Eur. Phys. J. C (2015) 75:528
DOI 10.1140/epjc/s10052-015-3760-7

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Realizations of κ-Minkowski space, Drinfeld twists, and related symmetry algebras

Tajron Jurić[a], **Stjepan Meljanac**[b], **Danijel Pikutić**[c]

Theoretical Physics Division, Ruđer Bošković Institute, Bijenička c.54, 10002 Zagreb, Croatia

Received: 18 June 2015 / Accepted: 25 October 2015 / Published online: 5 November 2015

Abstract Realizations of κ-Minkowski space linear in momenta are studied for time-, space- and light-like deformations. We construct and classify all such linear realizations and express them in terms of the $\mathfrak{gl}(n)$ generators. There are three one-parameter families of linear realizations for time-like and space-like deformations, while for light-like deformations, there are only four linear realizations. The relation between a deformed Heisenberg algebra, the star product, the coproduct of momenta, and the twist operator is presented. It is proved that for each linear realization there exists a Drinfeld twist satisfying normalization and cocycle conditions. κ-Deformed $\mathfrak{igl}(n)$-Hopf algebras are presented for all cases. The κ-Poincaré–Weyl and κ-Poincaré–Hopf algebras are discussed. The left–right dual κ-Minkowski algebra is constructed from the transposed twists. The corresponding realizations are nonlinear. All Drinfeld twists related to κ-Minkowski space are obtained from our construction. Finally, some physical applications are discussed.

Contents

1 Introduction

One of the biggest problems in fundamental theoretical physics is a great difficulty to reconcile quantum mechanics and general theory of relativity in order to formulate consistent theory of quantum gravity. It is argued that at very high energies the gravitational effects can no longer be neglected and that the spacetime is no longer a smooth manifold, but rather a fuzzy, or rather a non-commutative space [1,2]. Physical theories on such non-commutative manifolds require a new framework. This new framework is provided by non-commutative geometry [3]. In this framework, the search for generalized (quantum) symmetries that leave the physical action invariant leads to deformation of Poincaré symmetry,

[a] e-mail: tjuric@irb.hr
[b] e-mail: meljanac@irb.hr
[c] e-mail: dpikutic@irb.hr

with κ-Poincaré symmetry being one of the most extensively studied [4–10].

κ-Deformed Poincaré symmetry is algebraically described by the κ-Poincaré–Hopf algebra and is an example of deformed relativistic symmetry that can possibly describe the physical reality at the Planck scale. κ is the deformation parameter, usually interpreted as the Planck mass or some quantum gravity scale. It was shown that quantum field theory with κ-Poincaré symmetry emerges in a certain limit of quantum gravity coupled to matter fields after integrating out the gravitational/topological degrees of freedom [11–15]. This amounts to an effective theory in the form of a non-commutative field theory on the κ-deformed Minkowski space.

It is well known [16–19] that the deformations of the symmetry group can be realized through the application of the Drinfeld twist on that symmetry group [20–23]. The main virtue of the twist formulation is that the deformed (twisted) symmetry algebra is the same as the original undeformed one and that there is only a change in the coalgebra structure which then leads to the same free field structure as the corresponding commutative field theory.

In [24] it was shown that the coproduct of $D = 2$ and $D = 4$ quantum κ-Poincaré algebras in the classical basis cannot be obtained by the cochain twist depending only on the Poincaré generators (even if the coassociativity condition is relaxed). However, the deformation used in [24] is the so-called time-like type of deformation, and it is well known [25–27] that for light-like deformation such a twist indeed exists [28–32].

In this work, we work the other way round. Starting from κ-Minkowski space, we obtain its linear realizations, then coproducts of momenta from realizations, and, finally, we present a method for obtaining corresponding twists from those coproducts. We show that, for linear realizations, those twists are Drinfeld twists, satisfying normalization and cocycle conditions. The method for obtaining Drinfeld twists corresponding to each linear realization is elaborated and it is shown how these twists generate new Hopf algebras. The resulting symmetry algebras are κ-deformed $\mathfrak{igl}(n)$ Hopf algebras. In special cases we obtain κ-Poincaré–Weyl–Hopf algebra and κ-Poincaré–Hopf algebra, but the former is obtained only for the case of light-like deformation.

The paper is organized as follows. In Sect. 2, κ-Minkowski spacetime with deformation vector a_μ in various directions (time-like, space-like and light-like) is introduced. In Sect. 3, notion of linear realizations is introduced and all linear realizations in n dimensions for $n > 2$ are found. Those realizations are then expressed in terms of generators of $\mathfrak{gl}(n)$ algebra. In Sect. 4, the deformed Heisenberg algebra is presented, along with the star product and coproducts of the momenta. At the end of this section, the twist operator is introduced and the relation between star product, twist operator and coproduct of momenta is given. In Sect. 5 it is shown that the twist operator from the previous section is a Drinfeld twist, satisfying normalization and cocycle conditions. It is shown that initial linear realizations follow from these twists, which confirms the consistency of our approach. At the end of Sect. 5, the $\mathcal{R}$-matrix is presented. In Sect. 6, the κ-deformed $\mathfrak{igl}(n)$ Hopf algebra is presented, in the general and for four special cases. In Sect. 7, a left–right dual κ-Minkowski algebra is constructed from the transposed twists. Alternatively, the κ-Minkowski algebra is obtained from the transposed twists with $a_\mu \to -a_\mu$. The corresponding realizations are nonlinear. In Sect. 8, nonlinear realizations of κ-Minkowski space and related Drinfeld twists, known in the literature so far, are presented. Finally, in Sect. 9, an outlook and a discussion are given.

2 κ-Minkowski space

κ-Minkowski space is usually defined by [4–8,33]:

$$[\hat{x}_0, \hat{x}_i] = \frac{i}{\kappa}\hat{x}_i, \quad [\hat{x}_i, \hat{x}_j] = 0. \tag{1}$$

Equation (1) can be rewritten in a covariant way [34–37]:

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu), \tag{2}$$

where $a_\mu \in \mathbb{M}^n$ ($\mathbb{M}^n$ being undeformed n-dimensional Minkowski space) is a fixed deformation vector, which for the choice $a_0 = \kappa^{-1}$ and $a_i = 0$ corresponds to (1). The non-commutative coordinates $\hat{x}_\mu$ of κ-Minkowski space form a Lie algebra.

Note that the Lie algebra (2) is independent of metric. However, we point out that our physical requirement is that in the limit $a_\mu \to 0$, we get ordinary Minkowski spacetime. Hence, it is natural to assume and treat a_μ as a vector in undeformed Minkowski space. There are two possibilities. One is to fix real parameters a_μ and the other is when a_μ are not fixed (transforming together with non-commutative coordinates $\hat{x}_\mu$) [38]. In this paper we choose the first possibility.

Throughout the article, indices are raised and lowered by the Minkowski metric η, i.e. $a^\mu = \eta^{\mu\nu}a_\nu$ and $a_\mu = \eta_{\mu\nu}a^\nu$, where the convention with positive spatial eigenvalues of the metric is used, e.g. in $(3+1)$ dimensions $\eta = \mathrm{diag}(-1, 1, 1, 1)$. Also, indices are contracted the same way, i.e. $a \cdot b = a_\mu b^\mu = \eta^{\mu\nu}a_\mu b_\nu$ and $a^2 = a \cdot a = a_\mu a^\mu = \eta^{\mu\nu}a_\mu a_\nu$ for any vectors a_μ and b_μ.

The deformation vector can be time-like ($a^2 < 0$), light-like ($a^2 = 0$) and space-like ($a^2 > 0$), so it can be written like

$$a_\mu = \frac{1}{\kappa}u_\mu, \tag{3}$$

where κ^{-1} is an expansion parameter and $u^2 \in \{-1, 0, 1\}$, which corresponds to the previously mentioned three cases. Light-like deformation $a^2 = 0$ was first treated in context of null-plane quantum Poincaré algebra [39]. Depending on the sign of a^2, the κ-Minkowski Lie algebra is invariant under the following little groups:

- If a_μ is time-like ($a^2 < 0$), the little group is $SO(n-1)$.
- If a_μ is light-like ($a^2 = 0$), the little group is $E(n-2)$.
- If a_μ is space-like ($a^2 > 0$), the little group is $SO(n-2, 1)$.

It is useful to introduce an enveloping algebra $\hat{\mathcal{A}}$, generated by the elements $\hat{x}_\mu$ of the κ-Minkowski algebra.

3 Linear realizations

Commutative coordinates x_μ and momenta p_μ generate an undeformed Heisenberg algebra $\mathcal{H}$ given by

$$\begin{aligned} &[x_\mu, x_\nu] = 0, \\ &[p_\mu, x_\nu] = -i\eta_{\mu\nu}, \\ &[p_\mu, p_\nu] = 0. \end{aligned} \tag{4}$$

Analogously to $\hat{\mathcal{A}}$ in the previous section, commutative coordinates x_μ generate an enveloping algebra $\mathcal{A}$, which is subalgebra of undeformed Heisenberg algebra, i.e. $\mathcal{A} \subset \mathcal{H}$. Momenta p_μ generate algebra $\mathcal{T}$, which is also a subalgebra of the undeformed Heisenberg algebra, i.e. $\mathcal{T} \subset \mathcal{H}$. The undeformed Heisenberg algebra is, symbolically, $\mathcal{H} = \mathcal{A}\mathcal{T}$.

In general, the realization of an NC space is given by

$$\hat{x}_\mu = x_\alpha \varphi^\alpha{}_\mu(p), \tag{5}$$

where $\varphi^\alpha{}_\mu(p)$ is a function of p_μ, which should reduce to δ^α_μ in the limit when the deformation goes to zero [40–42].

It is important to note that different realizations $\hat{x}_\mu = x_\alpha \varphi^\alpha{}_\mu(p) = x'_\alpha \varphi'^\alpha{}_\mu(p')$ are related by similarity transformations, where (x_μ, p_ν) and (x'_μ, p'_ν) satisfy the undeformed Heisenberg algebra [43–46]. In this section, the additional label for x_μ and p_μ is omitted for the sake of simplicity.

We are looking for linear realizations of κ-Minkowski space, that is, the realizations where the function $\varphi^\alpha{}_\mu(p)$ is linear in p_μ. They can be written in the form

$$\hat{x}_\mu = x_\mu + l_\mu, \tag{6}$$

where l_μ is linear in momentum p_μ. It is given by

$$l_\mu = K_{\beta\mu}{}^\alpha x_\alpha p^\beta, \tag{7}$$

where $K_{\beta\mu}{}^\alpha \in \mathbb{R}$. Inserting it in (2) shows that $K_{\mu\nu}{}^\alpha$ has to satisfy

$$K_{\mu\nu}{}^\alpha - K_{\nu\mu}{}^\alpha = a_\mu \delta^\alpha_\nu - a_\nu \delta^\alpha_\mu, \tag{8}$$

$$K_{\gamma\mu}{}^\alpha K_{\beta\nu}{}^\gamma - K_{\gamma\nu}{}^\alpha K_{\beta\mu}{}^\gamma = a_\mu K_{\beta\nu}{}^\alpha - a_\nu K_{\beta\mu}{}^\alpha. \tag{9}$$

It also follows that l_μ satisfies the same commutation relations as $\hat{x}_\mu$:

$$[l_\mu, l_\nu] = i(a_\mu l_\nu - a_\nu l_\mu). \tag{10}$$

3.1 Classification of linear realizations

Since we assume that Eqs. (8) and (9) transform under a Lorentz algebra, the most general covariant ansatz for $K_{\mu\nu}{}^\alpha$ in terms of deformation vector a_μ for arbitrary number of dimensions[1] $n > 2$ is

$$K_{\mu\nu}{}^\alpha = A_0 a_\mu a_\nu a^\alpha + A_1 \eta_{\mu\nu} a^\alpha + A_2 \delta^\alpha_\mu a_\nu + A_3 a_\mu \delta^\alpha_\nu, \tag{11}$$

leading to the following l_μ:

$$\begin{aligned} l_\mu = {}& A_0 a_\mu (a\cdot x)(a\cdot p) + A_1 (a\cdot x) p_\mu + A_2 a_\mu (x\cdot p) \\ & + A_3 x_\mu (a\cdot p). \end{aligned} \tag{12}$$

From Eq. (8) it follows that

$$A_3 = A_2 + 1. \tag{13}$$

Using (9) in combination with (13) yields the following equations:

$$A_1(A_0 a^2 + A_1 + 1) = 0, \tag{14}$$

$$A_3(A_0 a^2 + A_3 + 1) = 0, \tag{15}$$

$$A_1 A_3 a^2 = 0. \tag{16}$$

Those equations have four solutions:

1. $A_1 = 0, A_2 = -1, A_3 = 0, a^2 A_0 = c$,
2. $A_1 = 0, A_2 = -c, A_3 = 1 - c, a^2 A_0 = c$,
3. $A_1 = -1 - c, A_2 = -1, A_3 = 0, a^2 A_0 = c$,
4. $A_1 = -1, A_2 = 0, A_3 = 1, a^2 = A_0 = 0$,

where $c \in \mathbb{R}$ is a free parameter. We will denote these four types of realizations by $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$, respectively.[2] Explicitly for the tensor $K_{\mu\nu\alpha}$ we have

$$\mathcal{C}_1 : K_{\mu\nu\alpha} = \begin{cases} \dfrac{c}{a^2} a_\mu a_\nu a_\alpha - \eta_{\mu\alpha} a_\nu, & \text{if } a^2 \neq 0, \\ -\eta_{\mu\alpha} a_\nu, & \text{if } a^2 = 0, \end{cases}$$

$$\mathcal{C}_2 : K_{\mu\nu\alpha} = \begin{cases} \dfrac{c}{a^2} a_\mu a_\nu a_\alpha - c\eta_{\mu\alpha} a_\nu + (1-c)\eta_{\nu\alpha} a_\mu, & \text{if } a^2 \neq 0 \\ \eta_{\nu\alpha} a_\mu, & \text{if } a^2 = 0, \end{cases}$$

[1] In two dimensions there are additional terms constructed with two dimensional Levi-Civita tensor $\epsilon_{\mu\nu}$. For example, there is a solution $K_{\mu\nu\lambda} = \frac{a_\mu a_\nu}{a^2}(c_1 a_\alpha + c_2 \epsilon_{\alpha\beta} a^\beta)$, where $c_1, c_2 \in \mathbb{R}$ are parameters and $a^2 \neq 0$.

