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Finite-width Gaussian sum rules for 0^{-+} pseudoscalar glueball based on correction from instanton–gluon interference to correlation function

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Abstract Based on a correction from instanton–gluon interference to the correlation function, the properties of the 0^{-+} pseudoscalar glueball are investigated in a family of finite-width Gaussian sum rules. In the framework of a semiclassical expansion for quantum chromodynamics in the instanton liquid background, the contribution arising from the interference between instantons and the quantum gluon fields is calculated, and it is included in the correlation function together with a pure-classical contribution from instantons and the perturbative one. The interference contribution turns out to be gauge-invariant, to be free from an infrared divergence, and to have a great role to play in restoring the positivity of the spectra of the full correlation function. The negligible contribution from vacuum condensates is excluded in our correlation function to avoid double counting. Instead of the usual zero-width approximation for the resonances, the usual Breit–Wigner form with a suitable threshold behavior for the spectral function of the finite-width resonances is adopted. Consistency between the subtracted and unsubtracted sum rules is very well justified. The values of the mass, decay width, and coupling constants for the 0^{-+} resonance in which the glueball fraction is dominant are obtained, and they agree with the phenomenological analysis.

1 Introduction

A significant issue in quantum chromodynamics (QCD) is to seek for the signal of the existence of glueballs. Because glueballs are bound states composed of only gluons in the quarkless world, such a signal may give a unique insight into the non-Abelian dynamics of QCD. Theoretical investigations including lattice simulations [1–3], model research [4–6], and sum rule analyses [7–12] have been going on for a

long time, but no decisive evidence of the existence of glueballs has been confirmed by experimental research up to now [13,14]. Further investigation on glueballs still makes sense.

One of the obstacles in theoretical research of glueballs is that non-perturbative dynamics of QCD, which is responsible for the formation of hadrons, is difficult to handle, and the QCD vacuum is recognized to be a medium with a complicated structure, and it may impact greatly on the attributes of hadrons. In particular, the tunneling effect between the degenerate vacua of QCD should be taken into account. In the leading order, this effect is described by instantons [15,16] and shown to be of great significance in generating the properties of the unusual hadrons, glueballs. Moreover, the glueball may be mixed with the usual mesons of the same quantum numbers, making the identification of the glueball more complicated [12,17].

Instantons, as the strong topological fluctuations of gluon fields in QCD, are widely believed to play an important role in the physics of the strong interaction (for reviews see [16,18]). In particular, instantons provide mechanisms for the violation of both $U(1)_A$ and the chiral symmetry in QCD, and they may therefore be important in determining hadron masses and in the resolution of the famous $U(1)_A$ problem. Furthermore, it was recently shown that instantons persist through the deconfinement transition, so that instanton–induced interactions between quarks and gluons may underlie the unusual properties of the so-called strongly coupled quark–gluon plasma recently discovered at RHIC [19].

In the instanton liquid model, in a precise sense one describes the QCD vacuum as a sum of independent instantons with radius $\bar{\rho} = (600\,\text{MeV})^{-1}$ and effective density $\bar{n} = (200\,\text{MeV})^4$ [20]. This model avoids the infrared problem caused by an infinite instanton density in the diluted gas model. The correctness of the instanton liquid model is still intensively being investigated. So far the model is essentially justified by its phenomenological success. The most

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important predictions are probably the breaking of the chiral symmetry (SBCS) in the axial triplet channel [21,22] and the absence of Goldstone bosons in the axial singlet channel.

The non-perturbative effects of QCD is commonly cast into the form of the description of a non-perturbative vacuum, such as quark and gluon condensates and the instanton configurations. When contributions from both condensates and the instanton are included into the correlation function together, it leads to a double counting in the sense that there can be two alternative ways for parametrizing the non-perturbative vacuum [23,24]. Moreover, the instanton distribution is closely connected with the vacuum condensates since the mean size and density of instantons can be deduced from the quark and gluon condensates, and, conversely, the values of the condensates can be reproduced from the instanton distribution [25–28]. The contributions of the instanton and those of the condensates should be equivalent to each other in some cases, and thus to include both contributions at the same time will cause a double counting problem. To avoid it, many authors invoke some special techniques in dealing with the two contributions [29–32]. This issue is, however, not settled really. Fortunately, the contribution of condensates to the correlation function in the glueball channels is unusually weak, as demonstrated in this work and by many other authors [33,34]. Therefore, it is assumed that the condensate contributions can be understood as a small fraction of the corresponding instanton one in the local limit. To thoroughly avoid the problem of double counting, we choose to work in the instanton vacuum model of QCD, and carry out a semiclassical expansion in instanton background fields as suggested in our previous work to analyze the properties of the lowest 0^{++} scalar glueball [35,36] and the 0^{-+} pseudoscalar one [37,38], where the correlation function of the glueball currents are calculated by just including the contributions from the pure instantons, the pure quantum gluons, and the interference between both, instead of working with both instantons and condensates at the same time.

Zhang and Steele [39] have reported a disparity between their Laplace sum rule and Gaussian sum rule for the pseudoscalar glueball. In this reference, the optimized parameters of the pseudoscalar glueball have been obtained from the Gaussian sum rule, while it fails to achieve satisfying the Laplace sum rule. This result seems strange, since both Laplace and Gaussian sum rules are derived from the same underlying dynamical theory, and should, at least approximately, be consistent. It reflects the inconsistency for including both condensate and instanton contributions in the correlation function and disregarding the important contribution from the interaction between instantons and their quantum counterparts at the same time.

Another serious problem in the 0^{-+} glueball sum rule approach is that the fundamental spectral positivity bound is violated when including the strong repulsive pure instanton contribution, and as a consequence, the signal for the pseudoscalar glueball disappears [39]. To cure this pathology of positivity violation, the topological charge screening effect in the QCD vacuum is added to the correlation function, and a suitable instanton-size distribution is taken into account [7,40]. However, comparing with the interference contribution, which we have recalculated in this paper, and the pure perturbative one in the considered energy region, the topological screening effect turns out to be negligible. The pathology of the positivity violation disappears when including the interference contribution (see below).

Phenomenologically, the identification of the pseudoscalar glueball has been a matter of debate since the Mark II experiment proposed glueball candidates [41]. Later, in the mass region of the first radial excitation of the η and η' mesons, a supernumerous candidate, the $\eta(1405)$ has been observed. It seems to be clear that $\eta(1405)$ is allowed as a glueball dominated state mixed with isoscalar $q\bar{q}$ states due to its behavior in production and decays, namely, it has comparably large branching ratios in the J/ψ radiative decay, but it has not been observed in $\gamma\gamma$ collisions [13,42,43]. A review on the experimental status of the $\eta(1405)$ is given in Ref. [13]. However, this state lies considerably lower than the theoretical expectations: the lattice QCD predictions suggest a glueball around 2.5 GeV [44,45]; the mass scale of the pseudoscalar glueball obtained in the QCD sum rule approach is above 2 GeV [7,37,38,40]. On the other hand, there are attractive arguments for the scalar and pseudoscalar glueballs being approximately degenerate in mass [46], and even the scenario that a pseudoscalar glueball may be lower in mass than the scalar one was recently discussed in Ref. [47]. The possibly non-vanishing gluonium content of the ground state η and η' mesons is discussed in [12,48–50]. Up to now, only the topological model of the glueball as a closed flux tube [46] predicts a degeneracy of the 0^{++} and 0^{-+} glueball masses and admits the region 1.3–1.5 GeV.

To reach a more realistic phenomenological situation, we now reexamine the correlation function in the instanton liquid vacuum, and we include all the resonances below and near the $\eta(1405)$ into the finite-width spectral function, and then we achieve a series of results in traditional Gaussian sum rule analyses which are consistent with the phenomenology. On the other hand, as a crosscheck, these results are almost the same as the Laplace ones (paper is accepted for publication in Physical Review D), because both Laplace and Gaussian sum rules are derived from the same underlying dynamical theory. The paper is organized as follows: in Sect. 2, we present systematically the calculation of the contribution to the correlation function due to the interference between instantons and quantum gluons. This interference between instantons and quantum gluons serves as a mechanism to keep the positivity of the spectral function in contrast with the so-called topological charge screening effect stressed in Ref. [7]. The effect of

the contribution of topological charge screening is found to be negligible as compared with the interference one in the considered energy region. The spectral function is constructed in a similar way to the case of the 0^{++} glueball [36,51] with a suitable threshold behavior in Sect. 3. In Sect. 4, a family of Gaussian sum rules are constructed. The numerical simulations are carried out in Sect. 5, and the results are consistent with various Gaussian sum rules and in accordance with the phenomenology. Finally, the main conclusions are given and a discussion of some open interesting issues is presented.

2 Correlation function

We are working in Euclidean QCD. The pseudoscalar glueball current is defined as

$$O_p(x) = \alpha_s G^a_{\mu\nu}(x)\widetilde{G}_{\mu\nu,a}(x) \tag{1}$$

where α_s is the strong coupling constant, $G^a_{\mu\nu}(x)$ is the gluon field strength tensor with the color index a and Lorentz indices μ and ν, and

$$\widetilde{G}_{\mu\nu,a}(x) = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^a_{\rho\sigma}(x) \tag{2}$$

is the dual of $G^a_{\mu\nu}(x)$. The current $O_p(x)$ is a Lorentz-irreducible, gauge-invariant, and local composite operator with the lowest dimension, and renormalization group-invariant at least to the leading order α_s in a quarkless world. It is noticed that the current $O_p(x)$ is anti-hermitian due to the imaginary time involved, while its analytic continuation to Minkowskian space-time is hermitian. The QCD correlation function is defined as

$$\Pi(q^2) = \int d^4x\, e^{iq\cdot x}\langle\Omega|T\{O_p(x)O_p^\dagger(0)\}|\Omega\rangle \tag{3}$$

where $O_p^\dagger$ is the hermitian conjugation of O_p. The advantages to use the hermitian conjugation in the definition are, first, the spectral functions both in Euclidean space-time and in its analytic continuation into Minkowskian space-time are, in principle, positively definite; and second, the relationship between the correlation functions both in Euclidean and Minkowskian formulations becomes very simple:

$$\Pi_E(Q^2 = q^2) \leftrightarrow \Pi_M(Q^2 = -q^2), \tag{4}$$

because the overall minus sign arising from the analytic continuation due to $(\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma})_E = -(\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma})_M$ is just canceled with another minus sign arising from the mentioned different hermiticity of the pseudoscalar glueball current. Therefore, the expressions for $\Pi_E(Q^2)$ and $\Pi_M(Q^2)$ are, in fact, the same function of Q^2.

In the framework of a semiclassical expansion, the glue potential field $B(x)$ can be decomposed into a summation of the classical instanton A and the corresponding quantum gluon field a:

$$B_\mu(x) = A_\mu(x) + a_\mu(x). \tag{5}$$

Consequently, the pure-glue Euclidean action can be expressed as

$$\begin{aligned}
S[B] &= S_0 - \int d^4x \left\{ L[A+a] + \frac{1}{2\xi}a^a_\mu D^{ab}_\mu D^{bc}_\nu a^c_\nu \right\} \\
&= S_0 - \frac{1}{2}\int d^4x \left\{ a^a_\mu \left[D^{ab}_\lambda D^{bc}_\lambda \delta_{\mu\nu} + 2gf^{abc}F^b_{\mu\nu} \right.\right. \\
&\quad \left. - \left(1 - \frac{1}{\xi}\right) D^{ab}_\mu D^{bc}_\nu \right] a^c_\nu - 2gf_{abc}a_{\mu b}a_{\nu c}D_{\mu,ad}a_{\nu d} \\
&\quad \left. - \frac{1}{2}g^2 f_{abc}a_{\mu b}a_{\nu c}f_{ade}a_{\mu d}a_{\nu e} \right\}
\end{aligned} \tag{6}$$

where $S_0 = 8\pi^2/g^2$ is the one-instanton contribution to the action, $F_{\mu\nu a}$ is the instanton field strength tensor

$$F_{\mu\nu,a}(A) = \partial_\mu A_{\nu,a} - \partial_\nu A_{\mu,a} + g_s f_{abc}A_{\mu,b}A_{\nu,c}, \tag{7}$$

and $D^{ab}_\mu(A)$ is the covariant derivative associated with the classical instanton field A^a_μ

$$D^{ab}_\mu(A) = \partial_\mu \delta_{ab} + gf_{acb}A^c_\mu. \tag{8}$$

In addition, the background field gauge

$$D^\mu(A)a_\mu = 0 \tag{9}$$

is used with ξ being the corresponding gauge parameter, and certainly the corresponding Faddeev–Popov ghosts according to the standard rule should be added to restore unitarity. We note here that the structure constants f_{abc} should be understood as ϵ_{abc} when any one of the color-indices a, b, and c is associated with an instanton field due to the property of the closure of any group.

