THE EUROPEAN
PHYSICAL JOURNAL C

CrossMark

Cosmic constraint on the unified model of dark sectors with or without a cosmic string fluid in the varying gravitational constant theory

Jianbo Lu[a], Yanfeng Xu, Yabo Wu

Department of Physics, Liaoning Normal University, Dalian 116029, People's Republic of China

Abstract Observations indicate that most of the universal matter is invisible and the gravitational constant $G(t)$ maybe depends on time. A theory of the variational G (VG) is explored in this paper, naturally producing the useful dark components in the universe. We utilize the following observational data: lookback time data, model-independent gamma ray bursts, growth function of matter linear perturbations, type Ia supernovae data with systematic errors, CMB, and BAO, to restrict the unified model (UM) of dark components in VG theory. Using the best-fit values of the parameters with the covariance matrix, constraints on the variation of G are $\left(\frac{G}{G_0}\right)_{z=3.5} \simeq 1.0015^{+0.0071}_{-0.0075}$ and $\left(\frac{\dot{G}}{G}\right)_{\text{today}} \simeq -0.7252^{+2.3645}_{-2.3645} \times 10^{-13}$ year^{-1}, with small uncertainties around the constants. The limit on the equation of state of dark matter is $w_{0\text{dm}} = 0.0072^{+0.0170}_{-0.0170}$, assuming $w_{0\text{de}} = -1$ in the unified model, and the dark energy is $w_{0de} = -0.9986^{+0.0011}_{-0.0011}$, assuming $w_{0\text{dm}} = 0$ a priori. The restrictions on the UM parameters are $B_s = 0.7442^{+0.0137+0.0262}_{-0.0132-0.0292}$ and $\alpha = 0.0002^{+0.0206+0.0441}_{-0.0209-0.0422}$ with 1σ and 2σ confidence level. In addition, the effects of a cosmic string fluid on the unified model in VG theory are investigated. In this case it is found that the ΛCDM ($\Omega_s = 0$, $\beta = 0$, and $\alpha = 0$) is included in this VG-UM model at 1σ confidence level, and larger errors are given: $\Omega_s = -0.0106^{+0.0312+0.0582}_{-0.0305-0.0509}$ (dimensionless energy density of cosmic string), $\left(\frac{G}{G_0}\right)_{z=3.5} \simeq 1.0008^{+0.0620}_{-0.0584}$, and $\left(\frac{\dot{G}}{G}\right)_{\text{today}} \simeq -0.3496^{+26.3135}_{-26.3135} \times 10^{-13}$ year^{-1}.

1 Introduction

Gravity theories are usually studied on the assumption that the Newton gravity constant G is constant. But some observations hint that G maybe depends on time [1], such as observations from white dwarf stars [2,3], pulsars [4], supernovae [5] and neutron stars [6]. In addition, cosmic observations predict that about 95 % of the universal matter is invisible, including dark matter (DM) and dark energy (DE). The unified models of two unknown dark sectors (DM and DE) have been studied in several theories, e.g. in the standard cosmology [7–9], in the Hořava–Lifshitz gravity [10], in the RS [11] and the KK higher-dimension gravity [12]. In this paper, we study the unified model of dark components in the theory of a varying gravitational constant (VG). The attractive point of this model is that the variation of G could result in the invisible components in universe, by relating the Lagrangian quantity of the generalized Born–Infeld theory to the VG theory. One source of DM and DE is introduced. In addition, cosmic strings have been studied in some fields, such as in emergent universe [13,14], in modified gravity [15], in inflation theory [16], and so on [17–20]. Here we discuss the effect of a cosmic string fluid on the cosmic parameters in VG theory. Using the Markov Chain Monte Carlo (MCMC) method [21], the cosmic constraints on a unified model of DM and DE with (or without) a cosmic string fluid are performed in the framework of a time-varying gravitational constant. The used cosmic data include the lookback time (LT) data [22,23], the model-independent gamma ray bursts (GRBs) data [24], the growth function (GF) of matter linear perturbations [25–32], the type Ia supernovae (SNIa) data with systematic errors [33], the cosmic microwave background (CMB) [34], and the baryon acoustic oscillation (BAO) data including the radial BAO scale measurement [35] and the peak-positions measurement [36–38].

[a] e-mail: lvjianbo819@163.com

Springer

2 A time-varying gravitational constant theory with unified dark sectors and a cosmic string fluid

We adopt the Lagrangian quantity of system

$$L = \sqrt{g}\left(\frac{R}{G(t)} + 16\pi\mathcal{L}_u\right) \tag{1}$$

with a parameterized time-varying gravitational constant $G = G_0 a(t)^{-\beta}$. t is the cosmic time, $a = (1+z)^{-1}$ is the cosmic scale factor, and z denotes the cosmic redshift. g is the determinant of metric, R is the Ricci scalar, and $\mathcal{L}_u = \mathcal{L}_b + \mathcal{L}_r + \mathcal{L}_d + \mathcal{L}_s$ corresponds to the Lagrangian density of universal matter including the visible ingredients: baryon $\mathcal{L}_b$ and radiation $\mathcal{L}_r$ and the invisible ingredients: dark sectors $\mathcal{L}_d$ and cosmic string (CS) fluid $\mathcal{L}_s$. Utilizing the variational principle, the gravitational field equation can be derived [39],

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$
$$+ G\left(\nabla_\mu\partial_\nu G^{-1} - g_{\mu\nu}\nabla_\sigma\partial^\sigma G^{-1}\right), \tag{2}$$

in which $R_{\mu\nu}$ is the Ricci tensor, $T_{\mu\nu}$ is the energy-momentum tensor of universal matter that comprises the pressureless baryon ($w_b = \frac{p_b}{\rho_b} = 0$), the positive-pressure photon ($w_r = \frac{p_r}{\rho_r} = \frac{1}{3}$), the CS fluid ($w_s = \frac{p_s}{\rho_s} = -\frac{1}{3}$), and the unknown dark components ($w_d = \frac{p_d}{\rho_d}$). w is for the equation of state (EoS), p is the pressure, and ρ denotes the energy density, respectively. Taking the covariant divergence for Eq. (2) and utilizing the Bianchi identity result in

$$3H\left(\frac{\dot{G}}{G}\right)^2 + 3\frac{\ddot{a}}{a}\frac{\dot{G}}{G} + 8\pi[\dot{G}\rho + G\dot{\rho} + 3HG(\rho+p)] = 0 \tag{3}$$

or its equivalent form

$$3H\beta[(\beta-1)H^2 - \dot{H}] + 8\pi G\left[\dot{\rho} + 3H(\rho+p) - \beta H\rho\right] = 0. \tag{3}$$

In the Friedmann–Robertson–Walker geometry, the evolution equations of the universe in VG theory are

$$H^2 = \frac{8\pi G_0}{3}a^{-\beta}\rho - \beta H^2, \tag{4}$$

$$2\frac{\ddot{a}}{a} + H^2 = -8\pi G_0 a^{-\beta}p - \beta H^2 - \beta^2 H^2 - \beta\frac{\ddot{a}}{a}. \tag{5}$$

From Eq. (4), we can see that a CS fluid can be equivalent to a curvature term in constant-G theory, while this fluid could not be equivalent to the curvature term in the VG theory due to the term $a^{-\beta}$ multiplying the density. Combing Eqs. (3), (4), and (5), we have

$$\dot{\rho} + 3H\left(\rho + \frac{2+2\beta}{2+\beta}p\right) = \frac{\beta - \beta^2}{2+\beta}H\rho. \tag{6}$$

A dot represents the derivative with respect to cosmic time t. Integrating Eq. (6) one obtains the energy density of baryon $\rho_b \propto a^{\frac{-\beta^2 - 2\beta - 6}{2+\beta}}$, the energy density of radiation $\rho_r \propto a^{\frac{-\beta^2 - 4\beta - 8}{2+\beta}}$, and the energy density of the cosmic string $\rho_s \propto a^{\frac{-\beta^2 - 4}{2+\beta}}$. Relative to the constant-G theory, the evolution equations of the energy densities are obviously modified in VG theory as regards the existence of the VG parameter β.

We concentrate on the Lagrangian density of the dark components in the form $\mathcal{L}_d = -A^{\frac{1}{1+\alpha}}\left[1 - (V'(\varphi))^{\frac{1+\alpha}{2\alpha}}\right]^{\frac{\alpha}{1+\alpha}}$ from the generalized Born–Infeld theory [40], in which $V(\varphi)$ is the potential. Relating this scalar field φ with the time-varying gravitational constant by $\varphi(t) = G(t)^{-1}$, it is then found that the dark ingredients can be induced by the variation of G. The energy density of the dark fluid in the VG frame complies with

$$\rho_d = \rho_{0d}\left[B_s + (1 - B_s)a^{(-3+\frac{\beta-\beta^2}{2+\beta})(1+\alpha)}\right]^{\frac{1}{1+\alpha}}, \tag{7}$$

here the parameter β reflects the variation of G; α and $B_s = \frac{6+6\beta}{\beta^2+2\beta+6}\frac{A}{\rho_{0VG-GCG}^{1+\alpha}}$ are model parameters. Equation (7) shows that the behavior of ρ_d is like cold DM at early time[1] (for $a \ll 1$, $\rho_d \approx \rho_{0d}(1-B_s)^{\frac{1}{1+\alpha}}a^{-3+\frac{\beta-\beta^2}{2+\beta}}$), and like cosmological-constant type DE at late time (for $a \gg 1$, $\rho_d \approx \rho_{0d}B_s^{\frac{1}{1+\alpha}}$). Then Eq. (7) introduces a unified model (UM) of dark sectors in VG theory (called VG-UM). The Hubble parameter H in the VG-UM model reads

$$H = \sqrt{\frac{H_0^2}{1+\beta}\left\{\Omega_{0d}[B_s a^{-\beta(1+\alpha)} + (1-B_s)a^{-(3+\frac{2\beta^2+\beta}{2+\beta})(1+\alpha)}]^{\frac{1}{1+\alpha}} + \Omega_b a^{\frac{-2\beta^2-4\beta-6}{2+\beta}} + \Omega_r a^{\frac{-2\beta^2-6\beta-8}{2+\beta}} + \Omega_s a^{\frac{-2\beta^2-2\beta-4}{2+\beta}}\right\}}, \tag{8}$$

with Hubble constant H_0 and dimensionless energy densities $\Omega_b = \frac{8\pi G_0\rho_{0b}}{3H_0^2}$, $\Omega_r = \frac{8\pi G_0\rho_{0r}}{3H_0^2}$, $\Omega_s = \frac{8\pi G_0\rho_{0s}}{3H_0^2}$, and $\Omega_{0d} + \Omega_b + \Omega_r + \Omega_s = 1 + \beta$. For $\beta = 0$, the above equations are reduced to the standard forms in the constant-G theory.

