Eur. Phys. J. C (2015) 75:544
DOI 10.1140/epjc/s10052-015-3792-z

THE EUROPEAN PHYSICAL JOURNAL C

CrossMark

Regular Article - Theoretical Physics

Slow-roll inflationary scenario in the maximally extended background

Ali A. Asgari, Amir H. Abbassi[a]

Department of Physics, School of Sciences, Tarbiat Modares University, P.O. Box 14155-4838, Tehran, Iran

Received: 28 April 2015 / Accepted: 11 November 2015 / Published online: 21 November 2015

Abstract During the inflationary epoch, the geometry of the universe may be described by a quasi-de Sitter space. On the other hand, the maximally extended de Sitter metric in the comoving coordinates accords with a special FLRW model with positive spatial curvature; therefore, the focus of the present paper is on the positively curved inflationary paradigm, for which we first of all derive the power spectra of comoving curvature perturbation and primordial gravitational waves in a positively curved FLRW universe according to the slowly rolling inflationary scenario. It can be shown that the curvature spectral index in this model automatically has a small negative running parameter, compatible with observational measurements. Afterwards, by taking into account the curvature factor, it investigates the relative amplitude of the scalar and tensor perturbations, clarifying that the tensor–scalar ratio for this model, against the spatially flat one, directly depends on the wavelength of the perturbative modes.

1 Introduction

Inflationary cosmology, which was proposed in the early 1980s, extends the standard Big-Bang model by postulating an early epoch of nearly exponential expansion in order to resolve a number of puzzles of the Big-Bang cosmology, such as the flatness, horizon, and monopole problems [1–3]. Inflation also explains the origin of the CMB anisotropies and the large scale structure of the cosmos. Indeed quantum vacuum fluctuations of the inflation field(s) got magnified to cosmic-sized classical perturbations after the horizon exit time and became the seeds for the growth of the structure and CMB anisotropies in the universe [4–7]. Before the advent of inflation the initial perturbations were *postulated* and their spectrum was supposed to be scalar-invariant in order to fit the observational data [8–10]. On the other hand, inflationary theory not only truly explains the origin of the primordial inhomogeneities but also predicts their spectrum. The spectrum of these inhomogeneities and the spectrum of the cosmological gravitational waves produced during the inflation are two observational tests of the inflationary theories. Cosmological observations are consistent with the simplest inflation model within the slow-roll paradigm [11,12]. According to this scenario, the curvature power of the spectrum is nearly flat [13–20], i.e.

$$\mathcal{R}_q^o \propto q^{-\frac{3}{2}-2\epsilon-\delta}, \tag{1}$$

where $\mathcal{R}_q$ is the Fourier component of a comoving curvature perturbation with comoving wave number q (the superscript “o” stands for “outside the Hubble horizon”). Furthermore, ϵ and δ are, respectively, the first and the second slow-roll parameters. According to the observational data $\epsilon \leq 0.008$ and $\delta \leq 0.018$ [11]. $\mathcal{R}$ characterizes the adiabatic scalar perturbations which for super-Hubble scales are roughly constant [21–23]. On the other hand, all inflationary models predict the existence of cosmological gravitational waves which produce a B-mode polarization pattern in the CMB anisotropies. Recently, this mode has been detected by the BICEP2 collaboration [24]. In the slow-roll approximation we have [13]

$$\mathcal{D}_q^o \propto q^{-\frac{3}{2}-\epsilon}, \tag{2}$$

where $\mathcal{D}_q$ is the amplitude of inflationary gravitational waves. The relative amplitude, characterized by the tensor–scalar ratio $r = 4|\frac{\mathcal{D}_q^o}{\mathcal{R}_q^o}|^2$, is a probe of the energy scale in the inflationary epoch. It can be shown that in a slow-roll approximation with a single scalar field $r = 16\epsilon$ [13]. The BICEP2 collaboration has reported $r \simeq 0.2$, which is greater than the upper limit $r < 0.11$ obtained by the Planck collaboration [11]. However, a joint analysis by the BICEP2/Keck Array team and the Planck collaboration shows that the BICEP2

[a] e-mail: ahabbasi@modares.ac.ir

detection of the B-mode is mainly due to the dust and cannot be attributed to primordial gravity waves, produced during inflation [25]. In addition to this inconsistency, there is another discrepancy which refers to the running parameter of the curvature spectral index. In the slow-roll single field inflation the running parameter is of the second order in terms of the slow-roll parameters, but the Planck data prefer $\mathfrak{N}_r = \frac{\partial \mathfrak{N}_s}{\partial lnq} \simeq -0.015$ [11], which is of the first order slow-roll parameters and, consequently has no justification in the slow-rolling inflationary model. In other words, the running parameter has a magnitude significantly greater than the slow-roll paradigm prediction in a spatially flat inflationary universe. On the other hand, there are some anomalies in the CMB power spectrum, such as suppression of the lowest CMB multiples [26] and lack of temperature correlations on scales beyond 70° [27], that may be pieces of evidence for a discrete spectrum and non-trivial spherical topologies [26,28]. In other words, some positive curvature models with a non-trivial topology can solve the problem of the CMB quadrupole and octopole suppression as well as the mystery of missing fluctuations which appear in the concordance model [29–33]. Furthermore, as well known, the inflationary universe background is described by a quasi-de Sitter space; however, the maximally extended de Sitter space, known as the *Lorentzian de Sitter space*, is included in the FLRW models with $K = +1$ and, therefore, has a positive spatial curvature [34]. Lorentzian de Sitter space is geodesically complete too. Besides, the last observational data do not rule out the $\Omega_K < 0$ case as well [35]. It is noteworthy that if the spatial curvature of the universe is positive, the curvature is dominant at the early stages of the inflationary era [28,36], so the curvature might be significant in primordial spectra of the perturbations and cannot be ignored. The dynamics of the inflationary universe with positive spatial curvature has been studied by Ellis et al. [36,37] who showed that however the number of e-foldings increases, the curvature parameter decreases and the universe would be closer to flat today [36]. On the other hand, Vilenkin discussed a cosmological model in which the inflationary universe is created by quantum tunneling from *nothing* [38–40]. This model does not include a Big-Bang singularity and predicts that the inflationary universe is positively curved. Although Linde has claimed that it is very difficult to obtain a realistic model of a closed inflationary universe [41], Ellis and Maartens constructed a single field inflationary model in the closed universe, known as the *eternal emergent universe scenario* [42,43]. This model is a nonsingular closed inflationary cosmology that begins from a meta-stable Einstein static state. Another closed inflationary model with positive curvature index has been introduced by Lasenby and Doran [44].

The calculation of the scalar (curvature) power spectrum during an inflationary epoch with nonzero spatial curvature was first performed by Starobinsky [45]. His work, however, did not include the tensor power spectrum. Moreover, the background of the inflationary universe was assumed to be exact de Sitter space. Moreover, Massó et al. [46] investigated the imprint of spatial curvature on the scalar power spectrum in a non-flat inflationary universe with an exact de Sitter space as the background. They evaluated the power spectrum at the horizon exit instant and discussed the effect of curvature on the angular power spectrum of CMB. In addition, Lyth and Stwart [47] and Rarita and Peebles [48] studied quasi-de Sitter models for spatially open universes. The present paper investigates the slow-rolling inflationary scenario in a spatially closed background with trivial topology, namely a positively curved FLRW universe, for which the curvature power spectrum has been obtained in an inflationary universe with positive spatial curvature by imposing appropriate initial conditions. The remarkable point of this work is that it investigates a *quasi-de Sitter* model by means of a slow-roll paradigm. Furthermore, tensor perturbations have been studied and finally the tensor–scalar ratio for spatially closed universe has been derived. The layout of the article is as follows: in Sect. 2, we derive the generalized Sasaki–Mukhanov equation, associated with the positively curved universe. Then the slow-roll parameters along with Sasaki–Mukhanov variable is generalized to the inflationary universe with positive curvature index. This section concludes by calculating the comoving power spectrum. Section 3 is for the investigation of a gravitational wave spectrum in the positively curved universe, while Sect. sec:4 contains the calculation of the tensor–scalar ratio in the FLRW universe with positive curvature index. Finally the last section is dedicated to the conclusion.