[2] Here $\mathcal{C}$ stands for *covariant*.

$$\mathcal{C}_3\colon K_{\mu\nu\alpha} = \begin{cases} \frac{c}{a^2} a_\mu a_\nu a_\alpha - (1+c)\eta_{\mu\nu}a_\alpha - \eta_{\mu\alpha}a_\nu, & \text{if } a^2 \neq 0 \\ -\eta_{\mu\nu}a_\alpha - \eta_{\mu\alpha}a_\nu, & \text{if } a^2 = 0, \end{cases}$$

$$\mathcal{C}_4\colon K_{\mu\nu\alpha} = -\eta_{\mu\nu}a_\alpha + \eta_{\nu\alpha}a_\mu, \quad \text{only for } a^2 = 0. \tag{17}$$

Inserting (17) into (6) and (7) gives

$$\mathcal{C}_1\colon \hat{x}_\mu = \begin{cases} x_\mu + a_\mu\left[\frac{c}{a^2}(a\cdot x)(a\cdot p) - (x\cdot p)\right], & a^2 \neq 0 \\ x_\mu - a_\mu(x\cdot p), & a^2 = 0, \end{cases}$$

$$\mathcal{C}_2\colon \hat{x}_\mu = \begin{cases} x_\mu\left[1+(1-c)(a\cdot p)\right] \\ \quad + a_\mu\left[\frac{c}{a^2}(a\cdot x)(a\cdot p) - c(x\cdot p)\right], & a^2 \neq 0 \\ x_\mu\left[1+(a\cdot p)\right], & a^2 = 0, \end{cases}$$

$$\mathcal{C}_3\colon \hat{x}_\mu = \begin{cases} x_\mu + a_\mu\left[\frac{c}{a^2}(a\cdot x)(a\cdot p) - (x\cdot p)\right] \\ \quad -(1+c)(a\cdot x)p_\mu, & a^2 \neq 0 \\ x_\mu - a_\mu(x\cdot p) - (a\cdot x)p_\mu, & a^2 = 0, \end{cases}$$

$$\mathcal{C}_4\colon \hat{x}_\mu = x_\mu\left[1+(a\cdot p)\right] - (a\cdot x)p_\mu, \quad \text{only for } a^2 = 0. \tag{18}$$

Linear realizations for κ-deformed Euclidean space were studied in [42]. However, in κ-Minkowski spacetime, we have found four new linear realizations corresponding to light-like deformations ($a^2 = 0$). Only one of them, $\mathcal{C}_4$, corresponds to a κ-Poincaré–Hopf algebra [30,31]. $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent for $c = 1$, while $\mathcal{C}_1$ and $\mathcal{C}_3$ are equivalent for $c = -1$. It is important to note that the first three solutions $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ are valid for all $a^2 \in \mathbb{R}$, and the fourth solution $\mathcal{C}_4$ is only valid in the case of a light-like deformation.

The inverse matrices $\varphi^{-1}_{\mu\nu}$, such that $\varphi_\mu{}^\alpha \varphi^{-1}_{\alpha\nu} = \eta_{\mu\nu}$, for $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, and $\mathcal{C}_4$ are

$$\mathcal{C}_1\colon \varphi^{-1}_{\mu\nu} = \begin{cases} \eta_{\mu\nu} + \frac{1}{1-(a\cdot p)+c}\left(p_\mu - c\frac{a_\mu}{a^2}\right)a_\nu, & a^2 \neq 0 \\ \eta_{\mu\nu} + \frac{p_\mu a_\nu}{1-(a\cdot p)}, & a^2 = 0, \end{cases}$$

$$\mathcal{C}_2\colon \varphi^{-1}_{\mu\nu} = \begin{cases} \frac{\eta_{\mu\nu}}{1+(1-c)(a\cdot p)} - \frac{c}{[1+(1-c)(a\cdot p)]^2} \\ \quad \times\left(\frac{a\cdot p}{a^2}a_\mu + p_\mu\right)a_\nu, & a^2 \neq 0 \\ \frac{\eta_{\mu\nu}}{1+(a\cdot p)}, & a^2 = 0, \end{cases}$$

$$\mathcal{C}_3\colon \varphi^{-1}_{\mu\nu} = \begin{cases} \eta_{\mu\nu} - \frac{1}{1-a\cdot p(1-c(a\cdot p))}\left[c(a\cdot p)\left(\frac{a_\mu}{a^2} + p_\mu\right) - p_\mu\right]a_\nu \\ \quad + \frac{(1+c)\left[(1-a\cdot p)a_\mu + a^2 p_\mu\right]p_\nu}{1-2a\cdot p+(1+c)((a\cdot p)^2 - a^2p^2)}, & a^2 \neq 0 \\ \eta_{\mu\nu} + \frac{p_\mu a_\nu}{1-a\cdot p} + \frac{\left[(1-a\cdot p)a_\mu + a^2p_\mu\right]p_\nu}{1-2a\cdot p+((a\cdot p)^2-a^2p^2)}, & a^2 \neq 0, \end{cases}$$

$$\mathcal{C}_4\colon \varphi^{-1}_{\mu\nu} = \frac{\eta_{\mu\nu} + a_\mu p_\nu}{1+(a\cdot p)}. \tag{19}$$

The special cases of $\mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}_3$ when $c = 0$, we denote $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$, respectively:

$$\begin{aligned} &\mathcal{S}_1\colon \hat{x}_\mu = x_\mu - a_\mu(x\cdot p), \\ &\mathcal{S}_2\colon \hat{x}_\mu = x_\mu\left[1+(a\cdot p)\right], \\ &\mathcal{S}_3\colon \hat{x}_\mu = x_\mu - a_\mu(x\cdot p) - (a\cdot x)p_\mu, \end{aligned} \tag{20}$$

where $a^2 \in \mathbb{R}$.

3.2 Symmetry algebra $\mathfrak{igl}(n)$

For fixed solution $K_{\mu\nu\lambda}$ we define the undeformed $\mathfrak{igl}(n)$ algebra generated by p_μ and $L_{\mu\nu}$:

$$\begin{aligned} &[L_{\mu\nu}, L_{\lambda\rho}] = \eta_{\nu\lambda}L_{\mu\rho} - \eta_{\mu\rho}L_{\lambda\nu}, \\ &[L_{\mu\nu}, p_\lambda] = -p_\nu\eta_{\mu\lambda}, \\ &[p_\mu, p_\nu] = 0. \end{aligned} \tag{21}$$

In addition to the commutation relations (21),

$$[L_{\mu\nu}, x_\lambda] = x_\mu\eta_{\nu\lambda} \tag{22}$$

also holds.

Linear realizations can be written in terms of $L_{\mu\nu}$:

$$\hat{x}_\mu = x_\mu - iK_{\beta\mu\alpha}L^{\alpha\beta}. \tag{23}$$

Particularly, for (18)

$$\mathcal{C}_1\colon \hat{x}_\mu = \begin{cases} x_\mu - ia_\mu\left(\frac{c}{a^2}a_\alpha a_\beta L^{\alpha\beta} - L_\alpha{}^\alpha\right), & a^2 \neq 0, \\ x_\mu + ia_\mu L_\alpha{}^\alpha, & a^2 = 0, \end{cases}$$

$$\mathcal{C}_2\colon \hat{x}_\mu = \begin{cases} x_\mu - i\frac{c}{a^2}a_\mu a_\alpha a_\beta L^{\alpha\beta} + icL_\alpha{}^\alpha a_\mu \\ \quad -i(1-c)a^\alpha L_{\mu\alpha}, & a^2 \neq 0, \\ x_\mu - ia^\alpha L_{\mu\alpha}, & a^2 = 0, \end{cases}$$

$$\mathcal{C}_3\colon \hat{x}_\mu = \begin{cases} x_\mu - i\frac{c}{a^2}a_\mu a_\alpha a_\beta L^{\alpha\beta} \\ \quad +i(1+c)a^\alpha L_{\alpha\mu} + iL_\alpha{}^\alpha a_\mu, & a^2 \neq 0, \\ x_\mu + ia^\alpha L_{\alpha\mu} + iL_\alpha{}^\alpha a_\mu, & a^2 = 0, \end{cases}$$

$$\mathcal{C}_4\colon \hat{x}_\mu = x_\mu + ia^\alpha(L_{\alpha\mu} - L_{\mu\alpha}) = x_\mu + ia^\alpha M_{\alpha\mu}, \quad \text{only for } a^2 = 0, \tag{24}$$

where $M_{\mu\nu} = L_{\mu\nu} - L_{\nu\mu}$ generate Lorentz algebra. Note that $\mathcal{C}_4$ is the only solution that can be written in terms of Lorentz generators.

Commutation relations between generators of $\mathfrak{igl}(n)$ algebra with $\hat{x}_\mu$ are

$$[p_\mu, \hat{x}_\nu] = -i\varphi_{\mu\nu} = -i(\eta_{\mu\nu} + K_{\alpha\nu\mu}p^\alpha), \tag{25}$$

$$[L_{\mu\nu}, \hat{x}_\lambda] = \hat{x}_\mu\eta_{\nu\lambda} + i(K_{\beta\lambda\alpha}\eta_{\nu\lambda} - K_{\beta\lambda\nu}\eta_{\alpha\mu} - K_{\mu\lambda\alpha}\eta_{\beta\nu})L^{\alpha\beta}. \tag{26}$$

The algebra generated by $L_{\mu\nu}$, p_μ, and $\hat{x}_\mu$ satisfies all the Jacobi relations. Only for solution $\mathcal{C}_4$ this is also true for an algebra generated by $M_{\mu\nu}$, p_μ, and $\hat{x}_\mu$.

At the end of this section let us introduce the anti-involution operator $\dagger$ by $\lambda^\dagger = \bar{\lambda}$, for $\lambda \in \mathbb{C}$ and a bar denoting ordinary complex conjugation, $(\hat{x}_\mu)^\dagger = \hat{x}_\mu$, $(x_\mu)^\dagger = x_\mu$, $(p_\mu)^\dagger = p_\mu$, and $(M_{\mu\nu})^\dagger = -M_{\mu\nu}$. Since $(a_\mu)^\dagger = a_\mu$ Eqs. (2), (4), (25), and (27) remain unchanged (i.e. they are invariant) under the action of $\dagger$. Note that the realizations $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ are generally not hermitian. In order to get the hermitian realizations, one has to make following substitutions: $\hat{x}_\mu \to \frac{1}{2}(\hat{x}_\mu + \hat{x}_\mu^\dagger)$, $l_\mu \to \frac{1}{2}(l_\mu + l_\mu^\dagger)$, $L_{\mu\nu} \to \frac{1}{2}(L_{\mu\nu} - (L_{\mu\nu})^\dagger)$ throughout the whole paper [47].

4 Deformed Heisenberg algebra, star product, and twist operator

Non-commutative κ-Minkowski coordinates $\hat{x}_\mu$ and momenta p_μ generate a deformed Heisenberg algebra $\hat{\mathcal{H}}$ given by [48]

$$\begin{aligned}
&[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu\hat{x}_\nu - a_\nu\hat{x}_\mu),\\
&[p_\mu, \hat{x}_\nu] = -i\varphi_{\mu\nu}(p) = -i(\eta_{\mu\nu} + K_{\alpha\nu\mu}p^\alpha),\\
&[p_\mu, p_\nu] = 0.
\end{aligned} \tag{27}$$

From the previous section, it follows that $\hat{\mathcal{H}}$ is isomorphic to $\mathcal{H}$. Algebra $\hat{\mathcal{A}}$ is a subalgebra of $\hat{\mathcal{H}}$, i.e. $\hat{\mathcal{A}} \subset \hat{\mathcal{H}}$. The deformed Heisenberg algebra is, symbolically, $\hat{\mathcal{H}} = \hat{\mathcal{A}}\mathcal{T}$.