---

[1] β describes the effect on the energy density of dark matter from a variation of G.

Table 1 The 38 data points of galaxy age [22,23]. The first six data are from Ref. [22]

z_i	0.10	0.25	0.60	0.70	0.80	1.27	0.1171	0.1174	0.222	0.2311	0.3559	0.452	0.575	0.644	0.676	0.833	0.836	0.922	1.179
t_i	10.65	8.89	4.53	3.93	3.41	1.60	10.2	10.0	9.0	9.0	7.6	6.8	7.0	6.0	6.0	6.0	5.8	5.5	4.6
z_i	1.222	1.224	1.225	1.226	1.34	1.38	1.383	1.396	1.43	1.45	1.488	1.49	1.493	1.51	1.55	1.576	1.642	1.725	1.845
t_i	3.5	4.3	3.5	3.5	3.4	3.5	3.5	3.6	3.2	3.2	3.0	3.6	3.2	2.8	3.0	2.5	3.0	2.6	2.5

3 Data fitting

3.1 Lookback time

References [41,42] define the LT as the difference between the current age t_0 of universe at $z = 0$ and the age t_z of a light ray emitted at z,

$$t_L(z) = \int_0^z \frac{\mathrm{d}z'}{(1+z')H(z')}. \tag{9}$$

Then the age $t(z_i)$ of an object at redshift z_i can be expressed by the difference between the age of universe at z_i and the age of universe at z_F (when the object was born) [22],

$$t(z_i) = \int_{z_i}^\infty \frac{\mathrm{d}z'}{(1+z')H(z')} - \int_{z_F}^\infty \frac{\mathrm{d}z'}{(1+z')H(z')}$$
$$= t_L(z_F) - t_L(z_i). \tag{10}$$

For an object at redshift z_i, the observed LT is subject to

$$t_L^{\mathrm{obs}} = t_L(z_F) - t_L(z_i) = [t_0^{\mathrm{obs}} - t(z_i)] - [t_0^{\mathrm{obs}} - t_L(z_F)]$$
$$= t_0^{\mathrm{obs}} - t(z_i) - \mathrm{d}f. \tag{11}$$

One defines

$$\chi_{\mathrm{age}}^2 = \sum_i \frac{[t_L(z_i) - t_L^{\mathrm{obs}}(z_i, \mathrm{d}f)]^2}{\sigma_T^2} + \frac{[t_0 - t_0^{\mathrm{obs}}]^2}{\sigma_{t_0^{\mathrm{obs}}}^2}, \tag{12}$$

with $\sigma_{t_0^{\mathrm{obs}}}^2 + \sigma_i^2 = \sigma_T^2$. $\sigma_{t_0^{\mathrm{obs}}}$ is the uncertainty of the total universe age, and σ_i is the uncertainty of the LT of galaxy i. Marginalizing the 'nuisance' parameter df results in [43]

$$\chi_{LT}^2(p_s) = -2\ln \int_0^\infty \mathrm{d}(\mathrm{d}f) \exp(-\chi_{\mathrm{age}}^2/2)$$
$$= A - \frac{B^2}{C} + \frac{[t_0 - t_0^{\mathrm{obs}}]^2}{\sigma_{t_0^{\mathrm{obs}}}^2}$$
$$- 2\ln\left[\sqrt{\frac{\pi}{2C}}\mathrm{erfc}\left(\frac{B}{\sqrt{2C}}\right)\right], \tag{13}$$

where $A = \sum_i \frac{\Delta^2}{\sigma_T^2}$, $B = \sum_i \frac{\Delta}{\sigma_T^2}$, $C = \sum_i \frac{1}{\sigma_T^2}$ and $\Delta = t_L(z_i) - [t_0^{\mathrm{obs}} - t(z_i)]$, respectively. p_s denotes the theoretical model parameters. $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$ is the complementary error function of x. The observational universal age at present, $t_0^{\mathrm{obs}} = 13.75 \pm 0.13$ Gyr [44], is used, and the observational data on the galaxies age are listed in Table 1.

3.2 Gamma ray bursts

In GRBs observation, the famous Amati's correlation is $\log \frac{E_{\mathrm{iso}}}{\mathrm{erg}} = a + b \log \frac{E_{p,i}}{300}$ keV [45,46], where $E_{\mathrm{iso}} = 4\pi d_L^2 S_{\mathrm{solo}}/(1+z)$ and $E_{p,i} = E_{p,\mathrm{obs}}(1+z)$ are the isotropic energy and the cosmological rest-frame spectral peak energy, respectively. d_L is the luminosity distance and S_{bolo} is the bolometric fluence of GRBs. Reference [47] introduced a model-independent quantity for a distance measurement,

$$\bar{r}_p(z_i) = \frac{r_p(z)}{r_p(z_0)}, \quad r_p(z) = \frac{(1+z)^{1/2}}{z}\frac{H_0}{c}r(z),$$
$$r(z) = \frac{d_L(z)}{1+z} \tag{14}$$

with z_0 being the lowest GRBs redshift. For the GRBs constraint, χ_{GRBs}^2 has the form

$$\chi_{\mathrm{GRBs}}^2(p_s) = [\Delta\bar{r}_p(z_i)] \cdot (\mathrm{Cov}_{\mathrm{GRBs}}^{-1})_{ij} \cdot [\Delta\bar{r}_p(z_i)], \tag{15}$$

in which $\Delta\bar{r}_p(z_i) = \bar{r}_p^{\mathrm{data}}(z_i) - \bar{r}_p(z_i)$, and $(\mathrm{Cov}_{\mathrm{GRBs}}^{-1})_{ij}$ is the covariance matrix. Using 109 GRBs data, Ref. [24] obtained five model-independent datapoints listed in Table 2, where $\sigma(\bar{r}_p(z_i))^+$ and $\sigma(\bar{r}_p(z_i))^-$ are the 1σ errors. The $\{\bar{r}_p(z_i)\}$ correlation matrix is [24]

$$(\overline{\mathrm{Cov}}_{\mathrm{GRB}}) = \begin{pmatrix} 1.0000 & 0.7780 & 0.8095 & 0.6777 & 0.4661 \\ 0.7780 & 1.0000 & 0.7260 & 0.6712 & 0.3880 \\ 0.8095 & 0.7260 & 1.0000 & 0.6046 & 0.5032 \\ 0.6777 & 0.6712 & 0.6046 & 1.0000 & 0.1557 \\ 0.4661 & 0.3880 & 0.5032 & 0.1557 & 1.0000 \end{pmatrix}, \tag{16}$$

Table 2 Distances calculated by using the 109 GRBs data via Amati's correlation [24]

Number	z	$\bar{r}_p^{data}(z)$	$\sigma(\bar{r}_p(z_i))^+$	$\sigma(\bar{r}_p(z_i))^-$
0	0.0331	1.0000	–	–
1	1.0000	0.9320	0.1711	0.1720
2	2.0700	0.9180	0.1720	0.1718
3	3.0000	0.7795	0.1630	0.1629
4	4.0480	0.7652	0.1936	0.1939
5	8.1000	1.1475	0.4297	0.4389

Table 3 The observational data of the growth function f_{obs}

z_i	0.15	0.22	0.32	0.35	0.41	0.55	0.60	0.77	0.78	1.4
f_{obs}	0.51 ± 0.11	0.60 ± 0.10	0.654 ± 0.18	0.70 ± 0.18	0.50 ± 0.07	0.75 ± 0.18	0.73 ± 0.07	0.91 ± 0.36	0.70 ± 0.08	0.90 ± 0.24
Refs.	[25,26]	[27]	[28]	[29]	[27]	[30]	[27]	[31]	[27]	[32]

with the covariance matrix

$$(\mathrm{Cov}_{\mathrm{GRB}})_{ij} = \sigma(\bar{r}_p(z_i))\sigma(\bar{r}_p(z_j))(\overline{\mathrm{Cov}}_{\mathrm{GRB}})_{ij}, \tag{17}$$

where $\sigma(\bar{r}_p(z_i)) = \sigma(\bar{r}_p(z_i))^+$, if $\bar{r}_p(z) \geq \bar{r}_p(z)^{\mathrm{data}}$; $\sigma(\bar{r}_p(z_i)) = \sigma(\bar{r}_p(z_i))^-$, if $\bar{r}_p(z) < \bar{r}_p(z)^{\mathrm{data}}$.

