

# Evolution of axially symmetric anisotropic sources in $f(R, T)$ gravity

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**Abstract** We discuss the dynamical analysis in  $f(R, T)$  gravity (where  $R$  is the Ricci scalar and  $T$  is the trace of the energy momentum tensor) for gravitating sources carrying axial symmetry. The self-gravitating system is taken to be anisotropic and the line element describes an axially symmetric geometry avoiding rotation about the symmetry axis and meridional motions (zero vorticity case). The modified field equations for axial symmetry in  $f(R, T)$  theory are formulated, together with the dynamical equations. Linearly perturbed dynamical equations lead to the evolution equation carrying the adiabatic index  $\Gamma$ , which defines the impact of a non-minimal matter to geometry coupling on the range of instability for Newtonian and post-Newtonian approximations.

## 1 Introduction

Recent developments in astrophysics and structure formation theories reveal that gravitating sources might deviate from the most commonly studied spherical symmetry. Such deviations in realistic scenarios appear incidentally, giving rise to the importance of non-spherical symmetries in gravitating objects. Herein, we intend to look into the implications of the restricted class of axially symmetric sources (avoiding reflection and rotation) on the gravitational evolution in the context of the  $f(R, T)$  theory of gravity. Consideration of the dynamic sources together with the angular momentum is a cumbersome task; however, observational data suggests that the lack of spherical symmetry prevails in the more practical and interesting situations. A viable  $f(R, T)$  model ( $\frac{df}{dr} \geq 0$ ,  $\frac{d^2f}{dr^2} \geq 0$ ) with a locally anisotropic matter distribution has been taken into account for the dynamical analysis.

The evolution of gravitating sources has been studied with a great deal of interest in the recent past. Stars tend to collapse when outward drawn pressure decreases because of continuous fuel consumption, leading to an imbalance in outward forces and inwardly acting gravitational pull. In such a situation, the gravitational force becomes the only governing force, massive stars burn nuclear fuel more rapidly, and so more become unstable as compared to the stars with relatively less mass. There are many factors other than the mass of the gravitating source that implicate intense modifications in the range of stability/instability such as isotropy, anisotropy, shear, dissipation, and radiation. Chandrasekhar [1] presented valuable explorations to set the instability range for spherically symmetric gravitating source in terms of the adiabatic index  $\Gamma$  comprising the pressure to density ratio with the time transition.

Hillebrandt and Steinmetz [2] presented the instability criterion for an anisotropic matter configuration of gravitating objects. Herrera et al. [3–7] published a major contribution to the establishment of the instability range of general relativistic fluids for different cases (isotropic, anisotropic, dissipative collapse etc.); they remarked that the pressure anisotropy largely participates in setting the dynamical instability. Moreover, they also worked out the imprints of axially and reflection symmetric static and dynamic sources by a general framework and some analytic models [8,9]. Axially symmetric shearing geodesic and shear-free dissipative fluids are discussed in [10,11], where the shearing geodesic case represents the zero radiation production.

General relativity (GR) is a self-consistent theory, it is adequate for the explanation of many gravitational phenomena up to cosmological scales. The scheme of GR appears to disagree with progressing observational data such as large scale structures ranging from galaxies to galaxy clustering, IA-type supernovae, the cosmic microwave background [12–15], etc. Alternatively, it can be said that GR is not the only definite gravitational theory that is suitable for all scales.

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Many attempts have been made to validate gravitational theories on large scales and agree with the cosmic acceleration [16–28], by introducing modified theories of gravity [29–34] for e.g.  $f(R)$ ,  $f(G)$ , Brans–Dicke theory,  $f(R, T)$ , and so on.

Since the introduction of  $f(R, T)$  theory in 2011 [35], people [36–39] worked on energy conditions along with its cosmological and thermodynamic implications. The  $f(R, T)$  theory represents a generalization of  $f(R)$  theory carrying non-minimal matter to geometry coupling. Extensive work has been done on the instability range of spherically symmetric stars in GR as well as in modified theories of gravity. The literature on the dynamical analysis of axially symmetric sources abounds in GR. However, it being a heavier task to handle modified dynamical equations in modified theories, very few attempts have been made to explore the axial symmetry.

