambda+Q)^2}$	$\frac{-Q}{\lambda+Q}$
(h)	Any	$\frac{\epsilon\lambda}{\sqrt{6}}$	0	0	$\sqrt{1 - \frac{\epsilon\lambda^2}{6}}$	0	$-1 + \frac{\epsilon\lambda^2}{3}$	1	$-1 + \frac{\epsilon\lambda^2}{3}$

The condition of existence, if any, is shown for the fixed point [point (a) exists only for $Q = 0$, for instance, while the point (b) does not exist for the phantom field]. The table shows the correspondent equation of state for the dark energy (16), the effective equation of state (17), and the density parameter for dark energy (13)

Table 2 Eigenvalues and stability of the fixed points for the quintessence (phantom) field

Point	μ_1	μ_2	μ_3
(a)	$\frac{9}{2}$	$-\frac{3}{2}$	0
(b)	$Q^2 - \frac{3}{2}$	$Q^2 - \frac{3}{2}$	0
(c)	–1	–1	0
(d)	$-1 + \frac{1}{2Q^2}$	–1	0
(e)	See the main text		
(f)	$3x_1^2 + \sqrt{6}Qx_1$	$3(x_1^2 - 1)$	$\mp x_1 \sqrt{1 - x_1^2}$
(g)	$-\frac{\lambda+4Q}{2(\lambda+Q)}$	$-\frac{3(\lambda+2Q)}{2(\lambda+Q)}$	0
(h)	$-3 + \epsilon\lambda(\lambda + Q)$	$-3 + \frac{\epsilon\lambda^2}{2}$	0
	μ_4	μ_5	Stability
(a)	$\frac{3}{2}$	$-\frac{1}{2}$	Saddle
(b)	$Q(Q + \lambda) + \frac{3}{2}$	$Q^2 + \frac{1}{2}$	Unstable or saddle
(c)	2	1	Saddle
(d)	$2 + \frac{\lambda}{2Q}$	$1 - \frac{Q^2}{2}$	Saddle
(e)	See the main text		
(f)	$3 - \frac{\sqrt{6}x_1\lambda}{2}$	1	Unstable or saddle
(g)	μ_{4d}	μ_{5d}	Saddle or stable
(h)	$-3 + \frac{\epsilon\lambda^2}{2}$	$-2 + \frac{\epsilon\lambda^2}{2}$	Saddle or stable

this limit implies that $x_2 \propto \phi\dot{\theta}/H \rightarrow \infty$ as well, provided that H is finite. This issue occurred for the points (b) and (d) to (h), as can be seen in Table 1 whose mathematical inconsistency indicates that these critical points are not physically acceptable.

Since all critical points are similar to the ones found in the literature [21,28,30] we reproduce the main results, which are valid for the case of real scalar field, in the appendix for the sake of completeness.

3.3 Summary

From the eight fixed points presented in the quintessence (phantom) case, only (a) and (c) are physically viable and they describe the sequence: radiation \rightarrow matter. Both of them are unstable, however, there does not exist a point that describes the dark-energy-dominated universe. Thus the extra degree of freedom due to the phase θ spoils the physically acceptable fixed points that exist for the case of real

scalar field, indicating that the dynamical system theory is not a good tool when one tries to analyze the complex quintessence (phantom).

4 Tachyon dynamics

The complex tachyon field $\Phi = \phi e^{i\theta}$, where ϕ is the absolute value of the tachyon field and θ is a phase, is described by the Born–Infeld Lagrangian

$$\mathcal{L}_{BI} = -\sqrt{-g}V(|\Phi|)\sqrt{1 - \partial^\mu \Phi \partial_\mu \Phi}, \tag{26}$$

where $V(\phi)$ is the tachyon potential, which depends only on the absolute value of the scalar field $\phi \equiv |\Phi|$. For a homogeneous field $\phi \equiv \phi(t)$ and $\theta \equiv \theta(t)$, in an expanding universe with Friedmann–Robertson–Walker metric, the Lagrangian becomes