3.3 Growth function of matter linear perturbations

The χ^2_{GF} can be constructed by the growth function of matter linear perturbations f

$$\chi^2_{\mathrm{GF}}(p_s) = \sum_i \frac{[f_{\mathrm{th}}(p_s, z_i) - f_{\mathrm{obs}}(z_i)]^2}{\sigma^2(z_i)}, \tag{18}$$

where the used the observational values of f_{obs} listed in Table 3. f is defined via $f(a) = \frac{aD'(a)}{D(a)}$, with $D = \frac{\frac{\delta\rho}{\rho}(a)}{\frac{\delta\rho}{\rho}(a=1)}$. A prime denotes the derivative with respect to a. So in theory, f can be obtained by solving the following differential equation in VG theory:

$$D''(a) + \left[\frac{H'(a)}{H(a)} + \frac{1}{a} + \frac{4 + 2\beta + 2\beta^2}{a(2+\beta)}\right]D'(a)$$
$$- \frac{6 + 2\beta + \beta^2}{(2+\beta)^2}\frac{H_0^2\Omega_{0m}}{H(a)^2 a^2}a^{\frac{-6-2\beta-\beta^2}{2+\beta}}D(a) = 0. \tag{19}$$

For $\beta = 0$, the above equation reduces to the constant-G theory. The derivation of the evolution equation for $D(a)$ in VG theory is shown in the appendix. Compared with the most popular ΛCDM model, the effective current matter density can be written $\Omega_{0m} = \Omega_b + (1 + \beta - \Omega_s - \Omega_b - \Omega_r)(1 - B_s)$ for VG-UM. Obviously, for $\beta = 0$ it is consistent with the form of Ω_{0m} in UM of constant-G theory [48–50].

3.4 Type Ia supernovae

We use the Union2 dataset of SNIa published in Ref. [33]. In VG theory, the theoretical distance modulus $\mu_{th}(z)$ is written as $\mu_{th}(z) = 5\log_{10}[D_L(z)] + \frac{15}{4}\log_{10}\frac{G}{G_0} + \mu_0$, where $D_L(z) = \frac{H_0}{c}(1+z)^2 D_A(z)$ and $\mu_0 = 5\log_{10}(\frac{H_0^{-1}}{Mpc}) + 25 = 42.38 - 5\log_{10}h$. h is a re-normalized quantity defined by $H_0 = 100h$ km s^{-1} Mpc^{-1}. $D_A(z) = \frac{c}{(1+z)\sqrt{|\Omega_k|}}$ $\mathrm{sinn}[\sqrt{|\Omega_k|}\int_0^z \frac{dz'}{H(z')}]$ is the proper angular diameter distance; here $\mathrm{sinn}(\sqrt{|\Omega_k|}x)$ denotes $\sin(\sqrt{|\Omega_k|}x)$, $\sqrt{|\Omega_k|}x$ and $\sinh(\sqrt{|\Omega_k|}x)$ for $\Omega_k < 0$, $\Omega_k = 0$ and $\Omega_k > 0$, respectively.

A cosmic constraint from the SNIa observations can be found by a calculation [51–60]:

$$\chi^2_{\mathrm{SNIa}}(p_s) = \sum_{\mathrm{SNIa}} \frac{\{\mu_{\mathrm{th}}(p_s, z_i) - \mu_{\mathrm{obs}}(z_i)\}^2}{\sigma_{\mu_i}^2}$$
$$= \sum_{\mathrm{SNIa}} \frac{\{5\log_{10}[D_L(p_s, z)] + \frac{15}{4}\log_{10}\frac{G}{G_0} - m_{\mathrm{obs}}(z_i) + M'\}^2}{\sigma_i^2}, \tag{20}$$

where $\mu_{\mathrm{obs}}(z_i) = m_{\mathrm{obs}}(z_i) - M$ is the observed distance moduli, with the absolute magnitude M. The nuisance parameter $M' = \mu_0 + M$ can be marginalized over analytically, $\bar{\chi}^2_{\mathrm{SNIa}}(p_s) = -2\ln\int_{-\infty}^{+\infty}\exp\left[-\frac{1}{2}\chi^2_{\mathrm{SNIa}}(p_s, M')\right]dM'$, resulting in [61–70]

$$\chi^2_{\mathrm{SNIa}}(p_s) = A - (B^2/C), \tag{21}$$

where

$$A = \sum_{\mathrm{SNIa}}\left\{5\log_{10}[D_L(p_s, z_i)] + \frac{15}{4}\log_{10}\frac{G}{G_0} - m_{\mathrm{obs}}(z_j)\right\} \cdot$$
$$C_{ij}^{-1} \cdot \left\{5\log_{10}[D_L(p_s, z_j)] + \frac{15}{4}\log_{10}\frac{G}{G_0} - m_{\mathrm{obs}}(z_j)\right\}$$
$$B = \sum_{\mathrm{SNIa}} C_{ij}^{-1} \cdot \left\{5\log_{10}[D_L(p_s, z_j)] + \frac{15}{4}\log_{10}\frac{G}{G_0} - m_{\mathrm{obs}}(z_j)\right\}$$
$$C = \sum_{\mathrm{SNIa}} C_{ii}^{-1}. \tag{22}$$

The inverse of the covariance matrix C_{ij}^{-1} with systematic errors can be found in Refs. [33,71].

3.5 Cosmic microwave background

χ^2_{CMB} has the form [72,73]

$$\chi^2_{\mathrm{CMB}}(p_s) = \triangle d_i[\mathrm{Cov}^{-1}(d_i(p_s), d_j(p_s))][\triangle d_i]^t, \tag{23}$$

with $\triangle d_i(p_s) = d_i^{\mathrm{theory}}(p_s) - d_i^{\mathrm{obs}}$. Nine-year WMAP gives $d_i^{\mathrm{obs}} = [l_A(z_*) = 302.04, R(z_*) = 1.7246, z_* = 1090.88]$, and the corresponding inverse covariance matrix [34]

$$\mathrm{Cov}^{-1} = \begin{pmatrix} 3.182 & 18.253 & -1.429 \\ 18.253 & 11887.879 & -193.808 \\ -1.429 & -193.808 & 4.556 \end{pmatrix}. \tag{24}$$

$z_* = 1048\left[1 + 0.00124(\Omega_b h^2)^{-0.738}\right]\left[1 + g_1(\Omega_{0m}h^2)^{g_2}\right]$ is the redshift at the decoupling epoch of the photons with $g_1 = 0.0783(\Omega_b h^2)^{-0.238}\left(1 + 39.5(\Omega_b h^2)^{0.763}\right)^{-1}$

and $g_2 = 0.560\left(1 + 21.1(\Omega_b h^2)^{1.81}\right)^{-1}$, $l_A(p_s; z_*) = (1 + z_*)\frac{\pi D_A(p_s; z_*)}{r_s(z_*)}$ is the acoustic scale, and $R(p_s; z_*) = \sqrt{\Omega_{0m} H_0^2}(1 + z_*)D_A(p_s; z_*)/c$ is the CMB shift parameter.

3.6 Baryon acoustic oscillation

The radial (line-of-sight) BAO scale measurement from galaxy power spectra can be described by

$$\Delta z_{BAO}(z) = \frac{H(z) r_s(z_d)}{c}. \tag{25}$$

Two observational values are $\Delta z_{BAO}(z = 0.24) = 0.0407 \pm 0.0011$ and $\Delta z_{BAO}(z = 0.43) = 0.0442 \pm 0.0015$, respectively [35]. Here $r_s(z)$ is the comoving sound horizon size $r_s = c \int_0^t \frac{c_s dt}{a}$. c_s is the sound speed of the photon–baryon fluid, $c_s^{-2} = 3 + \frac{4}{3} \times \left(\frac{\Omega_b}{\Omega_\gamma}\right) a$. z_d denotes the drag epoch, $z_d = \frac{1291(\Omega_{0m}h^2)^{-0.419}}{1+0.659(\Omega_{0m}h^2)^{0.828}}\left[1 + b_1(\Omega_b h^2)^{b_2}\right]$ with $b_1 = 0.313(\Omega_{0m}h^2)^{-0.419}\left[1 + 0.607(\Omega_{0m}h^2)^{0.674}\right]$ and $b_2 = 0.238(\Omega_{0m}h^2)^{0.223}$.

The measurement of BAO peak positions can be performed by the WiggleZ Dark Energy Survey [36], the Two Degree Field Galaxy Redshift Survey [37], and the Sloan Digitial Sky Survey [38]. Introducing $D_V(z) = \left[(1 + z)^2 D_A^2(z)\frac{cz}{H(z; p_s)}\right]^{1/3}$, one can exhibit the observational data from BAO peak positions thus:

$$X = \begin{pmatrix} \frac{r_s(z_d)}{D_V(0.106)} - 0.336 \\ \frac{r_s(z_d)}{D_V(0.2)} - 0.1905 \\ \frac{r_s(z_d)}{D_V(0.35)} - 0.1097 \\ \frac{r_s(z_d)}{D_V(0.44)} - 0.0916 \\ \frac{r_s(z_d)}{D_V(0.6)} - 0.0726 \\ \frac{r_s(z_d)}{D_V(0.73)} - 0.0592 \end{pmatrix},$$

$$V^{-1} = \begin{pmatrix} 4444 & 0 & 0 & 0 & 0 & 0 \\ 0 & 30318 & -17312 & 0 & 0 & 0 \\ 0 & -17312 & 87046 & 0 & 0 & 0 \\ 0 & 0 & 0 & 23857 & -22747 & 10586 \\ 0 & 0 & 0 & -22747 & 128729 & -59907 \\ 0 & 0 & 0 & 10586 & -59907 & 125536 \end{pmatrix} \tag{26}$$

where V^{-1} is the inverse covariance matrix shown in Ref. [74].

The χ^2_{BAO} can be constructed:

$$\chi^2_{BAO}(p_s) = \frac{[\Delta z_{BAO}(z = 0.24) - 0.0407]^2}{0.0011^2} + \frac{[\Delta z_{BAO}(z = 0.43) - 0.0442]^2}{0.0015^2} + X^t V^{-1} X. \tag{27}$$

X^t denotes the transpose of X.