2 Curvature power spectrum in the positively curved FLRW universe

2.1 The Sasaki–Mukhanov equation associated with the positively curved inflationary universe

In order to find the curvature power spectrum in a spatially closed universe, the ordinary Sasaki–Mukhanov equation [7,13,49,50] should be generalized to the case $K = +1$, in which K is the curvature index in the FLRW metric. This equation describes the evolution of the comoving curvature perturbation in the inflationary epoch. For this purpose, it is supposed that the homogeneous inflation field $\bar{\Phi}(t)$ has been perturbed by a small fluctuation $\delta\Phi(t, \mathbf{x})$ during the inflation era (hereafter a bar over any quantity stands for its unperturbed value). Such fluctuations are accompanied by the (scalar) perturbation in the FLRW metric (with $K = +1$), based on which the line element of the universe may be written as [13]

$$\mathrm{d}s^2 = -(1+E)\,\mathrm{d}t^2 + 2a\,(\partial_i F)\,\mathrm{d}t\mathrm{d}x^i + a^2\,(1+2\mathcal{R})\,\tilde{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j,$$
$$\tilde{g}_{ij} = \delta_{ij} + \frac{x^i x^j}{1-\mathbf{x}^2}, \tag{3}$$

which is the FLRW metric with $K = +1$ in the comoving quasi-Cartesian coordinates x^i plus the scalar linear perturbation in the *comoving gauge*. Here E and $\mathcal{R}$ are, respectively, the *lapse function* and *comoving curvature perturbation*. It can be shown that in the comoving gauge [13]

$$\delta\rho = \delta p = -\frac{1}{2}E\dot{\bar{\Phi}}^2, \tag{4}$$
$$\delta\Phi = 0, \tag{5}$$

where ρ and p are the energy density and pressure of the perfect fluid associated to the inflaton (the dot stands for the derivation respect to the cosmic time t). On the other hand, according to the perturbative field equations as well as the energy conservation law E, F, and $\mathcal{R}$ do not evolve independently and therefore [51]

$$\frac{4\mathcal{R}}{a^2} + Ha\nabla^2 F - 6H\dot{\mathcal{R}} - \ddot{\mathcal{R}} + \left(3H^2 + \dot{H}\right)E + \frac{1}{2}H\dot{E} + \nabla^2\mathcal{R} = 0, \tag{6}$$
$$2\dot{\mathcal{R}} - HE + \frac{2}{a}F = 0, \tag{7}$$
$$4HaF + 2a\dot{F} + E + 2\mathcal{R} = 0, \tag{8}$$
$$\delta\rho - \dot{\bar{\Phi}}^2\left(a\nabla^2 F - 3\dot{\mathcal{R}} + 3HE\right) = 0, \tag{9}$$

where Eq. (9) is the energy conservation law. Notice that $\nabla^2 = \frac{1}{a^2}\tilde{g}^{ij}\nabla_i\nabla_j$ is the *Laplace–Beltrami operator* with respect to $a^2\tilde{g}_{ij}$. After some tedious and lengthy calculations, these equations can be combined, and an explicit equation can be extracted in terms of $\mathcal{R}$,

$$\left[Ha^2\left(n^2-4\right) + \frac{1}{H} - \frac{\dot{H}a^2}{H}\right]\ddot{\mathcal{R}}_{\mathbf{n}} + \left[Ha^2\frac{\dot{\chi}}{\chi}\left(n^2-4\right) - \dot{H}a^2\left(2n^2-5\right) + 3H^2a^2\left(n^2-4\right) + 3\right]\dot{\mathcal{R}}_{\mathbf{n}} + \left[H\left(n^2-4\right)\left(n^2-5\right) + \frac{\dot{H}}{H}\left(n^2-3\right) + \frac{1}{Ha^2}\left(n^2-5\right) - \frac{\dot{\chi}}{\chi}\left(n^2-4\right)\right]\mathcal{R}_{\mathbf{n}} = 0, \tag{10}$$

where $\chi = \dot{H} - \frac{1}{a^2}$ and $\mathcal{R}_{\mathbf{n}}$ is the Fourier component of $\mathcal{R}$ with comoving canonical wave number n. Notice that $\mathbf{n} = (n, l, m)$ where $n = 3, 4, ..., 0 \le l \le n-1$ and $|m| \le l$ [51]. Here, due to the compactness of the spatial section of spacetime, the comoving wave number is discrete. Furthermore, wave numbers $n = 1, 2$ correspond to the pure gauge [52,53], hence they could be totally ignored. One can rewrite Eq. (10) in terms of the conformal time τ:

$$\left[\left(n^2-3\right)\mathcal{H} + \frac{1}{\mathcal{H}} - \frac{\mathcal{H}'}{\mathcal{H}}\right]\mathcal{R}''_{\mathbf{n}} + \left[pg2\left(n^2-3\right)\mathcal{H}^2 - 2\left(n^2-3\right)\mathcal{H}' + \left(n^2-4\right)\mathcal{H}\frac{\chi'}{\chi} + 2\right]\mathcal{R}'_{\mathbf{n}} + \left[\left(n^2-3\right)\left(n^2-5\right)\mathcal{H} + \left(n^2-3\right)\frac{\mathcal{H}'}{\mathcal{H}} + \left(n^2-5\right)\frac{1}{\mathcal{H}} - \left(n^2-4\right)\frac{\chi'}{\chi}\right]\mathcal{R}_{\mathbf{n}} = 0, \tag{11}$$

where the prime symbol indicates derivation with respect to the conformal time. Moreover, $\mathcal{H} = Ha$ is the comoving Hubble parameter and $\chi = \mathcal{H}^2 - \mathcal{H}' + 1 = 4\pi G\bar{\phi}'^2$ (indeed $\boldsymbol{\chi} = -\frac{\chi}{a^2}$). Equation (11) is the *generalized Sasaki–Mukhanov equation* for the inflationary universe with positive curvature index.