$$\mathcal{L}_{BI} = -a^3 V(\phi)\sqrt{1 - \dot{\phi}^2 - \phi^2 \dot{\theta}^2}, \tag{27}$$

where $a \equiv a(t)$ is the scale factor. The equations of motion for ϕ and θ are, respectively,

$$\frac{\ddot{\phi}}{1 - \phi^2 \dot{\theta}^2 - \dot{\phi}^2} + \frac{\ddot{\theta} + \dot{\phi} \dot{\theta} \phi^{-1}}{(1 - \phi^2 \dot{\theta}^2 - \dot{\phi}^2)} \frac{\phi^2 \dot{\phi} \dot{\theta}}{(1 - \phi^2 \dot{\theta}^2)} + \frac{3H\dot{\phi} - \phi \dot{\theta}^2}{1 - \phi^2 \dot{\theta}^2} + \frac{V'(\phi)}{V(\phi)} = 0, \tag{28}$$

$$\frac{\ddot{\theta}}{1 - \phi^2 \dot{\theta}^2 - \dot{\phi}^2} + 3H\dot{\theta} + \frac{2\dot{\phi} \dot{\theta}}{\phi(1 - \phi^2 \dot{\theta}^2 - \dot{\phi}^2)} = 0, \tag{29}$$

where the prime denotes time derivative with respect to ϕ . When the phase θ is zero, we recover the well-known equation of motion for the tachyon field.

The Friedmann equations for the complex tachyon, in the presence of barotherpic fluids, are

$$H^2 = \frac{1}{3} \left(\frac{V(\phi)}{\sqrt{1 - \phi^2 \dot{\theta}^2 - \dot{\phi}^2}} + \rho_m + \rho_r \right), \tag{30}$$

$$\dot{H} = -\frac{1}{2} \left(\frac{V(\phi)(\dot{\phi}^2 + \phi^2 \dot{\theta}^2)}{\sqrt{1 - \phi^2 \dot{\theta}^2 - \dot{\phi}^2}} + \rho_m + \frac{4}{3}\rho_r \right), \tag{31}$$

The equation of state for the tachyon field yields

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \dot{\phi}^2 + \phi^2 \dot{\theta}^2 - 1, \tag{32}$$

thus the tachyon behavior is between the cosmological constant one ($w_\phi = -1$) and matter one ($w_\phi = 0$).

The interaction between the tachyon and the dark matter is driven by the coupling $\mathcal{Q} = Q\rho_m\rho_\phi\dot{\phi}/H$, which in turn we consider that it modifies the right-hand side of Eq. (28). With

this form of coupling, the time dependence of the coupling is implicit in the Hubble parameter H . We are now ready to proceed the dynamical analysis of the system.

4.1 Autonomous system

The dimensionless variables for the case of tachyon field are

$$x_1 \equiv \dot{\phi}, \quad x_2 \equiv \phi \dot{\theta}, \quad x_3 \equiv \frac{1}{H\phi}, \quad y \equiv \frac{\sqrt{V(\phi)}}{\sqrt{3}H}, \tag{33}$$

$$z \equiv \frac{\sqrt{\rho_r}}{\sqrt{3}H}, \quad \lambda \equiv -\frac{V'}{V^{3/2}}, \quad \Gamma \equiv \frac{VV''}{V'^2}.$$

Since $\dot{\phi}$ and θ are dimensionless variables, ϕ has dimension of time.