4 Cosmic constraints on unified model of dark sectors with (or without) a CS fluid in VG theory

Multiplying the separate likelihoods $L_i \propto e^{-\chi_i^2/2}$, one can express the joint analysis of χ^2

$$\chi^2 = \chi^2_{LT} + \chi^2_{GRBs} + \chi^2_{GF} + \chi^2_{SNIa} + \chi^2_{CMB} + \chi^2_{BAO}. \tag{28}$$

4.1 The case with a CS fluid

In order to obtain a stringent constraint on VG theory, we utilize cosmic data different from Ref. [39] to calculate the joint likelihood. Concretely, the LT data, the GRBs data, the GF data, the SNIa data with systematic error, and the BAO data from radial measurement are not used in Ref. [39]. After calculation, the 1-dimension distribution and the 2-dimension contours of the parameters for the VG-UM model with a CS fluid are illustrated in Fig. 1. From Fig. 1 and Table 4, we can see that the restriction on dimensionless energy density of CS is $\Omega_s = -0.0106^{+0.0312+0.0582}_{-0.0305-0.0509}$ in the varying-G theory containing unified dark sectors. In the constant-G theory, one knows that a CS fluid with $w_s = -1/3$ is usually equivalent to a curvature term. But in the VG theory this equivalence is lost due to the term $a^{-\beta}$ multiplying the density, as shown in Eq. (4). Comparing the VG theory with the constant-G theory, it can be seen that the uncertainty of Ω_s in VG theory is larger than some results on Ω_k in constant-G theory. For example, using the same data to constrain other models we have $\Omega_k = -0.0002^{+0.0024+0.0052}_{-0.0024-0.0048}$ (with model parameter $\Omega_{0de} = 0.7098^{+0.0144+0.0265}_{-0.0140-0.0294}$) in ΛCDM model, $\Omega_k = -0.0001^{+0.0025+0.0052}_{-0.0025-0.0050}$ (with model parameters $B_s = 0.7665^{+0.0101+0.0194}_{-0.0099-0.0205}$ and $\alpha = 0.0209^{+0.0186+0.0401}_{-0.0189-0.0373}$) in constant-$G$ UM. Taking the ΛCDM model as a reference, we can see that the influence on the fitting value of Ω_k is small from the added parameter B_s and α as seen in the constant-G UM model, while the influence on the value of Ω_s is large by the added VG parameter β as indicated in the VG-UM model. From Table 4, one reads off the VG parameter $\beta = -0.0128^{+0.0394+0.0756}_{-0.0385-0.0718}$. The other parameters are $B_s = 0.7457^{+0.0147+0.0269}_{-0.0145-0.0299}$ and $\alpha = 0.0216^{+0.0757+0.1504}_{-0.0781-0.1466}$. We then find at 1σ confidence level, the flat ΛCDM model ($\Omega_s = 0$, $\beta = 0$ and $\alpha = 0$) is included in the VG-UM model with a CS fluid. This result in VG theory is the same as the popular feature that the complicated cosmological model is usually degenerate with the ΛCDM model.

4.2 The case without a CS fluid

For the case without a CS fluid, a stringent constraint on VG parameter is $\beta = 0.0007^{+0.0032+0.0062}_{-0.0033-0.0067}$, where a small uncertainty at 2σ regions for β is given. Still, it is shown that the value of β is around zero at 1σ confidence level for
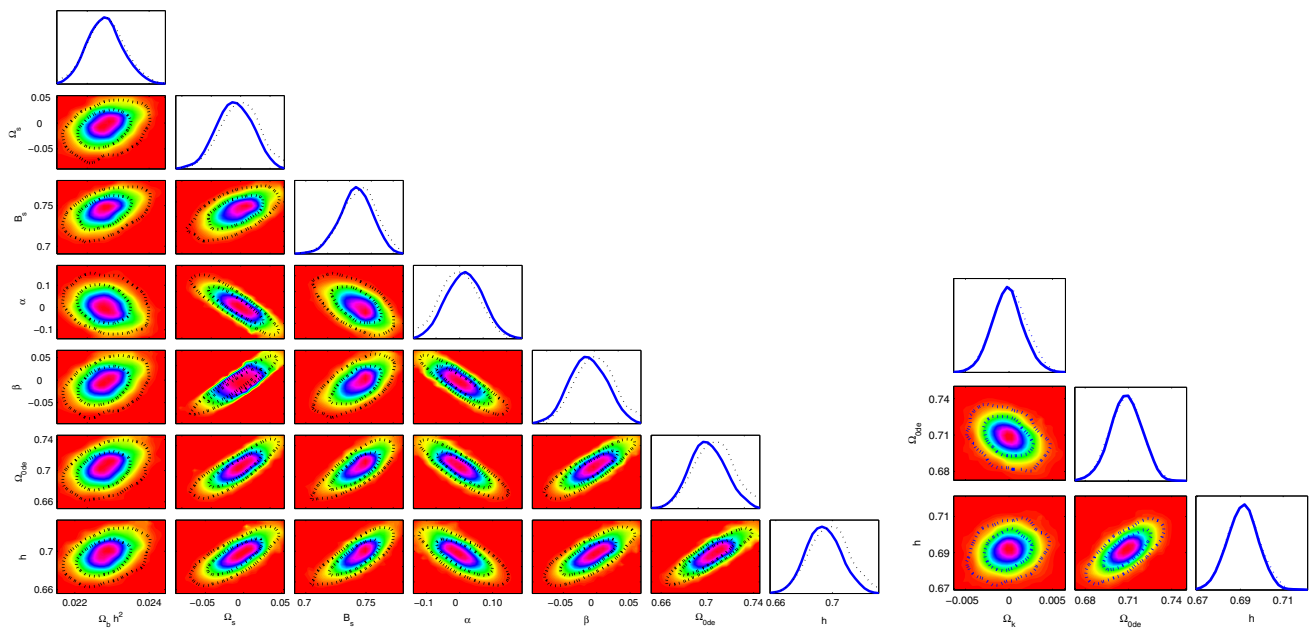
Fig. 1 1σ and 2σ contours of the parameters for the VG-UM model with a CS fluid (*left*) and the ΛCDM (*right*) model

Table 4 The mean values with limits and the best-fit values of the parameters for VG-UM model with a CS fluid

	Mean values with limits (VG-UM)	Best fit (VG-UM)	Mean values with limits (ΛCDM)	Best fit (ΛCDM)
–	$\Omega_s = -0.0106^{+0.0312+0.0582}_{-0.0305-0.0509}$	0.0006	$\Omega_k = -0.0002^{+0.0024+0.0052}_{-0.0024-0.0048}$	-0.0004
β	$-0.0128^{+0.0394+0.0756}_{-0.0385-0.0718}$	0.0005	0	0
B_s	$0.7457^{+0.0147+0.0269}_{-0.0145-0.0299}$	0.7520	–	–
α	$0.0216^{+0.0757+0.1504}_{-0.0781-0.1466}$	0.0004	0	0
h	$0.6922^{+0.0149+0.0306}_{-0.0149-0.0289}$	0.6981	$0.6916^{+0.0100+0.0197}_{-0.0101-0.0193}$	0.6930
$100\Omega_b h^2$	$2.2580^{+0.0555+0.1154}_{-0.0557-0.1051}$	2.2691	$2.2683^{+0.0412+0.0815}_{-0.0420-0.0776}$	2.266
Ω_{0de}	$0.6983^{+0.0165+0.0347}_{-0.0161-0.0309}$	0.7175	$0.7098^{+0.0144+0.0265}_{-0.0140-0.0294}$	0.7126

both cases: including or not including a CS fluid, and the case containing a CS fluid has a larger error for β than that not containing a CS fluid. In VG theory, the constraints on the UM model parameters are $B_s = 0.7442^{+0.0137+0.0262}_{-0.0132-0.0292}$, $\alpha = 0.0002^{+0.0206+0.0441}_{-0.0209-0.0422}$, $h = 0.6905^{+0.0098+0.0191}_{-0.0096-0.0203}$, and $100\Omega_b h^2 = 2.267^{+0.054+0.116}_{-0.051-0.102}$. At 1σ confidence level, the value of $\alpha = 0$ is not excluded, which demonstrates that the ΛCDM model cannot be distinguished from VG-UM model by the joint cosmic data. Besides the mean values with limits, the best-fit values of the VG-UM model parameters are determined and exhibited in Table 5 and Fig. 2, too. As a reference, the ΛCDM model is calculated by using the combined observational data appearing in Sect. III, and the best-fit values and the mean values with limits on ΛCDM model are shown in Table 5. In the ΛCDM model, one obtains $\Omega_{0de} = 0.7101^{+0.0126+0.0270}_{-0.0135-0.0282}$, that is, the result is compatible with the effective result of Ω_{0de} in the VG-UM model.

In order to agglomerate and form a structure of the universe, one uses that the baryonic (and DM) component must have a near zero pressure. Given that $w_b = \frac{p_b}{\rho_b} = \frac{-\beta(1-\beta)}{3(2+\beta)} \sim 0$, $\beta \sim 0$ or $\beta \sim 1$ could be solved. From the above constraint on the parameter β, one can see that the solution $\beta \sim 0$ is consistent with our fitting result for both cases: including or not including a CS fluid in the universe.