4.1 Actions $\blacktriangleright$ and $\triangleright$

The action $\blacktriangleright$ is a map $\blacktriangleright$: $\hat{\mathcal{H}} \otimes \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ satisfying the following properties:

$$\hat{f} \blacktriangleright \hat{g} = \hat{f}\hat{g}, \quad \forall \hat{f}, \hat{g} \in \hat{\mathcal{A}}, \tag{28}$$

$$p_\mu \blacktriangleright \hat{f} = [p_\mu, \hat{f}] \blacktriangleright 1, \quad \forall \hat{f} \in \hat{\mathcal{A}}, \tag{29}$$

$$p_\mu \blacktriangleright 1 = 0. \tag{30}$$

It follows that

$$\begin{aligned}
&\hat{\mathcal{H}} \blacktriangleright 1 = \hat{\mathcal{A}},\\
&\hat{\mathcal{A}} \blacktriangleright 1 = \hat{\mathcal{A}}.
\end{aligned} \tag{31}$$

In complete analogy, the action $\triangleright$ is a map $\triangleright$: $\mathcal{H}\otimes\mathcal{A} \to \mathcal{A}$ satisfying the following properties:

$$f \triangleright g = fg, \quad \forall f, g \in \mathcal{A}, \tag{32}$$

$$p_\mu \triangleright f = [p_\mu, f] \triangleright 1, \quad \forall f \in \mathcal{A}, \tag{33}$$

$$p_\mu \triangleright 1 = 0. \tag{34}$$

Also, it follows that

$$\begin{aligned}
&\mathcal{H} \triangleright 1 = \mathcal{A},\\
&\mathcal{A} \triangleright 1 = \mathcal{A}.
\end{aligned} \tag{35}$$

$\blacktriangleright$ and $\triangleright$ are actions, so they satisfy

$$(\hat{f}\hat{g}) \blacktriangleright \hat{h} = \hat{f} \blacktriangleright (\hat{g} \blacktriangleright \hat{h}), \tag{36}$$

$$(fg) \triangleright h = f \triangleright (g \triangleright h). \tag{37}$$

4.2 Star product

For a κ-Minkowski space, there exists an isomorphism (as vector spaces) between $\hat{\mathcal{A}}$ and $\mathcal{A}$, defined by

$$\begin{aligned}
&\hat{f} \triangleright 1 = f,\\
&f \blacktriangleright 1 = \hat{f},
\end{aligned} \tag{38}$$

where $\hat{f} \in \hat{\mathcal{A}}$, and also, using the realization for $\hat{x}_\mu$ (5), $\hat{f} \in \mathcal{H}$. Similarly, $f \in \mathcal{A}$, and also the inverting realization for $\hat{x}_\mu$, $f \in \hat{\mathcal{H}}$.

Using this identification, the star product $\star$: $\mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ is defined by

$$f \star g \equiv (\hat{f}\hat{g}) \triangleright 1 = \hat{f} \triangleright g. \tag{39}$$

For κ-Minkowski space, the star product is associative:

$$(f \star g) \star h = f \star (g \star h). \tag{40}$$

The star product defines an algebra $\mathcal{A}_\star$, which is defined like $\mathcal{A}$, but with a non-commutative star product instead of ordinary multiplication. The algebras $\mathcal{A}_\star$ and $\hat{\mathcal{A}}$ are isomorphic as algebras, not only as vector spaces. It follows that

$$(f \star g) \blacktriangleright 1 = \hat{f}\hat{g}, \tag{41}$$

$$\begin{aligned}
&\hat{\mathcal{H}} \triangleright 1 = \mathcal{A}_\star,\\
&\hat{\mathcal{A}} \triangleright 1 = \mathcal{A}_\star,
\end{aligned} \tag{42}$$

$$\begin{aligned}
&\mathcal{H} \blacktriangleright 1 = \hat{\mathcal{A}},\\
&\mathcal{A} \blacktriangleright 1 = \hat{\mathcal{A}}.
\end{aligned} \tag{43}$$

It can be shown that

$$e^{ik\cdot\hat{x}} \triangleright 1 = e^{iK(k)\cdot x}, \tag{44}$$

where $K_\mu(k)$ is an invertible function K_μ: $\mathbb{M}^n \to \mathbb{M}^n$, which is calculated in "Appendix A" for linear realizations of $\hat{x}_\mu$. The inverse relation is

$$e^{ik\cdot x} \blacktriangleright 1 = e^{iK^{-1}(k)\cdot\hat{x}}. \tag{45}$$

It can also be shown that there is a function $P_\mu(k_1, k_2)$, such that

$$e^{ik_1\cdot\hat{x}} \triangleright e^{ik_2\cdot x} = e^{iP(k_1,k_2)\cdot x}. \tag{46}$$

The star product between such exponentials is then given by

$$e^{ik_1\cdot x} \star e^{ik_2\cdot x} = e^{iK^{-1}(k_1)\cdot\hat{x}} \triangleright e^{ik_2\cdot x} = e^{iP(K^{-1}(k_1),k_2)\cdot x} \equiv e^{i\mathcal{D}(k_1,k_2)\cdot x}. \tag{47}$$

Note that $K_\mu(k) = P_\mu(k,0)$ and $\mathcal{D}_\mu(k_1,k_2) = P_\mu(K^{-1}(k_1),k_2)$. $\mathcal{D}_\mu(k_1,k_2)$ describes deformed addition of momenta $(k_1)_\mu \oplus (k_2)_\mu = \mathcal{D}_\mu(k_1,k_2)$ (for more details see [49]). Calculation of $P_\mu(k_1,k_2)$, $\mathcal{D}_\mu(k_1,k_2)$ and $K_\mu(k)$ for linear realizations (described in previous section) is given in "Appendix A".

For elements $f, g \in \mathcal{A}$, which can be Fourier transformed,

$$f = \int d^n k\ \tilde{f}(k) e^{ik\cdot x}, \tag{48}$$

$$g = \int d^n k\ \tilde{g}(k) e^{ik\cdot x}, \tag{49}$$

we find corresponding elements $\hat{f}, \hat{g} \in \hat{\mathcal{A}}$,

$$\hat{f} = f \blacktriangleright 1 = \int d^n k\ \tilde{f}(k) e^{iK^{-1}(k)\cdot\hat{x}}, \tag{50}$$

$$\hat{g} = g \blacktriangleright 1 = \int d^n k\ \tilde{g}(k) e^{iK^{-1}(k)\cdot\hat{x}}. \tag{51}$$

Then the star product $f \star g$ can be written in the following way:

$$f \star g = \hat{f}\hat{g} \triangleright 1 = \int d^n k_1 d^n k_2\ \tilde{f}(k_1)\tilde{g}(k_2) e^{i\mathcal{D}(k_1,k_2)\cdot x}. \tag{52}$$

4.3 Coproduct of momenta

The undeformed coproduct $\Delta_0 : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}$ for momentum p_μ is

$$\Delta_0 p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu. \tag{53}$$

The deformed coproduct for the momenta $\Delta : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}$ is [50–52]

$$\Delta p_\mu = \mathcal{D}_\mu(p \otimes 1, 1 \otimes p). \tag{54}$$

Using the results from "Appendix A", we have

$$\Delta p_\mu = p_\mu \otimes 1 + \Lambda^{-1}_{\alpha\mu} \otimes p^\alpha, \tag{55}$$

where

$$\Lambda_{\mu\nu} = (e^{\mathcal{K}})_{\mu\nu}, \tag{56}$$

$$\mathcal{K}_{\mu\nu} = -K_{\mu\alpha\nu}(K^{-1})^\alpha(p), \tag{57}$$

and

$$\Delta\Lambda_{\mu\nu} = \Lambda_{\mu\alpha} \otimes \Lambda^\alpha{}_\nu, \tag{58}$$

$$\Delta(\Lambda^{-1})_{\mu\nu} = (\Lambda^{-1})_{\alpha\nu} \otimes (\Lambda^{-1})_\mu{}^\alpha. \tag{59}$$

We also have

$$p_\mu \hat{f} = (p_\mu \blacktriangleright \hat{f}) + (\Lambda^{-1}_{\alpha\mu} \blacktriangleright \hat{f}) p^\alpha, \tag{60}$$

$$\Lambda_{\mu\nu}\hat{f} = (\Lambda_{\mu\alpha} \blacktriangleright \hat{f})\Lambda^\alpha{}_\nu, \tag{61}$$

$$(\Lambda^{-1})_{\mu\nu}\hat{f} = ((\Lambda^{-1})_{\alpha\nu} \blacktriangleright \hat{f})(\Lambda^{-1})_\mu{}^\alpha. \tag{62}$$

For example if $\hat{f} = \hat{x}_\lambda$ we have

$$[\Lambda_{\mu\nu}, \hat{x}_\lambda] = iK_{\mu\lambda}{}^\alpha \Lambda_{\alpha\nu}, \tag{63}$$

$$[\Lambda^{-1}_{\mu\nu}, \hat{x}_\lambda] = -i\Lambda^{-1}_{\mu\alpha} K^\alpha{}_{\lambda\nu}. \tag{64}$$

In order to specify Δp_μ, we have to express $K^{-1}_\mu(p) \equiv p^W_\mu$ in terms of momenta p_μ (see "Appendix A"). The momentum p_μ acts on $e^{ik\cdot x}$ and $e^{ik\cdot\hat{x}}$ with $\triangleright$ and $\blacktriangleright$, respectively, in the following way:

$$p_\mu \triangleright e^{ik\cdot x} = k_\mu e^{ik\cdot x}, \quad p_\mu \blacktriangleright e^{ik\cdot\hat{x}} = K_\mu(k) e^{ik\cdot\hat{x}}, \tag{65}$$

and the momentum p^W_μ acts as

$$p^W_\mu \blacktriangleright e^{ik\cdot\hat{x}} = k_\mu e^{ik\cdot\hat{x}}, \quad p^W_\mu \triangleright e^{ik\cdot x} = K^{-1}_\mu(k) e^{ik\cdot x}. \tag{66}$$

It is useful to introduce the shift operator Z, with the properties

$$[Z, \hat{x}_\mu] = ia_\mu Z, \tag{67}$$

$$Z = e^{-a\cdot p^W}. \tag{68}$$

Explicitly, for $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$, the coproducts of the momenta are

- Case $\mathcal{C}_1$:

$$\Delta p_\mu = p_\mu \otimes 1 + Z \otimes p_\mu + \frac{a_\mu}{a^2}(Z^{1-c} - Z) \otimes a \cdot p, \tag{69}$$

$$\Lambda^{-1}_{\mu\nu} = \left[\eta_{\mu\nu} + \frac{a_\mu a_\nu}{a^2}(Z^{-c} - 1)\right] Z, \tag{70}$$

$$\Lambda_{\mu\nu} = \left[\eta_{\mu\nu} + \frac{a_\mu a_\nu}{a^2}(Z^{c} - 1)\right] Z^{-1}, \tag{71}$$

$$p^W_\mu = \left[p_\mu - \frac{a_\mu}{a^2}(Z - 1 + a \cdot p)\right] \frac{\ln Z}{Z - 1}, \tag{72}$$

$$Z = [1 - (1 - c)a \cdot p]^{\frac{1}{1-c}}. \tag{73}$$

- Case $\mathcal{C}_2$:

$$\Delta p_\mu = p_\mu \otimes 1 + \left(Z^c - \frac{c}{1+c}\right) \otimes p_\mu + \left(c\frac{a_\mu}{a^2} + (c-1)\frac{p^W_\mu}{\ln Z}\right) \frac{Z^{-1} - Z^c}{1+c} \otimes a \cdot p, \tag{74}$$

$$\Lambda^{-1}_{\mu\nu} = \eta_{\mu\nu}\left(Z^c - \frac{c}{1+c}\right) + a_\mu \left(c\frac{a_\nu}{a^2} + (c-1)\frac{p^W_\nu}{\ln Z}\right) \frac{Z^{-1} - Z^c}{1+c}, \tag{75}$$

$$\Lambda_{\mu\nu} = \eta_{\mu\nu}\left(Z^{-c} - \frac{c}{1+c}\right)$$

$$+a_\mu\left(c\frac{a_\nu}{a^2}+(c-1)\frac{p_\nu^W}{\ln Z}\right)\frac{Z-Z^{-c}}{1+c}, \tag{76}$$

$$p_\mu^W=\left[p_\mu-\frac{a_\mu}{a^2}(1-Z^{-1}+a\cdot p)\right]\frac{\ln Z}{1-Z^{-1}} \tag{77}$$

$$Z=[1-(c-1)a\cdot p]^{\frac{1}{c-1}}. \tag{78}$$

- Case $\mathcal{C}_3$:

$$\Delta p_\mu=p_\mu\otimes 1+Z\otimes p_\mu+a_\mu\left((1+c)\frac{p_\alpha^W}{\ln Z}-c\frac{a_\alpha}{a^2}\right)\times(Z-1)Z\otimes p^\alpha, \tag{79}$$

$$\Lambda_{\mu\nu}^{-1}=\left[\eta_{\mu\nu}+\left((1+c)\frac{p_\mu^W}{\ln Z}-c\frac{a_\mu}{a^2}\right)a_\nu(Z-1)\right]Z, \tag{80}$$

$$\Lambda_{\mu\nu}=\left[\eta_{\mu\nu}+\left((1+c)\frac{p_\mu^W}{\ln Z}-c\frac{a_\mu}{a^2}\right)a_\nu(Z^{-1}-1)\right]Z^{-1}, \tag{81}$$

$$p_\mu^W=\left[p_\mu-\frac{a_\mu}{a^2}(Z-1+a\cdot p)\right]\frac{\ln Z}{Z-1}, \tag{82}$$

$$Z=\left[c+(1-c)\left((1-a\cdot p)^2-a^2p^2\right)\right]^{\frac{1}{2(1-c)}}. \tag{83}$$

- Case $\mathcal{C}_4$:

$$\Delta p_\mu=p_\mu\otimes 1+1\otimes p_\mu+p_\mu\otimes a\cdot p-a_\mu p_\alpha Z\otimes p^\alpha-\frac{a_\mu}{2}p^2Z\otimes a\cdot p, \tag{84}$$

$$\Lambda_{\mu\nu}^{-1}=\eta_{\mu\nu}+a_\mu p_\nu-\left(p_\mu+\frac{a_\mu}{2}p^2\right)a_\nu Z, \tag{85}$$

$$\Lambda_{\mu\nu}=\eta_{\mu\nu}+p_\mu a_\nu-a_\mu\left(p_\nu+\frac{a_\nu}{2}p^2\right)Z, \tag{86}$$

$$p_\mu^W=\left(p_\mu+\frac{a_\mu}{2}p^2\right)\frac{\ln Z}{1-Z^{-1}}, \tag{87}$$

$$Z=\frac{1}{1+a\cdot p}. \tag{88}$$

4.4 Relation between star product, twist operator, and coproduct

The star product is related to the twist operator $\mathcal{F}^{-1}$ in the following way:

$$f\star g=m[\mathcal{F}^{-1}(\triangleright\otimes\triangleright)(f\otimes g)], \tag{89}$$

where $f, g\in\mathcal{A}$. Furthermore,

$$\hat{f}=m[\mathcal{F}^{-1}(\triangleright\otimes 1)(f\otimes 1)],\quad f\in\mathcal{A}, \tag{90}$$

where $\hat{f}\in\hat{\mathcal{A}}$ is expressed in terms of x, $p\in\mathcal{H}$.

