

Anomalous Lense–Thirring precession in Kerr–Taub–NUT spacetimes

Chandrachur Chakraborty^a

Saha Institute of Nuclear Physics, Kolkata 700064, India

Received: 24 June 2015 / Accepted: 6 November 2015 / Published online: 1 December 2015
© The Author(s) 2015. This article is published with open access at Springerlink.com

Abstract Exact Lense–Thirring (LT) precession in Kerr–Taub–NUT spacetime is reviewed. It is shown that the LT precession does not obey the general inverse cube law of distance at strong gravity regime in Kerr–Taub–NUT spacetime. Rather, it becomes maximum just near the horizon, falls sharply and becomes zero near the horizon. The precession rate increases again and after that it falls obeying the general inverse cube law of distance. This anomaly is maximum at the polar region of this spacetime and it vanishes after crossing a certain ‘critical’ angle towards the equator from the pole. We highlight that this particular ‘anomaly’ also arises in the LT effect at the interior spacetime of the pulsars and such a signature could be used to identify a role of Taub–NUT solutions in the astrophysical observations or equivalently, a signature of the existence of a NUT charge in the pulsars. In addition, we show that if the Kerr–Taub–NUT spacetime rotates with the angular momentum $J = Mn$ (mass \times dual mass), the inner horizon goes to $r = 0$ and only the *event horizon* exists at the distance $r = 2M$.

1 Introduction

The Lense–Thirring (LT) precession [1] is an important phenomenon in General relativity as well as in relativistic astrophysics. In this phenomenon the locally inertial frames are dragged along the rotating spacetime due to the angular momentum of the stationary spacetime. The LT precession rate is proportional to the curvature as well as the angular velocity of the rotating spacetime. Thus, the effect will be larger for a massive and rapidly rotating spacetime around which the curvature effect is maximum. Perhaps Majumdar and myself [2] were first to argue for an investigation of the LT precession in a strong gravity situation. Without making any

preliminary assumption we have derived the exact LT precession rate of a gyroscope in the Kerr and Kerr–Taub–NUT (KTN) spacetimes. Now, the frame-dragging rate of a gyroscope can easily be obtained [3] just outside a Kerr spacetime [4] as well as a KTN spacetime [5,6]. Kerr and KTN spacetimes both are the vacuum solutions of Einstein equation. Kerr spacetime has two parameters: mass and Kerr parameter (angular momentum per unit mass) but there are three parameters to describe the KTN spacetime. The parameters are: the mass, the Kerr parameter and the NUT parameter. If the NUT parameter vanishes the KTN spacetime reduces to the Kerr spacetime and if the Kerr parameter vanishes the KTN spacetime reduces to the Taub–NUT spacetime. In the absence of the NUT parameter, the Taub–NUT spacetime reduces to pure Schwarzschild spacetime which is non-rotating. The Kerr spacetime is very well known to us and it is also physically reliable. We can describe the exterior geometry of many rotating astrophysical objects by the Kerr spacetime only in the approximation when the multipole momenta of the rotating matter are negligible. Otherwise, the metric receives corrections from higher gravitational multipole moments [7]. Thus, in the general sense the Kerr spacetime is astrophysically relevant but the KTN spacetime is quite different from the Kerr geometry. As it holds an additional parameter (NUT), this spacetime was not physically relevant till now.

Lynden-Bell and Nouri-Zonoz [8] were the first to initiate investigation of the observational possibilities for NUT charges or (gravito)magnetic monopoles. They have claimed that the signatures of such spacetime might be found in the spectra of supernovae, quasars, or active galactic nuclei. It has also been recently brought into focus by Kagramanova et al. [9] by a detailed and careful analysis of geodesics in the Taub–NUT spacetime. A rigorous analysis in extremal and non-extremal KTN spacetimes for timelike and spacelike geodesics has already been done by myself [10]. It should be noted that the (gravito)magnetic monopole spacetime with angular momentum (basically the KTN spacetime) admits relativistic thin accretion disks of a black hole in a galaxy or

The author is presently affiliated to Tata Institute of Fundamental Research, Mumbai 400005, India.

^ae-mail: chandrachur.chakraborty@saha.ac.in

quasars [11]. The accretion disks are basically formed just near the above mentioned astrophysical objects. In this sense the accretion phenomenon takes place in a very strong gravity regime where the frame-dragging effect is expected to be very high. Thus the frame-dragging effect should have greater impact on accretion disk phenomena. This provides us a strong motivation for studying the LT precession or frame-dragging effect in the KTN spacetime in more detail because it will affect the accretion in such spacetimes from massive stars and might offer novel observational prospects.

The KTN spacetime is a stationary and axisymmetric vacuum solution of Einstein equation. This spacetime consists of the Kerr and NUT parameters. The Kerr parameter is responsible for the rotation of the spacetime. In general sense the NUT charge should not be responsible explicitly for the rotation of the spacetime but implicitly this NUT charge can add a “rotational sense” in a non-rotating spacetime. The NUT charge is also called the ‘dual mass’ whose properties have been investigated in detail by Ramaswamy and Sen [12]. They also called the NUT parameter the “angular momentum monopole” [13], which is quite sound in the sense that it can give a “rotational sense” of the Taub–NUT spacetime even when the Kerr parameter vanishes. In this regard, though the Kerr parameter vanishes in the KTN spacetime, the Taub–NUT spacetime retains the rotational sense due to the NUT parameter. Due to the presence of the NUT parameter the spacetime still remains stationary and violates the time reflection symmetry. Time reflection changes the direction of rotation and thus does not restore one to the original configuration [14]. Thus, the failure of the hypersurface orthogonality (it also means that the spacetime preserves the time translation symmetry but violates the time reflection symmetry) condition implies that the neighbouring orbits of ξ^a (the timelike Killing vector which must exist in any stationary spacetime) “twist” around each other. In the Kerr spacetime, the presence of the Kerr parameter makes the spacetime stationary instead of static. Similarly, in the case of the Taub–NUT spacetime the NUT parameter compels the spacetime stationary instead of static. Thus, the Kerr and NUT parameters both are responsible to make the spacetime in rotation. Thus, it is needless to say that the KTN spacetime must be stationary.