The purpose of this manuscript is to work out the instability problem for axially symmetric (in absence of reflection and radiation) anisotropic sources in the context of  $f(R, T)$  gravity. The reason of avoiding reflection and rotation terms in axial symmetry is only to somehow reduce the complications in the analysis. The modified EH action in  $f(R, T)$  admits an arbitrary function of  $R$  and  $T$  to account for the exotic matter. The action in  $f(R, T)$  is given by [35]

$$\int dx^4 \sqrt{-g} \left[ \frac{f(R, T)}{16\pi G} + \mathcal{L}_{(m)} \right], \tag{1.1}$$

where  $\mathcal{L}_{(m)}$  represents the matter Lagrangian and  $g$  represents the metric. Several choices of  $\mathcal{L}_{(m)}$  can be considered, each of which stands for a specific form of fluid.

The article is organized as follows: The matter configuration and components of the field equations together with the dynamical equations are furnished in Sect. 2. Section 3 covers the information as regards the  $f(R, T)$  model and perturbed conservation equations leading to the collapse equation. Section 4 contains the range of stability of Newtonian (N) and post-Newtonian (pN) limits in terms of the adiabatic index. The last section consists of concluding remarks; it is followed by an appendix.

## 2 Interior spacetime and dynamical equations

The general line element for axially symmetric compact objects constituting five independent metric coefficients is given by

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + B^2 r^2 d\theta^2 + C^2 d\phi^2 + 2G dt d\theta \tag{2.2}$$

where the metric functions  $A, B, C, G, H$  have a dependence on time, and on radial and axial coordinates  $(t, r, \theta)$ . Here,

we ignore the meridional motions and rotation about the symmetry axis. The absence of  $dt d\theta$  and  $dt d\phi$  terms leads to a restricted character, i.e. the vorticity-free case. The modified equations are highly non-linear in nature, so it is a tough task to handle such equations with non-diagonal entries in the metric tensor, which is why we have taken the zero vorticity case to somehow manage the dynamical analysis by an analytic approach.

The reduced form of the general axial symmetry with three independent metric functions is [40]

$$ds^2 = -A^2(t, r, \theta) dt^2 + B^2(t, r, \theta) (dr^2 + r^2 d\theta^2) + C^2(t, r, \theta) d\phi^2. \tag{2.3}$$

Taking  $\mathcal{L}_{(m)} = -\rho$  and  $8\pi G = 1$ , varying the action (1.1) with respect to the metric tensor  $g_{uv}$ , leads to the following form for the modified field equations:

$$G_{uv} = \frac{1}{f_R} \left[ (f_T + 1) T_{uv}^{(m)} + \rho g_{uv} f_T + \frac{f - R f_R}{2} g_{uv} + (\nabla_u \nabla_v - g_{uv} \square) f_R \right], \tag{2.4}$$

where  $\square = \nabla^u \nabla_u$ ,  $f_R \equiv df(R, T)/dR$ ,  $f_T \equiv df(R, T)/dT$ ,  $\nabla_u$  is the covariant derivative and  $T_{uv}^{(m)}$  is the energy momentum tensor for the usual matter. The matter configuration is considered to be locally anisotropic [32], as given by

$$T_{uv}^{(m)} = (\rho + p_\perp) V_u V_v - \left( K_u K_v - \frac{1}{3} h_{uv} \right) (P_{zz} - P_{xx}) - \left( L_u L_v - \frac{1}{3} h_{uv} \right) (P_{zz} - P_{xx}) + P g_{uv} + 2K_{(u} L_{v)} P_{xy}, \tag{2.5}$$

where  $\rho$  is the energy density and

$$P = \frac{1}{3} (P_{xx} + P_{yy} + P_{zz}), \quad h_{uv} = g_{uv} + V_u V_v,$$

$P_{xx}, P_{yy}, P_{zz}$  and  $P_{xy}$  are the respective stresses causing pressure anisotropy, provided that  $P_{xy} = P_{yx}$  and  $P_{xx} \neq P_{yy} \neq P_{zz}$ .  $K_u$  and  $L_u$  represent the four vectors in radial and axial directions, respectively, and  $V_u$  is for the four-velocity; these quantities are linked as

$$V_u = -A \delta_u^0, \quad K_u = B \delta_u^1, \quad L_u = r B \delta_u^2. \tag{2.6}$$