2.2 Re-definition of the slow-roll parameters; generalized Sasaki–Mukhanov variable

Now let us consider the slow-roll inflation which guarantees slow variation of inflation by considering a Coleman–Weinberg type potential. In general, slow-roll inflation may be described by the two flatness conditions [7,13]

$$\dot{\bar{\phi}}^2 \ll V\left(\bar{\phi}\right), \tag{12}$$
$$|\ddot{\bar{\phi}}| \ll H|\dot{\bar{\phi}}|. \tag{13}$$

In the spatially flat case Eqs. (12) and (13) are reduced to

$$\epsilon := -\frac{\dot{H}}{H^2} \ll 1, \tag{14}$$
$$\delta := \frac{\ddot{H}}{2H\dot{H}} \ll 1, \tag{15}$$

where ϵ and δ are, respectively, the first and the second slow-roll parameters, considered as being roughly constant. On the other hand, for the positively curved inflationary universe, the flatness conditions may be written in the same way as Eqs. (14) and (15) by re-defining the slow-roll parameters

$$\epsilon := -\frac{\boldsymbol{\chi}}{H^2 + \frac{1}{a^2}} = \frac{\mathcal{H}^2 - \mathcal{H}' + 1}{\mathcal{H}^2 + 1} \ll 1, \tag{16}$$
$$\delta := \frac{1}{2H}\frac{\dot{\boldsymbol{\chi}}}{\boldsymbol{\chi}} = \frac{1}{2\mathcal{H}}\frac{\chi'}{\chi} - 1 \ll 1. \tag{17}$$

One can rewrite Eq. (16) as

$$\left(\frac{1}{\mathcal{H}}\right)' = -(1-\epsilon)\left(1+\frac{1}{\mathcal{H}^2}\right), \tag{18}$$

which results in

$$\mathcal{H} = -\cot\Big[(1-\epsilon)\,\tau - \cot^{-1} n\Big]. \tag{19}$$

Here it is assumed that $\tau = \tau_n = -\int_t^{t_n} \frac{\mathrm{d}\eta}{a(\eta)}$ where t_n is the horizon exit time for the inhomogeneity mode n $(n = \mathcal{H}(t_n))$. Furthermore, combination of Eqs. (17) and (19) results in

$$\frac{\chi'}{\chi} = -2(1+\delta)\cot\Theta, \tag{20}$$

where $\Theta = (1-\epsilon)\,\tau - \cot^{-1} n$. Now by substituting Eqs. (19) and (20) in Eq. (11) it can be deduced that

$$\begin{aligned}&\left[\left(n^2-4\right)+\frac{\epsilon}{\cos^2\Theta}\right]\mathcal{R}''_{\mathbf{n}}\\ &\quad-\left[4\left(n^2-4\right)\cot 2\Theta + 4\epsilon\frac{n^2-3}{\sin 2\Theta}\right.\\ &\quad\left.+2\delta\left(n^2-4\right)\cot\Theta\right]\mathcal{R}'_{\mathbf{n}}\\ &\quad+\left[\left(n^2-4\right)\left(n^2-5\right)+2\left(n^2-4\right)\tan^2\Theta\right.\\ &\quad\left.-\epsilon\frac{n^2-3}{\cos^2\Theta}-2\delta\left(n^2-4\right)\right]\mathcal{R}_{\mathbf{n}} = 0.\end{aligned} \tag{21}$$

Hereafter, only linear perturbations are investigated, i.e. terms such as $\epsilon^2, \delta^2, \epsilon\delta$, etc. shall be ignored. Now let us define the new variable $\mathcal{V}_{\mathbf{n}}$ as

$$\mathcal{V}_{\mathbf{n}} = \mathscr{T}\mathcal{R}_{\mathbf{n}}, \quad \mathscr{T} = \mathscr{C}\frac{\exp\left[-\frac{\epsilon}{2(n^2-4)\cos^2\Theta}\right]}{|\sin\Theta|^{1+2\epsilon+\delta}\,|\cos\Theta|}. \tag{22}$$

(Here $\mathscr{C}$ is a constant, to be obtained soon.) Thus, Eq. (21) may be written in terms of $\mathcal{V}_{\mathbf{n}}$:

$$\begin{aligned}&\mathcal{V}''_{\mathbf{n}} + \Bigg[\left(n^2-5\right) - 2\cot^2\Theta + \epsilon\left(1+\cot^2\Theta\right)\\ &\quad\times\left(2\frac{1-\cot^2\Theta}{\cot^2\Theta}+\frac{1}{n^2-4}\frac{3-\cot^2\Theta}{\cot^4\Theta}\right)\\ &\quad-\delta\left(1+3\cot^2\Theta\right)\Bigg]\mathcal{V}_{\mathbf{n}} = 0.\end{aligned} \tag{23}$$

Before solving Eq. (23), let us find the constant $\mathscr{C}$. For this purpose, one may invoke the relation

$$\frac{\epsilon'}{\epsilon} = 2\mathcal{H}(\epsilon+\delta), \tag{24}$$

which can be derived from the logarithmic derivation of Eq. (16). Provided that ϵ and δ are constant, Eq. (24) yields

$$\left(\frac{a}{\mathfrak{R}}\right)^{\epsilon+\delta} = \sqrt{\epsilon}. \tag{25}$$

Here $\mathfrak{R}$ is a characteristic scale, appearing as the integral constant in Eq. (25).

On the other hand, Eq. (19) results in

$$a = \frac{1}{\mathfrak{H}}|\sin\Theta|^{-(\epsilon+\delta)}, \quad \mathfrak{H} = \sqrt{\frac{8\pi G}{3}\bar{\rho}}. \tag{26}$$

Consequently,

$$|\sin\Theta|^{\epsilon+\delta} = \frac{(\mathfrak{R}\mathfrak{H})^{-(\epsilon+\delta)}}{\sqrt{\epsilon}}. \tag{27}$$

Meanwhile, using Eqs. (19), (22) as well as (16) and (17) one can show

$$\frac{\mathscr{T}'}{\mathscr{T}} = \frac{a'}{a} - \frac{\mathcal{H}'}{\mathcal{H}} + \frac{\bar{\phi}''}{\bar{\phi}'} + \frac{1}{n^2-4}\left(\frac{1}{\mathcal{H}} - \frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}^3}\right), \tag{28}$$

which results in

$$\mathscr{T} = \frac{a\bar{\phi}'}{\mathcal{H}}\exp\left[\frac{1}{n^2-4}\left(\frac{1}{2\mathcal{H}^2} + \int\frac{\mathcal{H}^2+1}{\mathcal{H}^3}d\tau\right)\right]. \tag{29}$$

Now let us suppose $n \longrightarrow +\infty$, thus Eq. (29) takes the form

$$\lim_{n\to+\infty}\mathscr{T} = \frac{\mathscr{C}}{|\sin\Theta|^{1+2\epsilon+\delta}\,|\cos\Theta|} = \frac{a\bar{\phi}'}{\mathcal{H}} = \mathscr{Z}. \tag{30}$$

For the severe sub-Hubble modes, the curvature has a negligible imprint and may be disregarded, so it coincides with the $K = 0$ case. Besides, it can be shown that

$$\begin{aligned}\frac{a\bar{\phi}'}{\mathcal{H}} &= \frac{a}{\mathcal{H}}\sqrt{\frac{\mathcal{H}^2-\mathcal{H}'+1}{4\pi G}}\\ &= a\sqrt{\left(1+\frac{1}{\mathcal{H}^2}\right)\frac{\epsilon}{4\pi G}}\\ &= \frac{1}{\mathfrak{H}}|\sin\Theta|^{-1-\epsilon}\,|\cos\Theta|^{-1}\sqrt{\frac{\epsilon}{4\pi G}}.\end{aligned} \tag{31}$$

Therefore, the combination of Eqs. (31), (30), and (27) yields

$$\mathscr{C} = \frac{1}{\sqrt{4\pi G}}\frac{1}{\mathfrak{H}\,(\mathfrak{R}\mathfrak{H})^{\epsilon+\delta}}. \tag{32}$$

So

$$\mathcal{R}_{\mathbf{n}} = \sqrt{4\pi G}\mathfrak{H}\,(\mathfrak{R}\mathfrak{H})^{\epsilon+\delta}\,|\sin\Theta|^{1+2\epsilon+\delta}\,|\cos\Theta|\exp\left[\frac{\epsilon}{2\left(n^2-4\right)\cos^2\Theta}\right]\mathcal{V}_{\mathbf{n}}. \quad (33)$$

Notice that $\mathcal{V}_{\mathbf{n}}$ is the *generalized Sasaki–Mukhanov variable* for the inflationary universe with positive curvature index.