The strong gravity LT precession in KTN spacetime has already been highlighted in [2] by Majumdar and myself but we could not studied it in detail. Though our target was to investigate LT precession in Kerr and KTN spacetimes, it had taken a turn into the investigation of LT precession in Taub–NUT spacetime which was really very interesting in that situation. We were busy to shown that the LT precession could not vanish even in *non-rotating* (as Kerr parameter vanishes) Taub–NUT spacetime. Later, Chakraborty et al. [15] have discovered that frame-dragging curves are not smooth along the equator and its surroundings inside a rotating neutron star. Rather, the frame-dragging effect shows an interest-

ing *anomaly* along the equator inside the pulsars. The frame-dragging rate is maximum at the centre and decreases initially away from the centre, tends to zero (not exactly zero but very small) before the surface of the neutron star, rises again and finally approaches small value on the surface as well as outside of the pulsars. We think that this may not be the only case where we see this *anomaly*. After that we started to hunt for this type of feature in other spacetimes which are the vacuum solutions of Einstein equation and we get an almost similar *anomaly* in the KTN spacetime (we note that there are many differences between the KTN spacetime and the spacetime of a rotating neutron star; they are not the same). Previously, the strong gravity LT precession in Plebański–Demiański (PD) spacetimes (most general axisymmetric and stationary spacetime till now) has been investigated by Pradhan and myself [16]. But close observation shows that due to the presence of the NUT charge this anomaly in the frame-dragging can also arise in the PD spacetime. In the present paper, we are now investigating only LT precession in KTN spacetime as it may be astrophysically sound in the near future.

The paper is organised accordingly as follows: in Sect. 2, we review the LT precession in KTN spacetime. We also discuss a very special case of the KTN spacetime in a subsection of Sect. 2. We discuss our result in Sect. 3 and, finally, we conclude in Sect. 4.

2 Lense–Thirring precession in Kerr–Taub–NUT spacetime

The KTN spacetime is a geometrically stationary and axisymmetric vacuum solution of Einstein equation. This spacetime is mainly determined by three parameters: the mass (M), the angular momentum (J) per unit mass or Kerr parameter ($a = J/M$) and the NUT charge (n) or dual mass. The metric of the KTN spacetime can be written as [17]

$$ds^2 = -\frac{\Delta}{p^2}(dt - Ad\phi)^2 + \frac{p^2}{\Delta}dr^2 + p^2d\theta^2 + \frac{1}{p^2}\sin^2\theta(adt - Bd\phi)^2 \tag{1}$$

with

$$\Delta = r^2 - 2Mr + a^2 - n^2, \quad p^2 = r^2 + (n + a \cos\theta)^2, \\ A = a \sin^2\theta - 2n \cos\theta, \quad B = r^2 + a^2 + n^2. \tag{2}$$

The exact LT precession rate in the KTN spacetime is (Eq. (20) of [2])

$$\bar{\Omega}_{LT} = \frac{\sqrt{\Delta}}{p} \left[\frac{a \cos\theta}{\rho^2 - 2Mr - n^2} - \frac{a \cos\theta + n}{p^2} \right] \hat{r} + \frac{a \sin\theta}{p} \left[\frac{r - M}{\rho^2 - 2Mr - n^2} - \frac{r}{p^2} \right] \hat{\theta} \tag{3}$$

where $\rho^2 = r^2 + a^2 \cos^2 \theta$. The modulus of the above LT precession rate is

$$\Omega_{LT} = |\vec{\Omega}_{LT}| = \frac{1}{p} \left[\Delta \left(\frac{a \cos \theta}{\rho^2 - 2Mr - n^2} - \frac{a \cos \theta + n}{p^2} \right)^2 + a^2 \sin^2 \theta \left(\frac{r - M}{\rho^2 - 2Mr - n^2} - \frac{r}{p^2} \right)^2 \right]^{\frac{1}{2}} \quad (4)$$

It can easily be seen that the above equation is valid only in a timelike region, we mean, outside the ergosphere which is located at $r_{\text{ergo}} = M + \sqrt{M^2 + n^2 - a^2 \cos^2 \theta}$.

We plot r vs. Ω_{LT} for $a < n$ (Fig. 1) and $a > n$ (Fig. 2); we see that the LT precession rate curve is smooth along the equator [panel (b)] but it is not smooth along the pole [panel (a)]. The LT precession rate along the pole is very high just outside the ergosphere and falls sharply and becomes zero, rises again and finally approaches a small value after crossing the very strong gravity regime. We will now discuss an interesting situation in which the Kerr parameter a is equal to the NUT parameter n .

Special case $a = n$: The horizons of the KTN spacetime are located at $r_{\pm} = M \pm \sqrt{M^2 + n^2 - a^2}$ [10]. One horizon is located at $r_+ > 0$ and another is located at $r_- < 0$ (if $n > a$) [9]. The Kerr parameter a takes any value but is less than or equal to $\sqrt{M^2 + n^2}$ in the case of the KTN spacetime whereas a takes its highest value as M in the case of the Kerr spacetime. Without this restriction (if $a^2 > M^2 + n^2$) the both spacetimes lead to show the naked singularities. There are two special cases in KTN spacetimes for which a can take the value M only and for the second case a can take the value n . For the first case the angular momentum of the KTN spacetime would be $J = M^2$ which is similar to the case of extremal Kerr spacetime. In this case the horizons will be located at the distances $r_+ = M + n$ and $r_- = M - n$. If the mass of the spacetime is greater than the dual mass of the spacetime ($M > n$), the two horizons could be located at positive distances ($r_{\pm} > 0$) but if the dual mass is greater than the mass of the spacetime ($M < n$) r_- will be located at a negative distance ($r_- < 0$).