According to the decomposition (5), the correlation function Π splits into three parts, namely the pure-classical part, the pure quantum part, and the interference part in the leading order

$$\Pi^{\mathrm{QCD}}(Q^2) = \Pi^{(\mathrm{cl})}(Q^2) + \Pi^{(\mathrm{qu})}(Q^2) + \Pi^{(\mathrm{int})}(Q^2) \tag{10}$$

where the superscript indicates that it is calculated in the underlying dynamical theory, QCD. It is important to note that every part in the r.h.s. of (10) is gauge-invariant because the decomposition (5), in principle, has no impact on the gauge-invariance of the correlation function. The pure instanton contribution $\Pi^{(\mathrm{cl})}(Q^2)$ and the perturbative contribution $\Pi^{(\mathrm{qu})}(Q^2)$ up to three-loop level in the chiral limit of QCD are shown in Eqs. (80) and (81), respectively, in Appendix A.

The contribution of the topological charge screening which is not included in the r.h.s. of (10) is given in Appendix B.

One of our main tasks in this work is to calculate the contribution $\Pi^{(\text{int})}(Q^2)$ in (10), which is arising from the interference between the classical instantons and quantum gluons in the framework of the semiclassical expansion for QCD with the instanton background. After imposing the background covariant Feynman gauge ($\xi = 1$) for the quantum gluon fields, we are still free to choose a gauge for the background field A. In the following, the singular gauge is chosen to the non-perturbative instanton field configurations as

$$A_{\mu a}(x) = \frac{2}{g_s} \eta_{a\mu\nu}(x - z)_\nu \phi(x - z), \tag{11}$$

with

$$\phi(x - z) = \frac{\rho^2}{(x - z)^2[(x - z)^2 + \rho^2]}, \tag{12}$$

and the corresponding field strength tensor is

$$F_{\mu\nu,a}(x) = -\frac{8}{g_s} \left[\frac{(x - z)_\mu (x - z)_\rho}{(x - z)^2} - \frac{1}{4}\delta_{\mu\rho} \right]$$
$$\times \eta_{a\nu\rho} \frac{\rho^2}{((x - z)^2 + \rho^2)^2} - (\mu \leftrightarrow \nu), \tag{13}$$

where z and ρ denote, respectively, the center and size of the instanton, called collective coordinates together with the color orientation, and $\eta_{a\mu\nu}$ is the 't Hooft symbol, which should be replaced with the anti-'t Hooft one $\bar{\eta}_{a\mu\nu}$ for an anti-instanton field. For the sake of simplicity, in practice the one most used is the spike size distribution $n(\rho) = \bar{n}\delta(\rho - \bar{\rho})$, where $\bar{n}$ is the overall instanton density and $\bar{\rho}$ is the average instanton size. The fact that the strong coupling constant g_s emerges in the denominator of the r.h.s. of (11) reveals the non-perturbative nature of these classical configurations. In fact, instantons play a quite important role in the QCD sum rule. Early QCD sum rules neglecting instanton–induced continuum contributions did not obtain reliable results in many cases, but these problems were then solved by including such instanton–induced effects [9,30].

Before starting with the contraction between the quantum fields, we note that the time-development of the instanton vacuum produces the pre-exponential factor for the distribution of the instantons [15,52,53], and $\Pi^{(\text{int})}$ is understood as taking an ensemble average over the collective coordinates besides taking the usual vacuum expectation value due to the separation (5),

$$\Pi^{(\text{int})}(x)$$
$$= \sum_{I,\bar{I}} \int \mathrm{d}\rho\, n(\rho) \int \mathrm{d}^4 z \langle \Omega | T\{O_p(x) O_p^\dagger(0)\}^{(\text{int})} | \Omega \rangle, \tag{14}$$

where the super index '(int)' indicates the corresponding quantity containing only the interference part between the quantum and classical ones. Using the spike distribution, (14) becomes

$$\Pi^{(\text{int})}(x) = 2\bar{n} \int \mathrm{d}^4 z \langle \Omega | T\{O_p(x) O_p^\dagger(0)\}^{(\text{int})} | \Omega \rangle, \tag{15}$$

where the factor 2 comes from the mutually equal contributions of both instanton and anti-instanton. The next important step is to specify the form of the gluon propagator which in the background field Feynman gauge can be read from the part of the $S[B]$ quadratic in a [54,55]

$$\mathcal{D}_{\mu\nu}^{ab}(x, y) = \langle \Omega | T\{a_\mu^a(x) a_\nu^b(y)\} | \Omega \rangle$$
$$= \left\langle x \left| \left(\frac{1}{P^2 \delta_{\mu\nu} - 2F_{\mu\nu}} \right)^{ab} \right| y \right\rangle \tag{16}$$

with $P_\mu^{ab} = -i D_\mu^{ab}$. Keeping only terms proportional to F, one has [56]

$$\int \mathrm{d}^4 x\, e^{iq\cdot x} \mathcal{D}_{\mu\nu}^{ab}(x, 0)$$
$$= e^{iq\cdot(y-z)} \left\{ \frac{1}{q^2}\delta_{\mu\nu} + g_s \frac{2}{q^4} F_{\mu\nu}(z) \right.$$
$$\left. - i g_s \frac{(y - z)_\rho F_{\rho\sigma}(z) q_\sigma}{q^4} \delta_{\mu\nu}(z) + \cdots \right\} \tag{17}$$

where the first term in the r.h.s. of the above equation is the pure-gluon propagator in the usual Feynman gauge, and the second and third ones are the leading contributions of the instanton field to the gluon propagator. For the short distance region, we assume that the contribution from a single instanton is dominant over multi-instantons [57]. At the leading loop level, the gluon propagator (17) becomes the pure-gluon one.

In the calculation, we expand the current O_p into terms which are the products of quantum gluon fields and their derivatives with the coefficients being composed of the instanton fields,

$$O_P(x) = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} \alpha_s \sum_{i=1}^{10} O_i(x), \tag{18}$$

where the operators O_i in terms of instanton and quantum gluon fields are listed in Appendix C. Equation (15) can be rewritten as

$$\Pi^{(\text{int})}(q^2) = -\frac{1}{2}\alpha_s^2 \bar{n}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu'\nu'\rho'\sigma'}$$

$$\times \sum_{i,j} \int d^4z \int d^4x e^{iq\cdot x} \langle \Omega|T\{O_i(x)O_j(0)\}|\Omega\rangle$$

$$= \sum_{i=1}^{12} \Pi_i^{(\text{int})}(q^2) + \cdots \qquad (19)$$

where the $\cdots$ denotes the contributions from the products of operators being proportional to g_s^3, and the expressions of $\Pi_i^{(\text{int})}(q^2)$ in terms of O_i are shown in Appendix D. The corresponding 12 kinds of Feynman diagrams are shown in Fig. 1, where the contributions from the first three diagrams are of the order of α_s, and the contributions of the remainders are superficially of the order of α_s^2, and those from the diagrams 4, 5, and 6 in Fig. 1, in fact, are vanishing because of violating the conservation of the color-charge, namely

$$\Pi_i^{(\text{int})}(q^2) = 0, \quad \text{for } i = 4, 5, 6 \qquad (20)$$

Now we are in the position to evaluate the contributions of the remaining diagrams in Fig. 1. Using the standard technique of regularizing the ultraviolet divergence in the modified minimal subtraction scheme, the result for the interference part of the correlation function is

$$\Pi^{\text{int}}(Q^2) = c_0\alpha_s\bar{n}\pi + \alpha_s^2\bar{n}\Big\{c_1 + c_2(Q\rho)^{-2}$$

$$+ \Big[c_3(Q\rho)^2 + c_4 + c_5(Q\rho)^{-2}\Big]\ln\frac{Q^2}{\mu^2}\Big\}, \qquad (21)$$

where we have ignored terms proportional to the positive powers of q^2, which vanish after a Borel transformation, and the dimensionless coefficients c_i are numerically determined to be

$$c_0 = -118.23, \quad c_1 = -3700.59\alpha_s, \quad c_2 = -2394.47\alpha_s,$$
$$c_3 = 11561.90\alpha_s, \quad c_4 = 1850.30\alpha_s, \quad c_5 = 1197.24\alpha_s, \qquad (22)$$

through a tedious calculation. It should be noted that there is no infrared divergence as expected by the instanton size being fixed in the liquid instanton vacuum model. Comparing Eq. (21) with our previous result [37,38], they differ not only in some coefficients but also in the logarithm structures due to the fact that the newly improved calculation is free from an infrared divergence, while our old ones were not, and it would need a corresponding cutoff to regularize the integral in the infrared limit.

Putting everything above together, our final correlation function for the pseudoscalar glueball current is of the form
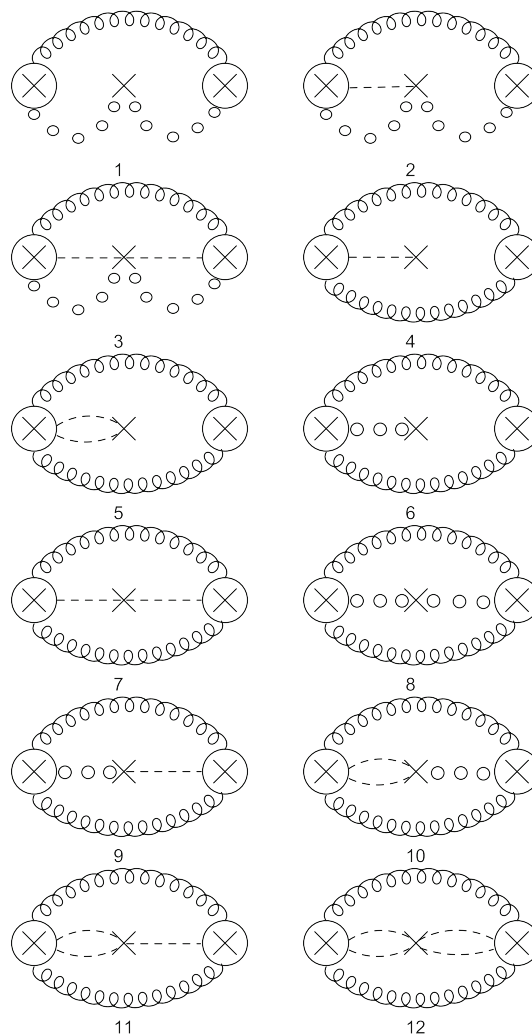

Fig. 1 Feynman diagrams for the interference contribution $\Pi^{(\text{int})}(Q^2)$ up to order α_s^2, where *spiral lines*, *dotted lines*, and the *lines* with *circles* denote gluons, instantons, and the instanton field strength tensor, respectively, and a cross stands for the position of the instantons

$$\Pi^{\text{QCD}}(Q^2) = -2^5\pi^2\bar{n}y^4 K_2^2(y)$$

$$+ \alpha_s\bar{n}\Big[c_0\pi + c_1 + c_2 y^{-2}$$

$$+ (c_3 y^2 + c_4 + c_5 y^{-2})\ln\frac{Q^2}{\mu^2}\Big]$$

$$+ \Big(\frac{\alpha_s}{\pi}\Big)^2 Q^4 \ln\frac{Q^2}{\mu^2}\Big[a_0 + a_1\ln\frac{Q^2}{\mu^2}$$

$$+ a_2\ln^2\frac{Q^2}{\mu^2}\Big] \qquad (23)$$

with $y = Q\bar{\rho}$.