2.3 Curvature power spectrum

Now let us find the solutions of Eq. (23). For this purpose, it is assumed that $x = \cos\Theta$, thus Eq. (23) reduces to

$$\left(1-x^2\right)\frac{\mathrm{d}^2\mathcal{V}_{\mathbf{n}}}{\mathrm{d}x^2} - x\frac{\mathrm{d}\mathcal{V}_{\mathbf{n}}}{\mathrm{d}x} + \left[\left(n^2-3\right)(1+2\epsilon)+2\delta-\frac{2+6\epsilon+3\delta}{1-x^2} +2\epsilon\left(1-\frac{2}{n^2-4}\right)\frac{1}{x^2}+\frac{3\epsilon}{n^2-4}\frac{1}{x^4}\right]\mathcal{V}_{\mathbf{n}} = 0. \quad (34)$$

The following solution may be proposed:

$$\mathcal{V}_{\mathbf{n}} = \mathcal{V}_{\mathbf{n}} + \epsilon\mathfrak{V}_{\mathbf{n}}, \quad (35)$$

where

$$\mathcal{V}_{\mathbf{n}} = \mathscr{A}\sqrt[4]{1-x^2}P_\nu^\mu(x) + \mathscr{B}\sqrt[4]{1-x^2}Q_\nu^\mu(x), \quad (36)$$

$$\begin{cases} \mu := \frac{3}{2}+2\epsilon+\delta, \\ \nu := (1+\epsilon)\sqrt{n^2-3}+\frac{\delta}{\sqrt{n^2-3}}-\frac{1}{2}. \end{cases} \quad (37)$$

Notice that P_ν^μ and Q_ν^μ are associated Legendre functions. By inserting the ansatz (35) in Eq. (34) and neglecting higher order infinitesimal terms, one may obtain a second order nonhomogeneous equation in terms of $\mathfrak{V}_{\mathbf{n}}$

$$\left(1-x^2\right)\frac{\mathrm{d}^2\mathfrak{V}_{\mathbf{n}}}{\mathrm{d}x^2} - x\frac{\mathrm{d}\mathfrak{V}_{\mathbf{n}}}{\mathrm{d}x} + \left[\left(n^2-3\right)(1+2\epsilon)+2\delta-\frac{2+6\epsilon+3\delta}{1-x^2}\right]\mathfrak{V}_{\mathbf{n}} = -\frac{1}{n^2-4}\left(2\frac{n^2-6}{x^2}+\frac{3}{x^4}\right) \times\left[\mathscr{A}\left(1-x^2\right)^{\frac{1}{4}}P_\nu^\mu(x)+\mathscr{B}\left(1-x^2\right)^{\frac{1}{4}}Q_\nu^\mu(x)\right], \quad (38)$$

which has the special solution

$$\mathfrak{V}_{\mathbf{n}} = \frac{1}{n^2-4}\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\sqrt[4]{1-x^2} \times\left\{\left[\mathscr{A}P_\nu^\mu(x)-\mathscr{B}Q_\nu^\mu(x)\right] \times\int_{x_0}^{x}\left(1-y^2\right)\left(2\frac{n^2-6}{y^2}+\frac{3}{y^4}\right) \times P_\nu^\mu(y)\,Q_\nu^\mu(y)\,\mathrm{d}y - \mathscr{A}Q_\nu^\mu(x) \times\int_{x_0}^{x}\left(1-y^2\right)\left(2\frac{n^2-6}{y^2}+\frac{3}{y^4}\right)\left[P_\nu^\mu(y)\right]^2\mathrm{d}y +\mathscr{B}P_\nu^\mu(x)\int_{x_0}^{x}\left(1-y^2\right)\left(2\frac{n^2-6}{y^2} +\frac{3}{y^4}\right)\left[Q_\nu^\mu(y)\right]^2\mathrm{d}y\right\}. \quad (39)$$

Here, x_0 is an arbitrary constant for which $|x_0| \leqslant 1$. Consequently, the general solution of Eq. (23) reduces to

$$\mathcal{V}_{\mathbf{n}}(\tau) = \sqrt{|\sin\Theta|}\left[\mathscr{A}P_\nu^\mu(\cos\Theta)+\mathscr{B}Q_\nu^\mu(\cos\Theta)\right] + \frac{\epsilon}{n^2-4}\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\sqrt{|\sin\Theta|} \times\left\{\left[-\mathscr{A}P_\nu^\mu(\cos\Theta)+\mathscr{B}Q_\nu^\mu(\cos\Theta)\right] \times\int_{\Theta_0}^{\Theta}\sin^3\Upsilon\left(2\frac{n^2-6}{\cos^2\Upsilon}+\frac{3}{\cos^4\Upsilon}\right) \times P_\nu^\mu(\cos\Upsilon)\,Q_\nu^\mu(\cos\Upsilon)\,d\Upsilon + \mathscr{A}Q_\nu^\mu(\cos\Theta)\int_{\Theta_0}^{\Theta}\sin^3\Upsilon\left(2\frac{n^2-6}{\cos^2\Upsilon}+\frac{3}{\cos^4\Upsilon}\right) \times\left[P_\nu^\mu(\cos\Upsilon)\right]^2 d\Upsilon \times -\mathscr{B}P_\nu^\mu(\cos\Theta)\int_{\Theta_0}^{\Theta}\sin^3\Upsilon\left(2\frac{n^2-6}{\cos^2\Upsilon} + \frac{3}{\cos^4\Upsilon}\right)\left[Q_\nu^\mu(\cos\Upsilon)\right]^2 d\Upsilon\right\}. \quad (40)$$

Hereafter, we put $\Theta_0 = -\cot^{-1} n$ (it is completely compatible with the conformal initial condition which is introduced below).

In order to determine the constants $\mathscr{A}$ and $\mathscr{B}$ one may use the *conformal (Bunch–Davies) initial condition*, which states [54,55]

$$\lim_{n\to+\infty}\mathcal{V}_{\mathbf{n}} = \frac{1}{\sqrt{2n}}\exp(-in\tau). \quad (41)$$

Thus, according to the asymptotic formulas of P_ν^μ and Q_ν^μ for large values of ν [56]

$$P_\nu^\mu(\cos\theta) \sim \frac{\Gamma(\mu+\nu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)}\sqrt{\frac{2}{\pi\sin\theta}}\sin\left[\left(\nu+\frac{1}{2}\right)\theta+\frac{\pi}{4}+\frac{\mu\pi}{2}\right]+\mathcal{O}\left(\nu^{-1}\right), \quad (42)$$

$$Q_\nu^\mu(\cos\theta) \sim \frac{\Gamma(\mu+\nu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)}\sqrt{\frac{\pi}{2\sin\theta}}\cos \times\left[\left(\nu+\frac{1}{2}\right)\theta+\frac{\pi}{4}+\frac{\mu\pi}{2}\right]+\mathcal{O}\left(\nu^{-1}\right), \tag{43}$$

and noting that $\lim_{n\to+\infty}\frac{(n+\alpha)!}{n!}\sim n^\alpha$, after a lot of lengthy but straightforward calculations, it can be shown that

$$\begin{cases} \mathscr{A}=\frac{i\sqrt{\pi}}{2}n^{-\frac{3}{2}-2\epsilon-\delta}, \\ \mathscr{B}=-\frac{1}{\sqrt{\pi}}n^{-\frac{3}{2}-2\epsilon-\delta}. \end{cases} \tag{44}$$