For the second case ($a = n$) the angular momentum of the KTN spacetime would be $J = Mn$. It is a very interesting situation. In this case the line element of the KTN spacetime would be

$$ds_n^2 = -\frac{\Delta_n}{p_n^2} (dt - A_n d\phi)^2 + \frac{p_n^2}{\Delta_n} dr^2 + p_n^2 d\theta^2 + \frac{1}{p_n^2} \sin^2 \theta (ndt - B_n d\phi)^2 \quad (5)$$

with

$$\Delta_n = r(r - 2M), \quad p_n^2 = r^2 + n^2(1 + \cos \theta)^2, \\ A_n = n(\sin^2 \theta - 2 \cos \theta), \quad B_n = r^2 + 2n^2. \quad (6)$$

It can easily be seen that this special rotating spacetime has an outer horizon at the distance $r_+ = 2M$ and an inner horizon at $r_- = 0$. The outer horizon at the distance $r_+ = 2M$ is just similar to the Schwarzschild spacetime where the event horizon is located at $r = 2M$. This spacetime can be treated as the rotating spacetime with the *event horizon* at $r = 2M$ and its angular momentum will be

$$J = Mn. \quad (7)$$

In other words it could be said that *the KTN spacetime rotating with the angular momentum $J = Mn$, possessed an outer horizon at $r = 2M$ and an inner horizon at $r = 0$* . There is an apparent similarity between Eq. (40) of Ref. [18] with our results but it is completely different situation. Furthermore, there should be an ergoregion in this special KTN spacetime. For this special case ($a = n$), the radius of the ergosphere for the KTN spacetime will be $M + \sqrt{M^2 + n^2 \sin^2 \theta}$. The LT precession rate in this special spacetime will be

$$\Omega_{LT|a=n} = \frac{n}{p_n} \left[\Delta_n \left(\frac{\cos \theta}{r^2 - 2Mr - n^2 \sin^2 \theta} - \frac{1 + \cos \theta}{p_n^2} \right)^2 + \sin^2 \theta \left(\frac{r - M}{r^2 - 2Mr - n^2 \sin^2 \theta} - \frac{r}{p_n^2} \right)^2 \right]^{\frac{1}{2}} \quad (8)$$

The above expression is valid outside the ergosphere as it diverges on the ergosphere and we also know that the LT precession is not defined in the spacelike surface. In Fig. 3, we plot r vs Ω_{LT} for $a = n = 1$. We see that the curve falls smoothly with increasing distance along the equator but it is not smooth along the pole. A similar behaviour is noticed in Figs. 1 and 2.

The LT precession rate along the pole is very high just outside the ergosphere and falls sharply and becomes zero, rises again and finally approaches a small value after crossing the strong gravity regime. This is really very peculiar and this was not observed in any other spacetimes which are the vacuum solutions of Einstein equation, previously.

3 Results

We know that the LT precession varies as $1/r^3$ in the weak gravity regime ('weak' Kerr metric) by the famous relation (Eq. 14.34 of Ref. [19])

$$\vec{\Omega}_{LT} = \frac{1}{r^3} [3(\vec{J} \cdot \hat{r})\hat{r} - \vec{J}], \quad (9)$$

where \hat{r} is the unit vector along r direction. We plot Ω_{LT} vs. r in the strong gravity situation (see Eq. (42) of Ref. [2])

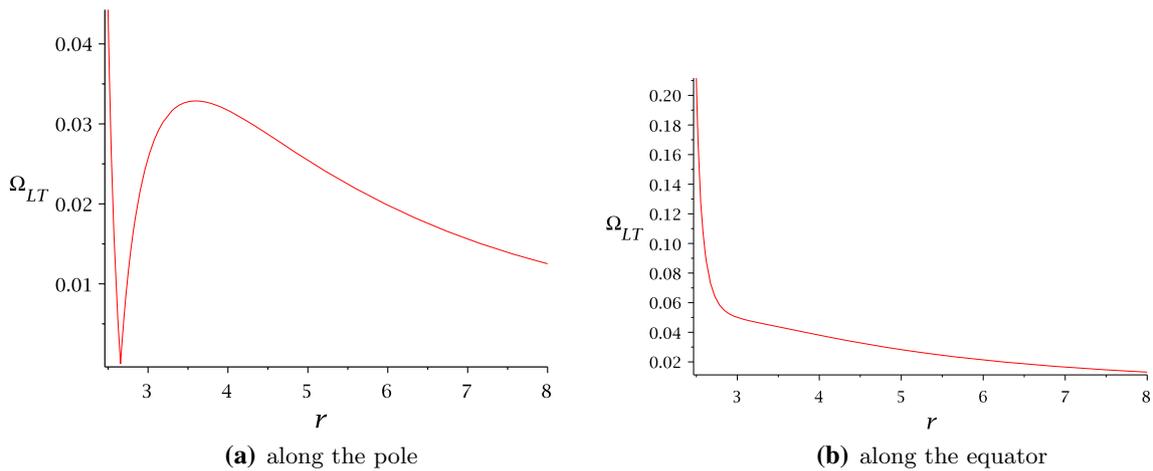


Fig. 1 Plot of Ω_{LT} vs. r in the KTN spacetime for $a = 0.1$ m, $n = 1$ m and $M = 1$ m

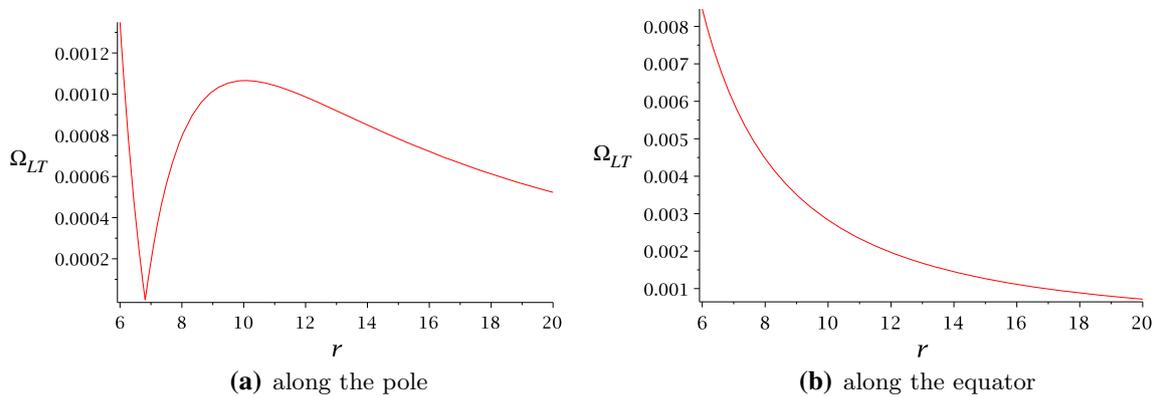


Fig. 2 Plot of Ω_{LT} vs. r in the KTN spacetime for $a = 0.7$ m, $n = 0.3$ m and $M = 1$ m

