

Effective gravitational field equations on m -brane embedded in n -dimensional bulk of Einstein and $f(\mathcal{R})$ gravity

Sumanta Chakraborty^{1,a} , Soumitra SenGupta^{2,b}

¹ IUCAA, Post Bag 4, Ganeshkhind, Pune University Campus, Pune 411 007, India

² Department of Theoretical Physics, Indian Association for the Cultivation of Science, Kolkata 700032, India

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Abstract We have derived effective gravitational field equations on a lower dimensional hypersurface (known as a brane), placed in a higher dimensional bulk spacetime for both Einstein and $f(\mathcal{R})$ gravity theories. We have started our analysis on the n -dimensional bulk from which the effective field equations on a $(n - 1)$ -dimensional brane have been obtained by imposing Z_2 symmetry. Subsequently, we have arrived at the effective equations in $(n - 2)$ dimensions starting from the effective equations for $(n - 1)$ -dimensional brane. This analysis has been carried out and is used to obtain the effective field equations for an $(n - m)$ -dimensional brane, embedded in a n -dimensional bulk. Having obtained the effective field equations in Einstein gravity, we have subsequently generalized the effective field equation for an $(n - m)$ -dimensional brane which is embedded in the n -dimensional bulk spacetime endowed with $f(\mathcal{R})$ gravity. We have also presented applications of our results in the context of Einstein and $f(\mathcal{R})$ gravity. In both cases we have discussed static spherically symmetric vacuum solutions as well as solutions in a cosmological context. Implications are also discussed.

1 Introduction

One of the fundamental problems in theoretical physics amounts to unifying all known interactions. A consistent unification of gravity with the other fundamental interactions has been an active area of research over many years. One of the leading contenders along this direction is superstring theory. In superstring theory the universe is assumed to be 11-dimensional of which seven are compactified leaving four non-compact dimensions, which we observe. Among various candidates, the 10-dimensional $E_8 \times E_8$ heterotic string the-

ory is a strong candidate as the theory is able to accommodate the standard model within it. Recently it has been shown that the 10-dimensional $E_8 \times E_8$ heterotic string theory is related to the 11-dimensional M theory (with an orbifold symmetry S^1/Z_2). This also confines the standard model particles to standard non-compact 4-dimensional spacetime while gravity can probe the higher dimensional spacetime [1–3].

A simplistic model which captures most of the key notions is a 5-dimensional model in which matter fields are confined to 4-dimensional spacetime while gravity *exists* in all the five spacetime dimensions. The first step in this direction with this simplified picture presented above was taken by Randall and Sundrum by introducing two 4-dimensional sub-manifolds (known as branes) in a 5-dimensional anti-de Sitter bulk [4]. This arrangement of two positive and negative tension branes separated by a finite distance (known as radion field [5–7]) can address the hierarchy problem by lowering the Higgs mass to the electroweak scale on the visible brane. Subsequently, they have introduced another model, which consists of a positive tension brane with an infinite extra dimension [8]. Due to the presence of an extra spacetime dimension it is expected that there would be deviation of various physical results from Einstein gravity, which would become more significant in the high energy limit. The brane world model subsequently was applied in the context of particle phenomenology [9–12], black hole physics [13–17], and cosmology [7, 18–21].

A natural extension of the Randall–Sundrum (RS) scenario to models with more than one warped extra dimension has been proposed, where several independent S^1/Z_2 orbifolds are considered along with the 4-dimensional non-compact manifold M_4 [22–26]. These multiply warped models are mainly motivated from various considerations, which include: radion stabilization, fermion mass hierarchy in the Standard Model, etc. There have been a significant number of follow-up works in these multiply warped models which involve: inflation, cosmic acceleration, matter field local-

^a e-mail: sumanta@iucaa.in; sumantac.physics@gmail.com

^b e-mail: tpssg@iacs.res.in

ization, Kaluza–Klein modes of graviton, gauge and scalar fields [27–32]. However, all these results depend on certain assumptions of bulk metric elements and the bulk Einstein equation under some simplified conditions, for example the presence of homogeneity and isotropy.

There exists an elegant way in which an effective field equation for gravity on a lower dimensional spacetime can be obtained. This involves the use of geometric quantities like the metric and the curvature induced on the brane from the bulk. Using these induced geometric quantities it is possible to arrive at a relation between the bulk curvature tensor and the brane curvature tensor, known as the Gauss–Codazzi equation [33,34]. From various contractions of the Gauss–Codazzi equation we can determine the effective field equation on the brane by relating the bulk Einstein tensor to that on the brane.

This approach was applied in order to obtain the gravitational field equation on the brane from a 5-dimensional bulk in [35]. The effective equation contains non-local terms inherited from the bulk. Subsequently starting from the effective field equation derived in [35] a class of vacuum solutions has been obtained in [16], which has been further generalized in [36].

It is strongly believed that general relativity is only a low energy approximation of some underlying fundamental high energy theory [37,38]. This suggests that the Einstein–Hilbert action in the high energy limit should be modified by the introduction of higher curvature terms. Such an alternative theory is the $f(\mathcal{R})$ theory, where \mathcal{R} is the bulk Ricci scalar. This theory has been studied extensively in the context of solar system tests, inflationary paradigm, and late time cosmic acceleration, along with the possibility of detecting gravitational waves [39–43]. Moreover, two-brane models, due to the presence of $f(\mathcal{R})$ term in the bulk, have also been a topic of discussion recently in collider physics [44–46] relating to the absence of graviton Kaluza–Klein Modes in LHC.

The above approach of obtaining an effective gravitational field equation is usually confined to Einstein gravity on the three-brane alone. This subsequently has been generalized to obtain an effective field equation on the three-brane when the bulk contains $f(\mathcal{R})$ gravity. This effective field equations has been used in [47] to obtain vacuum solution in $f(\mathcal{R})$ bulk.

In this work, we generalize the above setup by working in an n -dimensional bulk and then obtaining effective equation on an $(n - m)$ -dimensional brane where m can take any values. This seems natural from a string theory view point, where the universe is 11-dimensional, while our universe is a three-brane with four spacetime dimensions. Following this argument we have derived effective gravitational field equation on a $(n - m)$ -dimensional brane embedded in a n -dimensional bulk. We start with a n -dimensional bulk spacetime and then derive the effective field equation on a $(n - 1)$ -

dimensional brane, which can then be used as the starting point to derive the effective field equation on a $(n - 2)$ -dimensional brane. Having obtained the effective field equation on the $(n - 2)$ -dimensional brane one can easily recognize the pattern of obtaining the effective field equation in any arbitrary lower dimensional surfaces. Thus following the road from n -dimensional bulk to $(n - 2)$ -dimensional brane, we can obtain the effective field equation on an $(n - m)$ -dimensional brane as well. The above analysis can easily be generalized to the $f(\mathcal{R})$ model of gravity in the bulk following the analysis presented in [47].

The paper is organized as follows: in Sect. 2 we have elaborated the derivation of effective field equation for Einstein gravity. First, in Sect. 2.1 we present the derivation of effective field equation in $(n - 1)$ -dimensional brane, which is subsequently generalized in Sect. 2.2 to $(n - 2)$ -dimension. Following the pattern we obtain the effective gravitational field equation in $(n - m)$ -dimensional brane in Sect. 2.3. This result has been generalized for $f(\mathcal{R})$ model of gravity in Sect. 4. The effective equations so derived have been applied to obtain static spherically symmetric vacuum solutions and cosmological solutions along with their compatibility with standard cosmological evolutions. The results in the context of Einstein gravity have been discussed in Sect. 3 while those for $f(\mathcal{R})$ gravity is discussed in Sect. 5. Finally we conclude with a discussion of the results obtained.

Throughout the paper, we maintain the following convention: all Latin letters a, b, \dots run over all the n spacetime indices, Greek letters α, μ, \dots run over the $(n - 1)$ spacetime indices. Finally, capitalized Latin letters A, B, \dots run over the $(n - 2)$ spacetime indices.

2 Effective field equation in Einstein gravity: background and formulation

Hamiltonian formulation in general relativity comes up with the $(1+3)$ splitting of the spacetime, where a time direction is singled out. For this purpose we introduce a time coordinate $t = t(x^i)$ and foliate the spacetime with such $t = \text{constant}$ surfaces. Introducing a vector field $\mathbf{t} = \partial/\partial t$ we can move out of the surface by moving along the integral curves for \mathbf{t} . This involves a change in the t coordinate introducing the shift function N and change in the orientation captured by the lapse functions N^α . Then N, N^α yield four components of the Einstein tensor, while the remaining six components are being determined by induced metric on the surface $h_{\alpha\beta}$. The induced metric can be obtained from the bulk metric g_{ab} in two ways. First, we can introduce normalized normals to the surface given by $n_a = -N\nabla_a t$ and then define the induced metric as $h^{ab} = g^{ab} - n^a n^b$ (this formulation does not work for null surfaces, for which a complete treatment has been provided in [48]). Second, we can introduce coor-

dinates y^α on the $t = \text{constant}$ hypersurface on which we can introduce surface tetrads $e^\alpha_a = \partial x^a / \partial y^\alpha$ and obtain the induced metric: $h_{\alpha\beta} = g_{ab} e^\alpha_a e^\beta_b$. With the induced metric we can define covariant derivative on the surface and then using vectors on the $t = \text{constant}$ surface we can introduce curvature tensor on the spatial hypersurface. The relation between the 4-dimensional and the 3-dimensional curvature tensor is obtained through the Gauss–Codazzi equation. This can be manipulated in various ways in order to arrive at the effective Einstein equation on the $t = \text{constant}$ hypersurface. This procedure can be extended easily for timelike surfaces (i.e. spacelike normals) as well.

2.1 Effective field equation on $(n - 1)$ -dimensional brane

Having described the general setup let us now consider the effective field equation on a brane embedded in a higher dimensional bulk, which is the main motivation of this work. The procedure sketched above can then be applied for a generic situation where a single $(n - 1)$ -dimensional brane is embedded in a n -dimensional bulk spacetime with Einstein gravity. For this set up the effective gravitational field equation on the $(n - 1)$ -dimensional brane can be written as (see Appendix A.1):

$$\begin{aligned} &^{(n-1)}G_{\alpha\beta} \\ &= \kappa_n^2 \frac{n-3}{n-2} \left[T_{ab} e^\alpha_a e^\beta_b + h_{\alpha\beta} \left\{ \epsilon T_{ab} n^a n^b - \frac{1}{n-1} T \right\} \right] \\ &\quad - \epsilon E_{\alpha\beta} + \epsilon \left[K K_{\alpha\beta} - K_{\alpha\mu} K^\mu_\beta - \frac{1}{2} h_{\alpha\beta} (K^2 - K_{\mu\nu} K^{\mu\nu}) \right]. \end{aligned} \tag{1}$$

The above expression connects bulk energy-momentum tensor and the extrinsic curvature to the $(n - 1)$ -dimensional Einstein tensor. Given T_{ab} , along with surface contribution we can solve the above equation to derive $(n - 1)$ -dimensional solution.

The above effective equation is applicable to both spacelike and timelike surfaces. However, for our purpose, we will need only timelike surfaces, which are the branes. Thus the basic assumption to be followed in this section is that the n -dimensional spacetime can be projected on a $(n - 1)$ -dimensional brane with spacelike normal. The line element then takes the form

$$ds^2 = d\chi^2 + h_{\alpha\beta} dx^\alpha dx^\beta. \tag{2}$$

The brane is located at $\chi = 0$ on this n -dimensional spacetime. Using the standard assumption that normal matter can exist only on the brane, the bulk energy-momentum tensor can be written as

$$T_{ab} = -\Lambda_n g_{ab} + S_{\alpha\beta} e^\alpha_a e^\beta_b \delta(\chi) \tag{3}$$

where $S_{\alpha\beta}$ represents the brane energy-momentum tensor and Λ_n is the bulk cosmological constant. As the surface $\chi = 0$ contain surface energy-momentum tensor, the extrinsic curvature is not continuous. The discontinuity is related to the surface stress-energy tensor through the following relation:

$$S_{\alpha\beta} = \frac{1}{\kappa_n^2} ([K_{\alpha\beta}] - h_{\alpha\beta} [K]) \tag{4}$$

where $[K_{\alpha\beta}]$ and $[K]$ denote the jump in $K_{\alpha\beta}$ and K across the $\chi = 0$ surface. Thus we obtain the jump in the extrinsic curvature which, after imposing Z_2 symmetry, obeys the expression

$$K_{\alpha\beta}^+ = -K_{\alpha\beta}^- = \frac{1}{2} \kappa_n^2 \left(S_{\alpha\beta} - \frac{1}{n-2} h_{\alpha\beta} S \right). \tag{5}$$

From the extrinsic curvature we can contract the two indices to obtain a scalar. This scalar K corresponding to each side of the $\chi = 0$ hypersurface is represented by K^\pm . Therefore from Eq. (5) it obeys the expression: $K^+ = -K^- = -(\kappa_n^2 S / 2(n - 2))$. Using both the expressions for $K_{\alpha\beta}$ and for K we finally arrive at the following identities:

$$K K_{\alpha\beta} = \frac{\kappa_n^4}{4} \left(\frac{1}{(n-2)^2} h_{\alpha\beta} S^2 - \frac{1}{(n-2)} S S_{\alpha\beta} \right) \tag{6}$$

$$K_{\alpha\mu} K^\mu_\beta = \frac{\kappa_n^4}{4} \left(S_{\alpha\mu} S^\mu_\beta - \frac{2}{(n-2)} S S_{\alpha\beta} + \frac{1}{(n-2)^2} h_{\alpha\beta} S^2 \right) \tag{7}$$

$$K^2 = \frac{\kappa_n^4}{4} \frac{1}{(n-2)^2} S^2 \tag{8}$$

$$K_{\alpha\beta} K^{\alpha\beta} = \frac{\kappa_n^4}{4} \left(S_{\alpha\beta} S^{\alpha\beta} - \frac{2}{n-2} S^2 + \frac{n-1}{(n-2)^2} S^2 \right). \tag{9}$$

In order to proceed we need to decompose the brane stress-energy tensor in two parts. One of them involves the brane cosmological constant, while the other provides the matter stress-energy tensor on the brane. Implementation of this separation leads to

$$S_{\alpha\beta} = -\sigma h_{\alpha\beta} + \tau_{\alpha\beta} \tag{10}$$

where σ represents the brane tension. Next we substitute this decomposition of $S_{\alpha\beta}$ into the above expressions involving extrinsic curvature. This ultimately leads to

$$\begin{aligned} &K K_{\alpha\beta} - K_{\alpha\mu} K^\mu_\beta - \frac{1}{2} h_{\alpha\beta} (K^2 - K_{\mu\nu} K^{\mu\nu}) \\ &= \kappa_n^4 \left[-\frac{1}{4} \tau_{\alpha\mu} \tau^\mu_\beta + \frac{1}{8} h_{\alpha\beta} \tau_{\mu\nu} \tau^{\mu\nu} + \frac{1}{4(n-2)} \tau \tau_{\alpha\beta} \right] \end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{8(n-2)} h_{\alpha\beta} \tau^2 \Big] \\
 & + \kappa_n^4 \left[\frac{(n-3)}{4(n-2)} \sigma \tau_{\alpha\beta} - \frac{(n-3)}{8(n-2)} \sigma^2 h_{\alpha\beta} \right]. \tag{11}
 \end{aligned}$$

Equations (3) and (11) now need to be substituted in Eq. (1) in order to obtain the final form of the effective equation on the brane. After the substitution, the effective Einstein equation on the brane takes the form

$$\begin{aligned}
 {}^{(n-1)}G_{\alpha\beta} = & - \frac{(n-3)}{(n-1)} \kappa_n^2 \left(\Lambda_n + \frac{n-1}{8(n-2)} \kappa_n^2 \sigma^2 \right) h_{\alpha\beta} \\
 & + 8\pi \left(\sigma \kappa_n^4 \frac{n-3}{32\pi(n-2)} \right) \tau_{\alpha\beta} + \kappa_n^4 \pi_{\alpha\beta} - E_{\alpha\beta} \tag{12}
 \end{aligned}$$

where we have introduced the second rank tensor $\Pi_{\alpha\beta}$ as

$$\begin{aligned}
 \pi_{\alpha\beta} = & \left[-\frac{1}{4} \tau_{\alpha\mu} \tau_{\beta}^{\mu} + \frac{1}{8} h_{\alpha\beta} \tau_{\mu\nu} \tau^{\mu\nu} + \frac{1}{4(n-2)} \tau \tau_{\alpha\beta} \right. \\
 & \left. - \frac{1}{8(n-2)} h_{\alpha\beta} \tau^2 \right]. \tag{13}
 \end{aligned}$$

On rewriting, Eq. (12) can also be presented in a more compact form, which can be expressed as

$${}^{(n-1)}G_{\alpha\beta} = \kappa_{n-1}^2 T_{\alpha\beta} \tag{14}$$

where the energy-momentum tensor $T_{\alpha\beta}$ obeys the expression

$$T_{\alpha\beta} = -\Lambda_{n-1} h_{\alpha\beta} + \tau_{\alpha\beta} + \frac{\kappa_n^4}{\kappa_{n-1}^2} \pi_{\alpha\beta} - \frac{1}{\kappa_{n-1}^2} E_{\alpha\beta}. \tag{15}$$

In the above expression Λ_{n-1} represents the $(n-1)$ -dimensional cosmological constant which obeys the expression

$$\Lambda_{n-1} = \frac{(n-3)}{(n-1)} \frac{\kappa_n^2}{\kappa_{n-1}^2} \left(\Lambda_n + \frac{n-1}{8(n-2)} \kappa_n^2 \sigma^2 \right). \tag{16}$$

This generalizes the expression for the effective cosmological constant on a lower dimensional hypersurface embedded in a higher dimensional bulk. For $n = 5$ we retrieve the fine balancing relation of the Randall–Sundrum single brane model [35].