Thus

$$\begin{aligned} \mathcal{V}_{\mathbf{n}}(\tau) = \sqrt{|\sin\Theta|}n^{-\mu}\Bigg\{ & \frac{i\sqrt{\pi}}{2}P_\nu^\mu(\cos\Theta) \\ & -\frac{1}{\sqrt{\pi}}Q_\nu^\mu(\cos\Theta)-\frac{\epsilon}{n^2-4}\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)} \\ & \times\left[\frac{i\sqrt{\pi}}{2}P_\nu^\mu(\cos\Theta)+\frac{1}{\sqrt{\pi}}Q_\nu^\mu(\cos\Theta)\right] \\ & \times\int_0^{(1-\epsilon)\tau}\sin^3\Upsilon\left(2\frac{n^2-6}{\cos^2\Upsilon}+\frac{3}{\cos^4\Upsilon}\right) \\ & \times P_\nu^\mu(\cos\Upsilon)\,Q_\nu^\mu(\cos\Upsilon)\,\mathrm{d}\eta \\ & +\frac{i\sqrt{\pi}}{2}\frac{\epsilon}{n^2-4}\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}Q_\nu^\mu(\cos\Theta) \\ & \times\int_0^{(1-\epsilon)\tau}\sin^3\Upsilon\left(2\frac{n^2-6}{\cos^2\Upsilon}+\frac{3}{\cos^4\Upsilon}\right) \\ & \times\left[P_\nu^\mu(\cos\Upsilon)\right]^2\mathrm{d}\eta \\ & +\frac{1}{\sqrt{\pi}}\frac{\epsilon}{n^2-4}\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}P_\nu^\mu(\cos\Theta) \\ & \times\int_0^{(1-\epsilon)\tau}\sin^3\Upsilon\left(2\frac{n^2-6}{\cos^2\Upsilon}+\frac{3}{\cos^4\Upsilon}\right) \\ & \times\left[Q_\nu^\mu(\cos\Upsilon)\right]^2\mathrm{d}\eta\Bigg\}. \end{aligned} \tag{45}$$

Furthermore, by ignoring the non-linear terms, $\mathcal{R}_{\mathbf{n}}$ takes the form

$$\begin{aligned} \mathcal{R}_{\mathbf{n}}(\tau) = & \sqrt{4\pi G}\mathfrak{H}(\mathfrak{R}\mathfrak{H})^{\epsilon+\delta}\left|\frac{\sin\Xi}{n}\right|^\mu|\cos\Xi| \\ & \times\left[\frac{i\sqrt{\pi}}{2}P_\nu^\mu(\cos\Xi)-\frac{1}{\sqrt{\pi}}Q_\nu^\mu(\cos\Xi)\right] \\ & +\epsilon\sqrt{4\pi G}\mathfrak{H}(\mathfrak{R}\mathfrak{H})^{\epsilon+\delta}\left|\frac{\sin\Xi}{n}\right|^\mu|\cos\Xi| \\ & \times\Bigg\{-\frac{i\sqrt{\pi}}{2}\tau\frac{dP_\nu^\mu(\cos\Xi)}{d\tau}+\frac{1}{\sqrt{\pi}}\tau\frac{\mathrm{d}Q_\nu^\mu(\cos\Xi)}{d\tau} \\ & -\left[2\tau\cot 2\Xi+\frac{1}{2}\tau\cot\Xi-\frac{1}{2\left(n^2-4\right)\cos^2\Xi}\right] \\ & \times\left[\frac{i\sqrt{\pi}}{2}P_\nu^\mu(\cos\Xi)-\frac{1}{\sqrt{\pi}}Q_\nu^\mu(\cos\Xi)\right] \\ & -\frac{1}{n^2-4}\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)} \\ & \times\left[\frac{i\sqrt{\pi}}{2}P_\nu^\mu(\cos\Xi)+\frac{1}{\sqrt{\pi}}Q_\nu^\mu(\cos\Xi)\right] \\ & \times\int_0^{\tau}\sin^3\Upsilon\left(2\frac{n^2-6}{\cos^2\Upsilon}+\frac{3}{\cos^4\Upsilon}\right) \\ & \times P_\nu^\mu(\cos\Upsilon)\,Q_\nu^\mu(\cos\Upsilon)\,\mathrm{d}\eta \\ & +\frac{i\sqrt{\pi}}{2}\frac{1}{n^2-4}\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}Q_\nu^\mu(\cos\Xi) \\ & \times\int_0^{\tau}\sin^3\Upsilon\left(2\frac{n^2-6}{\cos^2\Upsilon}+\frac{3}{\cos^4\Upsilon}\right) \\ & \times\left[P_\nu^\mu(\cos\Upsilon)\right]^2\mathrm{d}\eta \\ & +\frac{1}{\sqrt{\pi}}\frac{1}{n^2-4}\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}P_\nu^\mu(\cos\Xi) \\ & \times\int_0^{\tau}\sin^3\Upsilon\left(2\frac{n^2-6}{\cos^2\Upsilon}+\frac{3}{\cos^4\Upsilon}\right) \\ & \times\left[Q_\nu^\mu(\cos\Upsilon)\right]^2\mathrm{d}\eta\Bigg\}, \end{aligned} \tag{46}$$

where $\Xi=\tau-\cot^{-1}n$ and $\Upsilon=\eta-\cot^{-1}n$.

It is important to evaluate the comoving curvature perturbation at the horizon exit time $\tau=0$ i.e. when the quantum fluctuations of the inflaton came to be classical perturbations. Besides, by inserting $\tau=0$ in Eq. (46) the arguments of P_ν^μ and Q_ν^μ become $\cos\left(\cot^{-1}n\right)=\frac{n}{\sqrt{n^2+1}}$; for $n\geq 3$, $0.94\leq\frac{n}{\sqrt{n^2+1}}<1$, so it may be plausible to use asymptotic formulas of the associated Legendre functions near 1, i.e. [57]

$$\theta\longrightarrow 0 \quad : \quad P_\nu^\mu(\cos\theta)\sim\frac{1}{\pi}\Gamma(\mu)\sin\mu\pi\left(\frac{2}{1-\cos\theta}\right)^{\frac{\mu}{2}}, \tag{47}$$

$$\theta\longrightarrow 0 \quad : \quad Q_\nu^\mu(\cos\theta)\sim\frac{1}{2}\Gamma(\mu)\cos\mu\pi\left(\frac{2}{1-\cos\theta}\right)^{\frac{\mu}{2}}. \tag{48}$$

So by doing some straightforward calculations, it can be shown

$$\mathcal{R}_{\mathbf{n}}^o=-\sqrt{G}\mathfrak{H}(\mathfrak{R}\mathfrak{H})^{\epsilon+\delta}\Gamma(\mu)\exp(-i\mu\pi)\frac{n^{1-\mu}}{\sqrt{n^2+1}} \times\left(2+\frac{2n}{\sqrt{n^2+1}}\right)^{\frac{\mu}{2}}\left(1+\frac{n^2+1}{2n^2\left(n^2-4\right)}\epsilon\right). \tag{49}$$

Let us approximate $\frac{n}{\sqrt{n^2+1}} \sim 1$, thus Eq. (49) takes the form

$$\mathcal{R}^{o}_{\mathbf{n}} \simeq -\sqrt{G}\mathfrak{H}\left(\mathfrak{R}\mathfrak{H}\right)^{\epsilon+\delta} 2^{\frac{3}{2}+2\epsilon+\delta}\Gamma\left(\mu\right)\exp\left(-i\mu\pi\right)n^{-\mu} \times\left(1+\frac{\epsilon}{2\left(n^2-4\right)}\right). \quad (50)$$