The components of the modified (effective) Einstein tensor are

$$G^{00} = \frac{1}{A^2 f_R} \rho + \frac{1}{A^2 f_R} \left[ \frac{f - R f_R}{2} - \frac{\dot{f}_R}{A^2} \left( \frac{2\dot{B}}{B} + \frac{\dot{C}}{C} \right) \right]$$

$$-\frac{f'_R}{B^2} \left( \frac{1}{r} + \frac{2B'}{B} - \frac{C'}{C} \right) - \frac{f_R^\theta}{r^2 B^2} \left( \frac{2B^\theta}{B} - \frac{C^\theta}{C} \right) + \frac{f''_R}{B^2}, \tag{2.7}$$

$$G^{01} = \frac{-1}{A^2 B^2 f_R} \left[ \frac{A'}{A} \dot{f}_R + \frac{\dot{B}}{B} f'_R - \dot{f}_R' \right], \tag{2.8}$$

$$G^{02} = \frac{-1}{r^2 A^2 B^2 f_R} \left[ \frac{A^\theta}{A} \dot{f}_R + \frac{\dot{B}}{B} f_R^\theta - \dot{f}_R^{\theta\theta} \right], \tag{2.9}$$

$$G^{11} = \frac{1}{B^2 f_R} \left[ P_{xx}(f_T + 1) + \rho f_T + \frac{\dot{f}_R}{A^2} \left( \frac{\dot{B}}{B} - \frac{\dot{A}}{A} - \frac{\dot{C}}{C} \right) - \frac{f - Rf_R}{2} - \frac{f_R^{\theta\theta}}{r^2 B^2} - \frac{\ddot{f}_R}{A^2} + \frac{f'_R}{B^2} \left( \frac{1}{r} - \frac{A'}{A} + \frac{B'}{B} - \frac{C'}{C} \right) + \frac{f_R^\theta}{r^2 B^2} \left( \frac{3B^\theta}{B} - \frac{A^\theta}{A} - \frac{C^\theta}{C} \right) \right], \tag{2.10}$$

$$G^{12} = \frac{1}{r^2 B^4 f_R} \left[ P_{xy}(f_T + 1) + f_R^{\theta\theta} - \frac{B^\theta}{B} f'_R - \frac{B'}{B} f_R^{\theta\theta} \right], \tag{2.11}$$

$$G^{22} = \frac{1}{r^2 B^4 f_R} \left[ P_{yy}(f_T + 1) + \rho f_T + \frac{\dot{f}_R}{A^2} \left( \frac{\dot{B}}{B} - \frac{\dot{A}}{A} + \frac{\dot{C}}{C} \right) + \frac{\ddot{f}_R}{A^2} - \frac{f - Rf_R}{2} - \frac{f''_R}{B^2} - \frac{f_R^\theta}{r^2 B^2} \times \left( \frac{A^\theta}{A} - \frac{B^\theta}{B} + \frac{C^\theta}{C} \right) - \frac{f'_R}{B^2} \left( \frac{A'}{A} - \frac{B'}{B} + \frac{C'}{C} \right) \right], \tag{2.12}$$

$$G^{33} = \frac{1}{C^2 f_R} \left[ P_{zz}(f_T + 1) + \frac{\ddot{f}_R}{A^2} - \frac{f_R^{\theta\theta}}{r^2 B^2} + \rho f_T - \frac{f - Rf_R}{2} - \frac{\dot{f}_R}{A^2} \left( \frac{\dot{A}}{A} - \frac{2\dot{B}}{B} \right) - \frac{f''_R}{B^2} - \frac{f'_R}{B^2} \left( \frac{A'}{A} - \frac{2B'}{B} - \frac{1}{r} \right) - \frac{f_R^\theta}{r^2 B^2} \left( \frac{A^\theta}{A} - \frac{2B^\theta}{B} \right) \right]. \tag{2.13}$$