Using the above expressions for the star product, Eqs. (52) and (89), the twist operator can be written as [49,53,54]

$$\mathcal{F}^{-1}=:\exp[i(tx_\alpha\otimes 1+(1-t)\otimes x_\alpha)(\Delta-\Delta_0)p^\alpha], \tag{91}$$

where $t\in\mathbb{R}$, generally defined up to the right ideal $\mathcal{I}_0\subset\mathcal{H}\otimes\mathcal{H}$ defined by

$$m\,(\mathcal{I}_0(\triangleright\otimes\triangleright)(\mathcal{A}\otimes\mathcal{A}))=0. \tag{92}$$

5 Drinfeld twists

Starting with Eq. (91) for the twist operator, we derive the Drinfeld twists [16–21] in "Appendix B". We have

$$\mathcal{F}=\exp(\mathcal{K}_{\beta\alpha}\otimes L^{\alpha\beta})=\exp(-ip_\alpha^W\otimes l^\alpha), \tag{93}$$

where p_μ^W is given in Sect. 4.3 after Eq. (64) and in "Appendix A", and we have $l_\mu=-iK_{\beta\mu\alpha}L^{\alpha\beta}$, where $K_{\beta\mu\alpha}$ satisfies (8) and (9), with solutions Eq. (17); l_μ generate the κ-Minkowski algebra

$$[l_\mu,l_\nu]=i(a_\mu l_\nu-a_\nu l_\mu), \tag{94}$$

and $L_{\mu\nu}$ generate the $\mathfrak{gl}(n)$ algebra; see Eq. (21).

The classical r-matrix r_{cl}, related to the twist (93) is

$$r_{cl}=p_\alpha\wedge l^\alpha=p_\alpha\otimes l^\alpha-l^\alpha\otimes p_\alpha. \tag{95}$$

For $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$, the twists are

$$\mathcal{F}_{\mathcal{C}_1}=\exp\left\{-\left(\eta_{\alpha\beta}-c\frac{a_\alpha a_\beta}{a^2}\right)\ln Z\otimes L^{\alpha\beta}\right\}=\exp\left\{-\ln Z\otimes\left(D-c\frac{a_\alpha a_\beta}{a^2}L^{\alpha\beta}\right)\right\}, \tag{96}$$

$$\mathcal{F}_{\mathcal{C}_2}=\exp\left\{-\left[c\left(\eta_{\alpha\beta}-\frac{a_\alpha a_\beta}{a^2}\right)\ln Z+(1-c)a_\beta p_\alpha^W\right]\otimes L^{\alpha\beta}\right\}, \tag{97}$$

$$\mathcal{F}_{\mathcal{C}_3}=\exp\left\{-\left[\left(\eta_{\alpha\beta}-c\frac{a_\alpha a_\beta}{a^2}\right)\ln Z-(1+c)a_\alpha p_\beta^W\right]\otimes L^{\alpha\beta}\right\}, \tag{98}$$

$$\mathcal{F}_{\mathcal{C}_4}=\exp\left\{\left(a_\alpha p_\beta^W-a_\beta p_\alpha^W\right)\otimes L^{\alpha\beta}\right\}=\exp\left\{a_\alpha p_\beta^W\otimes M^{\alpha\beta}\right\},\quad a^2=0. \tag{99}$$

Note that only for the case $\mathcal{C}_4$ ($a^2=0$), the corresponding twist operator can be expressed in terms of the Poincaré generators only [30,31,55].

Starting from the twist operator, the realization can be obtained using

$$\hat{x}_\mu=m[\mathcal{F}^{-1}(\triangleright\otimes 1)(x_\mu\otimes 1)]=x_\mu-iK_{\beta\mu\alpha}L^{\alpha\beta}. \tag{100}$$

Using the twists (96)–(99) yields the realizations $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$, respectively, which satisfy the κ-Minkowski algebra.

5.1 Undeformed $\mathfrak{igl}(n)$ Hopf algebra

The coproducts $\Delta_0 : \mathfrak{igl}(n) \to \mathfrak{igl}(n) \otimes \mathfrak{igl}(n)$ in undeformed $\mathfrak{igl}(n)$ Hopf algebra are

$$\Delta_0 p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu, \tag{101}$$

$$\Delta_0 L_{\mu\nu} = L_{\mu\nu} \otimes 1 + 1 \otimes L_{\mu\nu}, \tag{102}$$

the counit $\epsilon : \mathfrak{igl}(n) \to \mathbb{C}$ is

$$\epsilon(p_\mu) = \epsilon(p^W_\mu) = \epsilon(L_{\mu\nu}) = 0, \quad \epsilon(1) = 1, \tag{103}$$

and the antipode $S_0 : \mathfrak{igl}(n) \to \mathfrak{igl}(n)$ is

$$S_0(p_\mu) = -p_\mu, \quad S_0(L_{\mu\nu}) = -L_{\mu\nu}. \tag{104}$$

5.2 Normalization condition

Now we show that these twists satisfy the normalization condition and the cocycle condition, i.e. that they are Drinfeld twists.

The normalization condition

$$m(\epsilon \otimes 1)\mathcal{F} = 1 = m(1 \otimes \epsilon)\mathcal{F} \tag{105}$$

follows trivially, since the twist is of the form $\mathcal{F} = e^f$, where $f = -ip^W_\alpha \otimes l^\alpha$, therefore,

$$(\epsilon \otimes 1) f = (1 \otimes \epsilon) f = 0, \tag{106}$$

and from this follows

$$(\epsilon \otimes 1)\mathcal{F} = (\epsilon \otimes 1)e^f = 1 \otimes 1, \tag{107}$$

$$(1 \otimes \epsilon)\mathcal{F} = (1 \otimes \epsilon)e^f = 1 \otimes 1. \tag{108}$$

5.3 Cocycle condition

The cocycle condition is

$$(\mathcal{F} \otimes 1)(\Delta_0 \otimes 1)\mathcal{F} = (1 \otimes \mathcal{F})(1 \otimes \Delta_0)\mathcal{F}. \tag{109}$$

We shall prove it using the factorization properties of the twist $\mathcal{F}$,

$$(\Delta \otimes 1)\mathcal{F} = \mathcal{F}_{23}\mathcal{F}_{13}, \tag{110}$$

$$(1 \otimes \Delta_0)\mathcal{F} = \mathcal{F}_{13}\mathcal{F}_{12}, \tag{111}$$

where

$$\mathcal{F}_{12} = e^{\mathcal{K}_{\beta\alpha} \otimes L^{\alpha\beta} \otimes 1}, \tag{112}$$

$$\mathcal{F}_{13} = e^{\mathcal{K}_{\beta\alpha} \otimes 1 \otimes L^{\alpha\beta}}, \tag{113}$$

$$\mathcal{F}_{23} = e^{1 \otimes \mathcal{K}_{\beta\alpha} \otimes L^{\alpha\beta}}. \tag{114}$$

The first factorization property (110) can be proven to hold in the following way:

$$\begin{aligned}\mathcal{F}_{23}\mathcal{F}_{13} &= e^{1\otimes(-ip^W_\alpha)\otimes l^\alpha} e^{-ip^W_\alpha \otimes 1 \otimes l^\alpha} \\ &= (e^{ip^W_\alpha \otimes 1 \otimes l^\alpha} e^{1 \otimes ip^W_\alpha \otimes l^\alpha})^{-1} \\ &= e^{-i\mathcal{D}^W_\alpha (p^W \otimes 1, 1 \otimes p^W) \otimes l^\alpha}.\end{aligned} \tag{115}$$

This holds because l_μ generates the same algebra as $\hat{x}_\mu$ and

$$e^{ik_1\cdot\hat{x}} e^{ik_2\cdot\hat{x}} = e^{i\mathcal{D}^W(k_1,k_2)\cdot\hat{x}}. \tag{116}$$

Furthermore, since $\mathcal{D}^W_\mu(p^W \otimes 1, 1 \otimes p^W) = \Delta p^W_\mu$, it follows that

$$\mathcal{F}_{23}\mathcal{F}_{13} = e^{-i\Delta p^W_\alpha \otimes l^\alpha} = (\Delta \otimes 1)e^{-ip^W_\alpha \otimes l^\alpha} = (\Delta \otimes 1)\mathcal{F}. \tag{117}$$

The second factorization property (111) for our twist follows trivially:

$$\begin{aligned}(1 \otimes \Delta_0)\mathcal{F} &= (1 \otimes \Delta_0)e^{\mathcal{K}^{\beta\alpha} \otimes L_{\alpha\beta}} \\ &= e^{\mathcal{K}^{\beta\alpha} \otimes (\Delta_0 L_{\alpha\beta})} \\ &= e^{\mathcal{K}^{\beta\alpha} \otimes 1 \otimes L_{\alpha\beta}} e^{\mathcal{K}^{\beta\alpha} \otimes L_{\alpha\beta} \otimes 1} \\ &= \mathcal{F}_{13}\mathcal{F}_{12}.\end{aligned} \tag{118}$$

To see that the cocycle condition follows from the factorization properties, the first property, Eq. (110), should be multiplied by $\mathcal{F}_{12}$ from the right and second one, Eq. (111), by $\mathcal{F}_{23}$ from the left:

$$[(\Delta \otimes 1)\mathcal{F}](\mathcal{F} \otimes 1) = \mathcal{F}_{23}\mathcal{F}_{13}\mathcal{F}_{12}, \tag{119}$$

$$(1 \otimes \mathcal{F})(1 \otimes \Delta_0)\mathcal{F} = \mathcal{F}_{23}\mathcal{F}_{13}\mathcal{F}_{12}, \tag{120}$$

which implies

$$[(\Delta \otimes 1)\mathcal{F}](\mathcal{F} \otimes 1) = (1 \otimes \mathcal{F})(1 \otimes \Delta_0)\mathcal{F}. \tag{121}$$

Since $[(\Delta \otimes 1)\mathcal{F}](\mathcal{F} \otimes 1) = (\mathcal{F} \otimes 1)(\Delta_0 \otimes 1)\mathcal{F}$, this is the cocycle condition (109).

5.4 R-matrix

R-matrix is defined by [53,56]

$$\begin{aligned}\mathcal{R} &= \tilde{\mathcal{F}}\mathcal{F}^{-1} = e^{-il^\alpha \otimes p^W_\alpha} e^{ip^W_\beta \otimes l^\beta} \\ &= 1 \otimes 1 + r_{cl} + \mathcal{O}\left(\frac{1}{\kappa^2}\right),\end{aligned} \tag{122}$$

where $\tilde{\mathcal{F}} = \tau_0 \mathcal{F} \tau_0$ is a transposed twist; see Sect. 7. for details.

Up to the second order we have

$$\begin{aligned}\ln \mathcal{R} = i(p^W_\alpha \otimes l^\alpha - l^\alpha \otimes p^W_\alpha) - \frac{1}{2}([p^W_\alpha, l^\beta] \otimes l^\alpha p^W_\beta \\ - l^\alpha p^W_\beta \otimes [p^W_\alpha, l^\beta]) + \mathcal{O}(K^3),\end{aligned} \tag{123}$$

where

$$[p^W_\mu, l_\nu] = [p^W_\mu, \hat{x}_\nu - x_\nu] = [p^W_\mu, \hat{x}^\alpha](\eta_{\alpha\nu} - \varphi^{-1}_{\alpha\nu}(p)). \tag{124}$$

The commutator $[p^W_\mu, \hat{x}_\nu]$ is given in Eq. (A21) and the inverse matrices $\varphi^{-1}_{\mu\nu}$ are given in (19) and the relation between p_μ and p^W_μ is given in Eq. (A20).