The dark-energy density parameter is written in terms of these new variables as

$$\Omega_\phi \equiv \frac{\rho_\phi}{3H^2} = \frac{y^2}{\sqrt{1 - x_1^2 - x_2^2}}, \tag{34}$$

so that Eq. (30) can be written as

$$\Omega_\phi + \Omega_m + \Omega_r = 1, \tag{35}$$

where the matter and radiation density parameter are defined by $\Omega_i = \rho_i/(3H^2)$, with $i = m, r$. From Eqs. (34) and (35) we see that x_1, x_2 , and y are restricted in the phase plane by the relation

$$0 \leq x_1^2 + x_2^2 + y^4 \leq 1, \tag{36}$$

due to $0 \leq \Omega_\phi \leq 1$. In terms of these new variables the equation of state w_ϕ is

$$w_\phi = x_1^2 + x_2^2 - 1, \tag{37}$$

which is clearly a trivial extension for the complex scalar field. The total effective equation of state is

$$w_{\text{eff}} = \frac{p_\phi + p_r}{\rho_\phi + \rho_m + \rho_r} = -y^2 \sqrt{1 - x_1^2 - x_2^2} + \frac{z^2}{3}, \tag{38}$$

with an accelerated expansion for $w_{\text{eff}} < -1/3$. The dynamical system for the variables x_1, x_2, x_3, y, z and λ are

$$\frac{dx_1}{dN} = -(1 - x_1^2 - x_2^2) \times \left[3x_1 - \sqrt{3}y\lambda + 3Q \left(1 - z^2 - \frac{y^2}{\sqrt{1 - x_1^2 - x_2^2}} \right) \right] + x_2^2 x_3, \tag{39}$$

$$\frac{dx_2}{dN} = -x_1 x_2 x_3 - 3x_2(1 - x_1^2 - x_2^2), \tag{40}$$

Table 3 Critical points (x_1, x_2, x_3, y and z) of the Eq. (39), for the tachyon field

Point	x_1	x_2	x_3	y	z	w_ϕ	Ω_ϕ	w_{eff}
(a1)	± 1	0	0	0	0	0	0	0
(a2)	0	± 1	0	0	0	0	0	0
(a3)	1	0	$\frac{3}{2}$	0	0	0	0	0
(a4)	$-Q$	$\pm\sqrt{1-Q^2}$	0	0	0	0	0	0
(b)	Any	$\pm\sqrt{1-x_1^2}$	0	0	± 1	0	0	$\frac{1}{3}$
(c)	1	0	2	0	± 1	0	0	$\frac{1}{3}$
(d)	0	0	Any	0	± 1	-1	0	$\frac{1}{3}$
(e)	0	0	Any	1	0	-1	1	-1
(f1)	$\frac{\lambda y_c}{\sqrt{3}}$	0	0	y_c	0	$\frac{\lambda^2 y_c^2}{3} - 1$	1	w_ϕ
(f2)	$\frac{\lambda y_c}{\sqrt{3}}$	0	$\frac{\sqrt{3}\lambda y_c}{2}$	y_c	0	$\frac{\lambda^2 y_c^2}{3} - 1$	1	w_ϕ
(g)	$-Q$	0	0	0	0	$Q^2 - 1$	0	0
(h1)	x_f	0	0	y_f	0	$x_f^2 - 1$	$\frac{w_{\text{eff}}}{w_\phi}$	$\frac{x_f y_f \lambda}{\sqrt{3}} - 1$
(h2)	x_f	0	$\frac{\sqrt{3\lambda y_f}}{2}$	y_f	0	$x_f^2 - 1$	$\frac{w_{\text{eff}}}{w_\phi}$	$\frac{x_f y_f \lambda}{\sqrt{3}} - 1$

The table shows the correspondent equation of state for the dark energy (37), the effective equation of state (38) and the density parameter for dark energy (34)

$$\frac{dx_3}{dN} = -x_1 x_3^2 + \frac{x_3}{2} \left[3 + z^2 - \frac{3y^2(1-x_1^2-x_2^2)}{\sqrt{1-x_1^2-x_2^2}} \right], \tag{41}$$

$$\frac{dy}{dN} = \frac{y}{2} \left[-\sqrt{3}x_1 y \lambda + 3 + z^2 - \frac{3y^2(1-x_1^2-x_2^2)}{\sqrt{1-x_1^2-x_2^2}} \right], \tag{42}$$

$$\frac{dz}{dN} = -2z + \frac{z}{2} \left[3 + z^2 - \frac{3y^2(1-x_1^2-x_2^2)}{\sqrt{1-x_1^2-x_2^2}} \right], \tag{43}$$

$$\frac{d\lambda}{dN} = -\sqrt{3}\lambda x_1 y \left(\Gamma - \frac{3}{2} \right). \tag{44}$$