Before going on, let us compare the roles of the various parts of the contributions in correlation function. The imaginary part of the correlation function (23) can be worked out to be
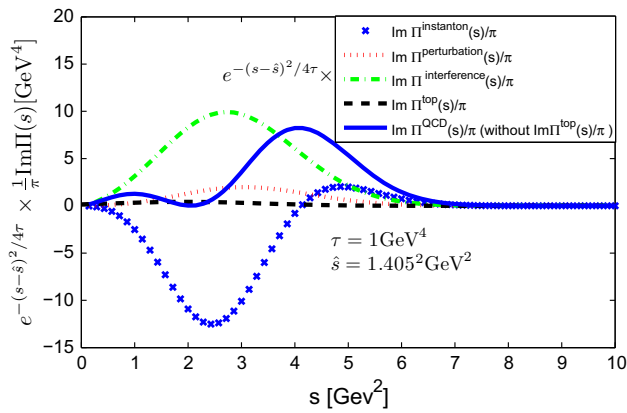
Fig. 2 The contributions to the imaginary part of the correlation function from the pure instanton (*cross line*), interference (*dashed–dotted line*), pure perturbative (*dotted line*) and topological charge screening (*dashed line*), and the total contribution without the topological charge screening one (*solid line*) versus s

$$
\frac{1}{\pi}\text{Im}\Pi^{\text{QCD}}(s) = 16\pi^3 s^2 \bar{n}\bar{\rho}^4 J_2(\bar{\rho}\sqrt{s})Y_2(\bar{\rho}\sqrt{s})
$$
$$
+ \alpha_s^2 \bar{n}\left[c_3\bar{\rho}^2 s - c_4 + c_5(\bar{\rho}^2 s)^{-1}\right]
$$
$$
- s^2 \left(\frac{\alpha_s}{\pi}\right)^2 \left[a_0 + 2a_1 \ln\frac{s}{\mu^2}\right.
$$
$$
\left. + \left(3\ln^2\frac{s}{\mu^2} - \pi^2\right)a_2\right] \tag{24}
$$

where the pure-classical contribution (the first term on the r.h.s. of (24)) is most dominant, and the contribution of the interference terms (the second term on the r.h.s.) comes on the second place, the pure perturbative contribution simply plays the role of the third place, as shown in Fig. 2, where the imaginary part of the correlation function is multiplied with a weight function $\exp(-(s-\hat{s})/4\tau)$, as required by the Gaussian sum rules, and in accordance with the spirit of the semiclassical expansion. We note that the contribution from the topological charge screening, from (84), is displayed in Fig. 2 as well, and its role is almost insignificant. Moreover, it is easy to see from Fig. 2 that the imaginary part of the correlation function is already positive from $s = 0.5\,\text{GeV}^2$ to $s = 10\,\text{GeV}^2$ without including the contribution of the topological charge screening, and the so-called positivity problem is no longer there. Therefore, the interference contribution to the correlation function is significant important not only in its magnitude but also in restoring the positivity to the spectral function.

We note here that the so-called condensate contribution to the correlation function of the pseudoscalar glueball current is proven to be very small on comparing with the one of (23), as shown in Appendix E, where the comparison between the real and imaginary parts of the correlation function and condensate contribution to it are made; they are shown in Figs. 8 and 9. For the reasons given above, the contributions

from the topological charge screening effect and the usual condensates are omitted in our sum rule analysis.

3 Spectral function

In the isosinglet channel there are five gauge-invariant composite operators with the quantum numbers of the 0^{-+}, which are bilinear in the fundamental quark, antiquark, and gluon fields, namely the pseudoscalar quark densities, the divergences of the axial quark currents, and the gluon anomaly:

$$
\hat{J}_5^{8,0} = i\bar{q}\gamma_5(\lambda^{8,0}/2)q, \tag{25}
$$
$$
\partial_\mu \hat{J}_{\mu 5}^{8,0} = \partial_\mu[\bar{q}\gamma_\mu\gamma_5(\lambda^{8,0}/2)q], \tag{26}
$$
$$
\hat{O}_p = \alpha_s G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a, \tag{27}
$$

where λ^8 is the flavor Gell-Mann matrix, and $\lambda^0 = \sqrt{2/3}I$ with I being the 3×3 flavor unit matrix. Only three of these operators are independent due to the two renormalization-invariant axial Ward identities

$$
\partial_\mu \hat{J}_{\mu 5}^8 = \frac{1}{3}(m_u + m_d + 4m_s)\hat{J}_5^8
$$
$$
+ \frac{1}{3}(4m_u + 4m_d + 2m_s)\hat{J}_5^0, \tag{28}
$$
$$
\partial_\mu \hat{J}_{\mu 5}^0 = \frac{\sqrt{2}}{3}(m_u + m_d - 2m_s)\hat{J}_5^8
$$
$$
+ \frac{2}{3}(m_u + m_d - 2m_s)\hat{J}_5^0 + \frac{\sqrt{3}}{4\sqrt{2\pi}}\hat{O}_p. \tag{29}
$$

Further, under renormalization and the flavor space rotation, we have

$$
(m\hat{J}_5^{8,0})_{(r)} = m\hat{J}_5^{8,0}, \tag{30}
$$

$$
(\partial_\mu \hat{J}_{\mu 5}^8)_{(r)} = \partial_\mu \hat{J}_{\mu 5}^8, \quad (\partial_\mu \hat{J}_{\mu 5}^0)_{(r)} = Z\partial_\mu \hat{J}_{\mu 5}^0, \tag{31}
$$

$$
(\hat{O}_p)_{(r)} = \hat{O}_p + \frac{4\sqrt{2\pi}}{\sqrt{3}}\partial_\mu \hat{J}_{\mu 5}^0, \tag{32}
$$

where the quantities with subscript (r) are the renormalized ones. As a consequence, the gluon anomaly operator $\hat{O}_p$, even though renormalization-invariant in the pure-gluon world, is a linear combination of the three operators $\hat{J}_5^8$, $\hat{J}_5^0$, and $\hat{O}_p$ after renormalization. Therefore, one assumes that there may be some isosinglet quark–antiquark pseudoscalar states mixed with the pseudoscalar glueball ground state G.

Now we construct the spectral function for the correlation function of the pseudoscalar glueball current. The usual lowest one-resonance plus a continuum model is used to saturate the phenomenological spectral function:

$$\frac{1}{\pi}\text{Im}\Pi^{\text{PHEN}}(s) = \rho^{\text{HAD}}(s) - \theta(s-s_0)\frac{1}{\pi}\text{Im}\Pi^{\text{QCD}}(s), \quad (33)$$

where s_0 is the QCD-hadron duality threshold, $\theta(s - s_0)$ the step function and $\rho^{\text{HAD}}(s)$ the spectral function for the lowest pseudoscalar glueball state. In the usual zero-width approximation, the spectral function for a single resonance is assumed to be

$$\rho^{\text{HAD}}(s) = F^2\delta(s-m^2), \quad (34)$$

where m is the mass of the lowest glueball, and F is the coupling constant of the current to the glueball defined as

$$\langle 0|O_p(0)|G\rangle = F. \quad (35)$$

The threshold behavior for $\rho^{\text{HAD}}(s)$ is known to be

$$\rho^{\text{HAD}}(s) \rightarrow \lambda_0 s, \quad \text{for } s \rightarrow 0 \quad (36)$$

from the low-energy theorem in a world without light quark flavors [58] or the one in a world with three light flavors and $m_{u,d} \ll m_s$ [59]. In fact, the threshold behavior (36) is only proven to be valid close to the chiral limit; it may not be extrapolated far away. Therefore, instead of considering the coupling F as a constant [7], we choose a model for F as

$$F = \begin{cases} \lambda_0 s, & \text{for } s < m_\pi^2, \\ fm^2, & \text{for } s \geq m_\pi^2, \end{cases} \quad (37)$$

where the λ_0 and f are some constants to be determined later in numerical simulation.

To go beyond the zero-width approximation, in facing the near-actual situation, the Breit–Wigner form for a single resonance is assumed for $\rho^{\text{HAD}}(s)$,

$$\rho^{\text{HAD}}(s) = \frac{F^2 m\Gamma}{(s-m^2+\Gamma^2/4)^2 + m^2\Gamma^2}, \quad (38)$$

where Γ is the width of the lowest glueball.

Further, the one-isolated lowest resonance assumption is, however, questioned from the admixture with quarkonium states, and it is known from the experimental data that there are five 0^{-+} pseudoscalar resonances up to and around the mass scale of 1.405 GeV (namely $\eta(548)$, $\eta(958)$, $\eta(1295)$, $\eta(1405)$, and $\eta(1475)$). The form of the spectral function for the five resonances is then taken to be

$$\rho^{\text{HAD}}(s) = \sum_{i=1}^{5}\frac{F_i^2 m_i\Gamma_i}{(s-m_i^2+\Gamma_i^2/4)^2 + m_i^2\Gamma_i^2}, \quad (39)$$

m_i and Γ_i being the mass and width of the ith resonance, respectively. For the sake of simplicity, all coupling constants F_i for $s < m_\pi^2$ are fixed with the same λ_0, as shown in (37).

It is noticed that there are other pseudoscalar resonances $\eta(1760)$ and $\eta(2225)$, which are omitted from the summary table of PDG. These two resonances are excluded in our consideration. The reasons may be listed in order: First, although $\eta(1760)$ and $\eta(2225)$ may certainly be coupled to the pseudoscalar glueball current via the gluon anomaly, such a coupling, however, contains a factor of the running coupling, as commonly seen in QCD, and it becomes weaker as Q^2 increases, as demonstrated in an effective QCD low-energy theory [60]. Second, the mixing between the considered pseudoscalar glueball and $\eta(1760)$ and $\eta(2225)$ is believed to be very small because the locations of $\eta(1760)$ and $\eta(2225)$ are far away from the scale of the lowest pseudoscalar glueball. Third, the continuum threshold s_0, determined in our sum rule approach, is only in an effective sense due to the accuracy level of the present calculation.

4 Finite-width Gaussian sum rules

In this section, we construct the appropriate sum rules of the 0^{-+} pseudoscalar glueball current, which has the form due to the dispersion relation

$$\Pi^{\text{QCD}}(Q^2) = \int_0^\infty ds\frac{1}{s+Q^2}\frac{1}{\pi}\text{Im}\Pi(s) \quad (40)$$

where $\text{Im}\Pi(s)$ could be simulated by the phenomenological one $\text{Im}\Pi^{\text{PHEN}}(s)$ within an assumed model of the spectral function in the spirit of the sum rule approach. Using the Borel transformation [61]

$$\hat{\mathcal{B}} \equiv \lim_{\substack{N\rightarrow\infty \\ Q^4\rightarrow\infty}}\Bigg|_{Q^4/N\equiv 4\tau}\frac{(-1)^N}{(N-1)!}(Q^4)^N\left(\frac{d}{dQ^4}\right)^N \quad (41)$$

to both sides of (40), a family of Gaussian sum rules can be formed to be [61]

$$\mathcal{G}_k^{\text{HAD}}(s_0,\hat{s},\tau) = \mathcal{G}_k^{\text{QCD}}(s_0,\hat{s},\tau) + \frac{1}{\sqrt{4\pi\tau}}\exp\left[-\frac{\hat{s}^2}{4\tau}\right]\Pi(0)\delta_{k,-1}, \quad (42)$$

where s_0 is the continuum threshold; hadronic physics is (locally) dual to QCD above it. We have

$$\Pi(0) = (8\pi)^2\frac{m_u m_d}{m_u+m_d}\langle\bar{q}q\rangle, \quad (43)$$

which comes from the low-energy theorem for QCD with three light flavors [59], and

$$\mathcal{G}_k^{\text{HAD}}(s_0,\hat{s},\tau) = \frac{1}{\sqrt{4\pi\tau}}\int_0^{s_0} ds\, s^k\exp\left[-\frac{(s-\hat{s})^2}{4\tau}\right]\frac{\rho^{\text{HAD}}(s)}{\pi}, \quad (44)$$

for the phenomenological contributions to the sum rules;

$$\mathcal{G}_k^{\text{QCD}}(s_0, \hat{s}, \tau) = \mathcal{G}_k^{\text{QCD}}(\hat{s}, \tau) - \mathcal{G}_k^{\text{CONT}}(s_0, \hat{s}, \tau), \quad (45)$$

for the theoretical contributions, where $\mathcal{G}_k^{\text{CONT}}(s_0, \hat{s}, \tau)$ is the contribution of the continuum being defined as

$$
\begin{aligned}
&\mathcal{G}_k^{\text{CONT}}(s_0, \hat{s}, \tau) \\
&= \frac{1}{\sqrt{4\pi\tau}} \int_{s_0}^{\infty} \mathrm{d}s s^k \exp\left[-\frac{(s-\hat{s})^2}{4\tau}\right] \frac{\text{Im}^{\text{QCD}}(s)}{\pi}, \quad (46)
\end{aligned}
$$

and $\mathcal{G}_k^{\text{QCD}}(\hat{s}, \tau)$ is defined as

$$
\begin{aligned}
\mathcal{G}_k^{\text{QCD}}(\hat{s}, \tau) = \frac{2\tau}{\sqrt{4\pi\tau}}\hat{\mathcal{B}}\Bigg[&\frac{(\hat{s}+iQ^2)^k \Pi^{\text{QCD}}(\hat{s}+iQ^2)}{iQ^2} \\
&- \frac{(\hat{s}-iQ^2)^k \Pi^{\text{QCD}}(\hat{s}-iQ^2)}{iQ^2}\Bigg]. \quad (47)
\end{aligned}
$$

