Eur. Phys. J. C (2015) 75:520
DOI 10.1140/epjc/s10052-015-3747-4

THE EUROPEAN PHYSICAL JOURNAL C

Comment

On the branching of the quasinormal resonances of near-extremal Kerr black holes

Shahar Hod[1,2,a]

[1] The Ruppin Academic Center, Emeq Hefer 40250, Israel
[2] The Hadassah Institute, Jerusalem 91010, Israel

Received: 1 June 2015 / Accepted: 19 October 2015 / Published online: 2 November 2015

Abstract It has recently been shown by Yang et al. (Phys Rev D 87:041502(R), 2013a; Phys Rev D 88:044047, 2013b) that rotating Kerr black holes are characterized by *two* distinct sets of quasinormal resonances. These two families of quasinormal resonances display qualitatively different asymptotic behaviors in the extremal ($a/M \to 1$) black-hole limit: the zero-damping modes are characterized by relaxation times which tend to infinity in the extremal black-hole limit ($\Im\omega \to 0$ as $a/M \to 1$), whereas the damped modes (DMs) are characterized by non-zero damping rates ($\Im\omega \to$ finite-values as $a/M \to 1$). In this paper we refute the claim made by Yang et al. that co-rotating DMs of near-extremal black holes are restricted to the limited range $0 \le \mu \lesssim \mu_c \approx 0.74$, where $\mu \equiv m/l$ is the dimensionless ratio between the azimuthal harmonic index m and the spheroidal harmonic index l of the perturbation mode. In particular, we use an analytical formula originally derived by Detweiler in order to prove the existence of DMs (damped quasinormal resonances which are characterized by *finite* $\Im\omega$ values in the $a/M \to 1$ limit) of near-extremal black holes in the $\mu > \mu_c$ regime, the regime which was claimed by Yang et al. not to contain DMs. We show that these co-rotating DMs (in the regime $\mu > \mu_c$) are expected to characterize the resonance spectra of rapidly rotating (near-extremal) black holes with $a/M \gtrsim 1 - 10^{-9}$.

1 Introduction

Perturbed black holes display a unique pattern of damped oscillations, known as quasinormal resonances, which characterize the relaxation phase of the black-hole spacetime. The spectrum of the quasinormal resonances reflects the physical parameters (such as mass, charge, and angular momentum) of the black-hole spacetime.

The complex quasinormal resonances correspond to perturbation fields which propagate in the black-hole spacetime with the physically motivated boundary conditions of purely outgoing waves at spatial infinity and purely ingoing waves crossing the black-hole horizon [1]. These boundary conditions single out a discrete set, $\{\omega^{\text{QNM}}(n; m, l)\}_{n=0}^{n=\infty}$, of complex black-hole resonances for each perturbation mode (here m and l are the azimuthal harmonic index and the spheroidal harmonic index of the wave field, respectively).

In a very interesting paper, Yang et al. [2,3] have recently studied numerically the quasinormal spectrum of near-extremal (rapidly rotating) Kerr black holes. The authors of [2,3] have reached the remarkable conclusion that these rapidly rotating black holes are characterized by *two* qualitatively distinct sets of quasinormal resonances:

- Zero-damping modes (ZDMs), which are characterized by the asymptotic property (for these unique black-hole resonances, see: [4–7])

$$\Im\omega^{\text{ZDM}}(\tau \to 0) \to 0, \tag{1}$$

and

- Damped modes (DMs), which are characterized by the asymptotic property

$$\Im\omega^{\text{DM}}(\tau \to 0) \to \text{finite values}. \tag{2}$$

Here

$$\tau \equiv \frac{r_+ - r_-}{r_+} \tag{3}$$

is the dimensionless Bekenstein–Hawking temperature of the black hole (The Bekenstein–Hawking temperatures of Kerr black holes are given by the relation $T_{\text{BH}} = \tau/8\pi M$. We shall use units in which $G = c = \hbar = 1$), where $r_\pm \equiv M \pm (M^2 - a^2)^{1/2}$ are the black-hole (event and inner) horizons. This

[a] e-mail: shaharhod@gmail.com

dimensionless temperature approaches zero in the extremal $a \to M$ ($r_- \to r_+$) limit of rapidly rotating black holes.