Thus this formalism with a negative n -dimensional cosmological constant can provide a solution to the cosmological constant problem by tuning the brane tension such that $\Lambda_{n-1} = 0$. In this case the brane tension can be obtained:

$$\sigma^2 = \frac{8(n-2)|\Lambda_n|}{(n-1)\kappa_n^2}. \tag{17}$$

Note that once the $(n-1)$ -dimensional cosmological constant has been set to zero it will remain zero in all the lower dimensional branes. Also G_{n-1} is the $(n-1)$ -dimensional gravitational constant having the expression

$$\kappa_{n-1}^2 = 8\pi G_{n-1} = \sigma \kappa_n^4 \frac{(n-3)}{4(n-2)}. \tag{18}$$

Having discussed the effective gravitational field equation in $(n-1)$ -dimension, we will now move one more step by deriving effective field equation in $(n-2)$ -dimensional spacetime.

2.2 Effective field equation on $(n-2)$ -dimensional brane

Similar analysis can be performed while obtaining effective field equation on the $(n-2)$ -dimensional brane starting from the $(n-1)$ -dimensional spacetime. The field equation in the $(n-1)$ -dimensional spacetime can be used, which ultimately leads to the following expression for effective field equation in $(n-2)$ dimensions (see Appendix A.2):

$$\begin{aligned}
 {}^{(n-2)}G_{AB} = & \frac{(n-4)}{(n-3)} \kappa_{n-1}^2 \\
 & \times \left[\mathcal{T}_{\alpha\beta} e_A^\alpha e_B^\beta + \left\{ \mathcal{T}_{\alpha\beta} s^\alpha s^\beta - \frac{1}{(n-2)} \mathcal{T} \right\} q_{AB} \right] \\
 & - \mathcal{E}_{AB} + \left[\mathcal{K}_{AB} \mathcal{K} - \mathcal{K}_{AC} \mathcal{K}_B^C - \frac{1}{2} q_{AB} (\mathcal{K}^2 - \mathcal{K}_{CD} \mathcal{K}^{CD}) \right]. \tag{19}
 \end{aligned}$$

This expression represents the effective Einstein equation in an $(n-2)$ -dimensional spacetime. Our next task is to rewrite the above equation in terms of various components of energy-momentum tensor, especially the extrinsic curvature components.

Let us now try to match our solutions to that of $(n-2)$ -dimensional brane energy-momentum tensor. As usual we assume the following form of metric ansatz:

$$\begin{aligned}
 ds^2 = & d\chi^2 + h_{\alpha\beta} dx^\alpha dx^\beta \\
 = & d\chi^2 + h_{\alpha\beta} e_\zeta^\alpha e_\zeta^\beta d\zeta^2 + q_{AB} dx^A dx^B \tag{20}
 \end{aligned}$$

where q_{AB} is the induced metric on the $(n-2)$ -dimensional brane. Following the previous derivation of effective field equation here also we divide the energy-momentum tensor, $\tau_{\alpha\beta}$, into two parts, $t_{\alpha\beta}$ pertaining to an $(n-1)$ -dimensional brane and ξ_{AB} as energy-momentum tensor for the $(n-2)$ -dimensional brane. Thus we have the following relation:

$$\begin{aligned}
 \tau_{\alpha\beta} = & t_{\alpha\beta} + \delta(\zeta) \xi_{AB} e_\alpha^A e_\beta^B \\
 = & t_{\alpha\beta} + \delta(\zeta) e_\alpha^A e_\beta^B (-\Sigma q_{AB} + \nu_{AB}) \tag{21}
 \end{aligned}$$

where Σ is the brane tension and ν_{AB} is the brane energy-momentum tensor. Then the tensor ξ_{AB} can be obtained from

the discontinuity of the extrinsic curvature from the relation

$$\xi_{AB} = \frac{1}{\kappa_{n-1}^2} ([\mathcal{K}_{AB}] - q_{AB} [\mathcal{K}]). \tag{22}$$

The jump in the extrinsic curvature by imposing Z_2 symmetry can be obtained in terms of ξ_{AB} as

$$\mathcal{K}_{AB}^+ = -\mathcal{K}_{AB}^- = \frac{1}{2\kappa_{n-1}^2} \left(\xi_{AB} - \frac{1}{n-3} q_{AB} \xi \right). \tag{23}$$

Having obtained the extrinsic curvature, the scalar \mathcal{K} can be obtained by contraction with q_{AB} . This has the expression $\mathcal{K}^+ = -\mathcal{K}^- = -(\kappa_{n-1}^2 \xi / 2(n-3))$. We therefore arrive at the following expressions for various combinations of ξ_{AB} as they appear in the effective field equation:

$$\xi \xi_{AB} = (n-2) \Sigma^2 q_{AB} - \nu \Sigma q_{AB} - (n-2) \Sigma \nu_{AB} + \nu \nu_{AB} \tag{24}$$

$$\xi_{AC} \xi_B^C = \Sigma^2 q_{AB} - 2 \Sigma \nu_{AB} + \nu_{AC} \nu_B^C \tag{25}$$

$$\xi^2 = (n-2)^2 \Sigma^2 - 2(n-2) \Sigma \nu + \nu^2 \tag{26}$$

$$\xi_{AB} \xi^{AB} = (n-2) \Sigma^2 - 2 \Sigma \nu + \nu_{AB} \nu^{AB}. \tag{27}$$

This enables us to write an expression for the extrinsic curvature contractions in terms of ξ_{AB} , which can further be simplified in terms of ν_{AB} following the above identities:

$$\begin{aligned} & \mathcal{K}_{AB} \mathcal{K} - \mathcal{K}_{AC} \mathcal{K}_B^C - \frac{1}{2} q_{AB} (\mathcal{K}^2 - \mathcal{K}_{CD} \mathcal{K}^{CD}) \\ &= \frac{\kappa_{n-1}^4}{4} \left[\frac{1}{n-3} \xi \xi_{AB} - \xi_{AC} \xi_B^C \right. \\ & \quad \left. - \frac{1}{2(n-3)} \xi^2 q_{AB} + \frac{1}{2} q_{AB} \xi^{CD} \xi_{CD} \right] \\ &= \kappa_{n-1}^4 \left[-\frac{1}{4} \nu_{AC} \nu_B^C + \frac{1}{8} q_{AB} \nu_{CD} \nu^{CD} \right. \\ & \quad \left. + \frac{1}{4(n-3)} \nu \nu_{AB} - \frac{1}{8(n-3)} q_{AB} \nu^2 \right] \\ & \quad - \kappa_{n-1}^4 \Sigma^2 \frac{(n-4)}{8(n-3)} q_{AB} + \kappa_{n-1}^4 \Sigma \frac{(n-4)}{4(n-3)} q_{AB}. \tag{28} \end{aligned}$$

It now just needs some rearrangement to write down the effective field equation for the $(n-2)$ -dimensional brane. After inserting all relevant expressions and then rearranging the effective Einstein equation for this $(n-2)$ -dimensional brane, we obtain

$$\begin{aligned} {}^{(n-2)}G_{AB} &= -\left(\frac{n-4}{n-2}\right) \Lambda_{n-1} q_{AB} \\ &+ \left(\frac{n-4}{n-3}\right) 8\pi G_{n-1} \left[t_{\alpha\beta} e_A^\alpha e_B^\beta + q_{AB} \left\{ t_{\alpha\beta} s^\alpha s^\beta - \frac{1}{n-2} t \right\} \right] \\ &+ \left(\frac{n-4}{n-3}\right) \kappa_n^4 \left[\Pi_{\alpha\beta} e_A^\alpha e_B^\beta + q_{AB} \left\{ \Pi_{\alpha\beta} s^\alpha s^\beta - \frac{1}{n-2} \Pi \right\} \right] \end{aligned}$$

$$\begin{aligned} & - \left(\frac{n-4}{n-3}\right) \left[E_{\alpha\beta} e_A^\alpha e_B^\beta + q_{AB} \left\{ E_{\alpha\beta} s^\alpha s^\beta - \frac{E}{n-2} \right\} \right] - \mathcal{E}_{AB} \\ & + \kappa_{n-1}^4 \Upsilon_{AB} - \kappa_{n-1}^4 \Sigma^2 \frac{(n-4)}{8(n-3)} q_{AB} + \kappa_{n-1}^4 \Sigma \frac{(n-4)}{4(n-3)} \nu_{AB} \tag{29} \end{aligned}$$

where we have introduced the following quantities:

$$\begin{aligned} \Pi_{\alpha\beta} &= \left[-\frac{1}{4} t_{\alpha\mu} t_\beta^\mu + \frac{1}{8} h_{\alpha\beta} t_{\mu\nu} t^{\mu\nu} \right. \\ & \quad \left. + \frac{1}{4(n-2)} t t_{\alpha\beta} - \frac{1}{8(n-2)} h_{\alpha\beta} t^2 \right] \tag{30} \end{aligned}$$

$$\begin{aligned} \Upsilon_{AB} &= \left[-\frac{1}{4} \nu_{AC} \nu_B^C + \frac{1}{8} q_{AB} \nu_{CD} \nu^{CD} \right. \\ & \quad \left. + \frac{1}{4(n-3)} \nu \nu_{AB} - \frac{1}{8(n-3)} q_{AB} \nu^2 \right]. \tag{31} \end{aligned}$$

The effective equation can be written in conventional form after some algebraic manipulations such that

$${}^{n-1}G_{AB} = \kappa_{n-2}^2 T_{AB} \tag{32}$$

where the energy-momentum tensor T_{AB} obeys the expression

$$\begin{aligned} T_{AB} &= -\Lambda_{n-2} q_{AB} + \nu_{AB} + \frac{\kappa_{n-1}^4}{\kappa_{n-2}^2} \Upsilon_{AB} - \frac{1}{\kappa_{n-2}^2} \mathcal{E}_{AB} \\ &+ \left(\frac{n-4}{n-3}\right) \frac{\kappa_{n-1}^2}{\kappa_{n-2}^2} \left[t_{\alpha\beta} e_A^\alpha e_B^\beta + q_{AB} \left\{ t_{\alpha\beta} s^\alpha s^\beta - \frac{1}{n-2} t \right\} \right] \\ &+ \left(\frac{n-4}{n-3}\right) \frac{\kappa_n^4}{\kappa_{n-2}^2} \left[\Pi_{\alpha\beta} e_A^\alpha e_B^\beta + q_{AB} \left\{ \Pi_{\alpha\beta} s^\alpha s^\beta - \frac{1}{n-2} \Pi \right\} \right] \\ &- \left(\frac{n-4}{n-3}\right) \frac{1}{\kappa_{n-2}^2} \left[E_{\alpha\beta} e_A^\alpha e_B^\beta + q_{AB} \left\{ E_{\alpha\beta} s^\alpha s^\beta - \frac{E}{n-2} \right\} \right]. \tag{33} \end{aligned}$$

Here, we have introduced the following two objects for notational simplicity:

$$\Lambda_{n-2} = \left(\frac{n-4}{n-2}\right) \frac{\kappa_{n-1}^2}{\kappa_{n-2}^2} \left[\Lambda_{n-1} + \frac{(n-2)}{8(n-3)} \kappa_{n-1}^2 \Sigma^2 \right], \tag{34}$$

$$\kappa_{n-2}^2 = 8\pi G_{n-2} = \kappa_{n-1}^4 \Sigma \frac{(n-4)}{4(n-3)} \tag{35}$$

where Λ_{n-2} and κ_{n-2} can be interpreted as the $(n-2)$ -dimensional cosmological constant and the $(n-2)$ -dimensional gravitational constant, respectively. However, for the case in which $t_{\alpha\beta} = 0$, i.e., the bulk energy-momentum tensor vanishes, the above equation gets simplified to

$${}^{(n-2)}G_{AB} = \kappa_{n-2}^2 T_{AB} \tag{36}$$

where T_{AB} has the simplified form

$$T_{AB} = -\Lambda_{n-2}q_{AB} + v_{AB} + \frac{\kappa_{n-1}^4}{\kappa_{n-2}^2}\Upsilon_{AB} - \frac{1}{\kappa_{n-2}^2}\mathcal{E}_{AB} - \left(\frac{n-4}{n-3}\right)\frac{1}{\kappa_{n-2}^2}\left[E_{\alpha\beta}e_A^\alpha e_B^\beta + q_{AB}\left\{E_{\alpha\beta}s^\alpha s^\beta - \frac{E}{n-2}\right\}\right]. \tag{37}$$

Thus the structure of effective field equation in $(n - 2)$ -dimensional brane for vanishing bulk energy-momentum tensor is mostly identical to the $(n - 1)$ -dimensional counterpart except for the Weyl curvature parts. The structure of the effective equation should be quite evident at this stage, which we are going to exploit later to obtain an effective field equation for an $(n - m)$ -dimensional brane.