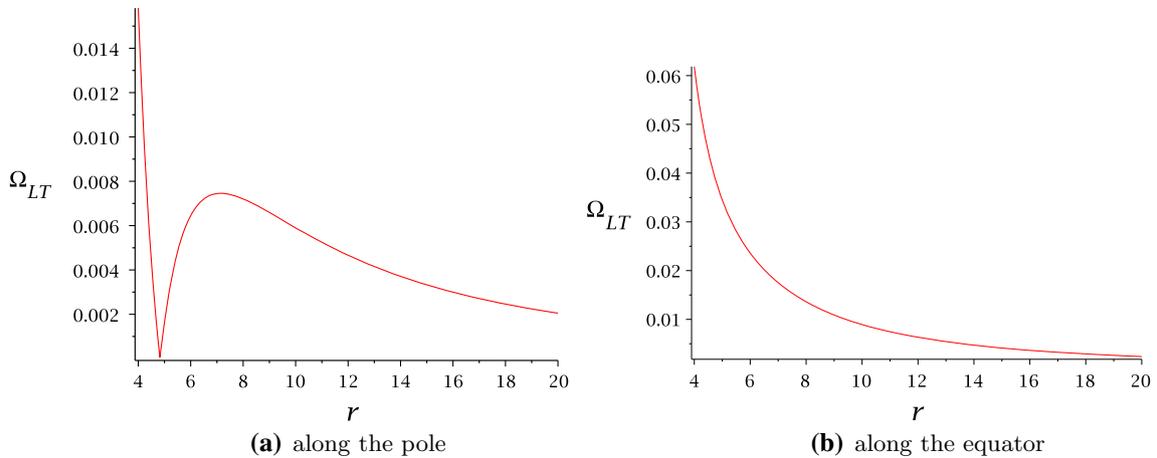


Fig. 3 Plot of Ω_{LT} (in m^{-1}) vs. r (in m) in the KTN spacetime for $a = n = 1$ m and $M = 1$ m

for maximally rotated Kerr spacetime along the pole [panel (a)] and the equator [panel (b)] in Fig. 4. Close observation reveals that the LT precession rates at the same distances (for a fixed r) along the equator and the pole are not the same. In the strong gravity regime Ω_{LT}^e is higher than Ω_{LT}^p as the

ratio (η) of the LT precession rate along the pole (Ω_{LT}^p) to the equator (Ω_{LT}^e) in the strong gravity regime is

$$\eta_K^{strong} = \frac{\Omega_{LT}^p}{\Omega_{LT}^e} = \frac{2r^3(r - 2M)}{(r^2 + a^2)^{\frac{3}{2}}(r^2 - 2Mr + a^2)^{\frac{1}{2}}}, \tag{10}$$

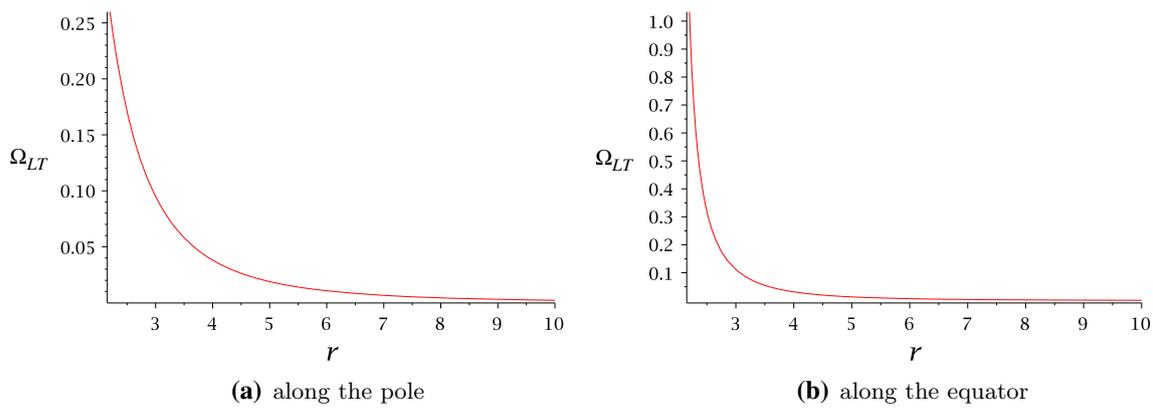


Fig. 4 Plot of strong gravity Ω_{LT} (in m^{-1}) vs. r (in m) in the Kerr spacetime for $a = M = 1$ m

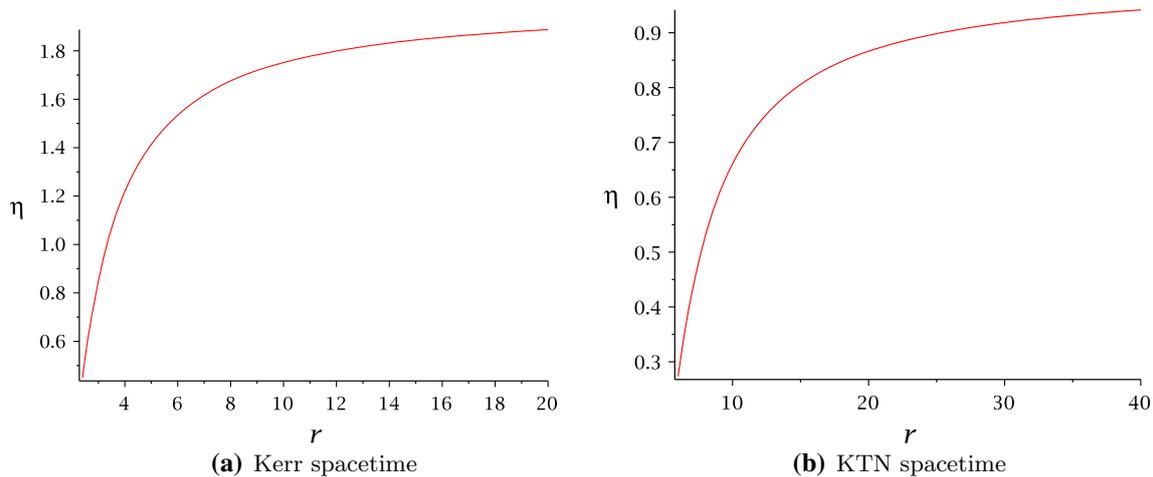


Fig. 5 Plot of η vs. r (in m) in the Kerr and KTN spacetimes for $a = n = 1$ m and $M = 1$ m