5 Behaviors of G with the confidence level in VG-UM theory with or without a CS fluid

In VG-UM theory with or without a CS fluid, the best-fit evolutions of $\frac{\dot{G}}{G}$ with their confidence level (the shadow region) are illustrated in Fig. 3 (one can also see in Table 6) by using the best-fit values of model parameters with their covariance matrix. A dot denotes the derivative with respect to t. In the

Table 5 The mean values with limits and the best-fit values of model parameters for VG-UM model without a CS fluid

	Mean values with limits (VG-UM)	Best fit (VG-UM)	Mean values with limits (ΛCDM)	Best fit (ΛCDM)
β	$0.0007^{+0.0032+0.0062}_{-0.0033-0.0067}$	0.0010	0	0
B_s	$0.7442^{+0.0137+0.0262}_{-0.0132-0.0292}$	0.7440	–	–
α	$0.0002^{+0.0206+0.0441}_{-0.0209-0.0422}$	0.0073	0	0
h	$0.6905^{+0.0098+0.0191}_{-0.0096-0.0203}$	0.6902	$0.6925^{+0.0094+0.0207}_{-0.0104-0.0198}$	0.6923
$100\Omega_b h^2$	$2.267^{+0.054+0.116}_{-0.051-0.102}$	2.256	$2.2647^{+0.0398+0.0789}_{-0.0394-0.0781}$	2.262
Ω_{0de}	$0.7093^{+0.0148+0.0296}_{-0.0150-0.0309}$	0.7095	$0.7101^{+0.0126+0.0270}_{-0.0135-0.0282}$	0.7106
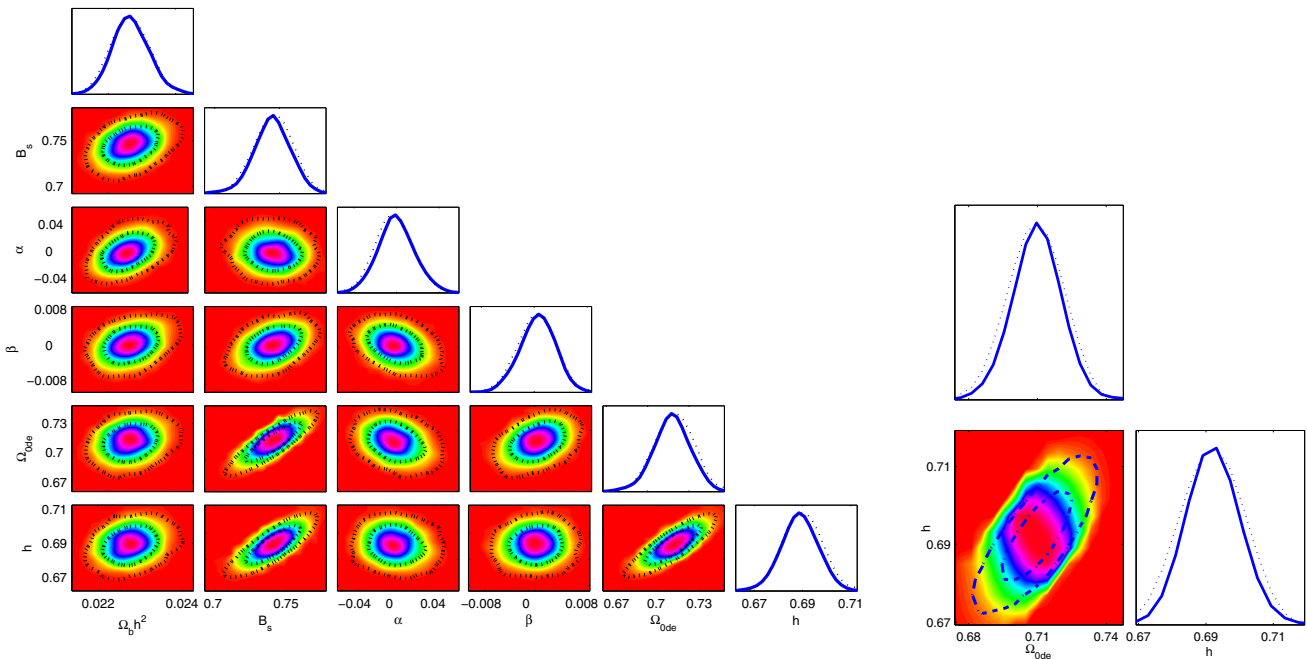
Fig. 2 1σ and 2σ contours of the parameters for the VG-UM model without a CS fluid (*left*) and the ΛCDM (*right*) model
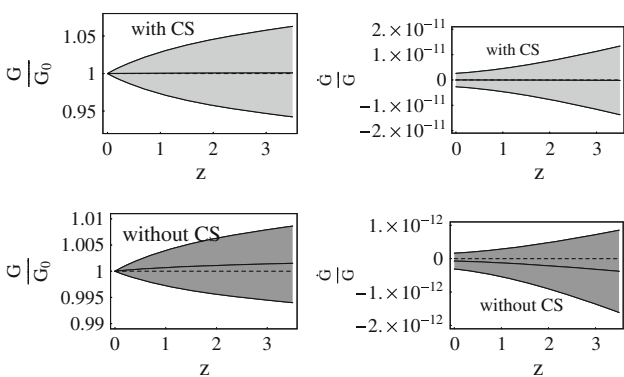
Fig. 3 The best-fit evolutions of $\frac{G}{G_0}$ and $\frac{\dot{G}}{G}$ with their confidence level in the VG-UM model containing (or not containing) a CS fluid

VG-UM model with a CS fluid, the limit on the variation of G at present is $\left(\frac{\dot{G}}{G}\right)_{\text{today}} \simeq -0.3496^{+26.3135}_{-26.3135} \times 10^{-13}$ year^{-1}, and at $z = 3.5$ we have $\left(\frac{G}{G_0}\right)_{z=3.5} \simeq 0.9917^{+0.0104}_{-0.0131}$ and

$\left(\frac{\dot{G}}{G}\right)_{z=3.5} \simeq -1.800^{+135.396}_{-135.396} \times 10^{-13}$ year^{-1}. For the case without a CS fluid, Fig. 3 shows the prediction that today's value is $\left(\frac{\dot{G}}{G}\right)_{\text{today}} \simeq -0.7252^{+2.3645}_{-2.3645} \times 10^{-13}$ year^{-1}. This restriction on $\left(\frac{\dot{G}}{G}\right)_{\text{today}}$ is more stringent than the other results in Table 7. Also, using the best-fit value of the parameter β with error, the shapes of $\frac{G}{G_0} = (1+z)^{\beta}$ are exhibited. Taking the high redshift $z = 3.5$ as another reference point, we find $\left(\frac{G}{G_0}\right)_{z=3.5} \simeq 1.0015^{+0.0071}_{-0.0075}$ and $\left(\frac{\dot{G}}{G}\right)_{z=3.5} \simeq -0.3792^{+1.2314}_{-1.2314} \times 10^{-12}$ year^{-1} in the VG-UM model without a CS fluid. It is important to rigorously constrain the value of β, since the monotonicity of $\frac{\dot{G}}{G} = -\beta H$ depends on the value of β. Figure 3 reveals that the behaviors of G and its derivative are around the constant-G theory for both cases: including or not including a CS fluid in universe.

Table 6 The best-fit values of $\frac{G}{G_0}$ and $\frac{\dot{G}}{G}$ with their confidence level in the VG-UM model containing (or not containing) a CS fluid

	With CS	Without CS
$\left(\frac{G}{G_0}\right)_{z=3.5}$	$1.0008^{+0.0620}_{-0.0584}$	$1.0015^{+0.0071}_{-0.0075}$
$\left(\frac{\dot{G}}{G}\right)_{\text{today}}$	$-0.3496^{+26.3135}_{-26.3135} \times 10^{-13}$ year^{-1}	$-0.7252^{+2.3645}_{-2.3645} \times 10^{-13}$ year^{-1}
$\left(\frac{\dot{G}}{G}\right)_{z=3.5}$	$-1.800^{+135.396}_{-135.396} \times 10^{-13}$ year^{-1}	$-3.792^{+12.314}_{-12.314} \times 10^{-13}$ year^{-1}

Table 7 Limits on the variation of G

Observations	Limits (year^{-1})
Pulsating white dwarf G117-B15A [2]	$\|\frac{\dot{G}}{G}\| \leq 4.1 \times 10^{-10}$
Nonradial pulsations of white dwarfs [3]	$-2.5 \times 10^{-10} \leq \frac{\dot{G}}{G} \leq 4 \times 10^{-11}$
Millisecond pulsar PSR J0437-4715 [4]	$\|\frac{\dot{G}}{G}\| \leq 2.3 \times 10^{-11}$
Type-Ia supernovae [5]	$\frac{\dot{G}}{G} \leq 10^{-11}$
Neutron star masses [6]	$\frac{\dot{G}}{G} = (-0.6 \pm 4.2) \times 10^{-12}$
Helioseismology [75]	$\|\frac{\dot{G}}{G}\| \leq 1.6 \times 10^{-12}$
Lunar laser ranging experiment [76]	$\frac{\dot{G}}{G} = (4 \pm 9) \times 10^{-13}$
Big Bang Nuclei-synthesis [77]	$-3.0 \times 10^{-13} < \frac{\dot{G}}{G} < 4.0 \times 10^{-13}$

6 Behaviors of EoS with the confidence level in VG-UM theory with or without a CS fluid

The EoS of UM in VG theory is demonstrated by

$$
w_{\text{VG-UM}}(z) = \frac{p_{\text{VG-UM}}}{\rho_{\text{VG-UM}}}
$$
$$
= \frac{\beta - 3}{3} \frac{B_s}{B_s + (1 - B_s)(1+z)^{(1+\alpha)(3-\beta)}}. \tag{29}
$$