Substituting the correlation function (23) of the 0^{-+} pseudoscalar glueball into (47), one can derive the Gaussian sum rules of $k = -1, 0$ and $+1$,

$$
\begin{aligned}
\mathcal{G}_{-1}^{\text{QCD}}(\hat{s}, \tau) = \hat{\mathcal{I}} \cdot{}& 16\pi^3\bar{n}\bar{\rho}^4 J_2\left(\bar{\rho}\sqrt{s}\right) Y_2\left(\bar{\rho}\sqrt{s}\right) s \\
&+ \bar{n}\pi\alpha_s c_0 \frac{1}{\sqrt{4\pi\tau}}\exp\left[-\frac{\hat{s}^2}{4\tau}\right] \\
&+ \bar{n}\alpha_s^2 \frac{1}{\sqrt{4\pi\tau}}\Bigg\{-(c_1+c_3\gamma)\exp\left[-\frac{\hat{s}^2}{4\tau}\right] \\
&+ c_2\bar{\rho}^2\sqrt{2\tau}e^{-\hat{s}^2/8\tau}D_{-1}(-\hat{s}/\sqrt{2\tau}) \\
&- (c_4/\bar{\rho}^2 - c_5(\gamma-1)/\bar{\rho}^2)\frac{\hat{s}}{2\tau}\exp\left[-\frac{\hat{s}^2}{4\tau}\right]\Bigg\} \\
&+ \frac{1}{\sqrt{4\pi\tau}}D_{-2}(-\hat{s}/\sqrt{2\tau})[a_0-(2\gamma-2)a_1 \\
&+ 0.5(6\gamma^2-12\gamma-\pi^2)a_2]2\tau e^{-\hat{s}^2/8\tau}, \quad (48)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}_0^{\text{QCD}}(\hat{s}, \tau) = \hat{\mathcal{I}} \cdot{}& 16\pi^3\bar{n}\bar{\rho}^4 J_2\left(\bar{\rho}\sqrt{s}\right) Y_2\left(\bar{\rho}\sqrt{s}\right) s^2 \\
&+ \bar{n}\alpha_s^2 \frac{1}{\sqrt{4\pi\tau}}\Bigg\{c_2\bar{\rho}^2 2\tau e^{-\hat{s}^2/8\tau}D_{-2}(-\hat{s}/\sqrt{2\tau}) \\
&- c_3\sqrt{2\tau}e^{-\hat{s}^2/8\tau}D_{-1}(-\hat{s}/\sqrt{2\tau}) \\
&+ (c_4/\bar{\rho}^2 - c_5\gamma/\bar{\rho}^2)\exp\left[-\frac{\hat{s}^2}{4\tau}\right]\Bigg\} \\
&+ \frac{1}{\sqrt{4\pi\tau}}D_{-3}(-\hat{s}/\sqrt{2\tau})[2a_0+(6-4\gamma)a_1 \\
&+ (6\gamma^2-18\gamma-\pi^2+6)a_2](2\tau)^{3/2}e^{-\hat{s}^2/8\tau}, \quad (49)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}_1^{\text{QCD}}(\hat{s}, \tau) = \hat{\mathcal{I}} \cdot{}& 16\pi^3\bar{n}\bar{\rho}^4 J_2\left(\bar{\rho}\sqrt{s}\right) Y_2\left(\bar{\rho}\sqrt{s}\right) s^3 \\
&+ \frac{\bar{n}\alpha_s^2}{\sqrt{4\pi\tau}}\Big\{2c_2\bar{\rho}^2(2\tau)^{3/2}e^{-\hat{s}^2/8\tau}D_{-3}(-\hat{s}/\sqrt{2\tau}) \\
&- c_3 2\tau e^{-\hat{s}^2/8\tau}D_{-2}(-\hat{s}/\sqrt{2\tau}) \\
&+ c_5\bar{\rho}^{-2}\sqrt{2\tau}e^{-\hat{s}^2/8\tau}D_{-1}(-\hat{s}/\sqrt{2\tau})\Big\} \\
&+ \frac{1}{\sqrt{4\pi\tau}}D_{-4}(-\hat{s}/\sqrt{2\tau})[6a_0+(22-12\gamma)a_1 \\
&+ (18\gamma^2-66\gamma-3\pi^2+36)a_2]4\tau^2 e^{-\hat{s}^2/8\tau}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (50)
\end{aligned}
$$

where

$$\hat{\mathcal{I}} = \frac{1}{\sqrt{4\pi\tau}}\int_0^{\infty}\mathrm{d}s\exp\left[-\frac{(s-\hat{s})^2}{4\tau}\right], \quad (51)$$

and the parabolic cylinder function $D_{-d-1}(\hat{s}\sqrt{\tau})$ is defined as

$$
\begin{aligned}
D_{-d-1}(z) &= \frac{\sqrt{2}(-1)^d e^{-z^2/4}}{d!}\frac{\mathrm{d}^d\left(e^{z^2/2}\int_{z/\sqrt{2}}^{\infty}\mathrm{d}y e^{-y^2}\right)}{(\mathrm{d}z)^d}, \\
&\qquad d \geq 0. \quad (52)
\end{aligned}
$$

5 Numerical simulation

The expressions for the three-loop running coupling constant $\alpha_s(Q^2)$ with three massless flavors ($N_f = 3$) at the renormalization scale μ [62],

$$\frac{\alpha_s(\mu^2)}{\pi} = \frac{\alpha_s^{(2)}(\mu^2)}{\pi} + \frac{1}{(\beta_0 L)^3}\left[L_1\left(\frac{\beta_1}{\beta_0}\right)^2 + \frac{\beta_2}{\beta_0}\right], \quad (53)$$

are used, where $\alpha_s^{(2)}(\mu^2)/\pi$ is the two-loop running coupling constant with ($N_f = 0$),

$$\frac{\alpha_s^{(2)}(\mu^2)}{\pi} = \frac{1}{\beta_0 L} - \frac{\beta_1}{\beta_0}\frac{\ln L}{(\beta_0 L)^2} \quad (54)$$

and

$$
\begin{aligned}
L &= \ln\left(\frac{\mu^2}{\Lambda^2}\right), \quad L_1 = \ln^2 L - \ln L - 1, \\
\beta_0 &= \frac{1}{4}\left[11 - \frac{2}{3}N_f\right], \\
\beta_1 &= \frac{1}{4^2}\left[102 - \frac{38}{3}N_f\right], \\
\beta_2 &= \frac{1}{4^3}\left[\frac{2857}{2} - \frac{5033}{18}N_f + \frac{325}{54}N_f\right], \quad (55)
\end{aligned}
$$

with the color number $N_c = 3$ and the QCD renormalization-invariant scale $\Lambda = 120$ MeV. We take $\mu^2 = \sqrt{\tau}$ after calculating the Borel transforms based on the renormalization group improvement for Gaussian sum rules [63]. The subtraction constant $\Pi(0)$ has been fixed as [7]

$$\Pi(0) \simeq -0.022 \, \text{GeV}^4, \tag{56}$$

and the values of the average instanton size and the overall instanton density are adopted from the instanton liquid model [25],

$$\bar{n} = 1 \, \text{fm}^{-4} = 0.0016 \, \text{GeV}^4,$$
$$\bar{\rho} = \frac{1}{3} \, \text{fm} \simeq 1.667 \, \text{GeV}^{-1}. \tag{57}$$

The resonance parameters in Eq. (39) could be estimated by matching both sides of the sum rules, Eq. (42), optimally in the fiducial domain. In doing so, the parameter $\hat{s}$ and the threshold s_0 should be determined on priority. Firstly, it is obvious that s_0 must be greater than the mass square of the highest lying isolated resonance considered, namely

$$s_0 \geq m_{\text{max}}^2 \tag{58}$$

in our multi-resonance assumption (or just the resonance mass itself in the case of a single-resonance assumption), and it should guarantee that there is a sum rule window for the Gaussian sum rules. Secondly, the peak positions of the $\mathcal{G}_k^{\text{QCD}}(s_0, \hat{s}, \tau)$ versus $\hat{s}$ curves should not change too much with moderate variation of s_0. Here we do not mention the values of τ in this condition, because the peak positions of these curves is not affected by the appropriate values of τ. It is found that the behavior of these curves can satisfy the above requirements as shown in Fig. 3 if s_0 lies in the interval of $(4 \, \text{GeV}^2, 5 \, \text{GeV}^2)$. It is also remarkable that the peak positions have already indicated the approximate mass of the hadron considered. Thus, we would expect that the mass of the pseudoscalar glueball should be near the value $\sqrt{\hat{s}} \simeq 1.449$ GeV. In another way, if one uses the curves of $\mathcal{G}_k^{\text{QCD}}(s_0, \hat{s}, \tau)$ versus τ with fixed $\hat{s}$ and s_0 to obtain the physical parameters through (43), then $\hat{s}$ should be set approximately to be $\hat{s}_{\text{peak}}$ of the curves of $\mathcal{G}_k^{\text{QCD}}(s_0, \hat{s}, \tau)$ versus τ, so as to highlight the underlying hadron state in our consideration and suppress the contributions from other states.