Generally, classical matrix $r_{cl} = \ln \mathcal{R}$ up to the first order in $\frac{1}{\kappa}$ and the classical r_{cl} matrices can be written in terms of $\mathfrak{igl}(n)$ generators as

$$r_{cl} = p_\mu \wedge l^\mu = -iK_{\beta\mu\alpha} p^\mu \wedge L^{\alpha\beta}, \tag{125}$$

where $K_{\beta\mu\alpha}$ are given in (17). Using (125) we find the classical r_{cl}-matrices for twists (96)–(99):

$$r_{cl}^{(\mathcal{C}_1)} = a \cdot p \wedge \left[\left(1 - \frac{c}{n}\right) D - c\frac{a_\alpha a_\beta}{a^2} S^{\alpha\beta}\right], \tag{126}$$

$$\begin{aligned} r_{cl}^{(\mathcal{C}_2)} &= a \cdot p \wedge \left[\left(c - \frac{1}{n}\right) D - c\frac{a^\alpha a^\beta}{a^2} S_{\alpha\beta}\right] \\ &\quad - (1-c)p^\alpha \wedge a^\beta \left(S_{\alpha\beta} + \frac{1}{2}M_{\alpha\beta}\right), \end{aligned} \tag{127}$$

$$\begin{aligned} r_{cl}^{(\mathcal{C}_3)} &= a \cdot p \wedge \left[\left(1 + \frac{1}{n}\right) D - c\frac{a_\alpha a_\beta}{a^2} S^{\alpha\beta}\right] \\ &\quad + (1+c)p^\alpha \wedge a^\beta \left(S_{\alpha\beta} - \frac{1}{2}M_{\alpha\beta}\right), \end{aligned} \tag{128}$$

$$r_{cl}^{(\mathcal{C}_4)} = a_\alpha P_\beta \wedge M^{\alpha\beta}, \tag{129}$$

where

$$S_{\mu\nu} = \frac{1}{2}(L_{\mu\nu} + L_{\nu\mu}) - \frac{1}{n} D\eta_{\mu\nu} \tag{130}$$

is the traceless symmetric part of $L_{\mu\nu}$, $D = L^\alpha_{\ \alpha}$ and $M_{\mu\nu} = L_{\mu\nu} - L_{\nu\mu}$. Note that for the case $\mathcal{C}_4$, $r_{cl}^{(\mathcal{C}_4)}$ in (129) coincides with the r_{cl} for the light-cone case discussed in [29]. Also $r_{cl}^{(\mathcal{C}_1)} = r_{cl}^{(\mathcal{C}_2)}$ for $c = 1$ and $r_{cl}^{(\mathcal{C}_1)} = r_{cl}^{(\mathcal{C}_3)}$ for $c = -1$, which is consistent with the discussion in Sect. 3.

6 Twisted symmetry algebras

The family of twists (93), applied to an undeformed $\mathfrak{igl}(n)$ Hopf algebra (Sect. 5.1) produces the corresponding κ-deformed $\mathfrak{igl}(n)$ Hopf algebras. For $h \in \mathfrak{igl}(n)$, the deformed coproduct Δh is related to the undeformed coproduct $\Delta_0 h$ via

$$\Delta h = \mathcal{F}\Delta_0 h \mathcal{F}^{-1}. \tag{131}$$

In the deformed $\mathfrak{igl}(n)$ Hopf algebra the coproduct Δ is

$$\Delta p_\mu = \mathcal{F}\Delta_0 p_\mu \mathcal{F}^{-1} = p_\mu \otimes 1 + \Lambda^{-1}_{\alpha\mu} \otimes p^\alpha, \tag{132}$$

$$\begin{aligned} \Delta L_{\mu\nu} &= \mathcal{F}\Delta_0 L_{\mu\nu}\mathcal{F}^{-1} \\ &= L_{\mu\nu} \otimes 1 + \left(\Lambda^{-1}_{\beta\gamma}\frac{\partial \Lambda^\gamma{}_\alpha}{\partial p^\mu} p_\nu + \Lambda^{-1}_{\beta\nu}\Lambda_{\mu\alpha}\right) \otimes L^{\alpha\beta}, \end{aligned} \tag{133}$$

$$\Delta \Lambda_{\mu\nu} = \Lambda_{\mu\alpha} \otimes \Lambda^\alpha{}_\nu, \tag{134}$$

$$\Delta(\Lambda^{-1})_{\mu\nu} = (\Lambda^{-1})_{\alpha\nu} \otimes (\Lambda^{-1})_\mu{}^\alpha, \tag{135}$$

where $\Lambda_{\mu\nu}$ and $\Lambda^{-1}_{\mu\nu}$ are given in Eqs. (71), (70), (75), (74), (81), (80), (86), and (85) for $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$, respectively.

We point out that generators l_μ (see Eq. (7)) close κ-Minkowski algebra (see Eq. (10)) and $[l_\mu, p_\nu] = iK_{\alpha\mu\nu}p^\alpha$. Note that the twists can be expressed in terms of l_μ and p_μ. From this it follows that $\Delta l_\mu = \mathcal{F}\Delta_0 l_\mu \mathcal{F}^{-1}$ (where $\Delta_0 l_\mu = l_\mu \otimes 1 + 1 \otimes l_\mu$) is closed in l_μ and p_μ.

The counit is unchanged:

$$\epsilon(p_\mu) = \epsilon(L_{\mu\nu}) = 0, \quad \epsilon(\Lambda_{\mu\nu}) = \epsilon(\Lambda^{-1}_{\mu\nu}) = \eta_{\mu\nu}, \tag{136}$$

and the antipode S, obtained from coproduct and counit via $m[(S \otimes 1)\Delta h] = m[(1 \otimes S)\Delta h] = \epsilon(h)$, is given by

$$S(p_\mu) = -\Lambda^{-1}_{\alpha\mu} p^\alpha, \tag{137}$$

$$S(L_{\mu\nu}) = -\left(\Lambda_\beta{}^\gamma \frac{\partial \Lambda^{-1}_{\gamma\alpha}}{\partial S(p^\mu)} S(p_\nu) + \Lambda_{\beta\nu}\Lambda^{-1}_{\mu\alpha}\right) L^{\alpha\beta}, \tag{138}$$

$$S(\Lambda_{\mu\nu}) = \Lambda^{-1}_{\mu\nu}, \tag{139}$$

$$S(\Lambda^{-1}_{\mu\nu}) = \Lambda_{\mu\nu}. \tag{140}$$

The deformed Hopf algebra acting on $\hat{x}_\mu \otimes 1$, i.e. using $g\hat{f} = m[\Delta g(\blacktriangleright \otimes 1)(\hat{f} \otimes 1)]$, $\forall g \in \mathfrak{igl}(n)$ and $\hat{f} \in \hat{\mathcal{A}}$ leads to

$$\begin{aligned} [L_{\rho\sigma}, \hat{x}_\nu] &= \eta_{\sigma\nu}\hat{x}_\rho + i\eta_{\sigma\nu}K_{\mu\rho\alpha}L^{\alpha\mu} \\ &\quad - iK_{\mu\nu\sigma}L_\rho{}^\mu + iK_{\rho\nu\alpha}L^\alpha{}_\sigma \end{aligned} \tag{141}$$

$$[p_\mu, \hat{x}_\nu] = -i(\eta_{\mu\nu} + K_{\beta\nu\mu}p^\beta), \tag{142}$$

which also leads to (2).

Let us consider special cases. For the case $\mathcal{S}_1$, the twist operator is

$$\mathcal{F}_{\mathcal{S}_1} = \exp\{-\ln(1 - a \cdot p) \otimes D\} \tag{143}$$

and coproducts and antipodes of p_μ, $D \equiv L_\alpha{}^\alpha$ and $M_{\mu\nu}$, obtained from the twist (143), are

$$\begin{aligned} \Delta p_\mu &= p_\mu \otimes 1 + Z \otimes p_\mu = \Delta_0 p_\mu - a \cdot p \otimes p_\mu, \\ \Delta D &= D \otimes 1 + Z^{-1} \otimes D, \\ \Delta M_{\mu\nu} &= \Delta_0 M_{\mu\nu} + (a_\mu p_\nu - a_\nu p_\mu) Z^{-1} \otimes D, \\ S(p_\mu) &= -Z^{-1} p_\mu, \\ S(D) &= -ZD = -D + (a \cdot p)D, \\ S(M_{\mu\nu}) &= -M_{\mu\nu} + (a_\mu p_\nu - a_\nu p_\mu). \end{aligned} \tag{144}$$

The coproduct and antipode of l_μ are

$$\begin{aligned} \Delta l_\mu &= l_\mu \otimes 1 + Z^{-1} \otimes l_\mu, \\ S(l_\mu) &= -Z l_\mu. \end{aligned} \tag{145}$$

The symmetry of this case is the Poincaré–Weyl symmetry. The case $\mathcal{S}_1$ corresponds to the right covariant realization $\hat{x}_\mu = x_\mu - a_\mu(x \cdot p)$, see Eq. (20), and is related to [57], but with interchanged left and right side in tensor product and with $a_\mu \to -a_\mu$.

For the case $\mathcal{S}_2$, the twist operator is

$$\mathcal{F}_{\mathcal{S}_2} = \exp\left\{-a_\beta p_\alpha \frac{\ln(1+a\cdot p)}{a\cdot p} \otimes L^{\alpha\beta}\right\} \tag{146}$$

and the coproducts and antipodes of p_μ and $L_{\mu\nu}$ are

$$\begin{aligned}
&\Delta p_\mu = \Delta_0 p_\mu + p_\mu \otimes a\cdot p = p_\mu \otimes Z^{-1} + 1 \otimes p_\mu,\\
&\Delta L_{\mu\nu} = \Delta_0 L_{\mu\nu} - a_\mu p^\alpha Z \otimes L_{\alpha\nu},\\
&S(p_\mu) = -Z p_\mu,\\
&S(L_{\mu\nu}) = -L_{\mu\nu} - a_\mu p^\alpha L_{\alpha\nu}.
\end{aligned} \tag{147}$$

The coproduct and antipode of l_μ are

$$\begin{aligned}
&\Delta l_\mu = \Delta_0 l_\mu + a_\mu p_\alpha Z \otimes l^\alpha,\\
&S(l_\mu) = -l_\mu - a_\mu (p\cdot l).
\end{aligned} \tag{148}$$

The case $\mathcal{S}_2$ corresponds to the left covariant realization $\hat{x}_\mu = x_\mu\,[1+(a\cdot p)]$; see Eq. (20).

For the case $\mathcal{S}_3$, the twist operator is

$$\mathcal{F}_{\mathcal{S}_3} = \exp\{-\ln Z \otimes D + a_\alpha p^W_\beta \otimes L^{\alpha\beta}\}, \tag{149}$$

where

$$p^W_\mu = \left(p_\mu + \frac{a_\mu p^2}{Z+1-a\cdot p}\right)\frac{\ln Z}{Z-1} \tag{150}$$

and

$$Z = \sqrt{(1-a\cdot p)^2 - a^2 p^2}, \tag{151}$$

and the coproduct and antipode of p_μ are

$$\begin{aligned}
\Delta p_\mu &= p_\mu \otimes 1 + \left[\eta_{\mu\alpha} + a_\mu\left(p_\alpha - \frac{a_\alpha p^2}{Z-1+a\cdot p}\right)\right]\\
&\quad \times Z \otimes p^\alpha,\\
S(p_\mu) &= -\left(p_\mu + a_\mu p^2 \frac{Z-1}{Z-1+a\cdot p}\right) Z.
\end{aligned} \tag{152}$$

Similarly one finds $\Delta L_{\mu\nu}$ and $S(L_{\mu\nu})$ using Eqs. (133) and (138), respectively. The coproduct and antipode of l_μ are

$$\begin{aligned}
\Delta l_\mu &= l_\mu \otimes 1 + Z \otimes l_\mu + \left[\frac{Z-1}{\ln Z} a_\mu p^W_\alpha + \frac{a_\alpha p_\mu}{Z^2}\right] \otimes l^\alpha,\\
S(l_\mu) &= -Z^{-1} l_\mu + \frac{1-Z^{-1}}{\ln Z} a_\mu (p^W\cdot l)\\
&\quad + \left(p_\mu + a_\mu p^2 \frac{Z-1}{Z-1+a\cdot p}\right) Z^3 (a\cdot l).
\end{aligned} \tag{153}$$

The case $\mathcal{S}_3$ corresponds to $\hat{x}_\mu = x_\mu - a_\mu(x\cdot p) - (a\cdot x)p_\mu$; see Eq. (20).

For the case $\mathcal{C}_4$, i.e. for the light-like κ deformation of Poincaré–Hopf algebra, the twist operator is

$$\mathcal{F}_{\mathcal{C}_4} = \exp\left\{a_\alpha p_\beta \frac{\ln(1+a\cdot p)}{a\cdot p} \otimes M^{\alpha\beta}\right\} \tag{154}$$

and the coproducts and antipodes of p_μ and $M_{\mu\nu}$, obtained from the twist (154), are

$$\begin{aligned}
\Delta p_\mu &= \Delta_0 p_\mu + \left[p_\mu a^\alpha - a_\mu\left(p^\alpha + \frac{1}{2}a^\alpha p^2\right) Z\right] \otimes p_\alpha,\\
\Delta M_{\mu\nu} &= \Delta_0 M_{\mu\nu} + (\delta^\alpha_\mu a_\nu - \delta^\alpha_\nu a_\mu)\left(p^\beta + \frac{1}{2}a^\beta p^2\right) Z \otimes M_{\alpha\beta},\\
S(p_\mu) &= \left[-p_\mu - a_\mu\left(p_\alpha + \frac{1}{2}a_\alpha p^2\right)p^\alpha\right] Z,\\
S(M_{\mu\nu}) &= -M_{\mu\nu} + (-a_\mu \delta^\beta_\nu + a_\nu \delta^\beta_\mu)\left(p^\alpha + \frac{1}{2}a^\alpha p^2\right) M_{\alpha\beta}.
\end{aligned} \tag{155}$$

The coproduct and antipode of l_μ are

$$\begin{aligned}
&\Delta l_\mu = \Delta_0 l_\mu + a_\mu p_\alpha \otimes l^\alpha,\\
&S(l_\mu) = -l_\mu + a_\mu Z (p\cdot l).
\end{aligned} \tag{156}$$

The case $\mathcal{C}_4$ corresponds to the natural realization $\hat{x}_\mu = x_\mu\,[1+(a\cdot p)] - (a\cdot x)p_\mu$; see Eq. (18). It is the only solution compatible with the κ-Poincaré–Hopf algebra [29–31,48,58].