From Fig. 4 (left), we can see that $w_{\text{VG-UM}} \sim 0$ (DM) at early time and $w_{\text{VG-UM}} \sim -1$ (DE) in the future for the VG-UM model with or without a CS fluid. If the dark sectors are thought to be separable, it is interesting to investigate the properties of both dark components in the VG-UM model. Supposing that the behavior of dark matter is known i.e. its EoS $w_{\text{dm}} = 0$ $\left(\rho_{\text{dm}} = \rho_{0\text{dm}} a^{\frac{-\beta^2 - 2\beta - 6}{2+\beta}}\right)$, the EoS of dark energy in the VG-UM model is subject to

$$
w_{\text{de}} = \frac{p_{\text{de}}}{\rho_{\text{de}}} = \frac{p_{\text{VG-UM}}}{\rho_{\text{VG-UM}} - \rho_{\text{dm}}} = \frac{-A}{\rho_{\text{VG-UM}}^{1+\alpha} - \rho_{\text{dm}}\rho_{\text{VG-UM}}^{\alpha}}. \tag{30}
$$

Using the best-fit values of model parameters and the covariance matrix, the evolutions of w_{de} with confidence level in

the VG-UM model containing (or not containing) a CS fluid are plotted in Fig. 4 (middle). If one deems the behavior of dark energy is the cosmological constant i.e. $w_{\Lambda} = -1$ ($p_{\Lambda} = -\rho_{\Lambda}$), the EoS of dark matter in VG-UM model obeys

$$
w_{\text{dm}} = \frac{p_{\text{dm}}}{\rho_{\text{dm}}} = \frac{p_{\text{VG-UM}} - p_{\Lambda}}{\rho_{\text{VG-UM}} - \rho_{\Lambda}} = \frac{\rho_{\Lambda}\rho_{\text{VG-UM}}^{\alpha} - A}{\rho_{\text{VG-UM}}^{1+\alpha} - \rho_{\Lambda}\rho_{\text{VG-UM}}^{\alpha}}, \tag{31}
$$

which is drawn in Fig. 4 (right) with the confidence level for two cases (with or without a CS fluid).

From Fig. 4, we get the current values $w_{0\text{dm}} = 0.0009^{+0.0304}_{-0.0304}$ in the VG-UM model with a CS fluid and $w_{0\text{dm}} = 0.0072^{+0.0170}_{-0.0170}$ in the VG-UM model without a CS fluid, which have the larger uncertainties than $w_{0\text{dm}} = 0.0010^{+0.0016}_{-0.0016}$ calculated by the non-unified model of constant-G theory by Ref. [78]. For the current value $w_{0\text{de}}$, it approximates -1 with the very small uncertainty for both VG-UM model with a CS fluid ($w_{0\text{de}} = -0.9998^{+0.0125}_{-0.0125}$) and VG-UM model without a CS fluid ($w_{0\text{de}} = -0.9986^{+0.0011}_{-0.0011}$). From the best-fit evolution in the VG-UM model with a CS fluid, we can see that both $w_{\text{de}}(\sim -1)$ and $w_{\text{dm}}(\sim 0)$ tend to be constant, but the uncertainties of them are much larger than that in model without a CS fluid. For the best-fit evolution in the VG-UM model without a CS fluid, w_{de} and w_{dm} are variable with the time and w_{dm} tends to have a small deviation from zero (small-positive pressure) at the recent time. In addition, at high redshift the uncertainty of w_{de} (or w_{dm}) is enlarged (or narrowed) for both VG-UM model with a CS fluid and VG-UM model without a CS fluid (Table 8).

7 Perturbation behaviors in structure formation for VG-UM theory

The study of the structure formation is necessary for a cosmological theory. We investigate the evolutions of the growth function f and the growth factor D in VG-UM theory. The derivations of the evolutionary equations for f and D are shown in the appendix. Using the definition $f(a) = \frac{aD'(a)}{D(a)} =$
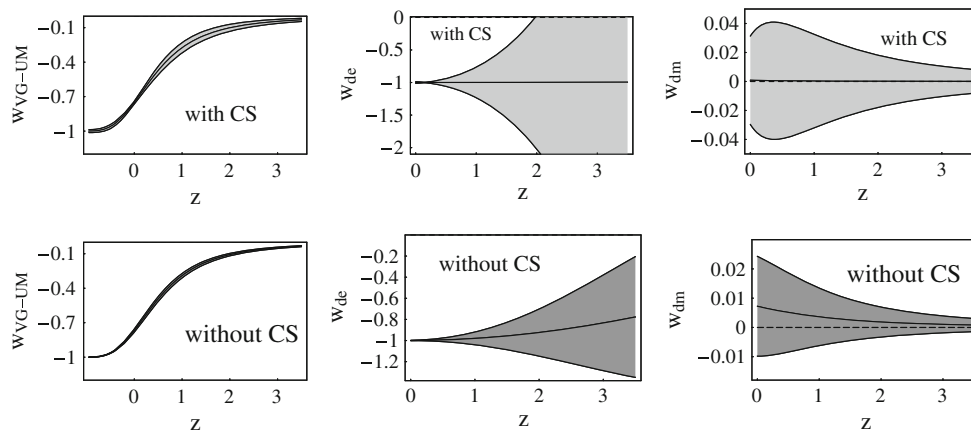
Fig. 4 The evolutions of EoS with confidence level in VG-UM model including (*lower*) or not including (*upper*) a CS fluid. The evolution of $w_{VG-UM}(z)$ (*left*), the evolution of $w_{de}(z)$ in the VG-UM model with assuming $w_{dm} = 0$ at prior (*middle*), and the evolution of $w_{dm}(z)$ in the VG-UM model assuming $w_{de} = -1$ a priori (*right*)

Table 8 The best-fit values of w_{0VG-UM}, w_{0dm}, and w_{0de} with their confidence level hinted by VG-UM model with or without a CS fluid

	w_{0VG-UM}	w_{0dm} (with $w_{0de} = -1$)	w_{0de} (with $w_{0dm} = 0$)
With CS	$-0.7519^{+0.0112}_{-0.0112}$	$0.0009^{+0.0304}_{-0.0304}$	$-0.9998^{+0.0125}_{-0.0125}$
Without CS	$-0.7438^{+0.0134}_{-0.0134}$	$0.0072^{+0.0170}_{-0.0170}$	$-0.9986^{+0.0011}_{-0.0011}$
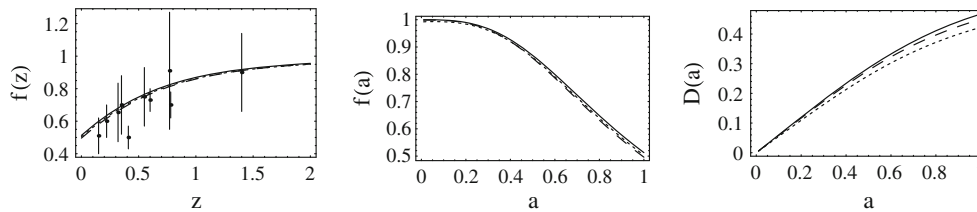
Fig. 5 The evolutions of $f(z)$, $f(a)$, and $D(a)$. The *solid lines* correspond to the ΛCDM model, the *short-dash lines* correspond to the VG-UM model with a CS fluid, and the *dot lines* correspond to the VG-UM model without a CS fluid

$\frac{d \ln \delta}{d \ln a}$, we obtain the dynamically evolutionary equation of f

$$(1 + z)f' - f^2 + (1 + z)f\frac{E'}{E} - \frac{4 + 2\beta + 2\beta^2}{2 + \beta}f$$
$$+ \frac{6 + 2\beta + \beta^2}{(2 + \beta)^2}\frac{\Omega_{0m}}{E^2}(1 + z)^{\frac{6+2\beta+\beta^2}{2+\beta}} = 0 \qquad (32)$$

where a prime denotes the derivative with respect to redshift z and $E(z) = H(z)/H_0$.

In Fig. 5, we use the best-fit values of cosmological parameters in Tables 4 and 5 to plot the evolutions of the growth function f and the growth factor D for the VG-UM model and the ΛCDM model by numerically solving Eqs. (19) and (32) with the initial conditions $a_i = 0.0001$, $D(a_i) = a_i$, $D'(a_i) = 0$, and $f(a_i) = 1$. We can see that the evolutions of $f(a)$ for the VG-UM model (including or not including a CS fluid) fit well as in the ΛCDM model, and the behavior of $f(z)$ are well consistent with the observational growth data listed in Table 3. In the VG-UM model with or without a CS

fluid, $D(a)$ evolves more slowly (slower growth of the perturbations) than that in the ΛCDM model. The current value of $D(a = 1)$ in the ΛCDM model is approximately 12 % larger than that in the VG-UM model without a CS fluid.

8 Conclusions

Observations anticipate that G may be variable and most of the universal energy density invisible. The attractive properties of this study is that the variation of G naturally results in the invisible components in the universe. The VG could provide a solution to the original problem of DM and DE. We apply recently observed data to constrain the unified model of the dark sectors with or without a CS fluid in the framework of VG theory. Using the LT, the GRBs, the GF, the SNIa with systematic error, the CMB from 9-year WMAP, and the BAO data from measurements of the radial and the

peak positions, the uncertainties of the VG-UM parameter space are obtained.

For the case without a cosmic string fluid, the constraint on the mean value of the VG parameter is $\beta = 0.0007^{+0.0032+0.0062}_{-0.0033-0.0067}$ with a small uncertainty around zero, and restrictions on the UM model parameters are $B_s = 0.7442^{+0.0137+0.0262}_{-0.0132-0.0292}$ and $\alpha = 0.0002^{+0.0206+0.0441}_{-0.0209-0.0422}$ with 1σ and 2σ confidence level. For the case with a cosmic string fluid, the restriction on the dimensionless density parameter of the CS fluid is $\Omega_s = -0.0106^{+0.0312+0.0582}_{-0.0305-0.0509}$ in the VG-UM theory. Obviously, the uncertainty of Ω_s is larger than some results on Ω_k in the framework of the G-constant theory. At 1σ confidence level the flat ΛCDM model ($\Omega_s = 0$, $\beta = 0$, and $\alpha = 0$) is included in the VG-UM model.