Besides, one needs a sum rule window in which the hadron physical properties should be stable in this region. For the upper limit τ_{max} of the sum rule window, the resonance contribution should be great than the continuum one,

$$\mathcal{G}_k^{\text{QCD}}(s_0, \hat{s}, \tau_{\text{max}}) \geq \mathcal{G}_k^{\text{CONT}}(s_0, \hat{s}, \tau_{\text{max}}) \tag{59}$$

according to the standard requirement due to the fact that in the energy region above τ_{max} the perturbative contribution
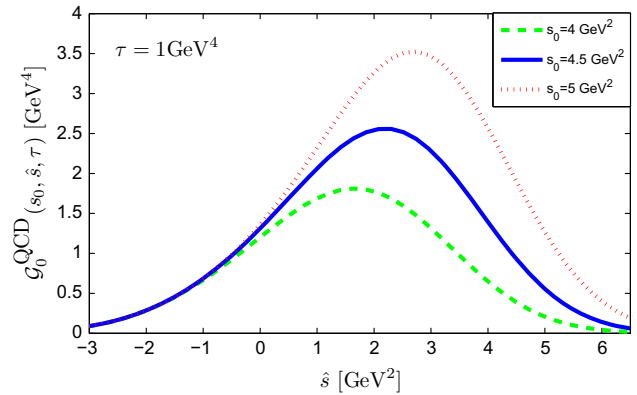
Fig. 3 The curves of $\mathcal{G}_0^{\text{QCD}}(s_0, \hat{s}, \tau)$ versus $\hat{s}$ with different s_0 at $\tau = 1 \, \text{GeV}^4$

is dominant. At τ_{min}, which lies in the low-energy region, we require that the single instanton contribution should be relatively large, so that

$$\frac{\mathcal{G}_k^{\text{inst}}(s_0, \hat{s}, \tau_{\text{min}})}{\mathcal{G}_k^{\text{QCD}}(s_0, \hat{s}, \tau_{\text{min}})} \geq 50 \, \%. \tag{60}$$

In the same time, to require that the multi-instanton corrections remain negligible, we simply adopt a rough estimate

$$\tau_{\text{min}} \geq 2\bar{\rho}^{-4} \simeq \left(\frac{2}{0.6 \, \text{GeV}} \right)^4. \tag{61}$$

According to the above requirements, we find that in the domain

$$\tau \in (0.5, 4.5) \, \text{GeV}^4, \tag{62}$$

our sum rules work very well. Finally, in order to measure the compatibility between both sides of the sum rules (42) in our numerical simulation, we divide the sum rule window $[\tau_{\text{min}}, \tau_{\text{max}}]$ into $N = 100$ segments of equal widths, $[\tau_i, \tau_{i+1}]$, with $\tau_0 = \tau_{\text{min}}$ and $\tau_N = \tau_{\text{max}}$, and we introduce a variation δ (called the matching measure) which is defined as

$$\delta = \frac{1}{N} \sum_{i=1}^{N} \frac{[L(\tau_i) - R(\tau_i)]^2}{|L(\tau_i)R(\tau_i)|}, \tag{63}$$

where $L(\tau_i)$ and $R(\tau_i)$ are the l.h.s. and r.h.s. of (42) evaluated at τ_i.

Let us first consider the case of the single-resonance plus continuum model (34) of the spectral function by excluding the interference contribution $\Pi^{\text{int}}(Q^2)$ from $\Pi^{\text{QCD}}(Q^2)$ (case A), in order to recover the earlier results. In this case, the imaginary part of $\Pi^{\text{QCD}}(Q^2)$, however, becomes negative for s below $3.9 \, \text{GeV}^2$, and the full interaction plays the role of a repulsive potential in the pseudoscalar channel. This

Table 1 The optimal fitting values of the mass m, width Γ, coupling constant f, continuum threshold s_0, $\hat{s}$, and matching measure δ for the possible 0^{-+} resonances in the sum rule window $[\tau_{\min}, \tau_{\max}]$ for the best matching between l.h.s. and r.h.s. of the sum rules (42) with $k = -1, 0, 1$ are listed, where in case A the correlation function $\Pi^{\mathrm{QCD}}(Q^2)$ contains only a pure perturbative contribution and a pure instanton one, and a single zero-width resonance plus continuum model is adopted for the spectral function, while all the contributions arising from a pure instanton, pure perturbative, and interference between both are included in the correlation function for cases B, C, and D, in which a single zero-width resonance plus continuum model of the spectral function is adopted for case B, and a single finite-width resonances plus continuum model for case C, and the five finite-width resonances plus continuum model for case D, respectively

| Case	k	Resonances	$\sqrt{\hat{s}}$ (GeV)	m (GeV)	Γ (GeV)	f (GeV)	s_0 (GeV^2)	$[\tau_{\min}, \tau_{\max}]$ (GeV^4)	$\delta/10^{-5}$
A	-1		2.050	1.950 ± 0.050	0	0.561	5.26 ± 0.14	$[0.8, 4.0]$	$22.60 + 1.46$
	0		2.100	2.000 ± 0.035	0	0.599	5.18 ± 0.12	$[0.5, 6.0]$	$1.99 + 0.13$
	1		2.110	2.080 ± 0.040	0	0.592	5.20 ± 0.10	$[0.8, 6.0]$	$7.56 + 0.45$
B	-1		1.405	1.682 ± 0.023	0	1.457	5.24 ± 0.15	$[1.0, 3.0]$	$49.72 + 2.98$
	0		1.490	1.620 ± 0.027	0	1.387	4.61 ± 0.09	$[0.8, 2.5]$	$3.67 + 0.18$
	1		1.380	1.631 ± 0.031	0	1.392	4.45 ± 0.11	$[1.4, 3.2]$	$8.97 + 0.43$
C	-1		1.405	1.405 ± 0.024	0.05	1.630	5.25 ± 0.14	$[0.8, 4.0]$	$2.44 + 0.12$
	0		1.350	1.400 ± 0.025	0.08	1.671	4.45 ± 0.12	$[0.5, 4.3]$	$4.47 + 0.21$
	1		1.380	1.416 ± 0.024	0.03	1.760	4.19 ± 0.11	$[0.5, 4.5]$	$2.19 + 0.11$
D	-1	$\eta(548)$	1.405	0.548 ± 0.008	1.3×10^{-6}	1.100	5.25 ± 0.12	$[0.5, 4.5]$	$4.88 + 0.23$
		$\eta(958)$		0.958 ± 0.014	1.9×10^{-3}	1.100			
		$\eta(1295)$		1.295 ± 0.020	0.055	1.200			
		$\eta(1405)$		1.405 ± 0.021	0.051	1.330			
		$\eta(1475)$		1.475 ± 0.023	0.085	1.010			
	0	$\eta(548)$	1.500	0.548 ± 0.009	1.3×10^{-6}	1.200	4.79 ± 0.11	$[0.5, 4.3]$	$3.38 + 0.17$
		$\eta(958)$		0.958 ± 0.016	1.9×10^{-3}	1.300			
		$\eta(1295)$		1.295 ± 0.021	0.055	1.195			
		$\eta(1405)$		1.405 ± 0.022	0.051	1.300			
		$\eta(1475)$		1.475 ± 0.025	0.085	1.011			
	1	$\eta(548)$	1.405	0.548 ± 0.010	1.3×10^{-6}	1.100	4.30 ± 0.10	$[1.0, 4.0]$	$3.56 + 0.17$
		$\eta(958)$		0.958 ± 0.017	1.9×10^{-3}	1.200			
		$\eta(1295)$		1.295 ± 0.022	0.055	1.210			
		$\eta(1405)$		1.405 ± 0.023	0.051	1.310			
		$\eta(1475)$		1.475 ± 0.024	0.085	1.050			

is the reason why the authors in Ref. [39] cannot find the signal for the pseudoscalar glueball. When we choose to work in the positively definite region, say $s \in (4, 10)$ GeV^2, the mass of the 0^{-+} glueball can be worked out by using the family of Gaussian sum rules (42). The fitting parameters are listed in the first three lines of Table 1 and the corresponding matching curves for $k = -1, 0$, and $+1$ are displayed in Fig. 4, respectively. The optical values of mass, coupling constant, and s_0 of the 0^{-+} pseudoscalar glueball are

$$m = 2.010 \pm 0.299 \, \text{GeV}, \quad f = 0.584 \pm 0.043 \, \text{GeV},$$
$$s_0 = 5.21 \pm 0.43 \, \text{GeV}^2, \tag{64}$$

where the errors are estimated from the uncertainties of the spread between the individual sum rules, and by varying the value of Λ in the region of

$$\Lambda = 120 \sim 200 \, \text{MeV}, \tag{65}$$

as assumed hereafter. The mass values of the 0^{-+} pseudoscalar glueball in (64) are reasonably consist with the one obtained in Ref. [7] by adding the topological charge screening effect and performing the so-called Gaussian-tail distribution for the instanton size. However, after performing the Gaussian-tail distribution for the instanton size, the mass of the 0^{++} glueball is lower, around 1.25 GeV [7], in contradiction with the lattice simulation [45,64] and phenomenology [65]. The mass scales in (64) are located at the strong repulsive potential region of the energy (below 3.9 GeV^2), where the bound state of a glueball cannot form when working in the spike distribution, which is motivated from the liquid instanton model of QCD vacuum in the large N_c limit. In fact, the fundamental spectral positivity bound can be traced back to the definition of the correlation function (3). The spectral function should be positive even before taking the average with any specific instanton-size distribution. It is difficult for us to understand that there is no artifice in changing the posi-
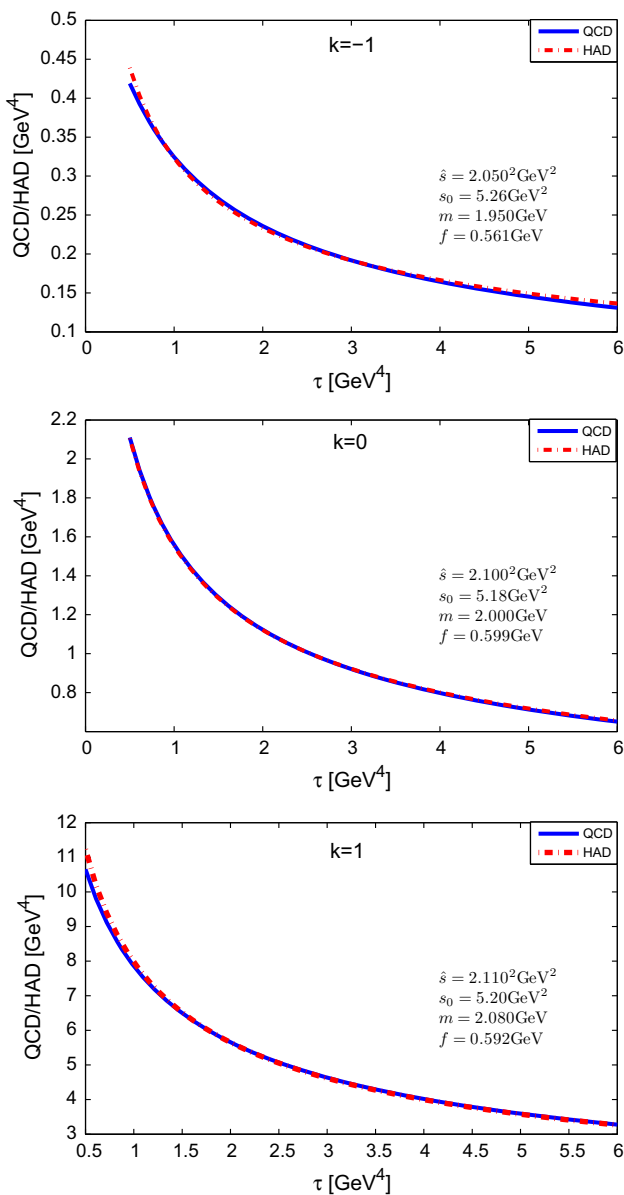
Fig. 4 The l.h.s. (*dashed line*) and r.h.s. (*solid line*) of the sum rules (42) with $k = -1, 0, 1$ versus τ in the case where the correlation function $\Pi^{\mathrm{QCD}}(Q^2)$ contains only a pure perturbative contribution and a pure instanton one, and a single zero-width resonance plus continuum model is adopted for the spectral function

tivity behavior of the spectral function just by performing the average with the Gaussian-tail distribution for the instanton size.

From now on, the full correlation function $\Pi^{\mathrm{QCD}}(Q^2)$ including the interference contribution $\Pi^{\mathrm{int}}(Q^2)$, (23), is used in our analysis of the Gaussian sum rules (42). In the case of single-resonance plus continuum models, specified, respectively, by (34) and (38), for the spectral function, the optimal parameters governing the sum rules with zero (case B) and finite (case C) widths are listed from the fourth to the

ninth line of Table 1 and the corresponding curves for the l.h.s. and r.h.s. of (42) with $k = -1, 0$, and $+1$ are displayed in Figs. 5 and 6, respectively. From Table 1, the optical values of the pseudoscalar glueball mass, width, coupling, and the duality threshold with the best matching are

$$m = 1.644 \pm 0.194\,\mathrm{GeV}, \quad f = 1.412 \pm 0.129\,\mathrm{GeV},$$
$$s_0 = 4.77 \pm 0.74\,\mathrm{GeV}^2, \tag{66}$$

for the one zero-width resonance model, and

$$m = 1.407 \pm 0.162\,\mathrm{GeV}, \quad \Gamma = 0.053 \pm 0.018\,\mathrm{GeV}$$
$$f = 1.687 \pm 0.145\,\mathrm{GeV}, \quad s_0 = 4.63 \pm 0.62\,\mathrm{GeV}^2, \tag{67}$$

for one finite-width resonance model. It is shown in Fig. 5 that the topological charge screening effect has little impact on Gaussian sum rules indeed.