2 The erroneous claim made in [2] and Detweiler's damped resonances

It has been asserted in Ref. [2,3] that the ZDMs (1) exist for all co-rotating modes ($m \geq 0$) (for these unique black-hole resonances, see: [4–7]), whereas the DMs (2) exist for counter-rotating modes ($m < 0$) and for co-rotating modes in the *limited* range

$$0 \leq \mu \lesssim \mu_c. \tag{4}$$

Here

$$\mu \equiv \frac{m}{l} \tag{5}$$

is the dimensionless ratio between the azimuthal harmonic index m and the spheroidal harmonic index l of the perturbation mode. The critical ratio, μ_c, is given by $\mu_c = \sqrt{\frac{15-\sqrt{193}}{2}} \simeq 0.74$ in the eikonal limit [2,3,8]. This critical value of the dimensionless ratio μ marks the boundary between perturbations modes (those with $\mu \leq \mu_c$) which are characterized by *imaginary* values of the angular-eigenvalue δ. [the parameter δ^2 is closely related to the angular-eigenvalue of the angular Teukolsky equation, see [9] for details (see, in particular, equations (2.7) and (6.3) of [9])] and perturbations modes (those with $\mu > \mu_c$) which are characterized by *real* values of the angular-eigenvalue δ.

Here we would like to point out that the assertion made in Ref. [2,3], according to which co-rotating DMs exist *only* in the limited range $0 \leq \mu \lesssim \mu_c$ [see Eqs. (2), (4)], is actually erroneous. In particular, we shall show that co-rotating DMs of near-extremal black holes [see Eq. (10) below] actually exist in the *entire* range

$$0 \leq \mu \leq 1. \tag{6}$$

In fact, Detweiler [10] has obtained an analytic expression for co-rotating DMs of near-extremal black holes which is valid in the regime $\mu > \mu_c$ (that is, as shown in [10], the expression (7) is valid for co-rotating modes with real δ eigenvalues):

$$\varpi_n \equiv M(\omega_n - m\Omega_H) = -\frac{e^{\theta/2\delta}}{4m}(\cos\phi + i\sin\phi) \times e^{-\pi n/\delta}\ , \tag{7}$$

where $\Omega_H \equiv a/2Mr_+$ is the angular-velocity of the black-hole horizon, and the integer n is the resonance parameter of the mode. Here we have used the definitions [10]

$$re^{i\theta} \equiv \left[\frac{\Gamma(2i\delta)}{\Gamma(-2i\delta)}\right]^2 \frac{\Gamma(1/2+s-im-i\delta)\Gamma(1/2-s-im-i\delta)}{\Gamma(1/2+s-im+i\delta)\Gamma(1/2-s-im+i\delta)};$$

$$\phi \equiv -\frac{1}{2\delta}\ln r. \tag{8}$$

It is worth emphasizing again that Eq. (7), originally derived in [10], describes DMs in the $\mu > \mu_c$ ($\delta^2 > 0$) regime, the regime which was claimed in [2,3] not to contain DMs.

3 The source of the erroneous claim made in [2]

It is important to understand the reason for the failure of Yang et. al. [2,3] to observe the DMs (7) of [10] in the regime $\mu > \mu_c$ (the failure in [2,3] to observe these resonances numerically is probably the reason behind the erroneous claim (see Eq. (4)) made in [2,3]). In order to understand the null result of [2,3] in finding numerically the DMs (7), one should examine the regime of validity of the analyzes presented in [9,10].

A careful check of these analyzes reveals that the expression (7) for the black-hole DMs [10] is valid in the regime

$$\tau \ll |\varpi| \ll x \ll 1\ , \tag{9}$$

where the dimensionless coordinate $x \equiv (r - r_+)/r_+$ belongs to an *overlapping* region in which two different expressions for the radial Teukolsky wave function (hypergeometric and confluent hypergeometric functions) can be matched; see [9,10] for details. Taking cognizance of the inequalities in (9), one realizes that the expression (7) for co-rotating DMs with $\mu > \mu_c$ is only valid in the regime of near-extremal (rapidly rotating) black holes.