Using the best-fit values of VG-UM parameters and their covariance matrix, the limits on today's value are $\left(\frac{\dot{G}}{G}\right)_{\text{today}} = -0.7252^{+2.3645}_{-2.3645} \times 10^{-13}$ or $\left(\frac{\dot{G}}{G}\right)_{\text{today}} \simeq -0.3496^{+26.3135}_{-26.3135} \times 10^{-13}\text{year}^{-1}$ for the universe with or without a CS fluid. Corresponding to these two cases, we find $\left(\frac{G}{G_0}\right)_{z=3.5} \simeq 1.0015^{+0.0071}_{-0.0075}$ and $\left(\frac{G}{G_0}\right)_{z=3.5} \simeq 1.0008^{+0.0620}_{-0.0584}$ at redshift $z = 3.5$. If one considers that the DM and the DE could be separable in the unified model, the EoS of DE and DM are discussed by combing with the fitting results. It is shown that $w_{0\text{dm}} = 0.0072^{+0.0170}_{-0.0170}$ or $w_{0\text{dm}} = 0.0009^{+0.0304}_{-0.0304}$ assuming $w_{0\text{de}} = -1$ for the VG-UM universe containing or not containing a CS fluid, while we have $w_{0\text{de}} = -0.9986^{+0.0011}_{-0.0011}$ or $w_{0\text{de}} = -0.9998^{+0.0125}_{-0.0125}$ assuming $w_{0\text{dm}} = 0$ a priori for the VG-UM model with or without a CS fluid.

Appendix A: The growth of structures in linear perturbation theory

In a sub-horizon region with length scale $r < H^{-1}$, the densities of DE and cold DM are expressed by $\tilde{\rho}_{\text{sde}}$ and $\tilde{\rho}_{\text{sdm}}$, respectively. We suppose that DE is not perturbed, and DM is perturbed in the sub-horizon region. So, we have $\tilde{\rho}_{\text{sde}} = \rho_{\text{de}}$ for the homogeneous DE in the whole universe and $\tilde{\rho}_{\text{sdm}} = \rho_{\text{dm}} + \delta\rho_{\text{dm}}$ for the perturbed DM, where ρ_{de} and ρ_{dm} denote the density of the DE and DM at background level, respectively. Obviously, the region of $\delta\rho_{\text{dm}} > 0$ will

cluster and form a structure. In analogy to the equation at the background level, the evolution of the matter density inside the perturbed region can be given by the following conservation equation:

$$\dot{\tilde{\rho}}_{\text{sdm}} + 3h\left(\frac{6 + 2\beta + \beta^2}{6 + 3\beta}\tilde{\rho}_{\text{sdm}} + \frac{2 + 2\beta}{2 + \beta}\tilde{p}_{\text{sdm}}\right) = 0. \quad (A1)$$

A tilde denotes the cosmological quantity in a perturbed region. In this region, the local expansion is described by $h = \dot{r}/r$ and the acceleration is

$$\frac{\ddot{r}}{r} = -\frac{8\pi G(t)}{3(2 + \beta)}(\tilde{\rho}_{\text{sdm}} + \rho_{\text{de}} + 3\tilde{p}_{\text{sdm}} + 3p_{\text{de}}) - \frac{\beta^2}{(2 + \beta)}h, \quad (A2)$$

which is the same as Eq. (5) for the background level. One can define the density contrast of DM,

$$1 + \delta_{\text{dm}} = \frac{\tilde{\rho}_{\text{sdm}}}{\rho_{\text{dm}}} \quad (A3)$$

with $\delta_{\text{dm}} > 0$. Differentiating Eq. (A3) with respect to t gives

$$\dot{\delta}_{\text{dm}} + \left(\frac{6 + 2\beta + \beta^2}{2 + \beta}\right)(1 + \delta_{\text{dm}})(h - H)$$
$$+ \frac{6 + 6\beta}{2 + \beta}(1 + \delta_{\text{dm}})(h\tilde{w}_{\text{sdm}} - Hw_{\text{dm}}) = 0 \quad (A4)$$

after using Eqs. (A1) and (6). Taking the time derivative in the above equation one obtains

$$\ddot{\delta}_{\text{dm}} - \frac{\dot{\delta}_{\text{dm}}^2}{1 + \delta_{\text{dm}}} + \left(\frac{6 + 2\beta + \beta^2}{2 + \beta}\right)(1 + \delta_{\text{dm}})(\dot{h} - \dot{H})$$
$$+ \frac{6 + 6\beta}{2 + \beta}\left(\dot{h}\tilde{w}_{\text{sdm}} + h\dot{\tilde{w}}_{\text{sdm}} - \dot{H}\tilde{w}_{sdm}\right.$$
$$\left. - H\dot{\tilde{w}}_{\text{sdm}})(1 + \delta_{\text{dm}})\right) = 0, \quad (A5)$$

where

$$\dot{h} - \dot{H} = -\frac{H^2}{2 + \beta}\Omega_{\text{dm}}\delta_{\text{dm}} - \frac{4 + 2\beta + 2\beta^2}{2 + \beta}(h - H)H$$
$$- \frac{3H^2}{2 + \beta}\left(\frac{\tilde{\rho}_{\text{sdm}}}{\tilde{\rho}_c}\tilde{w}_{\text{sdm}} - \Omega_{\text{dm}}w_{\text{dm}}\right) \quad (A6)$$

is given by substituting Eqs. (5) and (A2) into $\dot{H} = \frac{\ddot{a}}{a} - H^2$ and $\dot{h} = \frac{\ddot{r}}{r} - h^2$, respectively. In addition, in the calculation we used $\rho_c = 3H^2/8\pi G(t)$ and $h + H \simeq 2H$. Inserting (A6) into (A5) results in

$$\ddot{\delta}_{\text{dm}} - \frac{\dot{\delta}_{\text{dm}}^2}{1 + \delta_{\text{dm}}} + \frac{4 + 2\beta + \beta^2}{2 + \beta}H\dot{\delta}_{\text{dm}}$$
$$- \left(\frac{6 + 2\beta + \beta^2}{2 + \beta}\right)\frac{H^2\Omega_{\text{dm}}}{2 + \beta}(\delta_{\text{dm}} + \delta_{\text{dm}}^2)$$

$$+\left[\frac{4+2\beta+2\beta^2}{2+\beta}\frac{6+6\beta}{2+\beta}H(h\tilde{w}_{sdm}-Hw_{dm})\right.$$

$$+\frac{6+6\beta}{2+\beta}(\dot{h}w_{dm}+h\dot{\tilde{w}}_{sdm}-\dot{H}w_{dm}-H\dot{w}_{dm})$$

$$\left.+\left(\frac{6+2\beta+\beta^2}{2+\beta}\right)\frac{3H^2}{2+\beta}\left(\frac{\tilde{\rho}_{sdm}}{\tilde{\rho}_c}\tilde{w}_{sdm}-\Omega_{dm}w_{dm}\right)\right]$$

$$\times(1+\delta_{dm})=0. \tag{A7}$$

Neglecting square terms of δ_m in (A7), we obtain the evolution equation of the density contrast in a spherical overdense region,

$$\ddot{\delta}_{dm}+\frac{4+2\beta+2\beta^2}{2+\beta}H\dot{\delta}_{dm}-\left(\frac{6+2\beta+\beta^2}{2+\beta}\right)\frac{H^2\Omega_{dm}}{2+\beta}\delta_{dm}$$

$$+\left[\frac{4+2\beta+2\beta^2}{2+\beta}\frac{6+6\beta}{2+\beta}H(h\tilde{w}_{sdm}-Hw_{dm})\right.$$

$$+\frac{6+6\beta}{2+\beta}(\dot{h}\tilde{w}_{sdm}+h\dot{w}_{dm}-\dot{H}\tilde{w}_{sdm}-H\dot{w}_{sdm})$$

$$\left.+\left(\frac{6+2\beta+\beta^2}{2+\beta}\right)\frac{3H^2}{2+\beta}\left(\frac{\tilde{\rho}_{sdm}}{\tilde{\rho}_c}\tilde{w}_{sdm}-\Omega_{dm}w_{dm}\right)\right]$$

$$\times(1+\delta_{dm})=0. \tag{A8}$$

Taking $\beta=0$, the above equation reduces to the case of constant G given by Ref. [79]. Using the definition of the growth factor $D(a)$, we can rewrite Eq. (A8) as follows:

$$D''(a)+[\frac{E'(a)}{E(a)}+\frac{1}{a}+\frac{4+2\beta+2\beta^2}{a(2+\beta)}]D'(a)$$

$$-\frac{6+2\beta+\beta^2}{(2+\beta)^2}\frac{\Omega_{0dm}}{E(a)^2a^2}a^{\frac{-6-2\beta-\beta^2}{2+\beta}}D(a)$$

$$+[\frac{(4+2\beta+2\beta^2)(6+6\beta)}{(2+\beta)^2a^2H}(h\tilde{w}_{sdm}-Hw_{dm})$$

$$+\frac{6+6\beta}{(2+\beta)aH}(h'\tilde{w}_{sdm}+h\tilde{w}'_{sdm}-H'w_{dm}-Hw'_{dm})$$

$$+\frac{18+6\beta+3\beta^2}{(2+\beta)^2a^2}(\frac{\tilde{\rho}_{sdm}}{\tilde{\rho}_c}\tilde{w}_{sdm}-\Omega_{dm}w_{dm})]$$

$$\times[1+D(a)\delta_{dm}(a=1)]=0. \tag{A9}$$

The linear regime of cosmological perturbations is valid for all scales during the early radiation dominated era and for most scales during the matter dominated era. For $w_{dm}\simeq w_{sdm}\simeq 0$, the above equation reduces to

$$D''(a)+[\frac{E'(a)}{E(a)}+\frac{1}{a}+\frac{4+2\beta+2\beta^2}{a(2+\beta)}]D'(a)$$

$$-\frac{6+2\beta+\beta^2}{(2+\beta)^2}\frac{\Omega_{0dm}}{E(a)^2a^2}a^{\frac{-6-2\beta-\beta^2}{2+\beta}}D(a)=0. \tag{A10}$$