2.3 Generalization to $(n - m)$ -dimensional brane

Having discussed the effective gravitational field equation in $(n - 2)$ dimensions starting from n dimensions through a two step decomposition, it is now time to consider the generalization of the effective field equation for gravity to $(n - m)$ -dimension. This can be obtained following the analogy of previous results. The final expression for effective field equation in $(n - m)$ dimensions is

$${}^{(n-m)}G_{AB} = \kappa_{n-m}^2 T_{AB}^{(n-m)} \tag{38}$$

where the energy-momentum tensor has the following expression:

$$T_{AB}^{(n-m)} = -\Lambda_{n-m}q_{AB}^{(n-m)} + \tau_{AB}^{(n-m)} + \frac{\kappa_{n-m+1}^4}{\kappa_{n-m}^2}\Upsilon_{AB}^{(n-m)} - \frac{1}{\kappa_{n-m}^2}E_{AB}^{(n-m)} + \sum_{i=2}^m \left(\frac{n-i-2}{n-i-1}\right)\left[\frac{\kappa_{n-i+1}^2}{\kappa_{n-i}^2} \times \left(t_{\alpha\beta}^{(n-i+1)}e_A^\alpha e_B^\beta + q_{AB}\left\{t_{\alpha\beta}^{(n-i+1)}s^\alpha s^\beta - \frac{1}{n-2}t^{(n-i+1)}\right\}\right) + \frac{\kappa_{n-i+2}^4}{\kappa_{n-i}^2}\left(\Pi_{\alpha\beta}^{(n-i+1)}e_A^\alpha e_B^\beta + q_{AB} \times \left\{\Pi_{\alpha\beta}^{(n-i+1)}s^\alpha s^\beta - \frac{1}{n-2}\Pi^{(n-i+1)}\right\}\right) - \frac{1}{\kappa_{n-i}^2} \times \left(E_{\alpha\beta}^{(n-i+1)}e_A^\alpha e_B^\beta + q_{AB}\left\{E_{\alpha\beta}^{(n-i+1)}s^\alpha s^\beta - \frac{E^{(n-i+1)}}{n-2}\right\}\right)\right] \tag{39}$$

with the following identifications:

$$\Lambda_{n-m} = \left(\frac{n-m-2}{n-m}\right)\frac{\kappa_{n-m+1}^2}{\kappa_{n-m}^2} \times \left(\Lambda_{n-m+1} + \frac{(n-m)}{8(n-m-1)}\kappa_{n-m+1}\Sigma^2\right) \tag{40}$$

$$\kappa_{n-m}^2 = 8\pi G_{n-m} = \frac{(n-m-2)}{4(n-m-1)}\Sigma^{(n-m)}\kappa_{n-m+1}^4 \tag{41}$$

$$\Pi_{\alpha\beta} = \left[-\frac{1}{4}t_{\alpha\mu}t_\beta^\mu + \frac{1}{8}h_{\alpha\beta}t_{\mu\nu}t^{\mu\nu} + \frac{1}{4(n-m)}tt_{\alpha\beta} - \frac{1}{8(n-m)}h_{\alpha\beta}t^2\right] \tag{42}$$

$$\Upsilon_{AB} = \left[-\frac{1}{4}\tau_{AC}\tau_B^C + \frac{1}{8}q_{AB}\tau_{CD}\tau^{CD} + \frac{1}{4(n-m-1)}\tau\tau_{AB} - \frac{1}{8(n-m-1)}q_{AB}\tau^2\right]. \tag{43}$$

However, it may be observed that in the situation where there is only matter on the $(n - m)$ -dimensional brane but not on the other higher dimensional branes, all the additional terms vanishes except the Weyl tensor part. If the bulk spacetime is assumed to be anti de Sitter (AdS) then the Weyl tensor can taken to be zero, in which case the effective field equation takes a simplified form:

$${}^{(n-m)}G_{AB} = \kappa_{n-m}^2 \left(-\Lambda_{n-m}q_{AB}^{(n-m)} + \tau_{AB}^{(n-m)} + \frac{\kappa_{n-m+1}^4}{\kappa_{n-m}^2}\Upsilon_{AB}^{(n-m)}\right). \tag{44}$$

Thus in this simple situation the effective field equation has contribution from $(n - m)$ -dimensional cosmological constant, energy momentum tensor on the brane and higher order correction terms originating from Υ_{AB} . Having derived the effective field equation in Einstein gravity let us now consider the identical situation in $f(\mathcal{R})$ gravity as well.

3 Applications in general relativity

Having derived the effective field equation on an arbitrary lower dimensional hypersurface starting from a n -dimensional bulk, in this section we will consider two applications of our result. One of these applications would be for static, spherically symmetric configuration and the other in cosmology. As an illustration we will take the bulk to be six dimensional while the brane is obtained by two step decomposition and is 4-dimensional.

3.1 Static spherically symmetric vacuum brane

We start by considering the case for static, spherically symmetric spacetime with no matter source present. Thus the brane energy-momentum tensor vanishes, however, the electric part of the Weyl tensor survives. Hence the effective gravitational field equation on the 4-dimensional brane starting

from the six dimensional bulk can be obtained from Eqs. (38) and (39) as

$${}^{(4)}G_{AB} = -\mathcal{E}_{AB} - \frac{2}{3} \left[E_{\alpha\beta} e_A^\alpha e_B^\beta + q_{AB} \left(E_{\alpha\beta} s^\alpha s^\beta - \frac{1}{4} E \right) \right]. \tag{45}$$

In arriving at the above equation the cosmological constant has been neglected, since we are interested in a local space-time region. The electric part of the Weyl tensor in a static, spherically symmetric spacetime can be written using two unknown functions depending on the radial coordinate as

$$\mathcal{E}_{AB} = -\frac{\kappa_5^4}{\kappa_4^4} \left[U(r) \left(u_{AU} u_B + \frac{1}{3} \bar{q}_{AB} \right) + P(r) \left(r_A r_B - \frac{1}{3} \bar{q}_{AB} \right) \right] \tag{46}$$

where \bar{q}_{AB} is the induced metric on the $t = \text{constant}$ surface, such that $u^A \bar{q}_{AB} = 0$. The 5-dimensional electric part of the Weyl tensor would have similar decomposition depending on two separate radial functions $\bar{U}(r)$ and $\bar{P}(r)$. On projecting this 5-dimensional tensor $E_{\alpha\beta}$ on the 4-dimensional brane, the effective field equations reduce to

$$G_{AB} = \frac{\kappa_5^4}{\kappa_4^4} \left[U(r) \left(u_{AU} u_B + \frac{1}{3} \bar{q}_{AB} \right) + P(r) \left(r_A r_B - \frac{1}{3} \bar{q}_{AB} \right) \right] + \frac{2\kappa_6^4}{3\kappa_5^4} \left[\bar{U}(r) \left(u_{AU} u_B + \frac{1}{2} \bar{q}_{AB} \right) + \bar{P}(r) \left(r_A r_B - \frac{1}{2} \bar{q}_{AB} \right) \right]. \tag{47}$$

Now κ_n and κ_{n-1} are related through Eq. (18), such that $\kappa_6^4 = (16/3)\kappa_5^2(1/\sigma)$, $\kappa_5^4 = (6/\Sigma)\kappa_4^2$ and $\kappa_4^2 = 8\pi G$. Using these relations the above equation can be written in terms of the brane tensions, σ for the 5-dimensional brane, Σ for the 4-dimensional brane and the The gravitational constant G is

$$G_{AB} = \frac{3}{4\pi G \Sigma} \left[U(r) \left(u_{AU} u_B + \frac{1}{3} \bar{q}_{AB} \right) + P(r) \left(r_A r_B - \frac{1}{3} \bar{q}_{AB} \right) \right] + \frac{32}{9\sqrt{6} \sigma \sqrt{8\pi G}} \left[\bar{U}(r) \left(u_{AU} u_B + \frac{1}{2} \bar{q}_{AB} \right) + \bar{P}(r) \left(r_A r_B - \frac{1}{2} \bar{q}_{AB} \right) \right] \tag{48}$$

Now assuming a static, spherically symmetric metric ansatz of the form

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\Omega^2, \tag{49}$$

we obtain various components of the gravitational field equations:

$$-e^{-\lambda} \left(-\frac{\lambda'}{r} + \frac{1}{r^2} \right) + \frac{1}{r^2} = \frac{3}{4\pi G \Sigma} U(r) + \frac{32}{9\sqrt{6} \sigma \sqrt{8\pi G}} \bar{U}(r) \tag{50}$$

$$e^{-\lambda} \left(\frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \frac{1}{4\pi G \Sigma} [U(r) + 2P(r)] + \frac{16}{9\sqrt{6} \sigma \sqrt{8\pi G}} [\bar{U}(r) + \bar{P}(r)] \tag{51}$$

$$\frac{1}{2} e^{-\lambda} \left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu' \lambda'}{2} \right) = \frac{1}{4\pi G \Sigma} [U(r) - P(r)] + \frac{16}{9\sqrt{6} \sigma \sqrt{8\pi G}} [\bar{U}(r) - \bar{P}(r)]. \tag{52}$$

Hence the effect of multiple extra dimensions is manifest in the lower dimensional effective gravitational field equations through the appearance of the extra terms $\bar{U}(r)$ and $\bar{P}(r)$. The field equations have to be supplemented by the conservation equations, which will provide a differential relation connecting $U(r)$, $P(r)$, $\bar{U}(r)$, and $\bar{P}(r)$. The conservation equations turn out to be

$$\frac{1}{3} \left(\frac{dU}{dr} + 2 \frac{dP}{dr} \right) + \frac{\nu'}{3} (2U + P) + \frac{2P}{r} = 0 \tag{53}$$

$$\frac{1}{2} \left(\frac{d\bar{U}}{dr} + \frac{d\bar{P}}{dr} \right) + \frac{\nu'}{4} (3\bar{U} + \bar{P}) + \frac{2\bar{P}}{r} = 0. \tag{54}$$

Starting from Eq. (50) it is straightforward to obtain a solution for $e^{-\lambda}$. Since all the objects on the right hand side of Eq. (50) depend only on the radial coordinate we can integrate them to obtain the following solution for $e^{-\lambda}$:

$$e^{-\lambda} = 1 - \frac{2GM}{r} - \frac{Q(r)}{r};$$

$$Q(r) = \frac{3}{4\pi G \Sigma} \int dr r^2 U(r) + \frac{32}{9\sqrt{6} \sigma \sqrt{8\pi G}} \int dr r^2 \bar{U}(r). \tag{55}$$

Having obtained $e^{-\lambda}$ it is now pertinent to consider the other metric component e^ν as well. This involves all the extra contributions from dark radiation and dark pressure terms. In order to obtain solutions for these higher dimensional contributions, we need to impose certain relations among these higher dimensional objects. The most useful among them

being the relations $2U + P = 0$ and $3\bar{U} + \bar{P} = 0$. Then from Eqs. (53) and (54) we readily obtain,

$$P(r) = -2U(r) = \frac{P_0}{r^4};$$

$$\bar{P}(r) = -3\bar{U}(r) = \frac{\bar{P}_0}{r^6} \tag{56}$$

$$Q(r) = Q_0 + \frac{3\alpha P_0}{2r} + \frac{\bar{P}_0\beta}{9r^3};$$

$$\alpha = \frac{1}{4\pi G\Sigma}; \quad \beta = \frac{32}{9\sqrt{6}} \frac{\sqrt{\Sigma}}{\sigma\sqrt{8\pi G}}. \tag{57}$$

For these relations between dark radiation and dark pressure the static spherically symmetric solution corresponds to

$$e^{-\lambda} = e^{\nu} = 1 - \frac{2GM + Q_0}{r} - \frac{3\alpha P_0}{2r^2} - \frac{\bar{P}_0\beta}{9r^4}. \tag{58}$$

Note that this is an exact solution to the effective field equations, no assumptions or approximations have been made. Hence the effect of extra spatial dimension appears through the charge term, i.e., r^{-2} term and the r^{-4} term. Thus near $r = 0$, $1/r^4$ term would dominate, which originates from the existence of more than one extra spatial dimensions.

Having derived the static spherically symmetric solution there are two important aspects one can study: (a) stability of the solutions and (b) thermodynamics. We will briefly comment on these features. The stability of the solution can be obtained by considering perturbations, which again are of three types, scalar, vector, and tensor modes. These modes satisfy a set of decoupled wave equations with the structure

$$\left(\square - \frac{1}{f(r)}V\right)\Phi = 0. \tag{59}$$

Here \square represents d'Alembertian operator, $f(r) = e^{\nu} = e^{-\lambda}$, Φ stands for the perturbation modes and V is an appropriate potential associated with perturbation Φ . For stability, the potentials should be positive. If we assume that the corrections to the Schwarzschild solution are small, i.e., α and β are small compared to M , then the above solution is stable for all the modes.

Let us now discuss the case of black hole thermodynamics. For the location of the horizon we have the following equation:

$$1 - \frac{2GM + Q_0}{r_h} - \frac{3\alpha P_0}{2r_h^2} - \frac{\bar{P}_0\beta}{9r_h^4} = 0. \tag{60}$$

The mass of the black hole (or, equivalently the internal energy) can be obtained readily from the above equation as

$$M(r_h) = \frac{r_h}{2} - \frac{Q_0}{2} - \frac{3\alpha P_0}{4r_h} - \frac{\bar{P}_0\beta}{6r_h^3}. \tag{61}$$

Now entropy can be obtained from holography, i.e., $S = A/4$, such that $S = r_h^2$. Thus internal energy can be obtained in terms of the entropy density from Eq. (61) such that

$$M(S) = \frac{\sqrt{S}}{2} - \frac{Q_0}{2} - \frac{3\alpha P_0}{4\sqrt{S}} - \frac{\bar{P}_0\beta}{6S^{3/2}}. \tag{62}$$

Then the black hole temperature can be obtained in terms of entropy as,

$$T(S) = \left(\frac{\partial M}{\partial S}\right) = \frac{1}{4\sqrt{S}} + \frac{3\alpha P_0}{8S^{3/2}} + \frac{\bar{P}_0\beta}{4S^{5/2}} \tag{63}$$

which immediately leads to the following expression for the specific heat:

$$C_V = T \left(\frac{\partial S}{\partial T}\right) = -\frac{\frac{1}{4\sqrt{S}} + \frac{3\alpha P_0}{8S^{3/2}} + \frac{\bar{P}_0\beta}{4S^{5/2}}}{\frac{1}{8S^{3/2}} + \frac{9\alpha P_0}{16S^{5/2}} + \frac{5\bar{P}_0\beta}{8S^{7/2}}}. \tag{64}$$

Thus the specific heat of the black hole is negative, as it should, and there is no phase transition possible, since α and β are both positive quantities. Later we will show that inclusion of $f(\mathcal{R})$ gravity can alter the situation drastically, which we will discuss in the next section.

Having derived the static spherically symmetric solution and its various properties, let us now discuss some novel features of this solution

- Imposing the restrictions $2U(r) + P(r) = 0$ and $\bar{P}(r) + 3\bar{U}(r) = 0$, we have shown that the metric elements satisfy $e^{-\lambda} = e^{\nu}$. Thus the line element derived in Eq. (58) depends explicitly on the form of $U(r)$ and $\bar{U}(r)$. Thus the geometry of the spacetime is uniquely determined only when knowledge about multiple dimensions are obtained through $\bar{U}(r)$. Also it turns out that at small r the effect from $\bar{U}(r)$ dominates.
- The solution is also stable under perturbations. The potentials associated with respective wave equations are positive for scalar, vector, and tensor modes. Thus the black hole solution, will not change its character under perturbations.
- Finally, the specific heat associated with the black hole solution is negative. This shows that the black hole is thermodynamically unstable. Since all the additional parameters are positive there is no phase transition in this black hole system.

Hence the fact that spacetime has more than one extra dimensions is being reflected on the right hand side of the effective

gravitational field equations. These extra terms contribute to the energy-momentum tensor and modifies the metric components from just being $1 - (2GM/r)$.