In the numerical simulation for the case of the five finite-width resonances plus continuum model (39) for the spectral function (case D), we just choose the data in PDG as the fitting parameters for masses and width of the resonances $\eta(548)$, $\eta(985)$, $\eta(1295)$, and $\eta(1475)$, and the result of the single-resonance model (case C) as the fitting parameters for $\eta(1405)$; while the couplings of the five resonances to the current are chosen to be approximately the same as that for $\eta(1405)$ determined in case C because the pseudoscalar quarkonia can be directly coupled to the gluon anomaly, and, as a consequence, all five resonances should be coupled to the current with the strengths of almost the same magnitude of degree; finally, the optimal parameters are determined by adjusting the chosen parameters so that the matching measure δ for both sides of the Gaussian sum rules (42) is minimal. The optimal parameters governing the sum rules are listed in the remaining lines of Table 1. The corresponding curves for the l.h.s. and r.h.s. of (42) with $k = -1, 0$, and $+1$ are displayed in Fig. 7. Taking the average, the optical values of the widths of the five lowest 0^{-+} resonances in the world of QCD with three massless quarks and the corresponding optical fit parameters are predicted to be

$$m_{\eta(548)} = 0.548 \pm 0.022\,\mathrm{GeV},$$
$$f_{\eta(548)} = 1.133 \pm 0.167\,\mathrm{GeV},$$
$$\Gamma_{\eta(548)} = 1.3 \times 10^{-6} \pm 3.9 \times 10^{-8}\,\mathrm{GeV}, \tag{68}$$
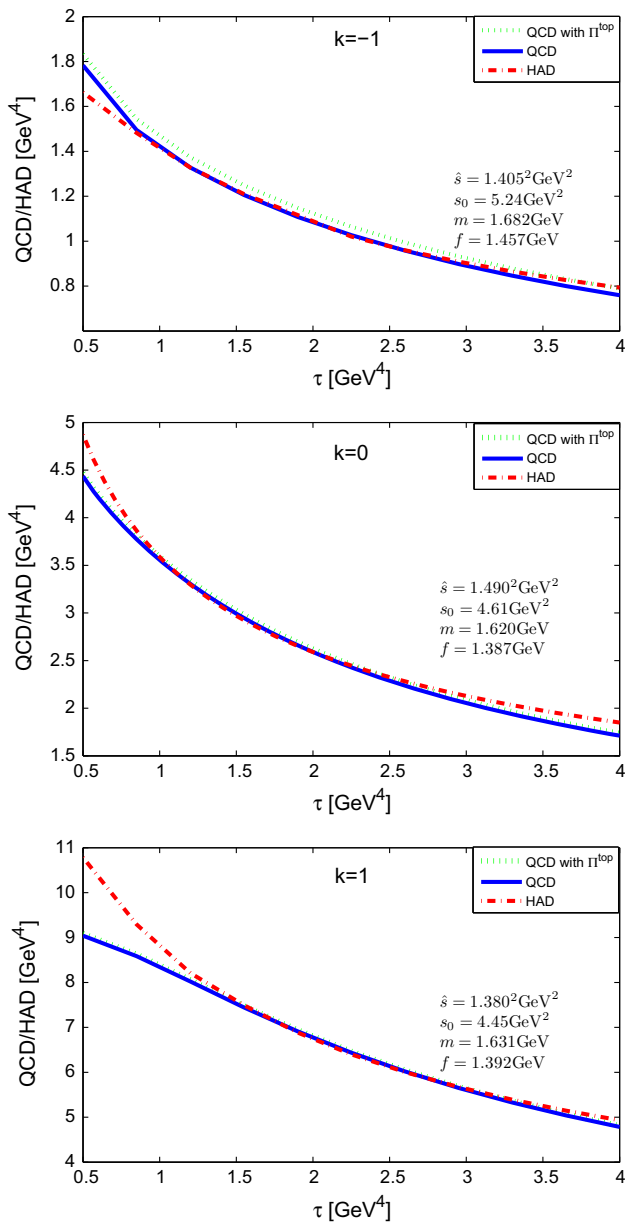
$$m_{\eta(985)} = 0.958 \pm 0.051\,\mathrm{GeV},$$
$$f_{\eta(985)} = 1.200 \pm 0.233\,\mathrm{GeV},$$
$$\Gamma_{\eta(985)} = 1.9 \times 10^{-3} \pm 5.7 \times 10^{-5}\,\mathrm{GeV}, \tag{69}$$

$$m_{\eta(1295)} = 1.295 \pm 0.075\,\mathrm{GeV},$$
$$f_{\eta(1295)} = 1.202 \pm 0.112\,\mathrm{GeV},$$

Fig. 5 The l.h.s. without the topological charge screening contribution (*dashed line*), the l.h.s. with the topological charge screening contribution (*dot line*) and the r.h.s. (*solid line*) of the sum rules (42) with $k = -1, 0, 1$ versus τ in the case where the interference contribution is included in the correlation function $\Pi^{\mathrm{QCD}}(Q^2)$, and a single zero-width resonance plus continuum model is adopted for the spectral function

$$\Gamma_{\eta(1295)} = 0.055 \pm 0.018 \, \mathrm{GeV}, \tag{70}$$

$$m_{\eta(1405)} = 1.405 \pm 0.081 \, \mathrm{GeV},$$
$$f_{\eta(1405)} = 1.313 \pm 0.105 \, \mathrm{GeV},$$
$$\Gamma_{\eta(1405)} = 0.051 \pm 0.017 \, \mathrm{GeV}, \tag{71}$$

$$m_{\eta(1475)} = 1.475 \pm 0.092 \, \mathrm{GeV},$$
$$f_{\eta(1475)} = 1.023 \pm 0.097 \, \mathrm{GeV},$$
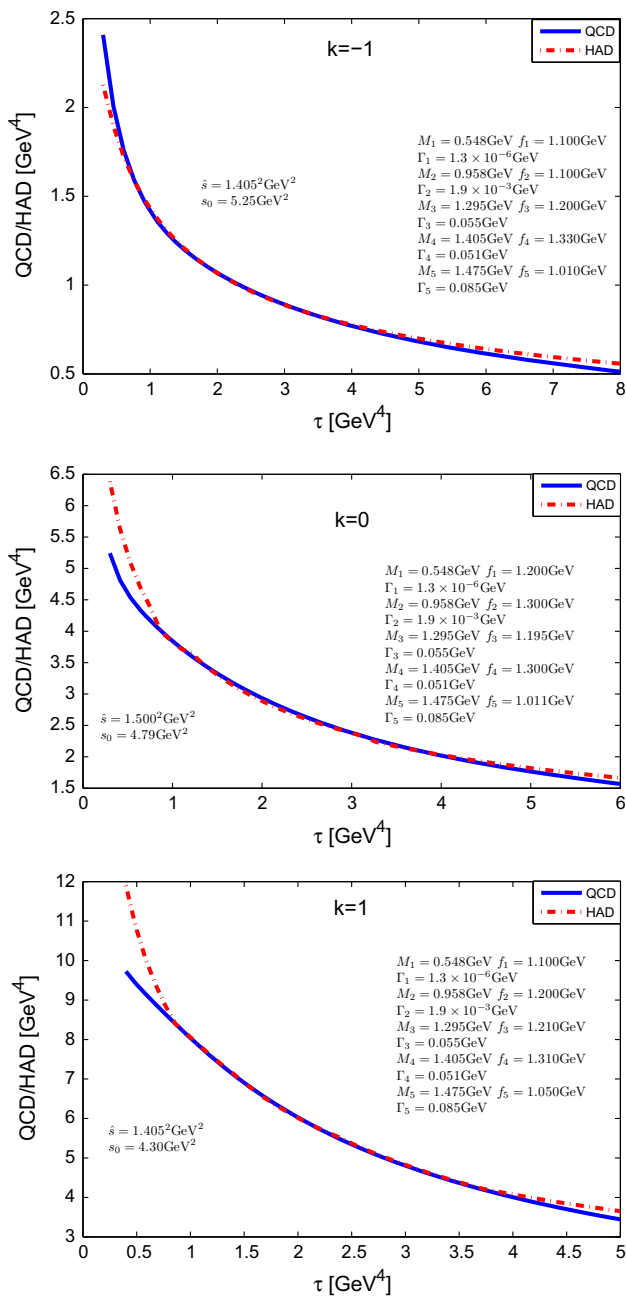$$\Gamma_{\eta(1475)} = 0.085 \pm 0.028 \, \mathrm{GeV}, \tag{72}$$
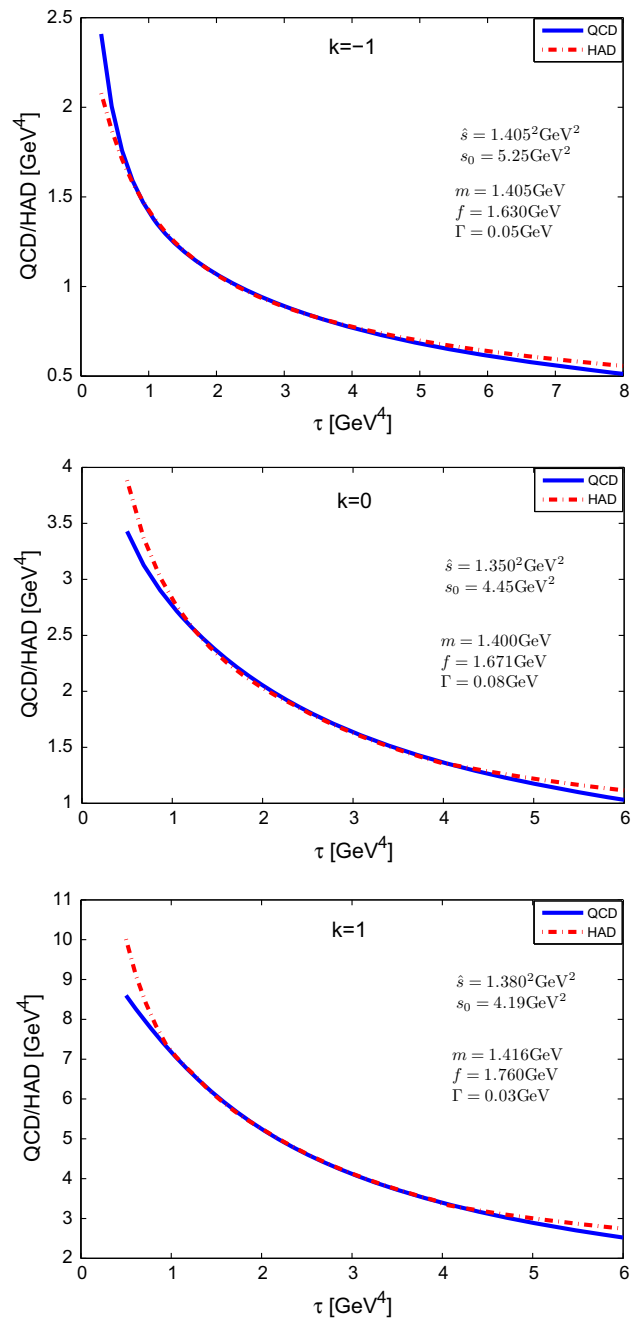
Fig. 6 The l.h.s. (*dashed line*) and r.h.s. (*solid line*) of the sum rules (42) with $k = -1, 0, 1$ versus τ in the case where the correlation function $\Pi^{\mathrm{QCD}}(Q^2)$ contains the pure instanton, interference, and pure perturbative contributions, and a single finite-width resonance plus continuum model is adopted for the spectral function

with

$$s_0 = 4.78 \pm 0.64 \, \mathrm{GeV}^2. \tag{73}$$

Figures 6 and 7 show the satisfactory compatibility between both sides of the sum rules over the whole fiducial region. These results are well in accordance with the experimental discovered resonance [14]