but in the weak gravity regime it follows from Eq. (9)

$$\eta_K^{weak} = \frac{\Omega_{LT}^p}{\Omega_{LT}^e} = \frac{2J}{r^3} = 2, \tag{11}$$

which is a constant. If we look for this ratio in the case of the KTN spacetime we find that

$$\eta_{KTN}^{strong} = \frac{\Omega_{LT}^p}{\Omega_{LT}^e} < 1. \tag{12}$$

It holds for ever, i.e. the LT precession rate along the equator (Ω_{LT}^e) is always higher than the LT precession rate along the pole (Ω_{LT}^p). In the weak gravity regime the ratio is only

$$\eta_{KTN}^{weak} = 1. \tag{13}$$

We can plot the ratio for the clear scenario.

The plot in Fig. 5 for the Kerr spacetime shows that Ω_{LT}^p and Ω_{LT}^e are the same at a distance $r_0 = 3.324$ m. For $r < r_0$, $\Omega_{LT}^p < \Omega_{LT}^e$ and for $r > r_0$, $\Omega_{LT}^p > \Omega_{LT}^e$.

We have already seen that the plots of Ω_{LT} vs. r along the pole and along the equator both are smooth for the Kerr spacetime but this is not the same for the KTN spacetime. In the KTN spacetime though the curve of Ω_{LT} vs. r along the equator is smooth, it is not smooth along the pole. We have studied here basically three cases. These are the following:

(i) $a = n$: In this case shown in Fig. 3, we take the Kerr parameter a is equal to the NUT parameter n ($a = n = 1$ m) and mass of the spacetime M is unity. Thus, the radius of the horizon is $r_+ \sim 2$ m. The LT precession rate along the pole [panel (a)] is tremendously high just outside the horizon. Then it falls sharply and becomes zero (local minimum) at $r_{min} \sim 4.8$ m. It rises again and gives a local maximum at $r_{max} \sim 7$ m. After that the curve of the LT precession rate follows the general inverse cube law and falls accordingly. We cannot see the same feature along the equator. We plot a 3-D picture of the LT precession rate in Fig. 6 where the Y axis represents the cosine of the colatitude ($\cos \theta$) and the X axis represents the distance (r) from the centre of the spacetime. The colours represent the value of the LT precession rate and the values of the same precession rates are also

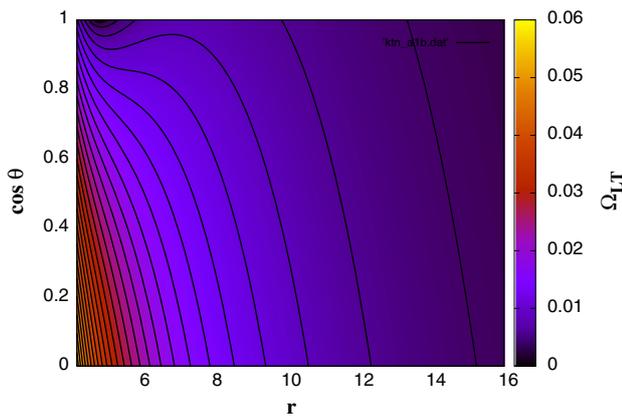


Fig. 6 3-D plot of $\Omega_{LT}(r, \theta)$ in the KTN spacetime for $a = n = 1$ m and $M = 1$ m

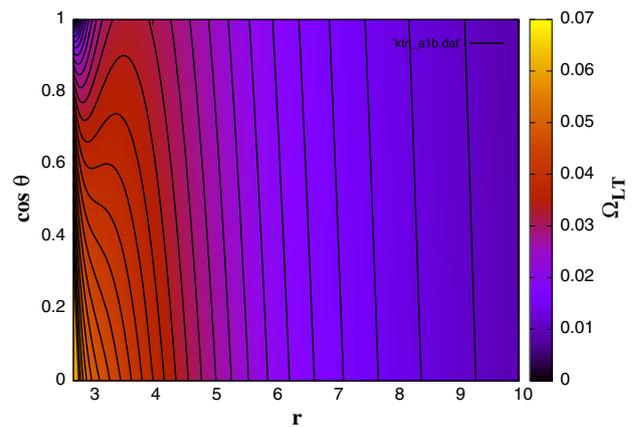


Fig. 8 3-D plot of $\Omega_{LT}(r, \theta)$ in the KTN spacetime for $a = 0.1$ m, $n = 1$ m and $M = 1$ m

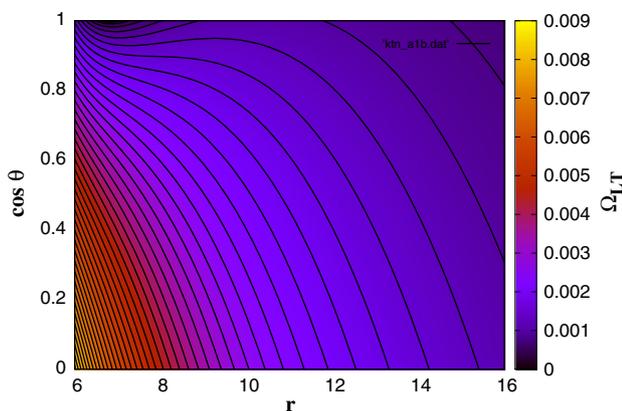


Fig. 7 3-D plot of $\Omega_{LT}(r, \theta)$ in the KTN spacetime for $a = 0.7$ m, $n = 0.3$ m and $M = 1$ m

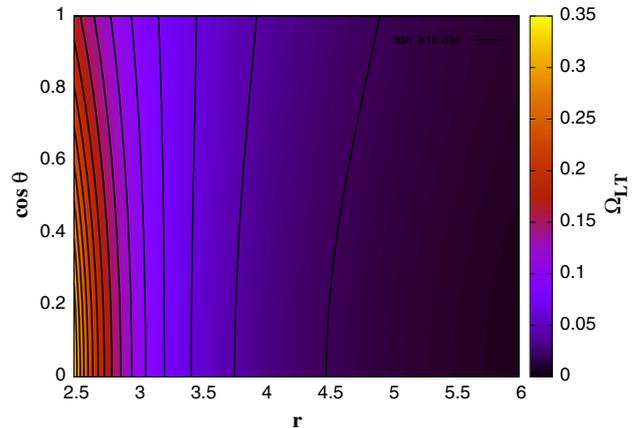


Fig. 9 3-D plot of $\Omega_{LT}(r, \theta)$ in the Kerr spacetime for $a = 1$ m and $M = 1$ m

separated by the isocurves. It shows that there is a local maximum and local minimum along the pole but it disappears after crossing a certain ‘critical’ angle. Here, it is around $\cos \theta \sim 0.6$.