3.2 Cosmological spacetime

After discussing the static spherically symmetric configuration it is natural to turn around and consider gravity theory on a lower dimensional hypersurface in a cosmological context. As already mentioned we will start with 6-dimensional bulk spacetime and then obtain cosmological solutions from the effective field equation on a 4-dimensional brane. In order to obtain analytical solution, we assume $E_{ab} = 0$ i.e. the contribution from the electric part of Weyl tensor vanishes. Then the effective field equation in four dimensions takes the following form:

$$G_{AB} = -\kappa_4^2 \Lambda_4 q_{AB} + \kappa_4^2 \tau_{AB} + \kappa_5^4 \Upsilon_{AB} + \frac{2}{3} \kappa_5^2 \left(t_{\alpha\beta} e_A^\alpha e_B^\beta + q_{AB} \left\{ t_{\alpha\beta} s^\alpha s^\beta - \frac{1}{4} t \right\} \right) + \frac{2}{3} \kappa_6^4 \left(\Pi_{\alpha\beta} e_A^\alpha e_B^\beta + q_{AB} \left\{ \Pi_{\alpha\beta} s^\alpha s^\beta - \frac{1}{4} \Pi \right\} \right). \tag{65}$$

In this expression along with brane energy-momentum tensor τ_{AB} and its higher order term Υ_{AB} , two additional contribution from the 5-dimensional matter energy-momentum tensor is present. We assume that the brane is filled with perfect fluid with the following form for the energy-momentum tensor on four dimensions:

$$\tau_B^A = \text{diag}(-\rho, p, p, p) \tag{66}$$

where ρ represents energy density of the perfect fluid and p yields its pressure. The energy-momentum tensor in five dimensions is taken to be pressure free dust, such that its energy-momentum tensor has the simple form

$$t_\beta^\alpha = \text{diag}(-\rho_5, 0, 0, 0, 0) \tag{67}$$

where the subscript ‘5’ indicates that it is from the 5-dimensional brane. Using which we arrive at the following expression for the higher order components:

$$\Pi_{tt} = \frac{3}{32} \rho_5^2; \quad \Pi = -\frac{3}{32} \rho_5^2. \tag{68}$$

Having obtained all the components of the matter energy-momentum tensor, the time-time and spatial components of the effective equations lead to (we will assume that the spacetime is spatially flat, i.e., $k = 0$):

$$3H^2 = \kappa_4^2 \Lambda_4 + \kappa_4^2 \rho + \frac{1}{12} \kappa_5^4 \rho^2 + \frac{1}{2} \kappa_5^2 \rho_5 + \frac{3}{64} \kappa_6^4 \rho_5^2 \tag{69}$$

$$H^2 + 2\frac{\ddot{a}}{a} = \kappa_4^2 \Lambda_4 - \kappa_4^2 p - \frac{1}{12} \kappa_5^4 (\rho^2 + 2p\rho) - \frac{1}{6} \kappa_5^2 \rho_5 - \frac{1}{64} \kappa_6^4 \rho_5^2. \tag{70}$$

Having obtained the Friedmann equations, let us explore the various phases of the universe. To get a handle, let us write the above expression in terms of relative abundances $\Omega_i = \rho_{i,0}/\rho_c$. Here $\rho_{i,0}$ stands for the present energy density of the i th component and ρ_c is the current total energy density of the universe. The energy density can be expanded in terms of baryonic and relativistic matter fields. We will assume matter in higher dimensions to be non-relativistic. Thus the Hubble parameter turns out to be

$$H^2 = H_0^2 \left[\Omega_\Lambda + (\Omega_B + \Omega_5) \left(\frac{a_0}{a}\right)^3 + \Omega_R \left(\frac{a_0}{a}\right)^4 \right] \tag{71}$$

where $\Omega_5 = (\kappa_5^2/16\pi G\ell)(\rho_{5,0}/\rho_c)$, with ℓ being a characteristic length associated with the extra dimension. Note that the above Hubble parameter can reproduce the standard Λ CDM model. For at late times Ω_Λ would dominate leading to accelerated expansion of the universe. At early times, the $1/a^4$ term would dominate, leading to radiation dominated universe. The intermediate region is dominated by non-relativistic matter such that $H^2 \sim a^{-3}$. However, the surprising fact about this model is that in the matter sector along with the baryonic matter, due to the extra dimension we have additional contribution, which can act as dark matter. In standard Λ CDM cosmology, the dark matter has to be included in the model by hand. However, in this case it appears in the Friedmann universe *naturally*, we need not have to invoke it by hand. Using the epoch of matter, radiation equality it is possible to get a numerical estimate about 5-dimensional gravitational constant as

$$1 + z_{eq} = 3.9 \times 10^4 \left(0.022 + \frac{\kappa_5^2 h^2}{16\pi G\ell} \frac{\rho_{5,0}}{\rho_c} \right) \tag{72}$$

where $h = 0.72$, ℓ stands for the length of the extra dimension. Thus knowing the redshift z_{eq} , we can determine the value of the higher dimensional gravitational constant κ_5^2 . Thus spacetime having more than one extra spatial dimensions can naturally provide additional matter components in the right hand side of the Friedmann equations. To have inflation we need to include an additional scalar field on top of this alike the standard Λ CDM model. Thus the effective field equations are capable to explain the phases of the universe.

Writing $\kappa_4^2 \Lambda_4 = \Lambda_{\text{eff}}$ and for the bulk matter we have $\rho_5 = (\rho_{5,0}/\ell)(1/a^3)$, where ℓ represents the finite length of the 5-dimensional brane coordinate. Assuming that the matter in the 4-dimensional brane is also pressure-less, we have the following expression for the Hubble parameter:

$$H^2 = \frac{\Lambda_{\text{eff}}}{3} + \frac{8\pi G}{3} \frac{\rho_0}{a^3} + \frac{1}{36} \kappa_5^4 \frac{\rho_0^2}{a^6} + \frac{1}{6} \kappa_5^2 \frac{\rho_{5,0}}{\ell} \frac{1}{a^3} + \frac{1}{64} \kappa_6^4 \frac{\rho_{5,0}^2}{\ell^2} \frac{1}{a^6} \tag{73}$$

where ρ_0 represents the value of energy density in the present epoch and $\rho_{5,0}$ represents the same for 5-dimensional matter. The above expression for Hubble parameter can be simplified and arranged properly leading to

$$H^2 = \frac{\Lambda_{\text{eff}}}{3} + \left\{ \frac{8\pi G}{3} \rho_0 + \frac{1}{6} \kappa_5^2 \frac{\rho_{5,0}}{\ell} \right\} \frac{1}{a^3} + \left\{ \frac{1}{36} \kappa_5^4 \rho_0^2 + \frac{1}{64} \kappa_6^4 \frac{\rho_{5,0}^2}{\ell^2} \right\} \frac{1}{a^6}. \tag{74}$$

Then following the techniques developed in [49] we can introduce three constants, namely:

$$C_\Lambda = \frac{\Lambda_{\text{eff}}}{3}; \quad C_\rho = \frac{8\pi G}{3} \rho_0 + \frac{1}{6} \kappa_5^2 \frac{\rho_{5,0}}{\ell};$$

$$C_{\text{Sq}} = \frac{1}{36} \kappa_5^4 \rho_0^2 + \frac{1}{64} \kappa_6^4 \frac{\rho_{5,0}^2}{\ell^2} \tag{75}$$

such that we obtain the following solution to the scale factor:

$$\exp[3\sqrt{C_\Lambda} (t - t_0)] = \frac{2\sqrt{C_\Lambda} \sqrt{C_\Lambda a^6 + C_\rho a^3 + C_{\text{Sq}}} + 2C_\Lambda a^3 + C_\rho}{2\sqrt{C_\Lambda} \sqrt{C_\Lambda + C_\rho + C_{\text{Sq}}} + 2C_\Lambda + C_\rho}; \quad C_\Lambda > 0 \tag{76}$$

$$3\sqrt{-C_\Lambda} (t_0 - t) = \sin^{-1} \left(\frac{2C_\Lambda a^3 + C_\rho}{\sqrt{C_\rho^2 - 4C_\Lambda C_{\text{Sq}}}} \right) - \sin^{-1} \left(\frac{2C_\Lambda + C_\rho}{\sqrt{C_\rho^2 - 4C_\Lambda C_{\text{Sq}}}} \right); \quad C_\Lambda < 0, C_\rho^2 > 4C_\Lambda C_{\text{Sq}}. \tag{77}$$

Though the solutions have a complicated appearance, we can impose certain conditions under which the solutions simplify substantially, yielding clearer physical insight. For that purpose we consider the situation $\Lambda_{\text{eff}} = 0$. This would be true if we assume the 5-dimensional cosmological constant to be negative, such that $\Lambda_5 = -(\kappa_5^2 \Sigma^2)/6$. Under this condition the Hubble parameter from Eq. (74) turns out to be

$$H^2 = \frac{8\pi G}{3} \rho \left[1 + \frac{\kappa_5^2}{16\pi G} \frac{\rho_5}{\rho} + \frac{\rho}{2\Sigma} \left(1 + \frac{9}{16} \frac{\kappa_6^4}{\kappa_5^4} \frac{\rho_5^2}{\rho^2} \right) \right]. \tag{78}$$

If we assume that ρ represents energy density of non-relativistic matter we arrive at the following differential equation:

$$H^2 = \left\{ \frac{8\pi G \rho_0}{3} + \frac{\kappa_5^2}{6} \frac{\rho_{5,0}}{\ell} \right\} \frac{1}{a^3} + \frac{4\pi G \rho_0^2}{3\Sigma} \times \left(1 + \frac{9}{16} \frac{\kappa_6^4}{\kappa_5^4} \frac{\rho_{5,0}^2}{\ell^2 \rho_0^2} \right) \frac{1}{a^6} \tag{79}$$

leading to the following solution for the scale factor:

$$a^3 = \left\{ 6\pi G \rho_0 + 3 \frac{\kappa_5^2}{8} \frac{\rho_{5,0}}{\ell} \right\} t^2 + \left\{ \sqrt{\frac{12\pi G \rho_0^2}{\Sigma} \left(1 + \frac{9}{16} \frac{\kappa_6^4}{\kappa_5^4} \frac{\rho_{5,0}^2}{\ell^2 \rho_0^2} \right)} \right\} t. \tag{80}$$

It is clear from the above expression that the universe undergoes a transition in the expansion rate. In the standard scenario there are three phases, the radiation dominated, the matter dominated and the accelerated expansion. If we treat the additional correction terms due to the existence of extra spatial dimensions with importance then even within matter dominated epoch we will have transition. The time scale when this happened depends on whether the brane matter energy dominates over the bulk matter or not. Below we present the result for the timescale of the transition for both situations:

$$t \sim \sqrt{\frac{1}{3\pi G \Sigma}} = \frac{4}{\Sigma \kappa_5^2} = -\frac{2}{3} \Lambda_5^{-1}; \quad (\rho_{5,0}/\ell \ll \rho_0) \tag{81}$$

$$t \sim \frac{\kappa_6^2}{\kappa_5^2} \left(1 - \frac{16\pi G \rho_0 \ell}{\kappa_5^2 \rho_{5,0}} \right) \quad (\rho_{5,0}/\ell \gg \rho_0). \tag{82}$$

Thus at early universe (within the matter dominated epoch) we have a high energy regime, where $a \sim t^{1/3}$, while at late time low energy regime the scale factor variation with time modifies to $a \sim t^{2/3}$, which is the standard evolution of the matter field. Note that the same should happen in radiation dominated epoch as well. Hence the scale factor at later times will behave as it should in a standard Λ CDM model. In a nutshell, the effect of extra dimensions on cosmological scenario can be seen from the following results:

- It is clear from the expression for Hubble parameter that the 5-dimensional matter adds to the 4-dimensional one and *enhances* the total non-relativistic matter content of the universe (4-dimensional brane we live in). Hence matter fields present in the extra dimension can act as a *viable* source of dark matter. The most important part of the above analysis corresponds to the fact that the dark matter contribution appears *naturally*, we need not have to add them by hand (a related issue has also been addressed in [50]).

- The existence of more than one extra spatial dimension introduces additional correction terms in the Friedmann equations. This results in transition within both matter dominated and radiation dominated epoch. This in principle can affect the structure formation in the universe and hence can be used to constrain various higher dimensional parameters.

Thus with matter present in the 5-dimensional brane we can have standard cosmology with proper scaling of the scale factor with time. However, the matter content and transition time scale will depend on the content in the extra dimensions.

The cosmological solutions in brane models have interesting holographic interpretations. It was shown in [54] that the usual 4-dimensional cosmological equations can be rewritten in an equivalent form of the entropy of 2-dimensional conformal field theory (CFT). In our case the Friedman equations are given by Eqs. (69) and (70), respectively. Now defining the Hawking, Bekenstein, and Bekenstein–Hawking entropies as,

$$S_H = \frac{(n-1)HV}{4G}, \quad S_{BH} = \frac{(n-1)V}{4Ga}, \quad S_B = \frac{2\pi a}{n} E \tag{83}$$

where $E = \rho V$ is the total energy within a volume V and $\rho_m \propto a^{-n}$. Also the total entropy should be bounded by the Hubble entropy S_H , which is the entropy of the black hole with the radius of the Hubble size. Using these definitions we can rewrite the Friedman equations in the form,

$$S_H = \frac{2\pi a}{n} \sqrt{E_{BH} (2E - KE_{BH})};$$

$$KE_{BH} = n(E + pV - T_H S_H) \tag{84}$$

with energy of the black hole being $E_{BH} = n(n-1)V/8\pi Ga^2$ and Hawking temperature being $T_H = -\dot{H}/2\pi H$. This is exactly the form one would have from the CV entropy relation. It explicitly shows the connection between 4-dimensional classical gravitational physics and 2-dimensional quantum 2-dimensional CFT [55]. This 2-dimensional CFT entropy representation of cosmological equations is called the generalized Cardy or Cardy–Verlinde (CV) formula. One can also consider dS or AdS black holes in the bulk, such that the brane equations of motion can be cast in the form of Friedman equations. By introducing brane matter the cosmological equations can be written in the CV form, which is related to the AdS black hole entropy (these and various other aspects have been discussed in [56] quite extensively). Now we will generalize these results to the $f(\mathcal{R})$ model of gravity.

4 Generalization to bulk $f(\mathcal{R})$ gravity

In recent years modifications of Einstein–Hilbert action by higher curvature terms is a subject of great interest. A very promising candidate among such modifications is the $f(\mathcal{R})$ gravity theory. The modifications introduced by the $f(\mathcal{R})$ term in the Lagrangian can address variety of problems. This theory also has the potential to survive all known tests of general relativity. It is therefore worthwhile to explore the nature of the effective gravitational field equation on a brane where the bulk is endowed with $f(\mathcal{R})$ gravity.