Fig. 7 The l.h.s. (*dashed line*) and the r.h.s. (*solid line*) of the sum rules (42) with $k = -1, 0, 1$ versus τ in the case where the correlation function $\Pi^{\mathrm{QCD}}(Q^2)$ contains the pure instanton, interference, and pure perturbative contributions, and the five finite-width resonances plus continuum model is adopted for the spectral function

$$m_{\eta(1405)} = 1409.8 \pm 2.5 \,\mathrm{MeV},$$
$$\Gamma_{\eta(1405)} = 51.1 \pm 3.4 \,\mathrm{MeV}. \tag{74}$$

6 Discussion and conclusion

The instanton–gluon interference and its role in finite-width Gaussian sum rules for the 0^{-+} pseudoscalar glueball are

analyzed in this paper. Our main results can be summarized as follows:

First, the contribution to the correlation function arising from the interference between the classical instanton fields and the quantum gluon ones is reexamined and derived in the framework of the a semiclassical expansion of the instanton liquid vacuum model of QCD. The resultant expression is gauge-invariant, and free of the infrared divergence, and it differs from our previous one not only in some coefficients but also in the logarithm structures [37,38]. Its magnitude is just between the larger contribution from pure classical instanton configurations and the smaller one from the pure quantum fields, and it plays a great role in sum rule analysis in accordance with the spirit of a semiclassical expansion. The imaginary part of the correlation function including this interference contribution turns out to be positive without including the topological charge screening effect, which is proven to be smaller than the perturbative contribution in the fiducial sum rule window and negligible in comparison with the interference effect. The so-called problem of positivity violations in the imaginary part of the correlation function, stressed by Forkel [7], disappears. Moreover, in the correlation function the traditional condensate contribution is excluded to avoid double counting [7], because condensates can be reproduced by the instanton distributions [25–28]; another reason to do so is that the usual condensate contribution is proven to be unusually weak, and it cannot fully reflect the non-perturbative nature of the low-lying gluonia [7,37,38,66]. In our opinion, the condensate contribution can be considered as a small fraction of the corresponding instanton one, so it is naturally taken into account already.

Second, the properties of the lowest lying 0^{-+} pseudoscalar glueball are systematically investigated in a family of Gaussian sum rules in five different cases. In case A, the correlation function $\Pi^{\mathrm{QCD}}(Q^2)$ contains only a pure perturbative contribution and a pure instanton one, and a single zero-width resonance plus continuum model is adopted for the spectral function, and, of course, the old results are recovered (even excluding the topological charge screening contribution), and some pathology is explored. To go beyond the above constraint, all the contributions arising from pure instanton, pure perturbative, and interference between both are included in the correlation function for cases B, C, and D, in which a single zero-width resonance plus continuum model of the spectral function is adopted for case B, and a single finite-width resonances plus continuum model for case C, and the five finite-width resonances plus continuum model for case D, respectively. The optimal fitting values of the mass m, width Γ, coupling constant f, continuum threshold s_0 for the possible 0^{-+} resonances are obtained, and quite consistent with each other. The main difference between this work and our previous one [37,38] is as follows. For the spectral function of the considered resonances,

instead of the zero-width approximation of one gluonic resonance plus other low-lying quark–antiquark ones, the finite-width Breit–Wigner form with a correct threshold behavior for the lowest five resonances with the same quantum numbers is used in this work. This is done in order to compare with the phenomenology. The resultant Gaussian sum rules with $k = -1, 0, +1$ are carried out with a few of the QCD standard input parameters, and it really is in accordance with the experimental data.

Let us now identify where the lowest lying 0^{-+} pseudoscalar glueball is. The results of the single-resonance plus continuum models B and C, namely Eqs. (66) and (67), imply that the meson $\eta(1405)$ may be the most favored candidate for the lowest lying 0^{-+} pseudoscalar glueball, because the difference between the two models is just the width of the resonances, and this is of course believed to be more in accordance with reality. This conclusion can further be justified by the result of the five-resonances plus continuum model, namely Eqs. (71), (72), and (73). Note that the first two resonances $\eta(548)$ and $\eta(985)$ are far away from the mass scale of $\eta(1405)$, and they usually are considered as the superposition of the fundamental flavor-singlet and octet pseudoscalar mesons composed of a quark–antiquark pair, thus having a dominant role as regards responsibility for the axial anomaly [67–69]. In order to explore the structures of the remaining three resonances $\eta(1295)$, $\eta(1405)$, and $\eta(1475)$, we would like to use the η–η'–G mixing formalism based on the anomalous Ward identity for the transition matrix elements [12,49] to relate the physical states η, η', and G to the fundamental flavor-singlet and octet quark–antiquark mesons and the lowest pure gluon state through a rotation,

$$\begin{pmatrix} \langle 0|\hat{O}_p|\eta_{1295}\rangle \\ \langle 0|\hat{O}_p|\eta_{1475}\rangle \\ \langle 0|\hat{O}_p|\eta_{1405}\rangle \end{pmatrix} = U \begin{pmatrix} \langle 0|\hat{O}_p|\eta_8\rangle \\ \langle 0|\hat{O}_p|\eta_1\rangle \\ \langle 0|\hat{O}_p|G\rangle \end{pmatrix}, \tag{75}$$

where the U is the mixing matrix [12,49]

$$U = \begin{pmatrix} \cos\varphi_p & -\sin\varphi_p & 0 \\ \cos\phi_G \sin\varphi_p & \cos\varphi_p \sin\phi_G & \sin\phi_G \\ -\cos\phi_G \sin\varphi_p & -\cos\varphi_p \sin\phi_G & \cos\phi_G \end{pmatrix} \tag{76}$$

where $\varphi_p \approx 40°$ and $\phi_G \approx 22°$ is the η–η' [49] mixing angle and the mixing angle of the pseudoscalar glueball with η'. Then one has

$$U = \begin{pmatrix} 0.766044 & -0.642788 & 0 \\ 0.595982 & 0.286965 & 0.374607 \\ -0.595982 & -0.286965 & 0.927184 \end{pmatrix}. \tag{77}$$

To be quantitative, the corresponding normalized couplings F to the three resonances η, η', and G with masses 1.295, 1.475, 1.405 GeV can be read from Table 1 to be

$$0.51 \,\text{GeV}^3, \quad 0.65 \,\text{GeV}^3, \quad 0.56 \,\text{GeV}^3, \tag{78}$$

after normalization, respectively. As a relatively rough estimation, from Eq. (78) the values of the couplings of $\hat{O}_p$ to the states η_1, η_8, and pseudoscalar glueball G are obtained,

$$\langle 0|\hat{O}_p|\eta_1\rangle = -0.117 \,\text{GeV}^3,$$
$$\langle 0|\hat{O}_p|\eta_8\rangle = 0.568 \,\text{GeV}^3,$$
$$\langle 0|\hat{O}_p|G\rangle = 0.813 \,\text{GeV}^3, \tag{79}$$

by reversing Eq. (75). This shows that the coupling of the 0^{-+} glueball current to the pure glueball state G is dominant, and the signs of the couplings $\langle 0|\hat{O}_p|\eta_1\rangle$ and $\langle 0|\hat{O}_p|\eta_8\rangle$ are similar to those predicted by the scalar glueball–meson coupling theorems [58,70].

Furthermore, $\eta(1405)$ and $\eta(1475)$ could originate from a single pole [see Amsler and Masoni's review for eta(1405) in PDG]. To check whether the single pseudoscalar meson assumption may be consistent with our sum rule approach, or the dependence of the results on the model selected for the spectral function, we add an analysis of the four-resonance model for the spectral function. The result is shown in Appendix F. From Fig. 10 and Table 2 it is easy to see that the four-resonance model for the spectral function is inconsistent with our sum rule analysis, because, first, the resultant mass scale of the fourth resonance is approximately 1.41 GeV in average, which is located outside of the sum rule window; second, the matching degree δ of the four-resonance model is obviously worse than the δ of the five-resonance model.

In summary, our result suggests that $\eta(1405)$ is a good candidate for the lowest 0^{-+} pseudoscalar glueball with some mixture with the nearby excited isovector and isoscalar $q\bar{q}$ mesons. This is a first theoretical support for the phenomenological estimation from the sum rule approach.

Appendix A: Pure instanton and perturbative contribution

In the instanton liquid model, the pure instanton contribution of 0^{-+} pseudoscalar glueball with a spike instanton distribution in Euclidean space-time is known [7,71–73],

$$\Pi_E^{(\text{cl})}(Q^2) = -2^5 \pi^2 \bar{n} \bar{\rho}^4 Q^4 K_2^2(Q\bar{\rho}) \tag{80}$$

where K_2^2 is the McDonald function and $n(\rho)$ is the size distribution of instantons, and the appearance of the overall minus sign in the r.h.s. of (80) is due to the anti-hermitian property of the current, i.e. $O_p^\dagger = -O_p$ in Euclidean space-time.

The second part of (10) has already been calculated up to three-loop level in the $\overline{\text{MS}}$ dimensional regularization scheme [11,71,74]

$$\Pi^{(\text{qu})}(Q^2) = \left(\frac{\alpha_s}{\pi}\right)^2 Q^4 \ln \frac{Q^2}{\mu^2} \left[a_0 + a_1 \ln \frac{Q^2}{\mu^2} \right. \\ \left. + a_2 \ln^2 \frac{Q^2}{\mu^2} \right], \tag{81}$$

where μ is the renormalization scale, and the coefficients with the inclusion of the correct threshold effect are

$$a_0 = -2 \left[1 + 20.75 \left(\frac{\alpha_s}{\pi}\right) + 305.95 \left(\frac{\alpha_s}{\pi}\right)^2 \right],$$

$$a_1 = 2 \left(\frac{\alpha_s}{\pi}\right) \left[\frac{9}{4} + 72.531 \left(\frac{\alpha_s}{\pi}\right) \right],$$

$$a_2 = -10.1250 \left(\frac{\alpha_s}{\pi}\right)^2 \tag{82}$$

for QCD with three massless quark flavors up to three-loop level.

Appendix B: Topological charge screening

The topological charge screening effect can be understood by tracing back to the anomaly of the axial-vector current $J_\mu^5 = \Sigma_i \bar{\psi}_i \gamma_5 \gamma_\mu \psi_i$ with ψ_i being the quark field of flavor i

$$\partial_\mu J_\mu^5 = 2N_f Q(x) \tag{83}$$

where the topological charge current $Q(x)$ relates to the pseudoscalar glueball current by $Q(x) = O_p(x)/(8\pi)$. Equation (83) indicates that instantons generate quark–antiquark pairs, which then may form a light meson to mediate the long-range multi-instanton interaction [16]. Kikuchi and Wudka [75] suggest that such a light meson, which can couple to the instanton in such a way that the instanton generates all light quark flavors with identical probability, being the meson η_0, and that it leads to an effective Lagrangian (see also [76,77]) which gives rise to the topological charge screening contribution Π^{top} to the correlation function of the pseudoscalar current [7],

$$\Pi^{\text{top}}(Q^2) = \frac{F_\eta^2}{Q^2 + m_\eta^2} + \frac{F_{\eta'}^2}{Q^2 + m_{\eta'}^2}, \tag{84}$$

where m_η and $m_{\eta'}$ are the masses of the mesons η and η', and the two constants F_η and $F_{\eta'}$ are evaluated to be $16\pi\bar{n}\xi \sin\phi$ and $16\pi\bar{n}\xi \cos\phi$; and ξ and $\phi \approx 22°$ are the coupling strength of η_0 to an instanton and the η–η' mixing angle, respectively.