(ii) $a > n$: In the second case shown in Fig. 7, the Kerr parameter $a = 0.7$ m and NUT parameter $n = 0.3$ m. Mass of the spacetime is $M = 1$ m. The radius of the horizon is $r_+ \sim 1.8$ m, the distance of local minimum is $r_{\min} \sim 7$ m and the distance of the local maximum is $r_{\max} \sim 10$ m. The ‘critical’ angle is around $\cos \theta \sim 0.8$.

(iii) $a < n$: For the third case exhibited in Fig. 8, the Kerr parameter $a = 0.1$ m and NUT parameter $n = 1$ m. Mass of the spacetime is $M = 1$ m. The radius of the horizon is $r_+ \sim 2.4$ m, the distance of the local minimum is $r_{\min} \sim 2.6$ m and the distance of the local maximum is $r_{\max} \sim 3.5$ m. The ‘critical’ angle is around $\cos \theta \sim 0.4$.

In all three cases, plots show the same feature but the numerical values are different depending on the values of a and n . For a fixed value of n , if a decreases the value of the LT precession rate at the local maximum increases and also the

distance of local minimum and maximum are shifted towards the horizon of the spacetime. If the NUT parameter vanishes (for the Kerr spacetime) there will be no local maximum and minimum as noticed in Figs. 4 and 9.

The local maximum along the pole in the KTN spacetime arises due to the NUT parameter and it is clear from Fig. 10 [panel (a)] that it is valid only for the Taub–NUT spacetime where the Kerr parameter vanishes but the NUT parameter does not vanish. The local minimum along the pole in KTN spacetime arises due to the Kerr parameter. It could not be seen directly from the Fig. 4 [panel (a)]. If we take Fig. 4 [panel (a)] of the Kerr spacetime and Fig. 10 [panel (a)] of the Taub–NUT spacetime and overlap these two figures with each other (as the KTN spacetime includes both the Kerr and the NUT parameters) just for our clear understanding, we can easily visualise the nature of the plots of the LT precession [panel (a) of Figs. 1, 2, 3] along the pole in the KTN spacetimes. Thus, the presence of the Kerr parameter is responsible for showing the local minimum along the pole in the KTN spacetime.

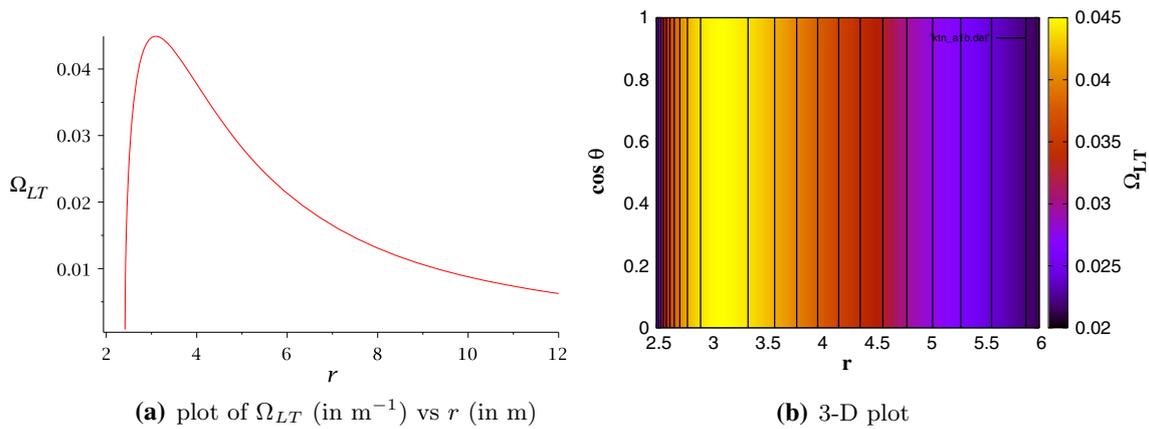


Fig. 10 Plot of Ω_{LT} in the Taub–NUT spacetime for $n = 1$ m and $M = 1$ m [basically, the expression of Ω_{LT} (see Eq. (25) of Ref. [2]) is independent of θ , thus the value (colour) of Ω_{LT} does not change with $\cos \theta$ in b]

Without the Kerr parameter the LT precession rate at a ‘local maximum’ in the Taub–NUT spacetime is higher than the LT precession rate at a ‘local maximum’ in the KTN spacetime. The presence of the Kerr parameter (or increasing the value of the Kerr parameter from 0 to a finite number) shifts the ‘local maximum’ and ‘local minimum’ away from the horizon and reduces the LT precession rate at the local maximum.

We note that the Taub–NUT spacetime is a stationary and spherically symmetric spacetime and the expression of Ω_{LT} (see Eq. (25) of Ref. [2]) is also independent of θ (the LT precession in the Taub–NUT spacetime has been discussed in detail in Sect. 3.2 of Ref. [2]). Thus the value (colour) of Ω_{LT} does not change with $\cos \theta$. It means that the LT precession rate is the same everywhere in that spacetime for a fixed distance r (no matter whether it is pole or equator) and the LT precession rate curve always shows a ‘peak’ as seen in panel (a) of Fig. 10 near the horizon. But if this Taub–NUT spacetime starts to rotate with an angular momentum $J (= aM, a$ is the Kerr parameter), it turns out to be the KTN spacetime. In this case, the LT precession rate curve shows a ‘peak’ (or ‘local maximum’) along the pole but disappears after crossing the ‘critical’ angle and we cannot see any ‘peak’ in the LT precession rate curve along the equator as discussed earlier. The ‘intrinsic’ angular momentum of the spacetime (J) is fully responsible for the no-show of ‘local maximum’ along the equator. The Kerr parameter is also responsible for reducing the LT precession rate at the ‘local maximum’ which has already been discussed in the previous paragraph. Thus, the ‘dual mass’ or the ‘angular momentum monopole’ n is only responsible for the ‘anomaly’ (appearance of local maximum and local minimum in the LT precession rate) and the Kerr parameter or the rotation of the spacetime tries to reduce this ‘anomaly’ as far as possible. The Kerr parameter is fully successful to reduce this effect along the equator but slowly it loses its power of reduction of this anomaly along the pole.