4.1 Effective field equation on the $(n-1)$ -dimensional brane

We consider a n -dimensional bulk spacetime endowed with $f(\mathcal{R})$ gravity. Following the standard procedure i.e., starting from the Gauss–Codazzi equation, subsequently using the Z_2 symmetry, we can obtain the effective gravitational field equation on the brane. This has already been derived earlier in the context of a 4-dimensional brane embedded in a 5-dimensional bulk (see [47] and the references therein). The effective gravitational field equation on the $(n-1)$ -dimensional brane turns out to be

$${}^{(n-1)}G_{\mu\nu} = \kappa_{n-1}^2 T_{\mu\nu}^{(n-1)} \tag{85}$$

where the stress-energy tensor $T_{\mu\nu}^{(n-1)}$ appearing on the right hand side can be decomposed into several small pieces. These include effective $(n-1)$ -dimensional cosmological constant induced from the n -dimensional cosmological constant, brane energy momentum tensor and its higher order contribution. Also due to the presence of $f(\mathcal{R})$ gravity in the bulk there is an extra term and all these terms along with the non-local bulk Weyl tensor turns out to be

$$T_{\alpha\beta}^{(n-1)} = -\Lambda_{n-1} h_{\alpha\beta} + \tau_{\alpha\beta} + \frac{\kappa_n^4}{\kappa_{n-1}^2} \pi_{\alpha\beta}$$

$$- \frac{1}{\kappa_{n-1}^2} E_{\alpha\beta} + \frac{1}{\kappa_{n-1}^2} Q_{\alpha\beta} \tag{86}$$

where the additional term due to $f(\mathcal{R})$ gravity is $Q_{\mu\nu}$. This additional term $Q_{\mu\nu}$ has the following expression:

$$Q_{\mu\nu} = \left[\frac{1}{4} \frac{f(\mathcal{R})}{f'(\mathcal{R})} - \frac{1}{4} \mathcal{R} - \frac{2}{3} \frac{\square f'(\mathcal{R})}{f'(\mathcal{R})} + \frac{2}{3} \frac{\nabla_a \nabla_b f'(\mathcal{R})}{f'(\mathcal{R})} \right]$$

$$\times h_{\mu\nu} + \frac{2}{3} \frac{\nabla_a \nabla_b f'(\mathcal{R})}{f'(\mathcal{R})} e_\mu^a e_\nu^b. \tag{87}$$

Note that in the Einstein–Hilbert limit i.e. $f(\mathcal{R}) \rightarrow \mathcal{R}$ the above term identically vanishes and the effective equation reduces to Eqs. (14) and (15). In the expression of the energy-momentum tensor, Λ_{n-1} is the $(n-1)$ -dimensional

brane cosmological constant (for an explicit expression, see Eq. (16)), $\tau_{\mu\nu}$ is the brane energy-momentum tensor, $\pi_{\alpha\beta}$ contain higher order terms like $\tau_{\alpha}^{\mu}\tau_{\beta\mu}$ etc. (see Eq. (13) for a detailed expression). Finally, $E_{\mu\nu}$ represents the non-local bulk effect.

In order to simplify the expression for $Q_{\mu\nu}$, we can impose some assumptions. The simplification can be significant with the choice: $\partial_{\mu}\mathcal{R} = 0$. Then using Taylor series expansion of bulk curvature \mathcal{R} around $y = 0$ hypersurface leads to: $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1 y + \mathcal{R}_2 y^2/2 + \mathcal{O}(y^3)$. As we have assumed that the bulk curvature depends only on the extra dimension y , all the coefficients appearing in the Taylor series expansion are constants. Hence we can conclude that all the derivatives calculated at $y = 0$ yield a constant contribution which does not depend on any of the brane coordinates. In this case we can rewrite the stress-energy tensor as

$$T_{\alpha\beta}^{(n-1)} = -\Lambda_{\text{eff}} h_{\alpha\beta} + \tau_{\alpha\beta} + \frac{\kappa_n^4}{\kappa_{n-1}^2} \pi_{\alpha\beta} - \frac{1}{\kappa_{n-1}^2} E_{\alpha\beta} \quad (88)$$

where we have an effective cosmological constant,

$$\Lambda_{\text{eff}} = \Lambda_{n-1} - \frac{F(\mathcal{R})}{\kappa_{n-1}^2}. \quad (89)$$

Thus the cosmological constant gets modified due to $f(\mathcal{R})$ gravity in the bulk leading to an additional constant term $F(\mathcal{R})$ to the induced cosmological constant Λ_{n-1} . This can provide a possible explanation to the cosmological constant problem by fine tuning Λ_{n-1} with $F(\mathcal{R})$ term where the quantity $F(\mathcal{R})$ has the following expression:

$$F(\mathcal{R}) = \left(g(\mathcal{R}) + \frac{2}{3} \frac{\nabla_A \nabla_B f'(\mathcal{R})}{f'(\mathcal{R})} n^A n^B \right)_{y=0}. \quad (90)$$

Having derived the effective gravitational field equation for an $(n - 1)$ -dimensional brane we will now write down the counterpart of this relation for an $(n - m)$ -dimensional brane as well.

4.2 Generalization to $(n - m)$ -dimensional brane

In this section we will generalize the above setup to an arbitrary lower dimensional hypersurface, namely, the $(n - m)$ -dimensional brane. This can be done easily by taking a cue from the discussion in Einstein gravity. The effective field equation in the $(n - m)$ -dimensional brane leads to the following expression:

$$^{(n-m)}G_{AB} = \kappa_{n-m}^2 T_{AB}^{(n-m)} \quad (91)$$

with the following expression for the energy-momentum tensor:

$$\begin{aligned} T_{AB}^{(n-m)} = & -\Lambda_{n-m} q_{AB}^{(n-m)} + \tau_{AB}^{(n-m)} + \frac{\kappa_{n-m}^4}{\kappa_{n-m}^2} \Upsilon_{AB}^{(n-m)} \\ & + \frac{1}{\kappa_{n-m}^2} Q_{AB}^{(n-m)} - \frac{1}{\kappa_{n-m}^2} E_{AB}^{(n-m)} + \sum_{i=2}^m \left(\frac{n-i-2}{n-i-1} \right) \\ & \times \left[\frac{\kappa_{n-i}^2}{\kappa_{n-i}^2} \left(t_{\alpha\beta}^{(n-i+1)} e_A^\alpha e_B^\beta + q_{AB} \left\{ t_{\alpha\beta}^{(n-i+1)} s^\alpha s^\beta \right. \right. \right. \\ & \left. \left. \left. - \frac{1}{n-2} t^{(n-i+1)} \right\} \right) + \frac{\kappa_{n-i}^4}{\kappa_{n-i}^2} \left(\Pi_{\alpha\beta}^{(n-i+1)} e_A^\alpha e_B^\beta + q_{AB} \right. \right. \\ & \left. \left. \times \left\{ \Pi_{\alpha\beta}^{(n-i+1)} s^\alpha s^\beta - \frac{1}{n-2} \Pi^{(n-i+1)} \right\} \right) + \frac{1}{\kappa_{n-i}^2} \right. \\ & \left. \times \left(Q_{\alpha\beta}^{(n-i+1)} e_A^\alpha e_B^\beta + q_{AB} \left\{ Q_{\alpha\beta}^{(n-i+1)} s^\alpha s^\beta - \frac{Q^{(n-i+1)}}{n-2} \right\} \right) \right. \\ & \left. - \frac{1}{\kappa_{n-i}^2} \left(E_{\alpha\beta}^{(n-i+1)} e_A^\alpha e_B^\beta + q_{AB} \right. \right. \\ & \left. \left. \times \left\{ E_{\alpha\beta}^{(n-i+1)} s^\alpha s^\beta - \frac{E^{(n-i+1)}}{n-2} \right\} \right) \right]. \quad (92) \end{aligned}$$

Here Λ_{n-m} is the induced brane cosmological constant, $\tau_{AB}^{(n-m)}$ represents the brane energy-momentum tensor, and $\Upsilon_{AB}^{(n-m)}$ is the higher order term. All the other extra terms originate from the effect of higher dimensional energy momentum tensor, $f(\mathcal{R})$ gravity and non-local effects from the bulk. However, just as in the previous section in this case also we can simplify the expression further by assuming that bulk curvature depends only on the extra dimension y . Also if we assume that there is no energy-momentum tensor on the higher dimensional branes and the bulk is AdS, then the Weyl tensor vanishes identically. The effective gravitational field equation, then takes a particularly simple form. In this case following [47], we arrive at:

$$\begin{aligned} ^{(n-m)}G_{AB} & = \kappa_{n-m}^2 \left(-\Lambda_{n-m} q_{AB}^{(n-m)} + \tau_{AB}^{(n-m)} + \frac{\kappa_{n-m}^4}{\kappa_{n-m}^2} \Upsilon_{AB}^{(n-m)} \right) \quad (93) \end{aligned}$$

where Λ_{n-m} has to be constructed from $\Lambda_{\text{eff}} = \Lambda_{n-1} - F(\mathcal{R})$ by standard procedure illustrated for Einstein gravity. Thus in this case the effective equation would be identical to that obtained from general relativity, except for a modified cosmological constant. Also κ_{n-m}^2 is related to $(n - m)$ -dimensional gravitational constant, $\tau_{AB}^{(n-m)}$ is the matter energy-momentum tensor on the $(n - m)$ -dimensional brane, with $\Upsilon_{AB}^{(n-m)}$ being its higher order extension.

This completes our discussion of the effective equation. However, as an illustration we will consider a simple application of our work in a cosmological context in the next section.

5 Applications to $f(\mathcal{R})$ gravity

Having discussed effective field equation in $f(\mathcal{R})$ gravity theory on a lower dimensional brane, alike general relativity we apply this in a static spherically symmetric and cosmological context. For that purpose we start with a 6-dimensional bulk spacetime and try to obtain a solutions from the effective field equations on a 4-dimensional brane.

5.1 Static spherically symmetric $f(\mathcal{R})$ brane

We start by considering the case for static, spherically symmetric spacetime with no matter source present. Thus the brane energy-momentum tensor vanishes, however, the electric part of the Weyl tensor survives. Hence the effective gravitational field equation on the 4-dimensional brane starting from the six dimensional bulk can be obtained from Eqs. (91) and (92) as

$$\begin{aligned}
 {}^{(4)}G_{AB} = & -\frac{1}{2}\kappa_5^2 \left(-\frac{1}{\kappa_5^2}F(\mathcal{R}) + \frac{1}{6}\kappa_5^2\Sigma^2 \right) q_{AB} - \mathcal{E}_{AB} \\
 & -\frac{2}{3} \left[E_{\alpha\beta}e_A^\alpha e_B^\beta + q_{AB} \left(E_{\alpha\beta}s^\alpha s^\beta - \frac{1}{4}E \right) \right] \quad (94)
 \end{aligned}$$

where $F(\mathcal{R})$ is given in Eq. (90). In arriving at the above equation the cosmological constant has been neglected. The electric part of the Weyl tensor in such a static, spherically symmetric spacetime can be written explicitly using two unknown functions depending on the radial coordinate as

$$\begin{aligned}
 \mathcal{E}_{AB} = & -\frac{\kappa_5^4}{\kappa_4^4} \left[U(r) \left(u_{AuB} + \frac{1}{3}\bar{q}_{AB} \right) \right. \\
 & \left. + P(r) \left(r_{ArB} - \frac{1}{3}\bar{q}_{AB} \right) \right] \quad (95)
 \end{aligned}$$

with \bar{q}_{AB} being the induced metric on the $t = \text{constant}$ surface, such that $u^A \bar{q}_{AB} = 0$. The 5-dimensional electric part of the Weyl tensor would have similar decomposition depending on two separate radial functions $\bar{U}(r)$ and $\bar{P}(r)$. On projecting this 5-dimensional tensor $E_{\alpha\beta}$ on the 4-dimensional brane, the effective field equations reduce to

$$\begin{aligned}
 G_{AB} = & -\frac{1}{2}\kappa_5^2 \left(-\frac{1}{\kappa_5^2}F(\mathcal{R}) + \frac{1}{6}\kappa_5^2\Sigma^2 \right) q_{AB} + \frac{\kappa_5^4}{\kappa_4^4} \\
 & \times \left[U(r) \left(u_{AuB} + \frac{1}{3}\bar{q}_{AB} \right) + P(r) \left(r_{ArB} - \frac{1}{3}\bar{q}_{AB} \right) \right] \\
 & + \frac{2}{3} \frac{\kappa_6^4}{\kappa_5^4} \left[\bar{U}(r) \left(u_{AuB} + \frac{1}{2}\bar{q}_{AB} \right) + \bar{P}(r) \left(r_{ArB} - \frac{1}{2}\bar{q}_{AB} \right) \right] \quad (96)
 \end{aligned}$$

Now κ_n and κ_{n-1} are related through Eq. (18), such that $\kappa_6^4 = (16/3)\kappa_5^2(1/\sigma)$, $\kappa_5^4 = (6/\Sigma)\kappa_4^2$, and $\kappa_4^2 = 8\pi G$. Using these relations the above equation can be written in terms of

the brane tensions, σ for the 5-dimensional brane, Σ for the 4-dimensional brane and the gravitational constant G as

$$\begin{aligned}
 G_{AB} = & -\left(-\frac{1}{2}F(\mathcal{R}) + 4\pi G\Sigma \right) q_{AB} + \frac{3}{4\pi G\Sigma} \\
 & \times \left[U(r) \left(u_{AuB} + \frac{1}{3}\bar{q}_{AB} \right) + P(r) \left(r_{ArB} - \frac{1}{3}\bar{q}_{AB} \right) \right] \\
 & + \frac{32}{9\sqrt{6}} \frac{\sqrt{\Sigma}}{\sigma\sqrt{8\pi G}} \\
 & \times \left[\bar{U}(r) \left(u_{AuB} + \frac{1}{2}\bar{q}_{AB} \right) + \bar{P}(r) \left(r_{ArB} - \frac{1}{2}\bar{q}_{AB} \right) \right]. \quad (97)
 \end{aligned}$$

Now assuming a static, spherically symmetric metric ansatz of the form,

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\Omega^2 \quad (98)$$

we obtain various components of the gravitational field equations to be

$$\begin{aligned}
 -e^{-\lambda} \left(-\frac{\lambda'}{r} + \frac{1}{r^2} \right) + \frac{1}{r^2} = & \left(-\frac{1}{2}F(\mathcal{R}) + 4\pi G\Sigma \right) \\
 + \frac{3}{4\pi G\Sigma} U(r) + \frac{32}{9\sqrt{6}} \frac{\sqrt{\Sigma}}{\sigma\sqrt{8\pi G}} \bar{U}(r) \quad (99)
 \end{aligned}$$

$$\begin{aligned}
 e^{-\lambda} \left(\frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = & -\left(-\frac{1}{2}F(\mathcal{R}) + 4\pi G\Sigma \right) \\
 + \frac{1}{4\pi G\Sigma} [U(r) + 2P(r)] + \frac{16}{9\sqrt{6}} \frac{\sqrt{\Sigma}}{\sigma\sqrt{8\pi G}} \\
 \times [\bar{U}(r) + \bar{P}(r)] \quad (100)
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{2}e^{-\lambda} \left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2} \right) \\
 = & -\left(-\frac{1}{2}F(\mathcal{R}) + 4\pi G\Sigma \right) + \frac{1}{4\pi G\Sigma} [U(r) - P(r)] \\
 + \frac{16}{9\sqrt{6}} \frac{\sqrt{\Sigma}}{\sigma\sqrt{8\pi G}} [\bar{U}(r) - \bar{P}(r)]. \quad (101)
 \end{aligned}$$

Hence the effect of multiple extra dimensions is manifested in the lower dimensional effective gravitational field equations through the appearance of extra terms $\bar{U}(r)$ and $\bar{P}(r)$. Also due to $f(\mathcal{R})$ gravity in the bulk there is one more additional term in the effective field equations, namely $F(\mathcal{R})$. In general, these equations has to be supplemented by the conservation equation, which will provide a differential relation connecting $U(r)$, $P(r)$, $\bar{U}(r)$, and $\bar{P}(r)$ which amounts to

$$\frac{1}{3} \left(\frac{dU}{dr} + 2\frac{dP}{dr} \right) + \frac{\nu'}{3} (2U + P) + \frac{2P}{r} = 0 \quad (102)$$

$$\frac{1}{2} \left(\frac{d\bar{U}}{dr} + \frac{d\bar{P}}{dr} \right) + \frac{\nu'}{4} (3\bar{U} + \bar{P}) + \frac{2\bar{P}}{r} = 0. \quad (103)$$

Alike the situation in Einstein–Hilbert action all the objects on the right hand side of Eq. (99) depend only on the radial coordinate which can be integrated leading to the following solution for $e^{-\lambda}$:

$$e^{-\lambda} = 1 - \frac{2GM}{r} - \frac{Q(r)}{r} - \left(-\frac{1}{2}F(\mathcal{R}) + 4\pi G\Sigma \right) \frac{r^2}{3} \tag{104}$$

$$Q(r) = \frac{3}{4\pi G\Sigma} \int dr r^2 U(r) + \frac{32}{9\sqrt{6}\sigma\sqrt{8\pi G}} \int dr r^2 \bar{U}(r). \tag{105}$$

In order to obtain the solution for e^ν we need to know the solutions for the dark radiation and dark pressure terms. For that we will use the equations of state, $2U + P = 0$ and $\bar{U} + \bar{P} = 0$, such that

$$P(r) = -2U(r) = \frac{P_0}{r^4}; \quad \bar{P}(r) = -3\bar{U}(r) = \frac{\bar{P}_0}{r^6} \tag{106}$$

$$Q(r) = Q_0 + \frac{3\alpha P_0}{2r} + \frac{\bar{P}_0\beta}{9r^3};$$

$$\alpha = \frac{1}{4\pi G\Sigma}; \quad \beta = \frac{32}{9\sqrt{6}\sigma\sqrt{8\pi G}}. \tag{107}$$

Then the full static spherically symmetric solution turns out to be

$$e^\nu = e^{-\lambda} = 1 - \frac{2GM + Q_0}{r} - \frac{3\alpha P_0}{2r^2} - \frac{\bar{P}_0\beta}{9r^4} - \left(-\frac{1}{2}F(\mathcal{R}) + 4\pi G\Sigma \right) \frac{r^2}{3}. \tag{108}$$

This is an exact solution to the effective field equations where no assumptions or approximations have been made. The effect of extra spatial dimension appears through the charge term, i.e., r^{-2} term and the r^{-4} term and the effect of modified gravity appears through the r^2 term. Note that if we assume $F(\mathcal{R}) = 8\pi G\Sigma$, then our solution will coincide with the Einstein–Hilbert action. Hence the addition of higher curvature terms to the bulk action can provide a potential solution to the cosmological constant problem.