Appendix C: The operators $O_i(x)$

The operators O_i in terms of instanton and quantum gluon fields are

$$O_1(x) = F_{\mu\nu,a}[A(x)]F_{\rho\sigma,a}[A(x)],$$
$$O_2(x) = 4F_{\mu\nu,a}[A(x)](\partial_\rho a_{\sigma a}[A(x)]),$$
$$O_3(x) = 4g_s f_{abc} F_{\mu\nu,a}[A(x)]A_{\rho b}(x)a_{\sigma c}(x),$$
$$O_4(x) = 4(\partial_\mu a_{\nu a}(x))(\partial_\rho a_{\sigma a}(x)),$$
$$O_5(x) = 8g_s f_{abc} A_{\rho b}(x)(\partial_\mu a_{\nu a}(x))a_{\sigma c}(x),$$
$$O_6(x) = 4g_s^2 f_{abc} f_{ade} A_{\mu b}(x)A_{\rho d}(x)a_{\nu c}(x)a_{\sigma e}(x),$$
$$O_7(x) = 2g_s f_{abc} F_{\mu\nu,a}[A(x)]a_{\rho b}(x)a_{\sigma c}(x),$$
$$O_8(x) = 4g_s f_{abc} a_{\mu b}(x)a_{\nu c}(x)(\partial_\rho a_{\sigma a}(x)),$$
$$O_9(x) = 4g_s^2 f_{abc} f_{ade} A_{\rho d}(x)a_{\mu b}(x)a_{\nu c}(x)a_{\sigma e}(x),$$
$$O_{10}(x) = g_s^2 f_{abc} f_{ade} a_{\mu b}(x)a_{\nu c}(x)a_{\rho d}(x)a_{\sigma e}(x), \tag{85}$$

where $F_{\mu\nu,a}[A(x)]$ is the instanton field strength associated with the instanton field A.

Appendix D: The interference contributions $\Pi_i^{(\text{int})}(q^2)$

The expressions of the interference contributions $\Pi_i^{(\text{int})}(q^2)$ in terms of O_i are

$$\Pi_1^{(\text{cl}+\text{qu})}(q^2) = \hat{T}\langle\Omega|O_2(x)O_2(0)|\Omega\rangle,$$
$$\Pi_2^{(\text{cl}+\text{qu})}(q^2) = \hat{T}\langle\Omega|O_2(x)O_3(0)|\Omega\rangle,$$
$$\Pi_3^{(\text{cl}+\text{qu})}(q^2) = \hat{T}\langle\Omega|O_3(x)O_3(0)|\Omega\rangle,$$
$$\Pi_4^{(\text{cl}+\text{qu})}(q^2) = \hat{T}\langle\Omega|O_4(x)O_5(0)|\Omega\rangle,$$
$$\Pi_5^{(\text{cl}+\text{qu})}(q^2) = \hat{T}\langle\Omega|O_4(x)O_6(0)|\Omega\rangle,$$
$$\Pi_6^{(\text{cl}+\text{qu})}(q^2) = \hat{T}\langle\Omega|O_4(x)O_7(0)|\Omega\rangle,$$
$$\Pi_7^{(\text{cl}+\text{qu})}(q^2) = \hat{T}\langle\Omega|O_5(x)O_5(0)|\Omega\rangle,$$
$$\Pi_8^{(\text{cl}+\text{qu})}(q^2) = \hat{T}\langle\Omega|O_7(x)O_7(0)|\Omega\rangle,$$
$$\Pi_9^{(\text{cl}+\text{qu})}(q^2) = \hat{T}\langle\Omega|O_5(x)O_7(0)|\Omega\rangle,$$
$$\Pi_{10}^{(\text{cl}+\text{qu})}(q^2) = \hat{T}\langle\Omega|O_6(x)O_7(0)|\Omega\rangle,$$
$$\Pi_{11}^{(\text{cl}+\text{qu})}(q^2) = \hat{T}\langle\Omega|O_5(x)O_6(0)|\Omega\rangle,$$
$$\Pi_{12}^{(\text{cl}+\text{qu})}(q^2) = \hat{T}\langle\Omega|O_6(x)O_6(0)|\Omega\rangle, \tag{86}$$

where

$$\hat{T} \equiv -\frac{1}{2}\alpha_s^2 \bar{n}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu'\nu'\rho'\sigma'} \int d^4z \int d^4x. \tag{87}$$

Appendix E: Comparison between the condensate contributions and the instanton–induced ones

The contributions arising from condensates up to the eighth dimensions are known to be as follows [7]:
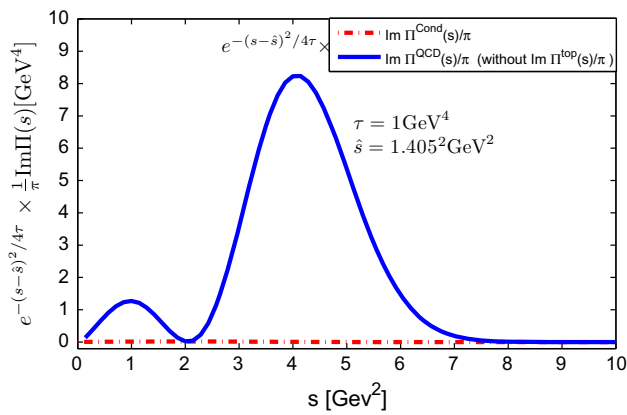
Fig. 8 The contributions to the imaginary part of the correlation function from the condensate (*dashed–dotted line*) and the total contribution (*solid line*) versus s
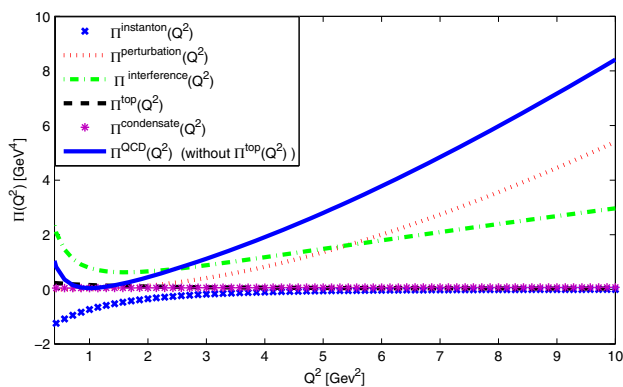
Fig. 9 The contributions to the correlation function arising from the pure instanton (*cross line*), interference (*dashed–dotted line*), pure perturbative (*dotted line*), topological charge screening (*dashed line*), condensates (*star line*) and the total contribution without the topological charge screening one (*solid line*) versus Q^2

$$\Pi^{\mathrm{cond}}(Q^2) = 4\alpha_s \langle \alpha_s G^2 \rangle + \frac{9}{\pi} \alpha_s^2 \langle \alpha_s G^2 \rangle \ln \frac{Q^2}{\mu^2}$$
$$- 8\alpha_s^2 \langle gG^3 \rangle \frac{1}{Q^2} + \frac{15\pi}{2} \alpha_s^2 \langle \alpha_s G^2 \rangle^2 \frac{1}{Q^4}, \quad (88)$$

where

$$\langle \alpha_s G^2 \rangle = 0.05\,\mathrm{GeV}^4,$$
$$\langle gG^3 \rangle = 0.27\,\mathrm{GeV}^2 \langle \alpha_s G^2 \rangle, \quad (89)$$

The imaginary part of condensates (88) has the form

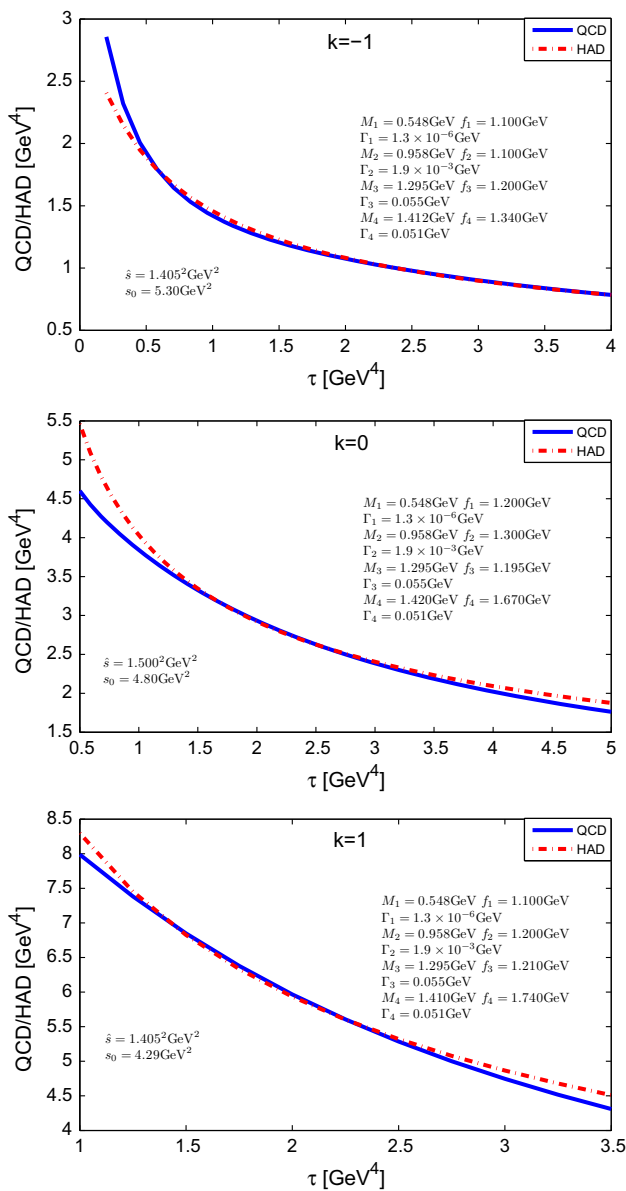$$\frac{1}{\pi} \mathrm{Im} \Pi^{\mathrm{Cond}}(s) = -\frac{9}{\pi} \alpha_s^2 \langle \alpha_s G^2 \rangle - 8\alpha_s^2 \langle gG^3 \rangle \delta(s)$$
$$+ \frac{15\pi}{2} \alpha_s^2 \langle \alpha_s G^2 \rangle^2 \delta'(s). \quad (90)$$

The comparison between the imaginary part of the correlation function, (24), and the condensate contribution to it are shown in Fig. 8, while the comparison between the various real parts of (23) (which altogether are related with the imaginary part by the dispersion relation (40)) and the condensate contribution to it are shown in Fig. 9.

Appendix F: The results for the four finite-width resonance model from the Gaussian sum rules

The optimal parameters of the four finite-width resonance model for the spectral function are listed in Table 2, and the corresponding plots are shown in Fig. 10.

Table 2 For the four finite-width resonances plus continuum model, the optimal fitting values of the mass m, width Γ, coupling constant f, continuum threshold s_0, $\hat{s}$, and matching measure δ for the possible 0^{-+} resonances in the sum rule window $[t_{\min}, t_{\max}]$ ($t = \tau^{-1}$) for the best matching between the l.h.s. and the r.h.s. of the sum rules with $k = -1, 0, 1$ are listed

k	Resonances	$\sqrt{\hat{s}}$ (GeV)	m (GeV)	Γ (GeV)	f (GeV)	s_0 (GeV^2)	$[t_{\min}, t_{\max}]$ (GeV^{-4})	$\delta/10^{-4}$
-1	$\eta(548)$	1.405	0.548	1.3×10^{-6}	1.100	5.30	[0.25, 1.00]	2.3
	$\eta(958)$		0.958	1.9×10^{-3}	1.100			
	$\eta(1295)$		1.295	0.055	1.200			
			1.412	0.051	1.340			
0	$\eta(548)$	1.500	0.548	1.3×10^{-6}	1.200	4.80	[0.29, 0.77]	2.2
	$\eta(958)$		0.958	1.9×10^{-3}	1.300			
	$\eta(1295)$		1.295	0.055	1.195			
			1.420	0.051	1.670			
1	$\eta(548)$	1.405	0.548	1.3×10^{-6}	1.100	4.29	[0.42, 0.83]	9.3
	$\eta(958)$		0.958	1.9×10^{-3}	1.200			
	$\eta(1295)$		1.295	0.055	1.210			
			1.410	0.051	1.740			

Fig. 10 The l.h.s. (*dashed line*) and r.h.s. (*solid line*) of the sum rules (43) with $k = -1, 0, 1$ versus τ in the case where the correlation function $\Pi^{QCD}(Q^2)$ contains the pure instanton, interference, and pure perturbative contributions, and the four finite-width resonance plus continuum model is adopted for the spectral function

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