Consequently, the curvature power spectrum in the maximally extended inflationary universe with single field reduces to

$$\mathcal{P}^{o}_{\mathcal{R}}\left(n\right) \propto n^{-3-4\epsilon-2\delta}\left(1+\frac{\epsilon}{n^2-4}\right). \quad (51)$$

Except the additional factor $1+\frac{\epsilon}{n^2-4}$, the spectrum (51) is similar to the nearly flat spectrum which can be deduced from the slow-rolling inflationary scenario with spatially flat background [13]. By defining the *curvature spectral index* as

$$\mathcal{P}^{o}_{\mathcal{R}}\left(n\right) \propto n^{\mathfrak{N}_s(n)-4}, \quad (52)$$

one can show that

$$\mathfrak{N}_s\left(n\right) = 1-4\epsilon-2\delta+\frac{2\epsilon}{\left(n^2-4\right)\ln n}. \quad (53)$$

Because $n \geq 3$,

$$1-4\epsilon-2\delta < \mathfrak{N}_s\left(n\right) \lesssim 1-3.64\epsilon-2\delta. \quad (54)$$

It means the curvature spectral index in the maximally extended universe shall be a bit larger than the $K=0$ corresponding model (for the $K=0$ case, $\mathfrak{N}_s\left(n\right)=1-4\epsilon-2\delta$). Moreover, $\mathfrak{N}_s$ directly depends on the comoving wave number (n) and so the spectrum is running. In other words, the *running parameter* of $\mathfrak{N}_s$ does not vanish,

$$\mathfrak{N}_r\left(n\right) = n\frac{\partial\mathfrak{N}_s}{\partial n} = -2\epsilon\frac{\left(n^2-4\right)+2n^2\ln n}{\left(n^2-4\right)^2\ln^2 n} < 0. \quad (55)$$

It is remarkable that the sign of $\mathfrak{N}_r$ coincides with the experimental data. Moreover, the running parameter in the maximally extended background inflationary model is proportional to ϵ i.e. $\mathfrak{N}_r$ is of the first order slow-roll parameters in full accordance with the reports [11], despite the spatially flat case in which against the Planck reports it is roughly zero.

3 Primordial gravitational waves power spectrum in the positively curved universe

The primordial gravitational waves during inflationary epoch can be treated in the same way as the comoving curvature perturbation, considered in the previous section. In fact, quantum fluctuations of the inflaton may result in tensorial perturbations, described by a symmetric traceless divergenceless tensor field $D_{ij}\left(t,\mathbf{x}\right)$, which perturbs the FLRW metric as [13]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2\left(\tilde{g}_{ij}+D_{ij}\right)\mathrm{d}x^i\mathrm{d}^j. \quad (56)$$

The propagation of D_{ij} in the positively curved FLRW universe is described by [51]

$$a^2\nabla^2 D_{ij} - 3a\dot{a}\dot{D}_{ij} - a^2\ddot{D}_{ij} - 2D_{ij} = -16\pi G a^2\Pi^T_{ij}. \quad (57)$$

Here $\Pi^T_{ij}\left(t,\mathbf{x}\right)$ is the anisotropic inertia tensor that vanishes for the scalar fields, so

$$a^2\nabla^2 D_{ij} - 3a\dot{a}\dot{D}_{ij} - a^2\ddot{D}_{ij} - 2D_{ij} = 0. \quad (58)$$

One may expand D_{ij} in terms of the t-t tensor spherical harmonics on $\mathbb{S}^3\left(a\right)$ [51]

$$D_{ij}\left(t,\mathbf{x}\right) = \sum_{nlm}\left[\mathcal{D}^{\mathbb{O}}_{nlm}\left(t\right)\left(T^{\mathbb{O}}_{ij}\right)_{nlm} + \mathcal{D}^{\mathbb{E}}_{nlm}\left(t\right)\left(T^{\mathbb{E}}_{ij}\right)_{nlm}\right], \quad (59)$$

where $\mathcal{D}^{\mathbb{O}}_{\mathbf{n}}$ and $\mathcal{D}^{\mathbb{E}}_{\mathbf{n}}$ correspond to two different polarizations of the gravitational waves. Notice that $\left\{\left(T^{\mathbb{O}}_{ij}\right)_{nlm}, \left(T^{\mathbb{E}}_{ij}\right)_{nlm}\right\}$ constitutes a complete orthonormal basis for the expansion of any symmetric traceless divergence-free covariant tensor field of rank 2 on $\mathbb{S}^3\left(a\right)$. Furthermore [51],

$$\nabla^2\left(T^{\mathbb{O}}_{ij}\right)_{nlm} = \frac{3-n^2}{a^2}\left(T^{\mathbb{O}}_{ij}\right)_{nlm}, \quad n=3,4,..., \quad (60)$$

$$\nabla^2\left(T^{\mathbb{E}}_{ij}\right)_{nlm} = \frac{3-n^2}{a^2}\left(T^{\mathbb{E}}_{ij}\right)_{nlm}, \quad n=3,4,... \quad (61)$$

Thus Eq. (59) reduces to two independent equations,

$$\begin{cases}\ddot{\mathcal{D}}^{\mathbb{O}}_{\mathbf{n}}\left(t\right)+3H\dot{\mathcal{D}}^{\mathbb{O}}_{\mathbf{n}}\left(t\right)+\frac{n^2-1}{a^2}\mathcal{D}^{\mathbb{O}}_{\mathbf{n}}\left(t\right)=0,\\ \ddot{\mathcal{D}}^{\mathbb{E}}_{\mathbf{n}}\left(t\right)+3H\dot{\mathcal{D}}^{\mathbb{E}}_{\mathbf{n}}\left(t\right)+\frac{n^2-1}{a^2}\mathcal{D}^{\mathbb{E}}_{\mathbf{n}}\left(t\right)=0.\end{cases} \quad (62)$$

Hereafter the superscripts $\mathbb{O}$ and $\mathbb{E}$ are omitted because both of $\mathcal{D}^{\mathbb{O}}_{\mathbf{n}}$ and $\mathcal{D}^{\mathbb{E}}_{\mathbf{n}}$ satisfy the equation

$$\ddot{\mathcal{D}}_{\mathbf{n}}\left(t\right)+3H\dot{\mathcal{D}}_{\mathbf{n}}\left(t\right)+\frac{n^2-1}{a^2}\mathcal{D}_{\mathbf{n}}\left(t\right)=0. \quad (63)$$

$\mathcal{D}_{\mathbf{n}}\left(t\right)$ is amplitude of the gravitational wave $D_{ij}\left(t,\mathbf{x}\right)$ as well as a tensor random field on $\mathbb{S}^3\left(a\right)$. By converting the cosmic time to the conformal time Eq. (63) takes the form

$$\mathcal{D}''_{\mathbf{n}}\left(\tau\right)+2\mathcal{H}\mathcal{D}'_{\mathbf{n}}\left(\tau\right)+\left(n^2-1\right)\mathcal{D}_{\mathbf{n}}\left(\tau\right)=0. \quad (64)$$

During the slow-rolling inflationary epoch it can be written that

$$\mathcal{D}''_{\mathbf{n}}(\tau) - 2\cot\Theta\mathcal{D}'_{\mathbf{n}}(\tau) + \left(n^2 - 1\right)\mathcal{D}_{\mathbf{n}}(\tau) = 0. \tag{65}$$