The static spherically symmetric solution being dS or AdS has three important aspects to explore, namely (a) stability of the solutions, (b) anti-evaporation, and (c) black hole thermodynamics. We will now comment on these features. The stability of the solution can be ascertained by studying perturbations around it. These are again of three types: scalar, vector, and tensor modes. These modes satisfy a set of decoupled wave equations with the following structure:

$$\left(\square - \frac{1}{f(r)}V \right) \Phi = 0. \tag{109}$$

Here \square stands for d'Alembertian operator, $f(r) = e^\nu = e^{-\lambda}$, Φ represents the perturbation modes and V is an appropriate potential connected to the perturbation Φ . For stability, we require the potentials to be positive. It turns out that for all choices of $F(\mathcal{R})$ the scalar and tensor modes have positive potentials. While for the vector mode we have stability for $F(\mathcal{R}) > 8\pi G\Sigma$, while the vector modes are unstable when $F(\mathcal{R}) < 8\pi G\Sigma$.

The anti-evaporation effect of this solution comes into picture when we set $P_0 = 0 = \bar{P}_0$. Then the spherically symmetric solution represents a Schwarzschild (A)de Sitter spacetime, which under proper limit can be mapped to the Nariai spacetime. A black hole in the Nariai spacetime has the peculiar property of increasing surface area due to quantum corrections as shown by Bousso and Hawking [51,52]. This phenomenon of anti-evaporation subsequently was generalized for Nariai black holes in the context of $f(\mathcal{R})$ gravity [53]. Here $f(\mathcal{R})$ gravity plays the role of the anomaly induced effective action leading to anti-evaporation of black holes. In our case as well with $P_0 = 0 = \bar{P}_0$, we have Nariai black hole as a limit. Thus from previous discussions it is clear that our solutions will also exhibit anti-evaporation effect. However, for P_0 and \bar{P}_0 being not equal to zero, our solution cannot be reduced to the Nariai form and thus in general the solution presented here will not exhibit anti-evaporation.

Finally, the case of black hole thermodynamics is at our hand. The horizon associated with the above spacetime structure has the location $r = r_h$, which one can determine by the following equation:

$$1 - \frac{2GM + Q_0}{r_h} - \frac{3\alpha P_0}{2r_h^2} - \frac{\bar{P}_0\beta}{9r_h^4} - \left(-\frac{1}{2}F(\mathcal{R}) + 4\pi G\Sigma \right) \frac{r_h^2}{3} = 0. \tag{110}$$

The mass of the black hole (or, equivalently the internal energy) can be obtained readily from the above equation as

$$M(r_h) = \frac{r_h}{2} - \frac{Q_0}{2} - \frac{3\alpha P_0}{4r_h} - \frac{\bar{P}_0\beta}{6r_h^3} - \left(-\frac{1}{2}F(\mathcal{R}) + 4\pi G\Sigma \right) \frac{r_h^3}{6}. \tag{111}$$

The entropy can be obtained from holography, i.e., $S = k_B A/4\hbar$. In this particular case of spherically symmetric spacetime choosing \hbar and Boltzmann constant appropriately we obtain, $S = r_h^2$. Thus internal energy of the system can be obtained in terms of the entropy from Eq. (111) as

$$M(S) = \frac{\sqrt{S}}{2} - \frac{Q_0}{2} - \frac{3\alpha P_0}{4\sqrt{S}} - \frac{\bar{P}_0\beta}{6S^{3/2}} - \left(-\frac{1}{2}F(\mathcal{R}) + 4\pi G\Sigma \right) \frac{S^{3/2}}{6}. \tag{112}$$

Then the black hole temperature can be obtained in terms of entropy as,

$$T(S) = \left(\frac{\partial M}{\partial S}\right) = \frac{1}{4\sqrt{S}} + \frac{3\alpha P_0}{8S^{3/2}} + \frac{\bar{P}_0\beta}{4S^{5/2}} - \left(-\frac{1}{2}F(\mathcal{R}) + 4\pi G\Sigma\right) \frac{\sqrt{S}}{4} \tag{113}$$

which immediately leads to the following expression for specific heat as

$$C_V = T \left(\frac{\partial S}{\partial T}\right) = \frac{\frac{1}{4\sqrt{S}} + \frac{3\alpha P_0}{8S^{3/2}} + \frac{\bar{P}_0\beta}{4S^{5/2}} + \left(\frac{1}{2}F(\mathcal{R}) - 4\pi G\Sigma\right) \frac{\sqrt{S}}{4}}{\left(\frac{1}{2}F(\mathcal{R}) - 4\pi G\Sigma\right) \frac{1}{8\sqrt{S}} - \frac{1}{8S^{3/2}} - \frac{9\alpha P_0}{16S^{5/2}} - \frac{5\bar{P}_0\beta}{8S^{7/2}}}. \tag{114}$$

It is clear from the above expression that the specific heat shows discontinuity in its behavior, thereby indicating a second order phase transition. The surface of discontinuity in the specific heat is given by

$$\frac{1}{2}F(\mathcal{R}) - 4\pi G\Sigma = \frac{1}{S} + \frac{9\alpha P_0}{2S} + \frac{5\bar{P}_0\beta}{S^{5/2}}. \tag{115}$$

In order to understand physics behind this phase transition, it is always illuminating to discuss some limiting cases. For example, if we assume pure Einstein gravity, the specific heat can be given by Eq. (64), which is everywhere negative. This may also be divergent, provided the entropy becomes negative which cannot be satisfied in general. Thus bulk terms with positive dark pressure cannot lead to a second order phase transition. In the other limit, where $P_0 = 0 = \bar{P}_0$, we get the divergence of specific heat to correspond to the condition: $S^{-1} = (1/2)F(\mathcal{R}) - 4\pi G\Sigma$. Thus for $F(\mathcal{R}) > 8\pi G\Sigma$ we have second order phase transition. Our calculations therefore confirm that the Schwarzschild anti-de Sitter solution shows a second order phase transition.

After deriving the static spherically symmetric solution along with its various properties, it is now time to discuss some novel features of this solution:

- The restrictions provided by $2U(r) + P(r) = 0$ and $\bar{P}(r) + 3\bar{U}(r) = 0$, show that the metric elements satisfy the condition $e^{-\lambda} = e^{\nu}$. The line element so derived depends explicitly on the form of $U(r)$, $\bar{U}(r)$, and the model of gravity $f(\mathcal{R})$. Thus the geometry of the spacetime is uniquely determined only when knowledge about multiple dimensions is obtained through $\bar{U}(r)$ and about the alternative gravity model.
- The solution is also stable under both scalar and tensor perturbations, since the potentials associated with respec-

tive wave equations are positive for scalar and tensor modes. Thus the black hole solution will not change its character under scalar and tensor perturbations.

- Finally, the specific heat associated with the black hole solution shows a second order phase transition. This shows that the black hole transforms from a thermodynamically unstable phase to a thermodynamically stable phase. Moreover, for vanishing of dark radiation we will obtain its mapping to a Nariai black hole and therefore it shows anti-evaporation.

Having discussed the effect on a black hole solution, we will now discuss the cosmological solutions and its implication in detail.

5.2 Cosmological scenario

As in the case of general relativity, in a cosmological scenario as well we can apply the effective field equations and obtain relevant solutions. Then we can check, where they are compatible with standard cosmological evolution. In the previous section we discussed vacuum solutions with the Weyl tensor providing a nonzero effect. Now we will consider exactly the opposite one—we will take $E_{ab} = 0$; i.e. the contribution from the electric part of the Weyl tensor is vanishing but there is matter in the spacetime. Then the effective field equation in four dimensions takes the following form:

$$G_{AB} = \left(-\frac{1}{2}F(\mathcal{R}) + 4\pi G\Sigma\right) q_{AB} + \kappa_4^2 \tau_{AB} + \kappa_5^4 \Upsilon_{AB} + \frac{2}{3}\kappa_5^2 \left(t_{\alpha\beta} e_A^\alpha e_B^\beta + q_{AB} \left\{t_{\alpha\beta} s^\alpha s^\beta - \frac{1}{4}t\right\}\right) + \frac{2}{3}\kappa_6^4 \left(\Pi_{\alpha\beta} e_A^\alpha e_B^\beta + q_{AB} \left\{\Pi_{\alpha\beta} s^\alpha s^\beta - \frac{1}{4}\Pi\right\}\right). \tag{116}$$

In this expression along with brane energy-momentum tensor τ_{AB} and its higher order term Υ_{AB} , two additional contribution from the 5-dimensional matter energy-momentum tensor is present. We assume that the brane is filled with perfect fluid with the following form for the energy-momentum tensor on four dimensions as:

$$\tau_B^A = \text{diag}(-\rho, p, p, p) \tag{117}$$

where ρ represents energy density of the matter fields and p yields its pressure. The energy-momentum tensor in five dimensions is taken to be pressure free dust, such that its energy-momentum tensor has the simple form

$$t_\beta^\alpha = \text{diag}(-\rho_5, 0, 0, 0, 0) \tag{118}$$

where, as before, the subscript ‘5’ indicates that it is matter from a 5-dimensional brane. Thus we arrive at the following expression for the higher order components:

$$\Pi_{tt} = \frac{3}{32}\rho_5^2; \quad \Pi = -\frac{3}{32}\rho_5^2. \tag{119}$$

Having obtained all the components in the above effective field equation, we can obtain the effective field equation itself. Both the time-time and spatial components of the effective equation lead to

$$\begin{aligned} 3H^2 &= -\left(-\frac{1}{2}F(\mathcal{R}) + 4\pi G\Sigma\right) + \kappa_4^2\Lambda_4 + \kappa_4^2\rho + \frac{1}{12}\kappa_5^4\rho^2 \\ &\quad + \frac{1}{2}\kappa_5^2\rho_5 + \frac{3}{64}\kappa_6^4\rho_5^2 \\ H^2 + 2\frac{\ddot{a}}{a} &= -\left(-\frac{1}{2}F(\mathcal{R}) + 4\pi G\Sigma\right) - \kappa_4^2 p \\ &\quad - \frac{1}{12}\kappa_5^4(\rho^2 + 2p\rho) - \frac{1}{6}\kappa_5^2\rho_5 - \frac{1}{64}\kappa_6^4\rho_5^2. \end{aligned} \tag{120}$$

For the bulk matter we have the following relation, namely, $\rho_5 = (\rho_{5,0}/\ell)(1/a^3)$, where ℓ represents the finite length of the 5-dimensional brane coordinate. After obtaining the Friedmann equations on the brane let us write them down in terms of relative abundances $\Omega_i = \rho_{i,0}/\rho_c$. The energy density appearing on the right hand side of Friedmann equations can be expanded in terms of baryonic and relativistic matter fields. For higher dimensional matter being non-relativistic the Hubble parameter turns out to be

$$H^2 = H_0^2 \left[\Omega_\Lambda + (\Omega_B + \Omega_5) \left(\frac{a_0}{a}\right)^3 + \Omega_R \left(\frac{a_0}{a}\right)^4 \right] \tag{121}$$

where $\Omega_5 = (\kappa_5^2/16\pi G\ell)(\rho_{5,0}/\rho_c)$ and Ω_Λ is from the combination $[-(1/2)F(\mathcal{R}) + 4\pi G\Sigma]$. Thus we have reproduced the standard Λ CDM model. At late times Ω_Λ would dominate leading to accelerated expansion of the universe, however, this acceleration is originating since the bulk spacetime is being described by $f(\mathcal{R})$ gravity theory and we are living on a brane with nonzero brane tension (the Σ term). Starting from the $1/a^4$ term, leading to a radiation dominated universe, we have a matter (i.e., non-relativistic matter) dominated epoch such that $H^2 \sim a^{-3}$ and finally the accelerating sector. The intriguing fact about the matter sector is that along with the baryonic matter, due to the existence of extra spatial dimensions we have an additional contribution. This additional contribution might act as a dark matter candidate. In standard Λ CDM cosmology, the dark matter has to be included by hand, while in our model it appears *naturally*. To get an estimate of 5-dimensional parameters we can use the epoch of matter–radiation equality, leading to

$$1 + z_{eq} = 3.9 \times 10^4 \left(0.022 + \frac{\kappa_5^2 h^2}{16\pi G\ell} \frac{\rho_{5,0}}{\rho_c} \right) \tag{122}$$

where $h = 0.72$, ℓ stands for the length of the extra dimension. From the redshift z_{eq} one can determine the value of the higher dimensional gravitational constant κ_5^2 . On the other hand, unlike the Einstein–Hilbert bulk, in this case as well by tuning the $f(\mathcal{R})$ model itself it is possible to have inflation. Thus the effective field equations on the brane are capable to explain the various phases of the universe.