4.2 Critical points

The fixed points of the system are obtained by setting $dx_1/dN = 0, dx_2/dN = 0, dx_3/dN = 0, dy/dN = 0, dz/dN,$ and $d\lambda/dN = 0$ in Eq. (39). When $\Gamma = 3/2,$ λ is constant the potential has the form found in [32,43] ($V(\phi) \propto \phi^{-2}$), known in the literature for both coupled [28,29] and uncoupled [17,18] dark energy³. The fixed points

³ The equation for λ is also equal zero when $x_1 = 0, y = 0$ or $\lambda = 0,$ so that λ should not necessarily be constant, for the fixed points with these values of x_1 or $y.$ However, for the case of dynamical $\lambda,$ the correspondent eigenvalue is equal zero, indicating that the fixed points is not hyperbolic.

are shown in Table 3. Notice that x_3 and y cannot be negative and recall that $\Omega_r = z^2.$

The fixed points $y_c, x_f,$ and y_f are shown below

$$y_c = \sqrt{\frac{\sqrt{\lambda^4 + 36} - \lambda^2}{6}}, \tag{45}$$

$$x_f = -\frac{Q}{2} \pm \frac{\sqrt{Q^2 + 4}}{2}, \tag{46}$$

$$y_f = \frac{-\lambda x_f + \sqrt{\lambda^2 x_f^2 + 12\sqrt{1-x_f^2}}}{\sqrt{12(1-x_f^2)}}. \tag{47}$$

The eigenvalues of the Jacobian matrix were found for each fixed point in Table 3. The results are shown in Table 4.

The points (a1)–(a4) correspond to a matter-dominated solution, since $\Omega_m = 1$ and $w_{\text{eff}} = 0.$ They are saddle points because at least one eigenvalue has an opposite sign. The point (a4) is actually the point (a1), with $Q = 1.$ Points (b), (c), and (d) are radiation-dominated solutions, with $\Omega_r = 1$ and $w_{\text{eff}} = 1/3.$ The difference between them is that (b) and (c) have $w_\phi = 0,$ while (d) has $w_\phi = -1$ and admits any value for $x_3.$ They are unstable [(b)] or saddle points [(c) or (d)].

The point (e) is in principle a dark-energy-dominated solution with $\Omega_\phi = 1$ and $w_{\text{eff}} = w_\phi = -1,$ whose existence is restrict to $\lambda = 0.$ However, a careful analysis shows that the Jacobian matrix for this critical point has zero eigenvector, thus it cannot be considered. Points (f1) and (f2) are also a