We assume $x = \cos\Theta$ Eq. (65) can be written as

$$\left(1 - x^2\right)\frac{d^2\mathcal{D}_{\mathbf{n}}}{d^2} + (1 + 2\epsilon)\, x\frac{d\mathcal{D}_{\mathbf{n}}}{d} + (1 + 2\epsilon)\left(n^2 - 1\right)\mathcal{D}_{\mathbf{n}}(x) = 0, \tag{66}$$

which has the general solution

$$\mathcal{D}_{\mathbf{n}}(x) = \left(1 - x^2\right)^{\frac{2\epsilon+3}{4}}\left[\mathscr{P}P^{\epsilon+\frac{3}{2}}_{n(1+\epsilon)-\frac{1}{2}}(x) + \mathscr{Q}Q^{\epsilon+\frac{3}{2}}_{n(1+\epsilon)-\frac{1}{2}}(x)\right]. \tag{67}$$

Here $\mathscr{P}$ and $\mathscr{Q}$ are two arbitrary constants. So the solution of Eq. (65) is

$$\mathcal{D}_{\mathbf{n}}(\tau) = |\sin\Theta|^{\iota}\left[\mathscr{P}P^{\iota}_{\kappa}(\cos\Theta) + \mathscr{Q}Q^{\iota}_{\kappa}(\cos\Theta)\right], \tag{68}$$

where

$$\begin{cases} \iota := \epsilon + \frac{3}{2}, \\ \kappa := n(1 + \epsilon) - \frac{1}{2}. \end{cases} \tag{69}$$

Besides, the initial condition that must be satisfied by $\mathcal{D}_{\mathbf{n}}$ is very similar to the Bunch–Davies initial condition applied to the Sasaki–Mukhanov variable [13]

$$\lim_{n\to+\infty}\mathcal{D}_{\mathbf{n}} = \frac{\sqrt{16\pi G}}{a(t)}\frac{1}{\sqrt{2n}}\exp(-in\tau), \tag{70}$$

which is applicable for both polarization modes distinctly. By considering the asymptotic formulas (42) and (44) and Eq. (26) as well, one can obtain

$$\begin{cases} \mathscr{P} = 2\pi i\mathfrak{H}\sqrt{G}n^{-\iota}, \\ \mathscr{Q} = -4\mathfrak{H}\sqrt{G}n^{-\iota}. \end{cases} \tag{71}$$

Thus

$$\mathcal{D}_{\mathbf{n}}(\tau) = 2\sqrt{G}\mathfrak{H}\left|\frac{\sin\Theta}{n}\right|^{\iota}\left[\pi i P^{\iota}_{\kappa}(\cos\Theta) - 2Q^{\iota}_{\kappa}(\cos\Theta)\right]. \tag{72}$$

$\mathcal{D}^o_{\mathbf{n}}$ may be determined by considering $\mathcal{D}_{\mathbf{n}}$ at the time of horizon crossing ($\tau = 0$),

$$\mathcal{D}^o_{\mathbf{n}} = -2\sqrt{G}\mathfrak{H}\Gamma(\iota)\exp(-i\iota\pi)\, n^{-\iota}\left(2 + \frac{2n}{\sqrt{n^2+1}}\right)^{\frac{\iota}{2}}. \tag{73}$$

Here again the asymptotic relations (47) and (48) have been used. By the approximation $\frac{n}{\sqrt{n^2+1}} \sim 1$, Eq. (73) acquires a simpler form,

$$\mathcal{P}^o_{\mathcal{D}}(n) \propto n^{-3-2\epsilon}, \tag{74}$$

So by the definition of the *tensor spectral index*,

$$\mathcal{P}^o_{\mathcal{D}} \propto n^{\mathfrak{N}_T - 3}, \tag{75}$$

one can obtain

$$\mathfrak{N}_T = -2\epsilon, \tag{76}$$

which is perfectly analogous to the tensor spectral index derived in the classical slow-rolling inflationary theory [13].

4 Tensor–scalar ratio in the positively curved universe

Tensor–scalar ratio in the positively curved FLRW universe may be defined as [13]

$$r_n := 4\frac{\mathcal{P}^o_{\mathcal{D}}(n)}{\mathcal{P}^o_{\mathcal{R}}(n)} = 4\left|\frac{\mathcal{D}^o_{\mathbf{n}}}{\mathcal{R}^o_{\mathbf{n}}}\right|^2. \tag{77}$$

Here the factor 4 refers to two different polarization modes of the gravitation waves. The significance of r_n comes from its measurability, indeed the tensor–scalar ratio can provide an assay for the inflationary scenarios and some inflation theories may be crossed out due to the contradiction with the observational value of r_n. According to the standard slow-rolling inflationary theory $r_q = 16\epsilon$ (q stands for the comoving wave number of perturbations in the spatially flat universe) [13], so if it is supposed that $\epsilon = 0.008$ [11], then $r = 0.128$, which is greater than the data result BICEP2/Keck Array and Planck released ($r_{0.05} < 0.12$) [25] whereby the question is brought up of whether it is possible to eliminate this flaw by considering a curvature factor. In order to answer, let us calculate r_n using Eqs. (49) and (73),

$$r_n = 16(\mathfrak{R}\mathfrak{H})^{-2(\epsilon+\delta)}\left[\frac{\Gamma\left(\epsilon+\frac{3}{2}\right)}{\Gamma\left(2\epsilon+\delta+\frac{3}{2}\right)}\right]^2 n^{2(\epsilon+\delta)} \times\left(2 + \frac{2n}{\sqrt{n^2+1}}\right)^{-(\epsilon+\delta)} \times\left(1 + \frac{1}{n^2}\right)\left(1 - \frac{n^2+1}{n^2(n^2-4)}\epsilon\right). \tag{78}$$

Besides, one can write

$$n^{2(\epsilon+\delta)} = \left(\mathcal{H}^2|_{\tau=0}\right)^{\epsilon+\delta} = \left(\cot^2\Theta|_{\tau=0}\right)^{\epsilon+\delta} = \left(\cos^2\Theta|_{\tau=0}\right)^{\epsilon+\delta}$$

$$\times\left(\sin^2\Theta|_{\tau=0}\right)^{-(\epsilon+\delta)} = \left(1+\frac{1}{n^2}\right)^{-(\epsilon+\delta)} \times(\mathfrak{RH})^{2(\epsilon+\delta)}\,\epsilon. \quad (79)$$

On the other hand, it is not hard to show that

$$\frac{\Gamma\left(2\epsilon+\delta+\frac{3}{2}\right)}{\Gamma\left(\epsilon+\frac{3}{2}\right)} = 1+(2-\gamma-\ln 2)(\epsilon+\delta)\simeq \exp \times\left[(2-\gamma-\ln 2)(\epsilon+\delta)\right] \simeq (2.074)^{\epsilon+\delta}, \quad (80)$$

where $\gamma\simeq 0.577$ is the Euler–Mascheroni constant. In order to derive Eq. (80) one can use the following relation [56]:

$$\frac{\Gamma(x+\epsilon+1)}{\Gamma(x+1)} = 1+\epsilon\left[-\gamma+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{x+n}\right)\right]. \quad (81)$$

By inserting Eqs. (79) and (80) in Eq. (78),

$$r_n = 16\epsilon\exp\left[(-4+2\gamma+2\ln 2)(\epsilon+\delta)\right]\left(1+\frac{1}{n^2}\right)^{1-(\epsilon+\delta)} \times\left(2+\frac{2n}{\sqrt{n^2+1}}\right)^{-(\epsilon+\delta)}\left(1-\frac{n^2+1}{n^2\left(n^2-4\right)}\epsilon\right). \quad (82)$$

For $n\gg 1$ Eq. (82) is reduced to

$$r_{n\gg 1}\simeq 16\epsilon\exp\left[(-4+2\gamma)(\epsilon+\delta)\right]\simeq 16\epsilon\,(0.058)^{\epsilon+\delta}. \quad (83)$$

If one chooses $\epsilon = 0.008$ and $\delta > -0.008$, obviously $r_{n\gg 1} > 16\epsilon$. On the other hand, in accordance with [25] let us consider $k_* = 0.05\,\mathrm{MPc}^{-1}$ as the pivot wave number, so the corresponding comoving wave number is

$$n_* = a_0 k_* = 7070,$$

where $a_0 = \frac{1}{H_0\sqrt{-\Omega_K}}$ according to the latest observational data [35] is $a_0 = 1.414\times 10^5$ MPc. Now by considering $n_* = 7070$ as the pivot comoving wave number and $\epsilon = 0.008$, one can show $r_* < 0.12$ provided that $0.0147 \lesssim \delta < 0.018$ (which is in the range permitted by the Planck data[1] [11]), so it may reduce the discrepancy between BICEP2/Keck Array team and Planck collaboration results and slow-rolling inflationary theory.