3.1 Appearance of the local maximum and local minimum: a comparison study of some well-known spacetimes

If we take the derivative of Eq. (4) with respect to r and plot $\frac{d\Omega_{LT}}{dr}|_{(r=R, \theta=\pi/2)}$ vs r we cannot find any positive real root in the region $r_{ergo} < r < \infty$. But the plot of $\frac{d\Omega_{LT}}{dr}|_{(r=R, \theta=0)}$ vs. r shows two positive real roots (which are basically a local maximum $R_1 = r_{max}$ and a local minimum $R_2 = r_{min}$) in the region $r_{ergo} < r < \infty$. It has a similarity with the case of the frame-dragging effect inside the rotating neutron stars (see the appendix of [15]).

It is very important to mention here that the LT precession rate (Eq. 4) reduces to

$$\Omega_{LT}|_{\theta=0} = \frac{-nr^2 + 2Mr(n+a) + n(n+a)^2}{(r^2 - 2Mr + a^2 - n^2)^{\frac{1}{2}}(r^2 + (n+a)^2)^{\frac{3}{2}}} \tag{14}$$

along the pole and it vanishes at

$$R_2 = \left(1 + \frac{a}{n}\right) \left(M + \sqrt{M^2 + n^2}\right) \tag{15}$$

or

$$\frac{R_2}{r_{+TN}} = \left(1 + \frac{a}{n}\right). \tag{16}$$

We note that the ‘event horizon’ of the Taub–NUT spacetime is located at $r_{+TN} = M + \sqrt{M^2 + n^2}$. It means when Kerr parameter is zero, the R_2 goes to on the horizon in the case of Taub–NUT spacetime. Thus, the local minimum of the Taub–NUT spacetime and the ‘horizon’ coincide at the same point [see panel (a) of Fig. 10]. As the value of a increases from 0 to a finite number, the position of local minimum shifts in outward direction. Thus, the rotation of the spacetime is responsible for shifting the local maximum

and local minimum. We can check whether the r_{\min} and r_{\max} are always outside the horizon (r_{+KTN}) or not. We can take the difference of r_{\min} and r_{+KTN} in KTN spacetime:

$$r_{\min} - r_{+KTN} = a/n(M + \sqrt{M^2 + n^2}) + \sqrt{M^2 + n^2} \cdot \left(1 - \sqrt{1 - a^2/(M^2 + n^2)}\right). \tag{17}$$

As $a^2 \geq (M^2 + n^2)$, the above relation reveals that r_{\min} is greater than r_{+KTN} which means r_{\min} always lies outside the horizon. This not only holds along the pole but also for all angles. For $\theta > 0$, r_{\min} lies outside the ergoregion. As it is difficult to calculate the position of the ‘local minimum’ analytically for all values of θ , we have plotted these for $0 \leq \theta \leq \pi/2$ and obtained the values numerically for a few cases, which has been described in the Results section. For the extremal KTN spacetime,

$$r_{\min} - r_{+KTN} = \sqrt{M^2 + n^2} \left[1 + \frac{M}{n} + \sqrt{1 + \frac{M^2}{n^2}}\right]. \tag{18}$$

Kerr spacetime does not show up this type of anomaly (Fig. 4). If we put $n = 0$ in Eq. (15) for Kerr spacetime, R_2 does not make any sense. Is this same ‘anomaly’ also appeared in the Kerr–Newman spacetime? To get the answer, we can write the exact LT precession rate in Kerr–Newman spacetime [16]:

$$\tilde{\Omega}_{LT}^{KN} = \frac{a}{\rho^3(\rho^2 - 2Mr + Q^2)} \times [\sqrt{\Delta}(2Mr - Q^2) \cos \theta \hat{r} + (M(2r^2 - \rho^2) + rQ^2) \sin \theta \hat{\theta}]. \tag{19}$$

In the Kerr–Newman spacetime,

$$\Delta = r^2 - 2Mr + a^2 + Q^2 \quad \text{and} \quad \rho^2 = r^2 + a^2 \cos^2 \theta \tag{20}$$

where Q is the total charge of the spacetime. The LT precession rate at the pole in Kerr–Newman spacetime is

$$|\tilde{\Omega}_{LT}^{KN}|_{\theta=0} = \frac{a(2Mr - Q^2)}{(r^2 + a^2)^{\frac{3}{2}}(r^2 - 2Mr + a^2 + Q^2)^{\frac{1}{2}}}. \tag{21}$$

The LT precession rate could vanish at $r = Q^2/2M$ and we may think that this particular point leads to a local minimum in Kerr–Newman spacetime like in KTN spacetime. To check this, we have to do a short calculation. First of all, we take

$$r_{\min} = \frac{Q^2}{2M} \tag{22}$$

and the radius of the horizon:

$$r_{+KN} = M + \sqrt{M^2 - Q^2 - a^2}. \tag{23}$$

We know that

$$a^2 \leq M^2 - Q^2, \tag{24}$$

$$\frac{a^2}{2M} \leq \frac{M}{2} - \frac{Q^2}{2M}. \tag{25}$$

Thus,

$$r_{\min} \leq M/2 - a^2/2M, \tag{26}$$

$$r_{\min} - r_{+KN} \leq -1/2M[(M^2 + a^2) + 2M(M^2 - Q^2 - a^2)^{1/2}]. \tag{27}$$

Equation (27) reveals that r_{\min} is always inside the horizon (r_{+KN}) of Kerr–Newman spacetime. This means that r_{\min} is in spacelike region where our basic formalism (Eq. (17) of [2]) of LT precession is not valid (as we know that the formalism is valid only in timelike surfaces, it has also been stated in previous sections) and to obtain the LT precession rate in this region is meaningless. Thus, r_{\min} of Kerr–Newman spacetime does not make any sense and LT precession do not show any ‘anomaly’ in this spacetime. Now, for extremal Kerr–Newman spacetime Eq. (27) reduces to

$$r_{\min} - r_{+KN} \leq -1/2M(2M^2 - Q^2) \tag{28}$$

As $M > Q$, r_{\min} in extremal Kerr–Newman spacetime is also inside the horizon and does not make sense. Thus, we can safely say that the ‘anomalous’ LT precession is absent in the Kerr–Newman spacetime.