In particular, since each inequality sign in (9) roughly corresponds to an order-of-magnitude difference between two variables (that is, $\tau/\varpi \lesssim 10^{-1}$, $\varpi/x \lesssim 10^{-1}$, and $x \lesssim 10^{-1}$), the expression (7) for the black-hole DMs [10] is not expected to be valid outside the regime (in order to be on the safe side, we have added an extra order of magnitude to the inequality (10))

$$\tau \lesssim 10^{-4}. \tag{10}$$

The inequality (10) corresponds to rapidly rotating black holes with [see Eq. (3)]

$$\frac{a}{M} \gtrsim 1 - 10^{-9}. \tag{11}$$

It is worth noting that the numerical analysis presented in [2,3] did not explore the deep near-extremal regime (11) of the rotating Kerr black holes. A. Zimmerman (private communication) has kindly updated me that he has not detected in his numerical studies DMs in the regime $\mu > \mu_c$ for a rapidly-rotating Kerr black hole with $a/M = 1 - 10^{-9}$. As a consequence, the co-rotating DMs (7) in the regime $\mu > \mu_c$ have not been observed in the numerical study of [2,3]. This simple fact has probably led Yang et al. [2,3] to the erroneous conclusion that co-rotating DMs are restricted to the limited range $0 \leq \mu \lesssim \mu_c$.

4 Summary

It is well known that rapidly rotating (near-extremal) black holes are characterized by *two* qualitatively distinct sets of quasinormal resonances: (1) zero-damping modes (ZDMs), which are characterized by the asymptotic property (for these unique black-hole resonances, see: [4–7]) $\Im\omega^{\text{ZDM}}(\tau \to 0) \to 0$, and (2) damped modes (DMs), which are characterized by the asymptotic property $\Im\omega^{\text{DM}}(\tau \to 0) \to$ finite values.

In this comment we have refuted the claim made in Ref. [2,3] that co-rotating DMs of near-extremal black holes are restricted to the limited range $0 \leq \mu \lesssim \mu_c$ [see Eqs. (2), (4)]. In particular, we have pointed out that the analytical expression (7), originally derived in [10], describes DMs in the $\mu > \mu_c$ regime (that is, as shown in [10], the expression (7) is valid for co-rotating modes with real δ eigenvalues), the regime which was claimed in [2,3] not to contain DMs.

Most importantly, we have emphasized the fact that the analytical expression (7) for the black-hole DMs is not expected to be valid outside the deep near-extremal regime (11) of rapidly rotating black holes.

Finally, it is worth emphasizing that rapidly rotating black holes in the regime (11) are probably of no astrophysical relevance [11]. However, these near-extremal black holes are very important from the point of view of quantum field theory. In particular, these black holes play a key role in the conjectured relation between the quantum states of near-extremal black holes and the corresponding quantum states of a two-dimensional conformal field theory [12–14] (For the physical relevance of these near-extremal black holes to the conjectured universal relaxation bound, see [4–7] and also [15–17]).

Acknowledgments This research is supported by the Carmel Science Foundation. I would like to thank Aaron Zimmerman for interesting correspondence. I would also like to thank Yael Oren, Arbel M. Ongo, Ayelet B. Lata, and Alona B. Tea for stimulating discussions.

Funded by SCOAP^3.

References

1. S.L. Detweiler, in *Sources of Gravitational Radiation*, ed. by L. Smarr (Cambridge University Press, Cambridge, 1979)
2. H. Yang, F. Zhang, A. Zimmerman, D.A. Nichols, E. Berti, Y. Chen, Phys. Rev. D **87**, 041502(R) (2013a)
3. H. Yang, A. Zimmerman, A. Zenginoglu, F. Zhang, E. Berti, Y. Chen, Phys. Rev. D **88**, 044047 (2013b)
4. S. Hod, Phys. Rev. D **75**, 064013 (2007). arXiv:gr-qc/0611004
5. S. Hod, Class. Quant. Grav. **24**, 4235 (2007). arXiv:0705.2306
6. S. Hod, Phys. Rev. D **78**, 084035 (2008). arXiv:0811.3806
7. S. Hod, Phys. Rev. D **80**, 064004 (2009). arXiv:0909.0314
8. S. Hod, Phys. Lett. B **715**, 348 (2012). arXiv:1207.5282
9. S.A. Teukolsky, W.H. Press, Astrophys. J. **193**, 443 (1974)
10. S. Detweiler, Astrophys. J. **239**, 292 (1980)
11. K.S. Thorne, Astrophys. J. **191**, 507 (1974)
12. J.M. Bardeen, G.T. Horowitz, Phys. Rev. D **60**, 104030 (1999)
13. M. Guica, T. Hartman, W. Song, A. Strominger, Phys. Rev. D **80**, 124008 (2009)
14. A. Strominger, C. Vafa, Phys. Lett. B **379**, 99 (1996)
15. A. Gruzinov. arXiv:0705.1725 [gr-qc]
16. A. Pesci, Class. Quantum Grav. **24**, 6219 (2007)
17. S. Hod, Phys. Lett. B **666**, 483 (2008). arXiv:0810.5419