Table 4 Eigenvalues and stability of the fixed points for the tachyon field

Point	μ_1	μ_2	μ_3	μ_4	μ_5	Stability
(a1)	$6(1 \pm Q)$	0	$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	Saddle
(a2)	0	6	$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	Saddle
(a3)	$6(1 + Q)$	$-\frac{3}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	Saddle
(a4)	0	$6(1 - Q^2)$	$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	Saddle
(b)	$6x_1^2$	$6x_2^2$	2	2	1	Unstable
(c)	6	-2	-2	2	1	Saddle
(d)	-3	-3	2	4	1	Saddle
(e)	-3	-3	0	-3	-2	Stable
(f1)	$\sqrt{3}Q\lambda y_c - 3\left(1 - \frac{\lambda^2 y_c^2}{3}\right)$	$-3\left(1 - \frac{\lambda^2 y_c^2}{3}\right)$	$\frac{\lambda^2 y_c^2}{2}$	$\frac{\lambda^2 y_c^2}{2} - 3$	$\frac{\lambda^2 y_c^2}{2} - 2$	Saddle
(f2)	$\sqrt{3}Q\lambda y_c - 3\left(1 - \frac{\lambda^2 y_c^2}{3}\right)$	$\frac{\lambda^2 y_c^2}{2} - 3$	$-\frac{\lambda^2 y_c^2}{2}$	$\frac{\lambda^2 y_c^2}{2} - 3$	$\frac{\lambda^2 y_c^2}{2} - 2$	Stable for $\lambda < 0$ or $Q = 0$
(g)	$-3(1 - Q^2)$	$-3(1 - Q^2)$	$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	Saddle
(h1)	$3\left(x_f^2 - \frac{x_f y_f \lambda}{\sqrt{3}}\right)$	$-3(1 - x_f)$	$\frac{3}{2}$	$\frac{3}{2}\left(\frac{x_f y_f \lambda}{\sqrt{3}} - 2\right)$	$\frac{3}{2}\left(\frac{x_f y_f \lambda}{\sqrt{3}} - \frac{4}{3}\right)$	Saddle
(h2)	$3\left(x_f^2 - \frac{x_f y_f \lambda}{\sqrt{3}}\right)$	$-\frac{\sqrt{3}}{2}\lambda x_f y_f - 3(1 - x_f)$	$-\frac{\sqrt{3}}{2}\lambda x_f y_f$	$\frac{3}{2}\left(\frac{x_f y_f \lambda}{\sqrt{3}} - 2\right)$	$\frac{3}{2}\left(\frac{x_f y_f \lambda}{\sqrt{3}} - \frac{4}{3}\right)$	Stable

dark-energy-dominated solution ($\Omega_\phi = 1$) whose equation of state depends on λ , which in turn can be either constant or zero. The case with constant λ are shown in Table 3 and an accelerated expansion occurs for $\lambda^2 < 2/\sqrt{3}$. For $\lambda = 0$ we recover the point (e). The eigenvalues μ_2, μ_4 , and μ_5 of the fixed point (f1) and (f2) are always negative. For these points $\frac{\lambda^2 y_c^2}{3} \leq 1$, then the first eigenvalue is also negative if $Q = 0, \lambda < 0$ or $\sqrt{3}Q\lambda y_c < 3$. Therefore, the point (f2) describes a dark-energy-dominated universe and can lead to a late-time accelerated universe if the requirement $\mu_1 < 0$ is satisfied. On the other hand, (f1) is a saddle point. The effective equation of state depends only on λ , so the coupling Q only changes the property of the fixed point.

The point (g) is also a saddle point with a matter-dominated solution, however, different from (a1)–(a4), the equation of state for the dark energy w_ϕ is no longer zero but depends on Q , leading to an universe with accelerated expansion for $Q^2 < 2/3$. For this point the coupling is restrict to values $0 \leq Q^2 \leq 1$.

The last fixed points (h1) and (h2) are valid for $x_f \neq 0$,⁴ for $Q \neq 0$ and for constant λ , and its behavior depends on Q . In order to have $x_f^2 \leq 1$, we must have $Q > 0$ for the case with plus sign in x_f (46), while we have $Q < 0$ for the minus sign case. We restrict our attention for the plus sign case. When $Q \rightarrow \infty, x_f \rightarrow 0$, and $y_f \rightarrow 1$, in agreement with the restriction (36). In addition, as pointed out in [39], the fixed points exist for some values of $\lambda > 0$ and Q , due to Eq. (36). The two fixed points have a similar behavior, however, (h1) is a saddle point, while (h2) is stable. Such a difference is

⁴ The case for $x_f = 0$ is the fixed point (e).

due to the eigenvalue μ_3 (Table 4). The eigenvalues μ_4 and μ_5 are always negative because w_{eff} is between 0 and -1 . The first eigenvalue is also negative because $\Omega_\phi \leq 1$, thus $x_f^2 - 1 \leq \frac{x_f y_f \lambda}{\sqrt{3}} - 1$, therefore $\frac{x_f y_f \lambda}{\sqrt{3}} \geq x_f^2$, since x_f is always positive. Therefore, the point (h2) can lead to a late-time accelerated universe, depending on the value of λ and Q .