Let us now explore the effect of additional correction terms on the Hubble parameter. To confine ourselves to one representative example, if we assume the matter in the 4-dimensional brane to be pressure-less, then we have

$$\begin{aligned} H^2 &= -\frac{1}{3} \left(-\frac{1}{2}F(\mathcal{R}) + 4\pi G\Sigma \right) + \frac{8\pi G}{3} \frac{\rho_0}{a^3} + \frac{1}{36}\kappa_5^4 \frac{\rho_0^2}{a^6} \\ &\quad + \frac{1}{6}\kappa_5^2 \frac{\rho_{5,0}}{\ell} \frac{1}{a^3} + \frac{1}{64}\kappa_6^4 \frac{\rho_{5,0}^2}{\ell^2} \frac{1}{a^6} \end{aligned} \tag{123}$$

where ρ_0 represents the value of the energy density in the present epoch and $\rho_{5,0}$ represents the same for 5-dimensional matter. The above expression for Hubble parameter can be simplified and arranged properly, leading to

$$\begin{aligned} H^2 &= -\frac{1}{3} \left(-\frac{1}{2}F(\mathcal{R}) + 4\pi G\Sigma \right) + \left\{ \frac{8\pi G}{3} \rho_0 + \frac{1}{6}\kappa_5^2 \frac{\rho_{5,0}}{\ell} \right\} \\ &\quad \times \frac{1}{a^3} + \left\{ \frac{1}{36}\kappa_5^4 \rho_0^2 + \frac{1}{64}\kappa_6^4 \frac{\rho_{5,0}^2}{\ell^2} \right\} \frac{1}{a^6}. \end{aligned} \tag{124}$$

Following the techniques developed in [49] we can introduce three constants, namely:

$$\begin{aligned} C_f &= -\frac{1}{3} \left(-\frac{1}{2}F(\mathcal{R}) + 4\pi G\Sigma \right); \\ C_\rho &= \frac{8\pi G}{3} \rho_0 + \frac{1}{6}\kappa_5^2 \frac{\rho_{5,0}}{\ell}; \\ C_{Sq} &= \frac{1}{36}\kappa_5^4 \rho_0^2 + \frac{1}{64}\kappa_6^4 \frac{\rho_{5,0}^2}{\ell^2} \end{aligned} \tag{125}$$

such that the following solution to the scale factor is obtained,

$$\begin{aligned} &\exp[3\sqrt{C_f}(t - t_0)] \\ &= \frac{2\sqrt{C_f}\sqrt{C_f a^6 + C_\rho a^3 + C_{Sq}} + 2C_f a^3 + C_\rho}{2\sqrt{C_f}\sqrt{C_f + C_\rho + C_{Sq}} + 2C_f + C_\rho}; \\ &C_f > 0 \end{aligned} \tag{126}$$

$$\begin{aligned} &3\sqrt{-C_f}(t_0 - t) \\ &= \sin^{-1} \left(\frac{2C_f a^3 + C_\rho}{\sqrt{C_\rho^2 - 4C_f C_{Sq}}} \right) - \sin^{-1} \left(\frac{2C_f + C_\rho}{\sqrt{C_\rho^2 - 4C_f C_{Sq}}} \right); \\ &C_f < 0, \quad C_\rho^2 > 4C_f C_{Sq}. \end{aligned} \tag{127}$$

Though the solutions are a bit complicated, we can impose certain conditions on them such that these solutions simplify substantially. For that purpose we consider the situation where $C_f = 0$, i.e., $F(\mathcal{R}) = 8\pi G \Sigma$. Since Σ is positive definite the above result suggests that $F(\mathcal{R})$ is positive. Applying this for the matter dominated epoch, the Hubble parameter from Eq. (124) turns out to be

$$H^2 = \frac{8\pi G}{3} \rho \left[1 + \frac{\kappa_5^2}{16\pi G} \frac{\rho_5}{\rho} - \frac{4\pi G \rho}{F(\mathcal{R})} \left(1 + \frac{9}{16} \frac{\kappa_6^4}{\kappa_5^4} \frac{\rho_5^2}{\rho^2} \right) \right]. \tag{128}$$

Assume ρ to be energy density of non-relativistic matter we arrive at

$$H^2 = \left\{ \frac{8\pi G \rho_0}{3} + \frac{\kappa_5^2}{6} \frac{\rho_{5,0}}{\ell} \right\} \frac{1}{a^3} + \frac{4\pi G \rho_0^2}{3\Sigma} \times \left(1 + \frac{9}{16} \frac{\kappa_6^4}{\kappa_5^4} \frac{\rho_{5,0}^2}{\ell^2 \rho_0^2} \right) \frac{1}{a^6} \tag{129}$$

leading to the following solution for the scale factor:

$$a^3 = \left\{ 6\pi G \rho_0 + 3 \frac{\kappa_5^2}{8} \frac{\rho_{5,0}}{\ell} \right\} t^2 + \left\{ \sqrt{\frac{12\pi G \rho_0^2}{\Sigma} \left(1 + \frac{9}{16} \frac{\kappa_6^4}{\kappa_5^4} \frac{\rho_{5,0}^2}{\ell^2 \rho_0^2} \right)} \right\} t. \tag{130}$$

It is clear from the above expression that the universe undergoes a transition in the expansion rate. The time scale when this happened depends on whether the brane matter energy dominates over the bulk matter or not. Below we present the result for the timescale of the transition for both the situations:

$$t \sim \sqrt{\frac{1}{3\pi G \Sigma}} = \sqrt{-\frac{8}{3F(\mathcal{R})}}; \quad (\rho_{5,0}/\ell \ll \rho_0) \tag{131}$$

$$t \sim \frac{\kappa_6^2}{\kappa_5^2} \left(1 - \frac{16\pi G \rho_0 \ell}{\kappa_5^2 \rho_{5,0}} \right) \quad (\rho_{5,0}/\ell \gg \rho_0). \tag{132}$$

Thus at early universe we have a high energy regime, where $a \sim t^{1/3}$, while at late time we have a low energy regime; the scale factor variation with time modifies to $a \sim t^{2/3}$, which is the standard evolution of the matter field. Note that in Einstein gravity as well we had a similar transition in the scale factor. But in this particular case, importantly the transition time depends crucially on the $f(\mathcal{R})$ model we are considering. A similar phenomenon will be observed for the radiation dominated epoch as well.

The key aspects of the above results are summarized as follows:

- First and foremost, the issue of dark energy in this particular case is resolved by the addition of higher curvature terms to the Einstein–Hilbert action and through the brane tension. The same $f(\mathcal{R})$ model can also explain the inflationary scenario without invoking any scalar field in the picture.
- The cosmology as presented by modified Friedmann equations automatically leads to an additional matter component originating from higher dimensions, which behaves as a dark matter candidate.
- Finally, note that due to the presence of additional correction terms in the effective equations on the brane few new cosmological phases enter the picture. For example, the matter dominated epoch gets divided into two parts and this depends explicitly on the $f(\mathcal{R})$ model under consideration. These modifications will have indirect effects on the structure formation, which can be used to constrain not only higher dimensional parameters but also the form of alternative gravity theories.

Thus with matter present in the 5-dimensional brane we can have standard cosmology with proper scaling of the scale factor with time. Moreover, from the effective field equation it is clear that the 5-dimensional matter adds to the 4-dimensional one and enhances the total non-relativistic matter content of the universe (the 4-dimensional brane we live in). The structure of the angular power spectrum, perturbation in this multiple extra-dimension scenario can lead to interesting and important results by providing constraints on various parameters in this models.

It is possible to ascribe the cosmological solutions so obtained holographic interpretations. Following [54] it suffices to show that the usual 4-dimensional cosmological equations can be rewritten as a 2-dimensional conformal field theory (CFT). For this we can define the Hawking, Bekenstein, and Bekenstein–Hawking entropies as

$$S_H = \frac{(n-1)HV}{4G}, \quad S_{BH} = \frac{(n-1)V}{4Ga}, \quad S_B = \frac{2\pi a}{n} E \tag{133}$$

where $E = \rho V$ is the total energy within a volume V and $\rho_m \propto a^{-n}$, but $\rho = \rho_m + F(\mathcal{R})/16\pi G$. This is how the effect of $f(\mathcal{R})$ gravity enters the picture, by altering the definition of the energy. Furthermore the total entropy is by the Hubble entropy S_H , which is the entropy of the black hole with the radius of the Hubble size. Using these definitions the Friedman equations can be rewritten as

$$S_H = \frac{2\pi a}{n} \sqrt{E_{BH} (2E - K E_{BH})}; \quad K E_{BH} = n (E + pV - T_H S_H) \tag{134}$$

with energy of the black hole being $E_{\text{BH}} = n(n-1)V/8\pi Ga^2$ and Hawking temperature being $T_{\text{H}} = -\dot{H}/2\pi H$. This is exactly the form one would have from the CV entropy relation. It explicitly shows the connection between 4-dimensional classical gravitational physics and 2-dimensional quantum 2-dimensional CFT [55]. One can also consider dS or AdS black holes in the bulk, such that the brane equations of motion can be cast in the form of Friedman equations. By introducing brane matter the cosmological equations can be written in the CV form which is related to AdS black hole entropy [56].

6 Discussion

Motivated by the success of extra dimensional models to explain various physical problems, e.g., hierarchy problem, cosmological constant problem, in this work we have constructed and worked within the premises of a very general higher dimensional model. Since gravity can propagate in the full bulk spacetime, the field equations for gravity on a lower dimensional brane is taken to be induced from the higher dimensional one. Previous work in this direction involved two assumptions, namely, the bulk is 5-dimensional while the brane is 4-dimensional and gravity in the bulk is originating from the Einstein–Hilbert action. However, in the spirit of string theory it is pertinent to ask what happens to this induced gravity program as the bulk spacetime is n -dimensional while the brane is $(n-m)$ -dimensional, for arbitrary m . Also from the recent success of $f(\mathcal{R})$ gravity it is interesting to discuss the situation when the bulk spacetime has $f(\mathcal{R})$ as the action, not \mathcal{R} . Below we summarize all the results obtained in this work along these lines:

- Starting from a n -dimensional bulk spacetime endowed with Einstein gravity, we first derive the effective field equations on $(n-1)$ and $(n-2)$ spacetime dimensions. Taking a cue from this analysis we could read off the effective field equations in $(n-m)$ dimensions for arbitrary m .
- As an application of this result, we have illustrated two situations. One, in which there is no matter but bulk effects populate the right hand side of effective field equations. The static spherically symmetric solution so obtained has the standard $1 - (2GM/r)$ term, with additional terms originating from the bulk Weyl tensor. These additional terms have contributions from both the 5- and 6-dimensional Weyl tensors (in particular its electric part). Second, the cosmological situation with matter but no bulk geometric effect. From the solutions so obtained it becomes evident that matter in higher dimensions can add to the matter content of the visible brane and has

the potential to act as dark matter source. More importantly, the effective Friedman equations can be cast in holographic language, such that it can be related to 2-dimensional CFT.

- Modifying Einstein–Hilbert action in the bulk by addition of $f(\mathcal{R})$ term is the next immediate generalization. In this context as well, we have first derived the effective field equations in $(n-1)$ -dimensional brane starting from n -dimensional bulk. Then we have obtained the effective field equations on a $(n-m)$ -dimensional brane, starting from the original n -dimensional bulk.
- Like the situation in general relativity, in this context as well we have discussed two cases. For the spherically symmetric, static vacuum spacetime, $f(\mathcal{R})$ gravity modifies the general relativity solution significantly. The solution also exhibits two remarkable effects. First of all, it shows a divergent specific heat, which points to the effect of a phase transition. Second, it can be written in Nariai form, which shows the anti-evaporation effect.
- In the cosmological context as well $f(\mathcal{R})$ gravity modifies the cosmological constant and the transition time scale depends explicitly on the $f(\mathcal{R})$ model of gravity under consideration. Also the $f(\mathcal{R})$ model of gravity exhibits the CV entropy relation, such that the Friedman equations can be rewritten in terms of 2-dimensional CFT.

Thus in a nutshell, starting from Einstein's theory in the bulk, which is taken to be n -dimensional we have constructed various induced objects on any arbitrary $(n-m)$ -dimensional brane providing a complete generalization of existing set up. We generalize this analysis further by incorporating $f(\mathcal{R})$ gravity in the bulk instead of Einstein gravity. In $f(\mathcal{R})$ gravity as well we follow the same procedure by relating geometrical objects like extrinsic curvature, Riemann curvature in the n -dimensional bulk with the $(n-m)$ -dimensional brane. In this case as well from the structure of the effective equation we can immediately generalize the set up on an arbitrary lower dimensional hypersurface as well.

Having derived all the necessary theoretical ingredients, we consider possible applications of our results in both the gravity theories. As an illustration, we took a 6-dimensional bulk spacetime in which a 4-dimensional brane is embedded. In static spherically symmetric vacuum spacetime we have additional terms from the bulk Weyl tensor as the source of gravity. For $f(\mathcal{R})$ gravity we have an equivalent of the cosmological constant making the solutions asymptotically (anti) de Sitter. In a cosmological context the 5-dimensional sub-manifold contains matter showing the standard cosmology to be an inherent and natural consequence of our model. Finally, we comment on the possibility of the bulk matter as a possible candidate of dark matter in the lower dimensional brane.

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A Appendix: Detailed calculations

In this section, we provide some detailed expressions, which have been used in the main text for arriving at various results.

A.1 Identities for the derivation of $(n - 1)$ -dimensional effective field equation

The starting point of obtaining the effective field equation is the Gauss–Codazzi equation, which relates the $(n - 1)$ -dimensional curvature tensor to the n -dimensional curvature tensor. For our purpose these two tensors are related through the following relation:

$$^{(n-1)}R_{\alpha\beta\mu\nu} = ^{(n)}R_{abcd}e_{\alpha}^ae_{\beta}^be_{\mu}^ce_{\nu}^d - \epsilon(K_{\alpha\nu}K_{\beta\mu} - K_{\alpha\mu}K_{\beta\nu}) \tag{135}$$

where we have used the definition

$$K_{\alpha\beta} = e_{\alpha}^ae_{\beta}^b\nabla_a n_b \tag{136}$$

with $\epsilon = n_i n^i$. Contraction of Eq. (135) with the $(n - 1)$ -dimensional metric $h_{\alpha\beta} = e_{\alpha}^ae_{\beta}^b(g_{ab} - \epsilon n_a n_b)$ leads to the connection between the $(n - 1)$ -dimensional Ricci tensor and the n -dimensional curvature tensor:

$$\begin{aligned} ^{(n-1)}R_{\alpha\mu} &= h^{\beta\nu} ^{(n-1)}R_{\alpha\beta\mu\nu} \\ &= ^{(n)}R_{abcd}e_{\alpha}^ae_{\beta}^be_{\mu}^ce_{\nu}^dh^{\beta\nu} \\ &\quad - \epsilon(K_{\alpha\nu}K_{\beta\mu} - K_{\alpha\mu}K_{\beta\nu})h^{\beta\nu} \\ &= ^{(n)}R_{abcd}e_{\alpha}^ae_{\mu}^c(g^{bd} - \epsilon n^b n^d) \\ &\quad - \epsilon(K_{\alpha\nu}K_{\mu}^{\nu} - K_{\alpha\mu}K) \\ &= ^{(n)}R_{ac}e_{\alpha}^ae_{\mu}^c - \epsilon ^{(n)}R_{abcd}e_{\alpha}^an^be_{\mu}^cn^d \\ &\quad - \epsilon(K_{\alpha\nu}K_{\mu}^{\nu} - K_{\alpha\mu}K). \end{aligned} \tag{137}$$

The Ricci scalar can now be obtained:

$$\begin{aligned} ^{(n-1)}R &= h^{\alpha\mu} ^{(n-1)}R_{\alpha\mu} \\ &= ^{(n)}R_{ac}e_{\alpha}^ae_{\mu}^ch^{\alpha\mu} - \epsilon ^{(n)}R_{abcd}e_{\alpha}^an^be_{\mu}^cn^dh^{\alpha\mu} \\ &\quad - \epsilon(K_{\alpha\nu}K_{\mu}^{\nu} - K_{\alpha\mu}K)h^{\alpha\mu} \end{aligned}$$

$$\begin{aligned} &= ^{(n)}R_{ac}(g^{ac} - \epsilon n^a n^c) \\ &\quad - \epsilon ^{(n)}R_{abcd}n^bn^d(g^{ac} - \epsilon n^a n^c) \\ &\quad - \epsilon(K_{\mu\nu}K^{\mu\nu} - K^2) \\ &= ^{(n)}R - 2\epsilon ^{(n)}R_{ac}n^an^c - \epsilon(K_{\mu\nu}K^{\mu\nu} - K^2). \end{aligned} \tag{138}$$