7 Transposed Drinfeld twists and left–right dual κ-Minkowski algebra

The transposed twist is $\tilde{\mathcal{F}} = \tau_0 \mathcal{F} \tau_0$, where $\tau_0 : \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}\otimes\mathcal{H}$ is a linear map such that $\tau_0(A\otimes B) = B\otimes A\ \forall A, B \in \mathcal{H}$. It is obtained from $\mathcal{F}$ by interchanging left and right side of the tensor product, and it is also a Drinfeld twist satisfying the normalization and cocycle condition. It is obtained from (91) by taking $t = 1$ and using the transposed coproduct $\tilde{\Delta} p_\mu = \tau_0 \Delta p_\mu \tau_0$ instead of Δp_μ.

From the transposed Drinfeld twist, a set of left–right dual generators of κ-Minkowski spacetime can be obtained:

$$\hat{y}_\mu = m[\tilde{\mathcal{F}}^{-1}(\triangleright \otimes 1)(x_\mu \otimes 1)] = x^\alpha \Lambda^{-1}_{\mu\alpha}, \tag{157}$$

where $\Lambda^{-1}_{\mu\nu}$ for $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, and $\mathcal{C}_4$ are given in (70), (75), (80), and (85), respectively. For example, for cases $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$, and $\mathcal{C}_4$, the generators $\hat{y}_\mu$ and $\hat{x}_\mu$ are

$$\mathcal{S}_1\colon\ \hat{y}_\mu = x_\mu(1-a\cdot p), \quad \hat{x}_\mu = x_\mu - a_\mu(x\cdot p), \tag{158}$$

$$\mathcal{S}_2\colon\ \hat{y}_\mu = x_\mu - (a\cdot x)p_\mu, \quad \hat{x}_\mu = x_\mu(1+a\cdot p), \tag{159}$$

$$\begin{aligned}
\mathcal{S}_3\colon\ \hat{y}_\mu &= \left[x_\mu + a_\mu\left(x\cdot p - \frac{(a\cdot x)p^2}{Z+1-a\cdot p}\right)\right] Z,\\
\hat{x}_\mu &= x_\mu - (a\cdot x)p - a_\mu(x\cdot p),
\end{aligned} \tag{160}$$

$$\begin{aligned}
\mathcal{C}_4\colon\ \hat{y}_\mu &= x_\mu + (a\cdot x)p_\mu - a_\mu\left(x\cdot p + \frac{a\cdot x}{2}p^2\right) Z,\\
\hat{x}_\mu &= x_\mu - (a\cdot x)p_\mu - x_\mu(a\cdot p), \quad a^2 = 0.
\end{aligned} \tag{161}$$

The generators $\hat{y}_\mu$ satisfy the κ-Minkowski algebra but with $-a_\mu$ instead of a_μ [48]:

$$[\hat{y}_\mu, \hat{y}_\nu] = -i(a_\mu \hat{y}_\nu - a_\nu \hat{y}_\mu). \tag{162}$$

The generators of κ-Minkowski space $\hat{x}_\mu$ commute with their duals $\hat{y}_\mu$ [48]:

$$[\hat{x}_\mu, \hat{y}_\nu] = 0. \tag{163}$$

Generally, the dual basis $\hat{y}_\mu$ is related to the basis $\hat{x}_\mu$ via

$$\hat{y}_\mu = \hat{x}^\alpha (e^{-\mathcal{C}})_{\mu\alpha}, \tag{164}$$

where $\mathcal{C}_{\mu\nu} = -C_{\mu\alpha\nu}(p^W)^\alpha$, where $C_{\mu\alpha\nu}$ are structure constants (see "Appendix A" in [55]). From this relation, and Eqs. (5) and (157) for $\hat{x}_\mu$ and $\hat{y}_\mu$, respectively, it follows that

$$\Lambda^{-1}_{\mu\nu} = (e^{-\mathcal{K}})_{\mu\nu} = (e^{-\mathcal{C}})_{\mu\alpha}\varphi_\nu{}^\alpha. \tag{165}$$

7.1 κ-Minkowski algebra from transposed twists with $a_\mu \to -a_\mu$

Starting with the family of twists (93), we define the related Drinfeld twists $\tilde{\mathcal{F}}\big|_{a_\mu\to -a_\mu}$. They lead to nonlinear realizations of $\hat{x}_\mu$, satisfying (2):

$$\begin{aligned}\hat{x}_\mu &= m\left[\tilde{\mathcal{F}}^{-1}\Big|_{a_\mu\to -a_\mu}(\triangleright\otimes 1)(x_\mu\otimes 1)\right]\\ &= x^\alpha\ \Lambda^{-1}_{\mu\alpha}\Big|_{a_\mu\to -a_\mu}.\end{aligned} \tag{166}$$

Then the corresponding dual generators $\hat{y}_\mu$ are given by

$$\hat{y}_\mu = x_\mu + iK_{\beta\mu\alpha}L^{\alpha\beta} = x_\mu - l_\mu. \tag{167}$$

Compared to the case with the transposed twists $\tilde{\mathcal{F}}$ in the beginning of this section, here the roles of $\hat{x}_\mu$ and $\hat{y}_\mu$ are interchanged, with $a_\mu \to -a_\mu$. With this new family of twists, $\hat{x}_\mu$ are nonlinear realizations of κ-Minkowski space, while $\hat{y}_\mu$ are linear realizations of dual κ-Minkowski space.

If we apply twists $\tilde{\mathcal{F}}\big|_{a_\mu\to -a_\mu}$ to the undeformed coproducts $\Delta_0 h$, we get coproducts $\tilde{\Delta}|_{a_\mu\to -a_\mu}h$, i.e. the left and right sides in coproducts Δh are interchanged and a_μ is replaced by $-a_\mu$.

We point out that solution $\mathcal{C}_4$, and its transposed case, are of special interest because they lead to a light-like κ-Poincaré–Hopf algebra. They are related to the result by Borowiec and Pachoł [29] (a comparison is given in Sect. 8.2).

8 Nonlinear realizations of κ-Minkowski space and related Drinfeld twists

We shall also present a few families of nonlinear realizations and the corresponding Drinfeld twist operators that have occurred in the literature so far.

8.1 Time-like deformations

The realizations we are considering are [41,49]:

$$\hat{x}_i = x_i\varphi(A), \tag{168}$$

$$\hat{x}_0 = x_0\psi(A) - a_0 x_k p^k \gamma(A), \tag{169}$$

where $A = -a\cdot p$ and functions $\varphi(A)$, $\psi(A)$ are such that $\varphi(0) = \psi(0) = 1$ and related to $\gamma(A)$ by

$$\gamma(A) = \frac{\psi(A)}{\varphi(A)}\frac{d\varphi(A)}{dA} + 1. \tag{170}$$

Generically, the symmetry algebra is κ-deformed $\mathfrak{igl}(n)$ Hopf algebra. We will present two cases.

(i) The first case is $\psi(A) = 1$, with arbitrary $\varphi(A)$ and $\gamma(A) = \frac{\varphi'(A)}{\varphi(A)} + 1$; see Eq. (170). The coproducts of the momenta are

$$\Delta p_0 = \Delta_0 p_0 = p_0\otimes 1 + 1\otimes p_0, \tag{171}$$

$$\Delta p_i = \varphi(A\otimes 1 + 1\otimes A)\left(\frac{p_i}{\varphi(A)}\otimes 1 + e^A\otimes\frac{p_i}{\varphi(A)}\right). \tag{172}$$

The twist operator is Abelian [54]

$$\begin{aligned}\mathcal{F}_\varphi = \exp\Bigg\{&(N\otimes 1)\ln\frac{\varphi(A\otimes 1 + 1\otimes A)}{\varphi(A\otimes 1)}\\ &+(1\otimes N)\left(A\otimes 1 + \ln\frac{\varphi(A\otimes 1 + 1\otimes A)}{\varphi(1\otimes A)}\right)\Bigg\},\end{aligned} \tag{173}$$

where $N = ix_i p^i$ and $[N, A] = 0$. Since this twist is Abelian, it automatically satisfies the cocycle condition, and therefore it is a Drinfeld twist. A special case is presented in [59–61].

(ii) In the second case, leading to Jordanian twists, given by Borowiec and Pachoł [62], $\psi(A)$ is a linear function, i.e. $\psi(A) = 1 + rA$, where $r\in\mathbb{R}$, and $\gamma(A) = 0$, which leads to

$$\varphi = \psi^{-\frac{1}{r}} = (1 + rA)^{-\frac{1}{r}}. \tag{174}$$

The coproducts of the momenta are

$$\Delta p_0 = p_0\otimes\varphi(A) + 1\otimes p_0, \tag{175}$$

$$\Delta p_i = p_i \otimes \psi(A) + 1 \otimes p_i. \quad (176)$$

The family of corresponding twist operators is

$$\mathcal{F}_r = \exp\left\{\left(-L^0{}_0 + \frac{1}{r}L^k{}_k\right) \otimes \ln\varphi(A)\right\}. \quad (177)$$

The special Jordanian twist was studied by Bu et al. [57] and corresponds to $\mathcal{S}_1$, but with interchanged left and right side of the tensor product, and with $a_0 \to -a_0$ and $c = r + 1$, i.e.

$$\mathcal{F}_r = \tilde{\mathcal{F}}_{\mathcal{S}_1}\Big|_{a_0 \to -a_0,\ c=r+1}. \quad (178)$$

8.2 Light-cone deformation

In the light-cone basis, the κ-Poincaré algebra was studied in [28,29]; the corresponding twist is an extended Jordanian twist, written in terms of two exponential factors. It is identical to the transposed twist of $\mathcal{F}_{\mathcal{C}_4}$ with $a_\mu \to -a_\mu$, i.e. $\tilde{\mathcal{F}}_{\mathcal{C}_4}\Big|_{a_\mu \to -a_\mu}$.

The extended Jordanian twist corresponding to a light-cone deformation is

$$\mathcal{F}_{LC} = e^{-iM_{+-}\otimes \ln\Pi_+} e^{-\frac{i}{\kappa}M_{+a}\otimes P^a \Pi_+^{-1}}, \quad (179)$$

where $[P_\mu, \hat{x}_\mu] = -i\eta_{\mu\nu}\,[1 + (a\cdot P)] + i a_\mu P_\nu$, see Eq. (18), and

$$\begin{aligned} \Pi_+ &= 1 + \frac{1}{\kappa}P_+ = 1 - a\cdot P, \\ a_0 &= a_1 = \frac{1}{\sqrt{2}\kappa}, \quad a_j = 0 \text{ for } j > 1, \\ P_\pm &= \frac{P_0 \pm P_1}{\sqrt{2}}, \\ M_{+-} &= iM_{01}, \quad M_{\pm j} = \frac{i}{\sqrt{2}}(M_{0j} \pm M_{1j}), \quad j > 1. \end{aligned} \quad (180)$$

If we define

$$\begin{aligned} \mathcal{A} &\equiv -iM_{+-} \otimes \ln\Pi_+, \\ \mathcal{B} &\equiv -\tfrac{i}{\kappa}M_{+a} \otimes P^a\Pi_+^{-1}, \end{aligned} \quad (181)$$

then $[\mathcal{A}, \mathcal{B}] = \alpha\mathcal{B}$, where

$$\alpha \equiv 1 \otimes \ln\Pi_+ \quad (182)$$

and $\mathcal{A}$, $\mathcal{B}$, and α generate the algebra (C1), given in “Appendix C”. Using the result (C2) from “Appendix C”, it follows that $\mathcal{F}_{LC}$, written as one exponential function, is given by

$$\mathcal{F}_{LC} = \exp\left\{-iM_{+-} \otimes \frac{(\Pi_+ - 1)\ln\Pi_+}{\Pi_+ - 1} - \frac{i}{\kappa}M_{+a} \otimes P^a \frac{\ln\Pi_+}{\Pi_+ - 1}\right\}. \quad (183)$$

Using our notation, this result is

$$\mathcal{F}_{LC} = \exp\left\{M_{\alpha\beta} \otimes a^\alpha P^\beta \frac{\ln(1 - a\cdot P)}{a\cdot P}\right\}, \quad (184)$$

which thus proves the relation

$$\mathcal{F}_{LC} = \tilde{\mathcal{F}}_{\mathcal{C}_4}\Big|_{a_\mu \to -a_\mu}. \quad (185)$$

The twist $\mathcal{F}_{LC}$ leads to the nonlinear realization (157) and the corresponding coproduct is a transposed coproduct with $a_\mu \to -a_\mu$.

9 Outlook and discussion

A full analysis of all possible linear realizations for κ-Minkowski space for time-, space-, and light-like deformations is given. These realizations can be expressed in terms of the generators of the $\mathfrak{gl}(n)$ algebra. The coproducts of the momenta for linear realizations are constructed. We have presented a method for constructing Drinfeld twist operators corresponding to each linear realization of κ-Minkowski space and proved that it satisfies the cocycle and normalization conditions. We have constructed a whole new class of Drinfeld twists compatible with κ-Minkowski space and linear realizations, denoted by $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$. The symmetries generated by the Drinfeld twists are described by κ-deformed $\mathfrak{igl}(n)$-Hopf algebras, and in the special case of $\mathcal{S}_1$ and $\mathcal{C}_4$ we get the Poincaré–Weyl–Hopf algebra and light-like κ-Poincaré–Hopf algebra, respectively. We further illustrate how our method also works for constructing Drinfeld twists for nonlinear realizations and we compared our results to the examples already known in the literature.

In this paper we were dealing mostly with linear realizations and the corresponding Drinfeld twists. However, for any realization, in general, one can construct a twist operator that does not have to satisfy the cocycle condition in the Hopf algebra sense (i.e. it is not a Drinfeld twist), rather it satisfies the cocycle condition in a more general sense (up to tensor exchange identities [44–46]), i.e. in the framework of Hopf algebroids [44–46]. It is crucial to notice that the κ-Minkowski space can be embedded into a Heisenberg algebra which has a natural Hopf algebroid structure. One can show that the star product resulting from this generalized twist operator is associative and that the corresponding symmetry algebra is a certain deformation of the $\mathfrak{igl}(n)$-Hopf algebra. This general framework is more suitable to address the questions of quantum gravity [63,64] and related new effects in Planck scale physics.