As in the case of quintessence and phantom, the fixed points that have $x_3 = 0$ [(a1), (a2), (a4), (b), (f1), (g), and (h1)] indicate that $\phi \rightarrow \infty$ and therefore $x_2 \equiv \phi^{\dot{\phi}} \rightarrow \infty$ as well. However, this limit is in contradiction to Table 3 for x_2 showing that these seven critical points are not physically acceptable.

All fixed points reproduce the previous results in the literature [28, 32, 39, 43] and they are generalizations of those analyses, with the same stability behavior for the critical points. This indicates that the degree of freedom due to the complex scalar has no effect on the stability and on the evolution of the system of equations, when compared with the case of real scalar field.

4.3 Summary

The critical points showed in the tachyonic case describe the three phases of the universe: the radiation-dominated era, the matter-dominated era, and the present dark-energy-dominated universe. The matter-dominated universe can be described by the saddle point (a3). There are two points that can represent the radiation-dominated era: (c) and (d). The two points are saddle, with the additional difference that the point (d) has an equation of state for dark energy equals -1 .

A tachyonic-dominated universe is described by the points (f2) and (h2). The point (f2) can be stable only if the coupling is zero or $\lambda < 0$. The last fixed point (h2) is stable and can describe an accelerated universe depending on the value of λ and Q .

From all the critical points, the cosmological transition radiation \rightarrow matter \rightarrow dark energy is achieved considering the following sequence of fixed points: (c) or (d) \rightarrow (a3) \rightarrow (f2) [λ dependent] or (h2) [Q and λ dependent]. Although the sequence is viable, the form of the potential dictates whether the fixed points are allowed or not. Among several possibilities in the literature, the potential $V(\phi) \propto \phi^{-n}$, for instance, leads to a dynamically changing λ (either if $\lambda \rightarrow 0$ for $0 < n < 2$, or $\lambda \rightarrow \infty$ for $n > 2$) [19]. A dynamically changing λ is allowed for the fixed points (a3), (c), and (d). On the other hand, points (f2) and (h2) require a constant λ , implying $V(\phi) \propto \phi^{-2}$ [17, 18, 28, 29].

5 Conclusions

In this paper we studied coupled dark energy using a complex scalar field, in the light of the dynamical system theory. There were analyzed three possibilities: quintessence, phantom, and tachyon field. All three possibilities are known in the literature for the real field [21, 28, 37, 39], and for uncoupled and complex quintessence field [33]. Thus a natural question that arises is how a complex scalar field changes the previous results and if there are new fixed points due to the complex field. Although some equations for the dimensionless variables are trivial extensions of the real field case (e.g. the equation of state for the scalar field), the differential equations were generalized. All fixed points found here are in agreement with the previous results, with no new fixed points, however, there are some crucial differences. For the quintessence and the phantom there is a contradiction between the fixed points x_2 and x_3 when the latter is zero. This situation occurs for almost all fixed points and the only two exceptions are unstable points that represent respectively the radiation and matter era, so the dark-energy-dominated universe is absent. Therefore the extra degree of freedom spoils the results known in the case of real scalar field. For the tachyon field all the critical points are also similar to the real field case, with the same stabilities. Therefore, the extra degree of freedom due to the complex tachyon field plays no role on the stability of the critical points. Although the results presented here enlarge the previous results found in the literature, with the generalization of the equations of motion, the dynamical system theory does not provide further information in what is already known for the case of real scalar field, letting open the possibility of studying complex scalar fields by other ways of analysis.

Acknowledgments I thank Elcio Abdalla and Giancarlo Camilo for various suggestions and comments, during all the steps of the work. I also thank an anonymous reviewer for his essential comments. This work is supported by FAPESP Grant no. 2013/10242-1.