We can reiterate here that we derived the exact frame-dragging rate inside the rotating neutron stars without making any assumption on the metric components as well as the energy-momentum tensor [15]. We discussed our results for two types of pulsars: (i) pulsars which rotate with their Kepler frequencies, and (ii) which rotate with a frequency lower than their Kepler frequencies. In the second case, we calculated the LT precession frequencies for three real pulsars: J1807-2500B, J0737-3039A and B1257+12. In both cases, it was shown that the frame-dragging rate monotonically decreases from the centre to the surface of the neutron star along the pole. In the case of the frame-dragging rate along the equatorial distance, it decreased initially away from the centre, became negligibly small well before the surface of the neutron star, rose again and finally approached a small value at the surface [15]. The appearance of a local maximum and minimum in this case was the result of the dependence of the frame-dragging frequency on the distance and angle. Moving from the equator to the pole, it was observed that this local maximum and minimum in the frame-dragging rate along the equator disappeared after crossing a critical angle. It was noted that the positions of local maximum and minimum of the frame-dragging rate along the equator depend on the rotation frequency and the central energy density of a particular pulsar. We have also estimated the LT precession frequencies at the centres of these pulsars without imposing any boundary conditions on them. The whole prescription revealed that the LT precession rate in the strong gravity regime depends

not only on the distance r but also on the colatitude θ of the gyroscope.

After the above discussion we can say that the local maxima and local minima appear only in the KTN spacetimes and the interior spacetimes of the rotating neutron stars. There may exist an indistinct relation between these two spacetimes. We discuss it in the next section. All other spacetimes which we have checked do not show this type of anomaly and it could be said easily that this anomalous LT precession in the KTN spacetime is appearing only due to the NUT charge (n).

4 Summary and discussion

We have shown that the LT precession in KTN and Taub–NUT spacetimes are quite different than the LT precession in other spacetimes. Other vacuum solutions of Einstein equation do not show this type of strange feature in the LT precession or frame-dragging effect. It has been discussed that this strangeness in the KTN spacetime is due to only the presence of the NUT parameter or (gravito)magnetic monopoles. Remarkably, it (i.e., the frame-dragging effect in the KTN spacetime) has an apparent similarity with the frame-dragging effect inside a rotating neutron star. The exact frame-dragging effect inside the rotating neutron star has recently been derived and discussed in detail by Chakraborty et al. [15] but this is the interior solution of the Einstein equation, not the vacuum solution. In the case of the interior of a pulsar, the LT precession shows the same ‘anomaly’ as the KTN spacetime but there is also a basic difference: the anomaly appears in the LT precession rate in the KTN spacetime along the pole but it appears along the equator in the case of a pulsar. The basic features of the plots are the same for the two cases. We do not know if there is any connection in these two spacetimes. We have already stated that Lynden-Bell and Nouri-Zonoz [8] first highlighted the observational possibilities for NUT charges or (gravito)magnetic monopoles and they claimed that the signatures of such a spacetime might be found in the spectra of supernovae, quasars, or active galactic nuclei. In our case, the resemblance of the anomalous LT precessions in the KTN spacetimes and the spacetime of the

pulsars could be the indirect proof of the existence of the NUT charge [(gravito)magnetic monopoles] inside the pulsars and we also suggest that such a signature could be used to identify a role of Taub–NUT solutions in the astrophysical observations. We hope that we shall be able to give a direct mathematical proof of the existence of (gravito)magnetic monopoles inside the pulsars in a future publication [20].

Acknowledgments Thanks are due to K. P. Modak and G. Das for valuable suggestions. I am grateful to the Department of Atomic Energy (DAE, Govt. of India) for the financial assistance.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made. Funded by SCOAP³.

References

1. J. Lense, H. Thirring, *Phys. Z.* **19**, 156–163 (1918)
2. C. Chakraborty, P. Majumdar, *Class. Quantum Gravity* **31**, 075006 (2014)
3. L.I. Schiff, *Am. J. Phys.* **28**, 340 (1960)
4. R.P. Kerr, *Phys. Rev. Lett.* **11**, 237–238 (1963)
5. A.H. Taub, *Ann. Math.* **53**, 3 (1951)
6. E. Newman, L. Tamburino, T. Unti, *J. Math. Phys.* **4**, 7 (1963)
7. N. Stergioulas, *Living Rev. Relativ.* **6**, 3 (2003)
8. D. Lynden-Bell, M. Nouri-Zonoz, *Rev. Mod. Phys.* **70**, 427–445 (1998)
9. V. Kagramanova, J. Kunz, E. Hackmann, C. Lämmerzahl, *Phys. Rev. D* **81**, 124044 (2010)
10. C. Chakraborty, *Eur. Phys. J. C* **74**, 2759 (2014)
11. C. Liu, S. Chen, C. Ding, J. Jing, *Phys. Lett. B* **701**, 285 (2011)
12. S. Ramaswamy, A. Sen, *Phys. Rev. Lett.* **57**, 8 (1986)
13. S. Ramaswamy, A. Sen, *J. Math. Phys. (N. Y.)* **22**, 2612 (1981)
14. R.M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984)
15. C. Chakraborty, K.P. Modak, D. Bandyopadhyay, *ApJ* **790**, 2 (2014)
16. C. Chakraborty, P. Pradhan, *Eur. Phys. J. C* **73**, 2536 (2013)
17. J.G. Miller, *J. Math. Phys.* **14**, 486 (1973)
18. C. Bunster, M. Henneaux, *PNAS* **104**, 12243 (2007)
19. J.B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Pearson, Delhi, 2009)
20. C. Chakraborty, In preparation