The above relation can be further simplified by using an identity for $^{(n)}R_{ab}n^an^b$, which can be written as

$$\begin{aligned} ^{(n)}R_{ab}n^an^b &= \nabla_a(n^b\nabla_b n^a) - \nabla_a n^b\nabla_b n^a \\ &\quad - \nabla_b(n^b\nabla_a n^a) + (\nabla_b n^b)^2 \\ &= -\nabla_i(Kn^i - a^i) - K_{\alpha\beta}K^{\alpha\beta} + K^2. \end{aligned} \tag{139}$$

Then we obtain $^{(n)}R$ in terms of $^{(n-1)}R$:

$$^{(n)}R = ^{(n-1)}R - \epsilon(K_{\mu\nu}K^{\mu\nu} - K^2) - 2\epsilon\nabla_i(Kn^i - a^i). \tag{140}$$

The relation between $(n - 1)$ -dimensional Einstein tensor and n -dimensional Riemann tensor is therefore given by

$$\begin{aligned} ^{(n-1)}G_{\alpha\mu} &= ^{(n-1)}R_{\alpha\mu} - \frac{1}{2}h_{\alpha\mu} ^{(n-1)}R \\ &= ^{(n)}R_{ac}e_{\alpha}^ae_{\mu}^c - \epsilon ^{(n)}R_{abcd}e_{\alpha}^an^be_{\mu}^cn^d \\ &\quad - \epsilon(K_{\alpha\nu}K_{\mu}^{\nu} - K_{\alpha\mu}K) - \frac{1}{2}h_{\alpha\mu} \\ &\quad \times [^{(n)}R - 2\epsilon ^{(n)}R_{ac}n^an^c - \epsilon(K_{\mu\nu}K^{\mu\nu} - K^2)] \\ &= \left(^{(n)}R_{ac} - \frac{1}{2}g_{ac} ^{(n)}R \right) e_{\alpha}^ae_{\mu}^c \\ &\quad + \epsilon ^{(n)}R_{ac}n^an^c h_{\alpha\mu} - \epsilon ^{(n)}R_{abcd}e_{\alpha}^an^be_{\mu}^cn^d \\ &\quad + \epsilon \left[K K_{\alpha\mu} - K_{\alpha\beta}K_{\mu}^{\beta} - \frac{1}{2}h_{\alpha\mu}(K^2 - K_{\mu\nu}K^{\mu\nu}) \right]. \end{aligned} \tag{141}$$

The Riemann tensor can be decomposed into a Ricci tensor, a Ricci scalar, and a Weyl curvature tensor in n dimensions:

$$\begin{aligned} ^{(n)}R_{abcd} &= ^{(n)}C_{abcd} + \frac{1}{n-2} \\ &\quad \times (g_{ac}R_{bd} - g_{ad}R_{bc} - g_{bc}R_{ad} + g_{bd}R_{ac}) \\ &\quad - \frac{1}{(n-1)(n-2)}(g_{ac}g_{bd} - g_{ad}g_{bc})R, \end{aligned} \tag{142}$$

which suggests

$$\begin{aligned} ^{(n)}R_{abcd}e_{\alpha}^an^be_{\mu}^cn^d &= ^{(n)}C_{abcd}e_{\alpha}^an^be_{\mu}^cn^d \\ &\quad + \frac{1}{n-2}(R_{ac}n^an^c h_{\alpha\mu} + \epsilon R_{ac}e_{\alpha}^ae_{\mu}^c) \\ &\quad - \frac{\epsilon}{(n-1)(n-2)}h_{\alpha\mu}R. \end{aligned} \tag{143}$$

Then substitution of the above equation in Eq. (141) leads to

$$\begin{aligned}
 {}^{(n-1)}G_{\alpha\beta} &= {}^{(n)}G_{ab}e_{\alpha}^a e_{\beta}^b + \epsilon \left(\frac{n-3}{n-2} \right) {}^{(n)} \\
 &\times R_{ab}n^a n^b h_{\alpha\beta} - \frac{1}{n-2} {}^{(n)}R_{ab}e_{\alpha}^a e_{\beta}^b \\
 &+ \frac{1}{(n-1)(n-2)} {}^{(n)}R h_{\alpha\beta} - \epsilon {}^{(n)}E_{\alpha\beta} \\
 &+ \epsilon \left[K K_{\alpha\beta} - K_{\alpha\mu} K_{\beta}^{\mu} - \frac{1}{2} h_{\alpha\beta} (K^2 - K_{\mu\nu} K^{\mu\nu}) \right]
 \end{aligned} \tag{144}$$

where we have defined ${}^{(n)}E_{\alpha\beta} = {}^{(n)}C_{abcd}e_{\alpha}^a n^b e_{\beta}^c n^d$. This yields the n -dimensional Einstein equation, with the following expressions:

$$\begin{aligned}
 {}^{(n)}R_{ab} - \frac{1}{2}g_{ab} {}^{(n)}R &= \kappa_n^2 T_{ab}; \quad {}^{(n)}R = -\frac{2}{n-2}\kappa_n^2 T; \\
 {}^{(n)}R_{ab} &= \kappa_n^2 \left(T_{ab} - \frac{1}{n-2}g_{ab}T \right).
 \end{aligned} \tag{145}$$

We therefore have the following expression:

$$\begin{aligned}
 {}^{(n)}G_{ab}e_{\alpha}^a e_{\beta}^b + \epsilon \left(\frac{n-3}{n-2} \right) {}^{(n)}R_{ab}n^a n^b h_{\alpha\beta} \\
 - \frac{1}{n-2} {}^{(n)}R_{ab}e_{\alpha}^a e_{\beta}^b + \frac{1}{(n-1)(n-2)} {}^{(n)}R h_{\alpha\beta} \\
 = \kappa_n^2 T_{ab}e_{\alpha}^a e_{\beta}^b + \epsilon \kappa_n^2 \left(\frac{n-3}{n-2} \right) \\
 \times \left(T_{ab} - \frac{1}{n-2}g_{ab}T \right) n^a n^b h_{\alpha\beta} \\
 - \kappa_n^2 \frac{1}{n-2} \left(T_{ab} - \frac{1}{n-2}g_{ab}T \right) e_{\alpha}^a e_{\beta}^b \\
 - \kappa_n^2 \frac{2}{(n-1)(n-2)^2} h_{\alpha\beta} T \\
 = \kappa_n^2 \frac{n-3}{n-2} \left[T_{ab}e_{\alpha}^a e_{\beta}^b + h_{\alpha\beta} \left\{ \epsilon T_{ab}n^a n^b - \frac{1}{n-1}T \right\} \right].
 \end{aligned} \tag{146}$$

This is the result used in Sect. 2.1.

A.2 Identities for the derivation of $(n-2)$ -dimensional effective field equation

We start with the connection between $(n-2)$ -dimensional curvature tensor and $(n-1)$ -dimensional curvature tensor:

$$\begin{aligned}
 {}^{(n-2)}R_{ABCD} &= {}^{(n-1)}R_{\alpha\beta\mu\nu}e_A^{\alpha}e_B^{\beta}e_C^{\mu}e_D^{\nu} \\
 &- (\mathcal{K}_{AD}\mathcal{K}_{BC} - \mathcal{K}_{AC}\mathcal{K}_{BD})
 \end{aligned} \tag{147}$$

where we have the extrinsic curvature on the $(n-2)$ -dimensional surface: $\mathcal{K}_{AB} = e_A^{\alpha}e_B^{\beta}\nabla_{\alpha}s_{\beta}$, with s_{α} being the normal to the surface. Then the induced $(n-2)$ -dimensional metric turns out to be $q_{AB} = e_A^{\alpha}e_B^{\beta}(h_{\alpha\beta} - s_{\alpha}s_{\beta})$. Contracting the above equation with q^{BD} leads to

$$\begin{aligned}
 {}^{(n-2)}R_{AC} &= {}^{(n-1)}R_{\alpha\beta\mu\nu}e_A^{\alpha}e_B^{\beta}e_C^{\mu}e_D^{\nu}q^{BD} \\
 &- (\mathcal{K}_{AD}\mathcal{K}_C^D - \mathcal{K}_{AC}\mathcal{K}) \\
 &= {}^{(n-1)}R_{\alpha\beta\mu\nu}e_A^{\alpha}e_C^{\mu}(h^{\beta\nu} - s^{\beta}s^{\nu}) \\
 &- (\mathcal{K}_{AD}\mathcal{K}_C^D - \mathcal{K}_{AC}\mathcal{K}) \\
 &= {}^{(n-1)}R_{\alpha\mu}e_A^{\alpha}e_C^{\mu} - {}^{(n-1)}R_{\alpha\beta\mu\nu}e_A^{\alpha}s^{\beta}e_C^{\mu}s^{\nu} \\
 &- (\mathcal{K}_{AD}\mathcal{K}_C^D - \mathcal{K}_{AC}\mathcal{K}).
 \end{aligned} \tag{148}$$

On further contraction we arrive at the following result:

$$\begin{aligned}
 {}^{(n-2)}R &= q^{AB} {}^{(n-2)}R_{AB} \\
 &= {}^{(n-1)}R - 2 {}^{(n-1)}R_{\alpha\mu}s^{\alpha}s^{\mu} - (\mathcal{K}_{AB}\mathcal{K}^{AB} - \mathcal{K}^2).
 \end{aligned} \tag{149}$$

Then we can use Eq. (139) to obtain an identical relation for ${}^{(n-1)}R_{\alpha\mu}s^{\alpha}s^{\mu}$. This leads to the relation

$$\begin{aligned}
 {}^{(n-1)}R &= {}^{(n-2)}R - (\mathcal{K}_{AB}\mathcal{K}^{AB} - \mathcal{K}^2) \\
 &- 2D_{\alpha}(\mathcal{K}s^{\alpha} - s^{\beta}D_{\beta}s^{\alpha}),
 \end{aligned} \tag{150}$$

which on using Eq. (140) yields

$$\begin{aligned}
 {}^{(n)}R &= {}^{(n-2)}R - (\mathcal{K}_{AB}\mathcal{K}^{AB} - \mathcal{K}^2) - 2D_{\alpha}(\mathcal{K}s^{\alpha} - s^{\beta}D_{\beta}s^{\alpha}) \\
 &+ (K_{\mu\nu}K^{\mu\nu} - K^2) + 2\nabla_i(Kn^i - a^i).
 \end{aligned} \tag{151}$$

In order to obtain the effective equation on this $(n-2)$ -dimensional surface we need various contractions of the Riemann tensor. For that purpose we start with the following identity:

$$\begin{aligned}
 {}^{(n-1)}R_{\alpha\beta\mu\nu}e_A^{\alpha}s^{\beta}e_B^{\mu}s^{\nu} \\
 = \left\{ {}^{(n-1)}C_{\alpha\beta\mu\nu} + \frac{1}{n-3}(h_{\alpha\mu} {}^{(n-1)}R_{\beta\nu} - h_{\alpha\nu} {}^{(n-1)}R_{\beta\mu}) \right. \\
 - h_{\beta\mu} R_{\alpha\nu}^{(n-1)} + h_{\beta\nu} R_{\alpha\mu}^{(n-1)} \\
 \left. - \frac{1}{(n-2)(n-3)} {}^{(n-1)}R(h_{\alpha\mu}h_{\beta\nu} - h_{\alpha\nu}h_{\beta\mu}) \right\} \\
 \times e_A^{\alpha}s^{\beta}e_B^{\mu}s^{\nu} = \mathcal{E}_{AB} + \frac{1}{(n-3)} \times (q_{AB} {}^{(n-1)}R_{\alpha\beta}s^{\alpha}s^{\beta} \\
 + {}^{(n-1)}R_{\alpha\beta}e_A^{\alpha}e_B^{\beta}) - \frac{1}{(n-2)(n-3)} {}^{(n-1)}Rq_{AB}
 \end{aligned} \tag{152}$$

where we have defined the tensor $\mathcal{E}_{AB} = {}^{(n-1)}C_{\alpha\beta\mu\nu}e_A^{\alpha}s^{\beta}e_B^{\mu}s^{\nu}$. Then the effective equation on the $(n-2)$ -dimensional hypersurface turns out to be

$$\begin{aligned}
 {}^{(n-2)}G_{AB} &= {}^{(n-1)}G_{\alpha\beta}e_A^\alpha e_B^\beta + {}^{(n-1)}R_{\alpha\beta} s^\alpha s^\beta q_{AB} - \mathcal{E}_{AB} \\
 &\quad - \frac{1}{(n-3)}(q_{AB} {}^{(n-1)}R_{\alpha\beta} s^\alpha s^\beta + {}^{(n-1)}R_{\alpha\beta} e_A^\alpha e_B^\beta) \\
 &\quad + \frac{1}{(n-2)(n-3)}q_{AB} {}^{(n-1)}R \\
 &\quad + \left[\mathcal{K}_{AB}\mathcal{K} - \mathcal{K}_{AC}\mathcal{K}_B^C - \frac{1}{2}q_{AB}(\mathcal{K}^2 - \mathcal{K}_{AB}\mathcal{K}^{AB}) \right] \\
 &= {}^{(n-1)}G_{\alpha\beta}e_A^\alpha e_B^\beta + \frac{(n-4)}{(n-3)}q_{AB} {}^{(n-1)}R_{\alpha\beta} s^\alpha s^\beta \\
 &\quad - \frac{1}{(n-3)} {}^{(n-1)}R_{\alpha\beta} e_A^\alpha e_B^\beta \\
 &\quad + \frac{1}{(n-2)(n-3)}q_{AB} {}^{(n-1)}R - \mathcal{E}_{AB} \\
 &\quad + \left[\mathcal{K}_{AB}\mathcal{K} - \mathcal{K}_{AC}\mathcal{K}_B^C - \frac{1}{2}q_{AB}(\mathcal{K}^2 - \mathcal{K}_{CD}\mathcal{K}^{CD}) \right]. \tag{153}
 \end{aligned}$$

Now instead of ${}^{(n-1)}R_{\alpha\beta}$, we can use the relation ${}^{(n-1)}R_{\alpha\beta} = {}^{(n-1)}G_{\alpha\beta} + \frac{1}{2}h_{\alpha\beta} {}^{(n-1)}R$ to replace the Ricci tensors in terms of Ricci scalar and Einstein tensors. This leads to

$$\begin{aligned}
 {}^{(n-2)}G_{AB} &= \frac{(n-4)}{(n-3)} \left[{}^{(n-1)}G_{\alpha\beta}e_A^\alpha e_B^\beta \right. \\
 &\quad \left. + \left\{ {}^{(n-1)}G_{\alpha\beta} s^\alpha s^\beta + \frac{(n-3)}{2(n-2)} {}^{(n-1)}R \right\} q_{AB} \right] \\
 &\quad - \mathcal{E}_{AB} + \left[\mathcal{K}_{AB}\mathcal{K} - \mathcal{K}_{AC}\mathcal{K}_B^C - \frac{1}{2}q_{AB}(\mathcal{K}^2 - \mathcal{K}_{CD}\mathcal{K}^{CD}) \right]. \tag{154}
 \end{aligned}$$

Now introducing the Einstein equation in $(n-1)$ dimensions we arrive at

$${}^{(n-1)}G_{\alpha\beta} = \kappa_{n-1}^2 \mathcal{T}_{\alpha\beta}; \quad {}^{(n-1)}R = -\frac{2}{(n-3)}\kappa_{n-1}^2 \mathcal{T}. \tag{155}$$

Using this expression in Eq. (154) leads to the effective field equation in $(n-2)$ dimensions, which is presented in Sect. 2.2.

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