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Appendix

In this appendix we reproduce the results in the literature [21, 28, 30] regarding the fixed points presented in Table 1, for the sake of completeness.

The fixed point (a) is a saddle point which describes a matter-dominated universe, however, it is valid only for $Q = 0$. The other possibility of matter-dominated universe with $Q \neq 0$ arises from the fixed point (b). This point is called “ ϕ -matter-dominated epoch” (ϕ MDE) [21] and it can be either unstable or a saddle point. However, due to $\Omega_\phi = 2Q^2/3 \ll 1$, the condition $Q^2 \ll 1$ should hold in order to the point be responsible for the matter era. Thus μ_1 and μ_2 are negative, while μ_5 is always positive and μ_4 is positive for $Q(\lambda + Q) > -3/2$. Therefore (b) is a saddle point.

The radiation-dominated universe is described by the critical points (c), (d), and (e), only for the quintessence field. The first two points are saddle, as is easily seen in Table 2, and the last one had its stability described numerically in [21]. However, both (d) and (e) are not suitable to describe the universe we live in, due to nucleosynthesis constraints [44, 45]. The nucleosynthesis bound $\Omega_\phi^{BBN} < 0.045$ [46] implies $Q^2 > 3.7$ for the point (d) and $\lambda^2 > 88.9$ for the point (e). Thus the requirement for the point (d) is not consistent with the condition of point (b) and the constraint on λ^2 does not allow a scalar field attractor, as we will see soon. Therefore, the only viable cosmological critical point for the radiation era is (c).

The point (f) is an unstable or saddle point and it does not describe an accelerated universe. The last possibility for the matter era is the point (g), with eigenvalues showed in Table 2 and Eq. (25). Since $w_{\text{eff}} \simeq 0$ for $|\lambda| \gg |Q|$, the fixed point is either stable or stable spiral, hence the universe would not exit from the matter dominance.

On the other hand, the point (g) can lead to an accelerated universe, for the quintessence field case ($\epsilon = +1$), provided that $3 < \lambda(\lambda + Q)$, because $\Omega_\phi \leq 1$, and $Q > \lambda/2$, from $w_{\text{eff}} < -1/3$. Regarding $\lambda > 0$, the two eigenvalues μ_1 and μ_2 are always negative and since $Q > 3/\lambda - \lambda$, the behavior of $\mu_{4d,5d}$ depends on the second term in the square root of (25)

$$A \equiv \frac{8[3 - \lambda(\lambda + Q)][3\epsilon + 2Q(\lambda + Q)]}{3(\lambda + 2Q)^2}. \quad (48)$$

From the condition $3 < \lambda(\lambda + Q)$ we have $A < 0$, and if $A < 1$ the fixed point is stable. Otherwise, i.e. $A > 1$, the critical point is a stable spiral. Thus the value of the coupling dictates which behavior the fixed point will have: stable for $3/\lambda - \lambda < Q < Q_*$ or stable spiral for $Q > Q_*$, where Q_* is the solution of $A = 1$. However, even in the case where one can get $\Omega_\phi \simeq 0.7$ [45, 47], there are no allowed region in the (Q, λ) plane corresponding to the transition from ϕ MDE to scaling attractor [21]. Thus it is hard to gather the conditions for the point ϕ MDE and the point (g). For the case of the phantom field ($\epsilon = -1$), the condition $y^2 > 0$ implies $2Q(Q + \lambda) > 3$. Hence, $\mu_4 < 0$ and $\mu_5 > 0$, and (g) is a saddle point.

The last fixed point (h) leads to an accelerated universe provided that $\lambda^2 < 2$. With this condition, the eigenvalues μ_2, μ_4 , and μ_5 are always negative. The first eigenvalue μ_1 is also always negative for the phantom field, and it is for the quintessence field with the condition $\lambda(\lambda + Q) < 3$. Therefore, the point is stable if the previous conditions are satisfied.

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