The problem of finding all possible linear realizations is closely related to a classification of bicovariant differential calculi on κ-Minkowski space [55]. Namely, the requirement that the differential calculus is bicovariant leads to finding all possible algebras between NC coordinates and NC one-forms that are closed (linear) in these NC one-forms. The corresponding equations for the structure constants (from the super-Jacobi identities) are exactly the same as Eqs. (8) and (9). The linear realizations elaborated in this paper are expressed in terms of the Heisenberg algebra, but one can extend this to a super-Heisenberg algebra, by introducing Grassmann coordinates and momenta. This way one can construct the extended twists [55,65] which have the same desired properties, but which also give the whole differential calculi.

With a linear realization it is much easier to understand and to perform a practical calculation in the NC space. In [66,67] it is proposed that the NC metric should be a central element of the whole differential algebra (generated by NC coordinates and NC one-forms). This NC metric should encode some of the main properties of the quantum theory of gravity. We hope that using the tool of linear realizations one can perform such calculations for a large class of deformations, and for all types of bicovariant differential calculi and predict new contributions to the physics of quantum black holes and the quantum origin of the cosmological constant [68,69].

Recently [32], the Drinfeld twist corresponding to $\mathcal{C}_4$ was analyzed and the corresponding scalar field theory was discussed. We are planing to further analyze the properties of quantum field theories [70], especially the gauge theories that arise from this twist, but we are also interested in pursuing further investigations of the physical aspects of the $\mathcal{C}_{1,2,3}$ cases.

Acknowledgments The authors would like to thank S. Mignemi, A. Pachoł, and Z. Škoda for comments. This work has been fully supported by Croatian Science Foundation under the Project (IP-2014-09-9582). The authors would also like to thank the anonymous referees for valuable remarks.

Funded by SCOAP^3.

Appendix A: Derivation of coproduct Δp_μ

Here we present construction of equations for $K_\mu(k)$ and $P_\mu(k_1, k_2)$ and their solutions for linear realizations.

From Eq. (46) we find

$$e^{-i\lambda k_1\cdot\hat{x}} p_\mu e^{i\lambda k_1\cdot\hat{x}} \triangleright e^{ik_2\cdot x} = P_\mu(\lambda k_1, k_2) e^{ik_2\cdot x}, \tag{A1}$$

where $(k_1)_\mu, (k_2)_\mu \in \mathbb{M}^n$ and $p_\mu \in \mathcal{T}$. Differentiating both sides by $\frac{\partial}{\partial\lambda}$ and using $\hat{x}_\mu = x^\alpha \varphi_{\alpha\mu}(p)$ we get the relation between $P_\mu(\lambda k_1, k_2)$ and the realization $\varphi_{\mu\nu}(P(\lambda k_1, k_2))$:

$$\frac{\partial P_\mu(\lambda k_1, k_2)}{\partial\lambda} = \varphi_{\mu\alpha}(P(\lambda k_1, k_2))k_1^\alpha. \tag{A2}$$

Note that for $\lambda = 0$ the boundary condition is

$$P_\mu(0, k) = k_\mu. \tag{A3}$$

The coproduct for momentum p_μ is calculated by [50–52]:

$$\Delta p_\mu = \mathcal{D}_\mu(p \otimes 1, 1 \otimes p), \tag{A4}$$

where the function $\mathcal{D}_\mu(k_1, k_2)$ is given by

$$\mathcal{D}_\mu(k_1, k_2) = P_\mu(K^{-1}(k_1), k_2), \tag{A5}$$

and $K_\mu^{-1}(k_1)$ is the inverse function of $P_\mu(k_1, 0) = K_\mu(k_1)$.

The function $\varphi_{\mu\alpha}(p)$ describes the choice of realization in the following way:

$$\hat{x}_\mu = x^\alpha \varphi_{\alpha\mu}(p). \tag{A6}$$

In the case of linear realizations $\varphi_{\alpha\mu}(p)$ is

$$\varphi_{\alpha\mu} = \eta_{\alpha\mu} + K_{\beta\mu\alpha} p^\beta; \tag{A7}$$

therefore

$$\frac{\partial P_\mu(\lambda k_1, k_2)}{\partial\lambda} = (k_1)_\mu + K^\alpha{}_{\beta\mu} P_\alpha(\lambda k_1, k_2) k_1^\beta. \tag{A8}$$

This can be solved by expanding $P(\lambda k_1, k_2)$ in terms of λ,

$$P_\mu(\lambda k_1, k_2) = \sum_{n=0}^{\infty} P_\mu^{(n)}(k_1, k_2)\lambda^n, \tag{A9}$$

which, comparing the terms with the same power of λ, leads to

$$P_\mu^{(1)}(k_1, k_2) = (k_1)_\mu + K^\alpha{}_{\beta\mu} P_\alpha^{(0)}(k_1, k_2) k_1^\beta, \tag{A10}$$

$$P_\mu^{(n+1)}(k_1, k_2) = \frac{1}{n+1} K^\alpha{}_{\beta\mu} P_\alpha^{(n)}(k_1, k_2) k_1^\beta, \quad \text{for } n \geq 1. \tag{A11}$$

The boundary condition (A3) leads to

$$P_\alpha^{(0)}(k_1, k_2) = (k_2)_\mu. \tag{A12}$$

For the sake of brevity, let us define

$$\mathcal{K}_{\mu\nu}(k) \equiv -K_{\mu\alpha\nu} k^\alpha. \tag{A13}$$

Using (A12) and (A13), Eqs. (A10) and (A11) become

$$P_\mu^{(1)}(k_1, k_2) = (k_1)_\mu - \mathcal{K}_{\alpha\mu}(k_1) k_2^\alpha, \tag{A14}$$

$$P_\mu^{(n+1)}(k_1, k_2) = -\frac{1}{n+1} \mathcal{K}^\alpha{}_\mu(k_1) P_\alpha^{(n)}(k_1, k_2), \quad \text{for } n \geq 1. \tag{A15}$$

The solution for $P_\mu(k_1, k_2)$ is

$$P_\mu(k_1, k_2) = \left(\frac{\eta - e^{-\mathcal{K}(k_1)}}{\mathcal{K}(k_1)}\right)_{\alpha\mu} k_1^\alpha + \left(e^{-\mathcal{K}(k_1)}\right)_{\alpha\mu} k_2^\alpha. \tag{A16}$$

The solution for $K_\mu(k) = P_\mu(k, 0)$ is simply

$$K_\mu(k) = \left(\frac{\eta - e^{-\mathcal{K}(k)}}{\mathcal{K}(k)}\right)_{\alpha\mu} k^\alpha. \tag{A17}$$

It is useful to define

$$k_\mu^W = K_\mu^{-1}(k). \tag{A18}$$

Inserting this definition and the solution (A16) into (A5) we get

$$\mathcal{D}_\mu(k_1, k_2) = P_\mu(k_1^W, k_2) = (k_1)_\mu + \left(e^{-\mathcal{K}(k_1^W)}\right)_{\alpha\mu} k_2^\alpha. \tag{A19}$$

The momentum $p_\mu^W = K_\mu^{-1}(p)$, introduced in Sect. 4.3 (see Eq. (66)), is related to p_μ via

$$p_\mu = \left(\frac{\eta - e^{-\mathcal{K}}}{\mathcal{K}}\right)_{\alpha\mu} (p^W)^\alpha \tag{A20}$$

and is given in closed form in (72), (77), (82), and (87) for the solutions $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, and $\mathcal{C}_4$, respectively. The momentum p_μ^W corresponds to a Weyl symmetric ordering [40,41,49],

$$[p_\mu^W, \hat{x}_\nu] = \eta_{\mu\nu} \frac{a \cdot p^W}{\mathrm{e}^{-a \cdot p^W} - 1} + \frac{a_\nu p_\mu^W}{a \cdot p^W}\left(1 + \frac{a \cdot p^W}{\mathrm{e}^{-a \cdot p^W} - 1}\right). \tag{A21}$$

For p_μ^W, it is useful to define

$$\mathcal{K}_{\mu\nu} \equiv \mathcal{K}_{\mu\nu}(p^W) = -K_{\mu\alpha\nu}(p^W)^\alpha. \tag{A22}$$

Using this definition and the solution (A19) with Eq. (A4) finally leads to the coproduct:

$$\Delta p_\mu = p_\mu \otimes 1 + \left(e^{-\mathcal{K}}\right)_{\alpha\mu} \otimes p^\alpha. \tag{A23}$$

Appendix B: Construction of the twist operator from the coproduct of momenta

It can be shown that for any $A_{\mu\nu}$ such that $[A_{\mu\nu}, L_{\sigma\rho}] = 0$ and $[A_{\mu\nu}, A_{\sigma\rho}] = 0$, the following holds:

$$: e^{L_\beta{}^\alpha A_\alpha{}^\beta} := e^{L_\beta{}^\alpha [\ln(1+A)]_\alpha{}^\beta}. \tag{B1}$$

This identity is a generalization of the result presented in [49] in Eqs. (A. 16) and (A. 17). See also Sect. 2 in [71].

Twists can be calculated from the known coproducts of the momenta using Eq. (91). We would like to write the twist in the following form:

$$\mathcal{F} = e^f, \tag{B2}$$

where

$$f = \sum_{s=1}^\infty f_s, \tag{B3}$$

and $f_s \in \mathcal{U}[\mathfrak{igl}(n)] \otimes \mathcal{U}[\mathfrak{igl}(n)]$ is the contribution to f in sth order of $\frac{1}{\kappa}$.

Inserting (55) into (91), for $t = 0$ we get

$$\mathcal{F}^{-1} =: \exp\left\{(\Lambda^{-1} - 1)^\beta{}_\alpha \otimes L^\alpha{}_\beta\right\} : . \tag{B4}$$

From Eq. (B1) it follows

$$\mathcal{F}^{-1} = \exp\left\{(\ln \Lambda^{-1})^\beta{}_\alpha \otimes L^\alpha{}_\beta\right\}. \tag{B5}$$

Since $\Lambda_{\mu\nu} = \left(e^{\mathcal{K}}\right)_{\mu\nu}$, we find the twist:

$$\mathcal{F} = \exp\left(\mathcal{K}_{\beta\alpha} \otimes L^{\alpha\beta}\right). \tag{B6}$$

Since $\mathcal{K}_{\mu\nu} = -K_{\mu\alpha\nu}(p^W)^\alpha$ and $l_\mu = -i K_{\beta\mu\alpha} L^{\alpha\beta}$, the result can also be written as

$$\mathcal{F} = \exp(-i p_\alpha^W \otimes l^\alpha). \tag{B7}$$

Appendix C: A special case of the BCH formula

Let us consider the algebra generated by $\mathcal{A}$, $\mathcal{B}$, and α:

$$[\mathcal{A}, \mathcal{B}] = \alpha\mathcal{B}, \quad [\mathcal{A}, \alpha] = [\mathcal{B}, \alpha] = 0, \tag{C1}$$

then

$$e^{\mathcal{A}} e^{\mathcal{B}} = e^{\mathcal{A} + \mathcal{B}\{(\alpha)}, \tag{C2}$$

where

$$f(\alpha) = \frac{\alpha}{1 - e^{-\alpha}}. \tag{C3}$$

This can be proved by representing $\mathcal{A}$ and $\mathcal{B}$ with 2×2 matrices:

$$\mathcal{A} = \frac{\alpha}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{C4}$$

These matrices satisfy the algebra (C1). Their exponentials are

$$e^{\mathcal{A}} = \begin{pmatrix} e^{\frac{\alpha}{2}} & 0 \\ 0 & e^{-\frac{\alpha}{2}} \end{pmatrix}, \quad e^{\mathcal{B}} = 1 + \mathcal{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{C5}$$

leading to

$$e^{\mathcal{A}} e^{\mathcal{B}} = \begin{pmatrix} e^{\frac{\alpha}{2}} & e^{\frac{\alpha}{2}} \\ 0 & e^{-\frac{\alpha}{2}} \end{pmatrix}. \tag{C6}$$

On the other hand, since

$$\mathcal{A} + \mathcal{B}\{(\alpha) = \begin{pmatrix} \frac{\alpha}{2} & f(\alpha) \\ 0 & -\frac{\alpha}{2} \end{pmatrix}, \tag{C7}$$

it follows that

$$e^{\mathcal{A}+\mathcal{B}\{(\alpha)} = \begin{pmatrix} e^{\frac{\alpha}{2}} & \frac{f(\alpha)}{\alpha}(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}) \\ 0 & e^{-\frac{\alpha}{2}} \end{pmatrix}. \tag{C8}$$

Comparing (C6) and (C8) gives $f(\alpha)$ in (C3).