Transferring the function from D to f in the above equation, we get

$$f'(a)+\frac{f^2(a)}{a}+[\frac{E'(a)}{E(a)}+\frac{2}{a}(\frac{2+\beta+\beta^2}{(2+\beta)})]f(a)$$

$$-\frac{6+2\beta+\beta^2}{(2+\beta)^2}\frac{\Omega_{0dm}}{E(a)^2a}a^{\frac{-6-2\beta-\beta^2}{2+\beta}}=0. \tag{A11}$$

References

1. K.I. Umezu, K. Ichiki, M. Yahiro, Phys. Rev. D **72**, 044010 (2005)
2. M. Biesiada, B. Malec, Mon. Not. Roy. Astron. Soc. **350**, 644 (2004). arXiv:astro-ph/0303489
3. O.G. Benvenuto et al., Phys. Rev. D **69**, 082002 (2004)
4. J.P.W. Verbiest et al., Astrophys. J. **679**, 675 (2008). arXiv:0801.2589
5. E. Gaztanaga et al., Phys. Rev. D **65**, 023506 (2002). arXiv:astro-ph/0109299
6. S.E. Thorsett, Phys. Rev. Lett. **77**, 1432 (1996). arXiv:astro-ph/9607003
7. L.X. Xu, J.B. Lu, Y.T. Wang, Eur. Phys. J. C **72**, 1883 (2012). arXiv:1204.4798
8. P.X. Wu, H.W. Yu, Phys. Lett. B **644**, 16 (2007)
9. K. Zhang, P.X. Wu, H.W. Yu, JCAP **01**, 048 (2014)
10. A. Ali, S. Dutta, E. N. Saridakis, A. A. Sen. arXiv:1004.2474
11. C. Ranjit, P. Rudra, S. Kundu. arXiv:1304.6713
12. S. Ghose, A. Saha, B.C. Paul. arXiv:1203.2113
13. S. Mukherjee, B.C. Paul, N.K. Dadhich, S.D. Maharaj, A. Beesham, Class. Quant. Grav. **23**, 6927 (2006)
14. S. Ghose, P. Thakur, B.C. Paul, Mon. Not. R. Astron. Soc. **421**, 20 (2012). arXiv:1105.3303
15. T. Harko, M. J. Lake. arXiv:1409.8454
16. F. Niedermann, R. Schneider, Phys. Rev. D **91**, 064010 (2015). arXiv:1412.2750
17. S. Kumar, A. Nautiyal, A. A. Sen. arXiv:1207.4024
18. O. S. Sazhina, D. Scognamiglio, M. V. Sazhin. arXiv:1312.6106
19. M. van de Meent, Phys. Rev. D **87**, 025020 (2013). arXiv:1211.4365
20. P. A. R. Ade, et al. arXiv:1303.5085
21. A. Lewis, S. Bridle, Phys. Rev. D **66**, 103511 (2002)
22. S. Capozziello, V.F. Cardone, M. Funaro, S. Andreon, Phys. Rev. D **70**, 123501 (2004)
23. J. Simon, L. Verde, R. Jimenez, Phys. Rev. D **71**, 123001 (2005)
24. L. Xu, JCAP **04**, 025 (2012). arXiv:1005.5055
25. L. Verde et al., Mon. Not. Roy. Astron. Soc. **335**, 432 (2002). arXiv:astro-ph/0112161
26. E.V. Linder, Astropart. Phys. **29**, 336–339 (2008). arXiv:0709.1113
27. C. Blake et al., Mon. Not. Roy. Astron. Soc. **415**, 2876 (2011). arXiv:1104.2948
28. R. Reyes et al., Nature. **464**, 256 (2010). arXiv:1003.2185
29. M. Tegmark et al., SDSS Collaboration. Phys. Rev. D **74**, 123507 (2006). arXiv:astro-ph/0608632
30. N.P. Ross et al., Mon. Not. Roy. Astron. Soc. **381**(2), 573–588, (2007). arXiv:astro-ph/0612400
31. L. Guzzo et al., Nature **451**, 541 (2008). arXiv:0802.1944
32. J. da Angela et al., Mon. Not. Roy. Astron. Soc. **383**(2), 565–580, (2008). arXiv:astro-ph/0612401
33. R. Amanullah et al., Supernova Cosmology Project Collaboration. arXiv:1004.1711
34. G. Hinshaw et al. arXiv:astro-ph/1212.5226
35. E. Gaztanaga, R. Miquel, E. Sanchez, Phys. Rev. Lett. **103**, 091302 (2009)
36. C. Blake et al. arXiv:1108.2635
37. F. Beutler et al. arXiv:1106.3366
38. W.J. Percival et al., Mon. Not. R. Astron. Soc. **401**, 2148 (2010). arXiv:astro-ph/0907.1660

39. J.B. Lu, L.X. Xu, H.Y. Tan, S.S. Gao, Phys. Rev. D **89**, 063526 (2014)
40. M.C. Bento, O. Bertolami, A.A. Sen, Phys. Rev. D **66**, 043507 (2002)
41. A. Sandage, Ann. Rev. Astron. Astrophys. **26**, 561 (1988)
42. N. Pires, Z. Zhu, J.S. Alcaniz, Phys. Rev. D **73**, 123530 (2006)
43. M.A. Dantas, J.S. Alcaniz, D. Jain, A. Dev, Astron. Astrophys. **467**, 421 (2007)
44. E. Komatsu et al., WMAP Collaboration. arXiv:1001.4538
45. B.E. Schaefer, Astrophys. J. **660**, 16 (2007). arXiv:astro-ph/0612285
46. L. Amati et al., Mon. Not. Roy. Astron. Soc. **391**, 577 (2008). arXiv:0805.0377
47. Y. Wang, Phys. Rev. D **78**, 123532 (2008)
48. M.C. Bento, O. Bertolami, A.A. Sen, Phys. Lett. B **575**, 172 (2003)
49. M. Li, X.D. Li, X. Zhang, Sci. China. Ser. G **53**, 1631 (2010)
50. P.T. Silva, O. Bertolami, Astrophys. J. **599**, 829 (2003)
51. A.C.C. Guimaraes, J.V. Cunha, J.A.S. Lima, JCAP **0910**, 010 (2009)
52. M. Szydlowski, W. Godlowski, Phys. Lett. B **633**, 427 (2006)
53. R.G. Cai, Q. Su, H.B. Zhang, JCAP. arXiv:astro-ph/1001.2207
54. R.G. Cai, Q.P. Su, Phys. Rev. D **81**, 103514 (2010)
55. Y.G. Gong, R.G. Cai, Y. Chen, Z.H. Zhu, JCAP **01**, 019 (2010)
56. Y.G. Gong, B. Wang, R.G. Cai, JCAP **04**, 019 (2010)
57. R. Gannouji, D. Polarski, JCAP **0805**, 018 (2008)
58. J.B. Lu, L.X. Xu, M.L. Liu, Phys. Lett. B **699**, 246–250 (2011)
59. Z.X. Li, P.X. Wu, H.W. Yu et al., Sci. China Phys. Mech. Astron. **57**, 381 (2014)
60. J.F. Zhang, L. Zhao, X. Zhang, Sci. China Phys. Mech. Astron. **57**, 387 (2014)
61. S. Nesseris, L. Perivolaropoulos, Phys. Rev. D **72**, 123519 (2005)
62. L. Perivolaropoulos, Phys. Rev. D **71**, 063503 (2005)
63. E. Di Pietro, J.F. Claeskens, Mon. Not. Roy. Astron. Soc. **341**, 1299 (2003)
64. C. Cheng, Q.G. Huang, Sci China Phys. Mech. Astron. **58**, 099801 (2015)
65. J.B. Lu et al., Sci China Phys. Mech. Astron. **57**, 796 (2014)
66. V. Acquaviva, L. Verde, JCAP **0712**, 001 (2007)
67. E. Garcia-Berro, E. Gaztanaga, J. Isern, O. Benvenuto, L. Althaus. arXiv:astro-ph/9907440
68. A. Riazuelo, J. Uzan, Phys. Rev. D **66**, 023525 (2002)
69. L.X. Xu, Phys. Rev. D **91**, 063008 (2015)
70. L.X. Xu, JCAP **02**, 048 (2014)
71. N. Suzuki, et al., Astrophys. J. **746**, 85 (2012). http://supernova.lbl.gov/Union/
72. J.B. Lu, D.H. Geng, L.X. Xu, Y.B. Wu, M.L. Liu, JHEP **02**, 071 (2015)
73. Z.X. Li, P.X. Wu, H.W. Yu, Z.H. Zhu, Phys. Rev. D **87**, 103013 (2013)
74. S. Nesseris, J. Garcia-Bellido, JCAP **11**, 033 (2012). arXiv:astro-ph/1205.0364
75. D.B. Guenther, L.M. Krauss, P. Demarque, Astrophys. J. **498**, 871 (1998)
76. J.G. Williams, S.G. Turyshev, D.H. Boggs, Phys. Rev. Lett. **93**, 261101 (2004). arXiv:gr-qc/0411113
77. C.J. Copi, A.N. Davies, L.M. Krauss, Phys. Rev. Lett. **92**, 171301 (2004)
78. L.X. Xu, Y.D. Chang, Phys. Rev. D **88**, 127301 (2013). arXiv:1310.1532
79. T. Padmanabhan, *Structure formation in the universe*. (Cambridge University Press, 1993)