[1] In the Planck collaboration paper, the slow-roll parameters are ϵ_V and η_V, which are to be compared to the definitions, given in this paper, i.e. $\epsilon_V = \epsilon$ and $\eta_V = \epsilon - \delta$.

5 Conclusion and summary

In this article we investigated an inflationary model with positive curvature index and calculated scalar and tensor perturbations power spectra associated with it. For the severe super-Hubble scales (i.e. $n \gg 1$) it seems that both spectra are completely similar to the spatially flat corresponding case. It is shown that this model yields a natural resolution of the running number problem. We also calculated the tensor–scalar ratio to show that it directly depends on the wave number of the perturbative modes. In addition, we showed that the tensor–scalar ratio in the positively curved universe against the flat case explicitly depends on the second slow-roll parameter.

Funded by SCOAP^3.

References

1. A.H. Guth, Phys. Rev. D **23**, 347 (1981)
2. A.D. Linde, Phys. Lett. B **108**, 389 (1982)
3. A. Albrecht, P.J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982)
4. A.H. Guth, S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982)
5. J.M. Bardeen, P.J. Steinhardt, M.H. Turner, Phys. Rev. D **28**, 679 (1983)
6. P. Peter, J.P. Uzan, *Primordial Cosmology* (Oxford University Press, Oxford, 2009)
7. D.H. Lyth, A.R. Liddle, *The Primordial Density Perturbation: Cosmology, Inflation and the Origin of Structure* (Cambridge University Press, Cambridge, 2009)
8. E.R. Harrison, Phys. Rev. D **1**, 2726 (1970)
9. Y.B. Zel'dovich, Mon. Not. R. Astron. Soc. **160**, 1P (1972)
10. P.J.E. Peebles, J.T. Yu, Astrophys. J. **162**, 815 (1970)
11. P.A.R. Ade et al., Astron. Astrophys. **571**, A22 (2014)
12. G. Hinshaw et al., Astrophys. J. Suppl. Ser. **208**, 19 (2013)
13. S. Weinberg, *Cosmology* (Oxford University Press, Oxford, 2008)
14. P.J. Steinhardt, M.S. Turner, Phys. Rev. D **29**, 2162 (1984)
15. D.S. Salopek, J.R. Bond, Phys. Rev. D **42**, 3936 (1990)
16. D.S. Salopek, J.R. Bond, Phys. Rev. D **43**, 1005 (1991)
17. A.R. Liddle, P. Parsons, J.D. Barrow, Phys. Rev. D **50**, 7222 (1994)
18. A.R. Liddle, D.H. Lyth, Phys. Lett. B **291**, 391 (1992)
19. K.A. Olive, Phys. Rep. **190**, 307 (1990)
20. E.D. Stewart, D.H. Lyth, Phys. Lett. B **302**, 171 (1993)
21. D.H. Lyth, Phys. Rev. D **31**, 1792 (1985)
22. J.M. Bardeen, Phys. Rev. D **22**, 1882 (1980)
23. A.A. Asgari, A.H. Abbassi, J. Cosmol. Astropart. Phys. **09**, 042 (2014)
24. P.A.R. Ade et al., Phys. Rev. Lett. **112**, 241101 (2014)
25. P.A.R. Ade et al., Phys. Rev. Lett. **114**, 101301 (2015)
26. M. Tegmark, A. de Oliveira-Costa, A.J.S. Hamilton, Phys. Rev. D **68**, 123523 (2003)
27. A. Kogut et al., Astrophys. J. Suppl. Ser. **148**, 161 (2003)

28. J.P. Uzan, U. Kirchner, G.F.R. Ellis, Mon. Not. R. Astron. Soc. **344**, L65 (2003)
29. J.P. Luminet, J.R. Weeks, A. Riazuelo, R. Lehoucq, J.P. Uzan, Nature **425**, 593 (2005)
30. R. Aurich, S. Lustig, F. Steiner, Class. Quantum Gravity **22**, 2061 (2005)
31. R. Aurich, S. Lustig, F. Steiner. Class. Quantum Gravity **22**, 3443 (2005)
32. R. Aurich, S. Lustig, Mon. Not. R. Astron. Soc. **424**, 1556 (2012)
33. R. Aurich, S. Lustig, Class. Quantum Gravity **29**, 235028 (2012)
34. S.W. Hawking, G.F.R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1973)
35. P.A.R. Ade et al., arXiv:1303.5086
36. G.F.R. Ellis, W. Stoeger, P. McEwan, P. Dunsby, Gen. Relativ. Gravit. **34**, 1445 (2002)
37. G.F.R. Ellis, P. McEwan, W. Stoeger, P. Dunsby, Gen. Relativ. Gravit. **34**, 1461 (2002)
38. A. Vilenkin, Phys. Rev. D **27**, 2848 (1983)
39. A. Vilenkin, Phys. Lett. B **117**, 25 (1982)
40. A. Vilenkin, Nucl. Phys. B **252**, 141 (1985)
41. A. Linde, J. Cosmol. Astropart. Phys. **05**, 002 (2003)
42. G.F.R. Ellis, R. Maartens, Class. Quantum Gravity **21**, 223 (2004)
43. G.F.R. Ellis, J. Murugan, C.G. Tsagas, Class. Quantum Gravity **21**, 233 (2004)
44. A. Lasenby, C. Doran, Phys. Rev. D **71**, 063502 (2005)
45. A.A. Starobinsky, arXiv:astro-ph/9603075
46. E. Massó, S. Mohanty, A. Nautiyal, G. Zsembinszki, Phys. Rev. D **78**, 043534 (2008)
47. D.H. Lyth, E.D. Stewart, Phys. Lett. B **252**, 336 (1990)
48. B. Ratra, P.J.E. Peebles, Phys. Rev. D **52**, 1837 (1995)
49. V. Mukhanov, JETP Lett. **41**, 493 (1985)
50. M. Sasaki, Prog. Theor. Phys. **76**, 1036 (1986)
51. A.A. Asgari, A.H. Abbassi, J. Khodagholizadeh, Eur. Phys. J. C **74**, 2917 (2014)
52. E.M. Lifshitz, J. Phys. (USSR) **10**, 116 (1946)
53. E.M. Lifshitz, I.M. Khalatnikov, Adv. Phys. **12**, 185 (1963)
54. T.S. Bunch, P.C.W. Davies, Proc. R. Soc. A **360**, 117 (1978)
55. J.E. Lidsey et al., Rev. Mod. Phys. **69**, 373 (1997)
56. M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables* (Courier Dover Publications, Mineola, 2012)
57. N.O. Virchenko, I. Fedotova, *Generalized Associated Legendre Functions and Their Applications* (World Scientific, Singapore, 2001)