References

1. S. Doplicher, K. Fredenhagen, J.E. Roberts, Spacetime quantization induced by classical gravity. Phys. Lett. B **331**, 39 (1994)
2. S. Doplicher, K. Fredenhagen, J.E. Roberts, The quantum structure of spacetime at the Planck scale and quantum fields. Commun. Math. Phys. **172**, 187 (1995)
3. A. Connes, *Noncommutative Geometry* (Academic Press, New York, 1994)
4. J. Lukierski, H. Ruegg, Quantum kappa Poincare in any dimension. Phys. Lett. B **329**, 189 (1994). arXiv:hep-th/9310117
5. J. Lukierski, A. Nowicki, H. Ruegg, V.N. Tolstoy, Q-deformation of Poincaré algebra. Phys. Lett. B **264**, 331 (1991)
6. J. Lukierski, A. Nowicki, H. Ruegg, New quantum Poincaré algebra and κ-deformed field theory. Phys. Lett. B **293**, 344 (1992)
7. S. Majid, H. Ruegg, Bicrossproduct structure of kappa Poincare group and noncommutative geometry. Phys. Lett. B **334**, 348 (1994). arXiv:hep-th/9405107
8. J. Kowalski-Glikman, S. Nowak, Doubly special relativity theories as different bases of kappa Poincare algebra. Phys. Lett. B **539**, 126 (2002). arXiv:hep-th/0203040
9. G. Amelino-Camelia, Doubly-special relativity: first results and key open problems. Int. J. Mod. Phys. D **11**, 35 (2002). arXiv:gr-qc/0210063
10. G. Amelino-Camelia, Testable scenario for relativity with minimum-length. Phys. Lett. B **510**, 255 (2001). arXiv:hep-th/0012238
11. F.A. Bais, N.M. Muller, B.J. Schroers, Quantum group symmetry and particle scattering in (2 + 1)-dimensional quantum gravity. Nucl. Phys. B **640**, 3–45 (2002). arXiv:hep-th/0205021
12. C. Meusburger, B. Schroers, Generalised Chern–Simons actions for 3d gravity and κ-Poincare symmetry. Nucl. Phys. B **806**, 462–488 (2009). arXiv:0805.3318 [gr-qc]
13. G. Amelino-Camelia, L. Smolin, A. Starodubtsev, Quantum symmetry, the cosmological constant and Planck scale phenomenology. Class. Quantum Gravity **21**, 3095 (2004). arXiv:hep-th/0306134
14. L. Freidel, J. Kowalski-Glikman, L. Smolin, 2 + 1 gravity and doubly special relativity. Phys. Rev. D **69**, 044001 (2004). arXiv:hep-th/0307085
15. L. Freidel, E.R. Livine, 3D quantum gravity and effective noncommutative quantum field theory. Phys. Rev. Lett. **96**, 221301 (2006). arXiv:hep-th/0512113
16. S. Majid, *Foundations of Quantum Group Theory* (Cambridge University Press, Cambridge, 1995)
17. M. Chaichian, A. Demichev, *Introduction to Quantum Groups* (World Scientific, Singapore, 1996)
18. J. Madore, *An Introduction to Noncommutative Differential Geometry and its Physical Applications (London Mathematical Society. Lecture Note Series)* (Cambridge University Press, Cambridge, 1999)
19. V.G. Drinfeld, *Quantum Groups* (Proceedings of the ICM, Rhode Island, 1987)
20. V.G. Drinfeld, Hopf algebras and the quantum Yang–Baxter equation. Sov. Math. Dokl. **32**, 254 (1985)
21. V.G. Drinfeld, Quasi Hopf algebras. Algebra Anal. **1**(6), 114 (1989). [Leningr. Math. J. **1**(6), 1419 (1990)]
22. P. Aschieri, F. Lizzi, P. Vitale, Twisting all the way: from classical mechanics to quantum fields. Phys. Rev. D **77**, 025037 (2008). arXiv:0708.3002
23. P. Aschieri, M. Dimitrijevic, P. Kulish, F. Lizzi, J. Wess, *Noncommutative Spacetimes: Symmetries in Noncommutative Geometry and Field Theory (Lecture Notes in Physics)* (Springer, New York, 2009)
24. A. Borowiec, J. Lukierski, A. Pachoł, Twisting and kappa-Poincare. J. Phys. A Math. Theor. **47** 405203 (2014). arXiv:1312.7807 [math-ph]
25. K. Przanowski, The bicovariant differential calculus on the kappa-Poincare and kappa-Weyl groups. Czech J. Phys. **47**, 107 (1997). arXiv:q-alg/9606022
26. P. Podles, Solutions of Klein–Gordon and Dirac equations on quantum Minkowski spaces. Commun. Math. Phys. **181**, 569–586 (1996). arXiv:q-alg/9510019
27. P. Kosinski, P. Maslanka, J. Lukierski, A. Sitarz, *Generalized kappa-Deformations and Deformed Relativistic Scalar Fields on Noncommutative Minkowski Space*. Proceedings of the Conference Topics in Mathematical Physics, General Relativity and Cosmology (World Scientific, Singapore, 2003). arXiv:hep-th/0307038
28. P.P. Kulish, V.D. Lyakhovsky, A.I. Mudrov, Extended jordanian twists for Lie algebras. J. Math. Phys. **40**, 4569 (1999). arXiv:math/9806014
29. A. Borowiec, A. Pachoł, Unified description for kappa-deformations of orthogonal groups. Eur. Phys. J. C **74**(3) (2014). arXiv:1311.4499 [math-ph]
30. T. Juric, S. Meljanac, R. Strajn, Universal κ-Poincare covariant differential calculus over κ-Minkowski space. Int. J. Mod. Phys. A **29**, 1450121 (2014). arXiv:1312.2751 [hep-th]
31. S. Meljanac, D. Pikutic, *Light-Like kappa-Poincare/Conformal Hopf Algebra and Covariant Drinfeld Twist*. Proceeding of XXXIII M. Born Symposium, Wrocław, 2014
32. T. Juric, S. Meljanac, A. Samsarov, Light-like κ-deformations and scalar field theory via Drinfeld twist. J. Phys. Conf. Ser. **634**(1), 012005 (2015). arXiv:1506.02475 [hep-th]
33. J. Lukierski, H. Ruegg, W.J. Zakrzewski, Classical and quantum mechanics of free κ relativistic systems. Ann. Phys. **243**, 90 (1995)
34. J. Lukierski, V. Lyakhovsky, M. Mozrzymas, κ-deformations of $D = 4$ Weyl and conformal symmetries. Phys. Lett. B **538**, 375 (2002). arXiv:hep-th/0203182
35. P. Kosiński, J. Lukierski, P. Maślanka, κ-deformed Wigner construction of relativistic wave functions and free fields on κ-Minkowski space. Nucl. Phys. Proc. Suppl. **102**, 161–168 (2001). arXiv:hep-th/0103127
36. P. Kosinski, P. Maslanka, in *From Field Theory to Quantum Groups*, eds. by B. Jancewicz, J. Sobczyk (World Scientific, Singapore, 1996), p. 41. arXiv:q-alg/9512018
37. J. Lukierski, in *Proc of Alushta Conf. on Recent Problems in QFT*, eds. by D. Shirkov, D.I. Kazakov, A.A. Vladimirov (Dubna, 1996), p. 82
38. S. Meljanac, A. Samsarov, Scalar field theory on kappa-Minkowski spacetime and translation and Lorentz invariance. Int. J. Mod. Phys. A **26**, 1439 (2011). arXiv:1007.3943
39. A. Ballesteros, F.J. Herranz, M.A. del Olmo, M. Santander, A new null-plane quantum Poincare algebra. Phys. Lett. B **351**, 137–145 (1995). arXiv:q-alg/9502019
40. S. Meljanac, S. Krešić-Jurić, M. Stojić, Covariant realizations of kappa-deformed space. Eur. Phys. J. C **51**, 229 (2007). arXiv:hep-th/0702215

41. S. Meljanac, M. Stojić, New realizations of Lie algebra kappa-deformed Euclidean space. Eur. Phys. J. C **47**, 531 (2006). arXiv:hep-th/0605133
42. M. Dimitrijevic, L. Möller, E. Tsouchnika, Derivatives, forms and vector fields on the kappa-deformed Euclidean space. J. Phys. A **37**, 9749–9770 (2004). arXiv:hep-th/0404224
43. S. Meljanac, S. Kresic-Juric, Generalized kappa-deformed spaces, star-products, and their realizations. J. Phys. A **41**, 235203 (2008). arXiv:0804.3072 [hep-th]
44. T. Juric, S. Meljanac, R. Strajn, κ-Poincare–Hopf algebra and Hopf algebroid structure of phase space from twist. Phys. Lett. A **377**, 2472 (2013). arXiv:1303.0994 [hep-th]
45. T. Juric, S. Meljanac, R. Strajn, Twists, realizations and Hopf algebroid structure of kappa-deformed phase space. Int. J. Mod. Phys. A **29**(5), 1450022 (2014). arXiv:1305.3088 [hep-th]
46. T. Juric, D. Kovacevic, S. Meljanac, κ-deformed phase space, Hopf algebroid and twisting. SIGMA **10**, 106 (2014) (18 pages). arXiv:1402.0397 [math-ph]
47. D. Kovačević, S. Meljanac, A. Samsarov, Z. Škoda, Hermitian realizations of kappa-Minkowski spacetime. Int. J. Mod. Phys. A **30**, 1550019 (2015) (26 pages). arXiv:1307.5772 [hep-th]
48. D. Kovacevic, S. Meljanac, Kappa-Minkowski spacetime, kappa-Poincaré Hopf algebra and realizations. J. Phys. A Math. Theor. **45**, 135208 (2012). arXiv:1110.0944
49. S. Meljanac, A. Samsarov, M. Stojić, K.S. Gupta, Kappa-Minkowski space-time and the star product realizations. Eur. Phys. J. C **53**, 295 (2008). arXiv:0705.2471
50. S. Meljanac, D. Meljanac, A. Samsarov, M. Stojic, Lie algebraic deformations of Minkowski space with Poincare algebra. arXiv:0909.1706 [math-ph]
51. S. Meljanac, D. Meljanac, A. Samsarov, M. Stojic, Kappa-deformed Snyder spacetime. Mod. Phys. Lett. A **25**, 579 (2010). arXiv:0912.5087 [hep-th]
52. S. Meljanac, D. Meljanac, A. Samsarov, M. Stojic, Kappa Snyder deformations of Minkowski spacetime, realizations and Hopf algebra. Phys. Rev. D **83**, 065009 (2011). arXiv:1102.1655 [math-ph]
53. T.R. Govindarajan, K.S. Gupta, E. Harikumar, S. Meljanac, D. Meljanac, Deformed oscillator algebras and QFT in the κ-Minkowski spacetime. Phys. Rev. D **80**, 025014 (2009). arXiv:0903.2355
54. T.R. Govindarajan, K.S. Gupta, E. Harikumar, S. Meljanac, D. Meljanac, Twisted statistics in κ-Minkowski spacetime. Phys. Rev. D **77**, 105010 (2008). arXiv:0802.1576 [hep-th]
55. T. Jurić, S. Meljanac, D. Pikutić, R. Štrajn, Toward the classification of differential calculi on κ-Minkowski space and related field theories. JHEP **1507**, 055 (2015). arXiv:1502.02972 [hep-th]
56. S. Meljanac, A. Samsarov, R. Strajn, Kappa-deformation of phase space; generalized Poincare algebras and R-matrix. JHEP **08**, 127 (2012). arXiv:1204.4324
57. J.G. Bu, J.H. Yee, H.C. Kim, Differential structure on κ-Minkowski spacetime realized as module of twisted Weyl algebra. Phys. Lett. B **679**, 486 (2009). arXiv:0903.0040 [hep-th]
58. A. Borowiec, A. Pachol, Classical basis for kappa-Poincare algebra and doubly special relativity theories. J. Phys. A **43**, 045203 (2010). arXiv:0903.5251 [hep-th]
59. J.G. Bu, H.C. Kim, Y. Lee, C.H. Vac, J.H. Yee, kappa-deformed spacetime from twist. Phys. Lett. B **665**, 95–99 (2008). arXiv:hep-th/0611175
60. H.C. Kim, Y. Lee, C. Rim, J.H. Yee, Differential structure on the κ-Minkowski spacetime from twist. Phys. Lett. B **671**, 398 (2009). arXiv:0808.2866 [hep-th]
61. H.C. Kim, Y. Lee, C. Rim, J.H. Yee, Scalar field theory in κ-Minkowski spacetime from twist. J. Math. Phys. **50**, 102304 (2009). arXiv:0901.0049 [hep-th]
62. A. Borowiec, A. Pachol, kappa-Minkowski spacetime as the result of Jordanian twist deformation. Phys. Rev. D **79**, 045012 (2009). arXiv:0812.0576 [math-ph]
63. P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp, J. Wess, A gravity theory on noncommutative spaces. Class. Quantum Gravity **22**, 3511 (2005). arXiv:hep-th/0504183
64. P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess, Noncommutative geometry and gravity. Class. Quantum Gravity **23**, 1883 (2006). arXiv:hep-th/0510059
65. T. Juric, S. Meljanac, R. Strajn, Differential forms and κ-Minkowski spacetime from extended twist. Eur. Phys. J. C **73**, 2472 (2013). arXiv:1211.6612 [hep-th]
66. E. Beggs, R. Makki, The Majid–Ruegg model and the Planck scales. arXiv:1306.4518 [hep-th]
67. E. Beggs, S. Majid, Gravity induced from quantum space time. Class. Quantum Gravity **31**, 035020 (2014). arXiv:1305.2403 [gr-qc]
68. S. Majid, W. Tao, Noncommutative fifferentials on Poisson–Lie groups and pre-Lie algebras. arXiv:1412.2284 [math.QA]
69. S. Majid, W. Tao, Cosmological constant from quantum spacetime. arXiv:1412.2285 [gr-qc]
70. S. Meljanac, A. Samsarov, J. Trampetic, M. Wohlgenannt, Scalar field propagation in the ϕ^4 kappa-Minkowski model. JHEP **1112**, 010 (2011). arXiv:1111.5553 [hep-th]
71. S. Meljanac, Z. Skoda, D. Svrtan, Exponential formulas and Lie algebra type star products. SIGMA **8**, 013 (2012). arXiv:1006.0478 [math.QA]