Here the dot, prime, and  $\theta$  indicate the derivatives w.r.t.  $t, r,$  and  $\theta$  coordinates, respectively. The expression for the Ricci scalar is

$$R = \frac{2}{A^2} \left[ \frac{\dot{A}}{A} \left( \frac{2\dot{B}}{B} + \frac{\dot{C}}{C} \right) - \frac{\dot{B}}{B} \left( \frac{\dot{B}}{B} + \frac{2\dot{C}}{C} \right) - \frac{2\ddot{B}}{B} - \frac{\ddot{C}}{C} \right] + \frac{2}{B^2} \left[ \frac{A''}{A} + \frac{A'C'}{AC} + \frac{B''}{B} - \frac{1}{r} \left( \frac{A'}{A} - \frac{B'}{B} - \frac{C'}{C} \right) - \frac{B'^2}{B^2} + \frac{C''}{C} + \frac{1}{r^2} \left( \frac{A^{\theta\theta}}{A} + \frac{B^{\theta\theta}}{B} + \frac{C^{\theta\theta}}{C} - \left( \frac{B^\theta}{B} \right)^2 + \frac{A^\theta C^\theta}{AC} \right) \right]. \tag{2.14}$$

In order to explore the stellar evolution, one needs to arrive at the dynamical equations, which can be obtained by employing the contracted Bianchi identities. Conservation laws play a significant part in the establishment of the instability range by a more generic analytic approach; the dynamical equations in our case are

$$G^{uv}_{;v} V_u = 0 \Rightarrow \left[ \frac{1}{f_R} T^{0v} + \frac{1}{f_R} T^{(D)0v} \right]_{;v} (-A) = 0, \tag{2.15}$$

$$G^{uv}_{;v} K_u = 0 \Rightarrow \left[ \frac{1}{f_R} T^{1v} + \frac{1}{f_R} T^{(D)1v} \right]_{;v} (B) = 0, \tag{2.16}$$

$$G^{uv}_{;v} L_u = 0 \Rightarrow \left[ \frac{1}{f_R} T^{2v} + \frac{1}{f_R} T^{(D)2v} \right]_{;v} (rB) = 0, \tag{2.17}$$

but, on simplification, we have

$$G^{00}_{,0} + G^{01}_{,1} + G^{02}_{,2} + G^{00} \left( \frac{2\dot{A}}{A} + \frac{2\dot{B}}{B} + \frac{\dot{C}}{C} \right) + G^{01} \left( \frac{3A'}{A} + \frac{2B'}{B} + \frac{C'}{C} + \frac{1}{r} \right) + G^{02} \left( \frac{3A^\theta}{A} + \frac{2B^\theta}{B} + \frac{C^\theta}{C} \right) + G^{11} \frac{B\dot{B}}{A^2} + G^{22} \frac{r^2 B\dot{B}}{A^2} + G^{33} \frac{C\dot{C}}{A^2} = 0, \tag{2.18}$$

$$G^{01}_{,0} + G^{11}_{,1} + G^{12}_{,2} + G^{00} \frac{AA'}{B^2} + G^{01} \left( \frac{\dot{A}}{A} + \frac{\dot{C}}{C} + \frac{4\dot{B}}{B} \right) + G^{11} \left( \frac{A'}{A} + \frac{3B'}{B} + \frac{C'}{C} + \frac{1}{r} \right) + G^{12} \left( \frac{A^\theta}{A} + \frac{4B^\theta}{B} + \frac{C^\theta}{C} \right) - G^{22} \left( r + \frac{r^2 B'}{B} \right) + G^{33} \frac{CC'}{B^2} = 0, \tag{2.19}$$

$$G^{02}_{,0} + G^{12}_{,1} + G^{22}_{,2} + G^{00} \frac{AA^\theta}{r^2 B^2} + G^{02} \left( \frac{\dot{A}}{A} + \frac{4\dot{B}}{B} + \frac{\dot{C}}{C} \right) - \frac{B^\theta}{r^2 B} G^{11} + \left( \frac{A'}{A} + \frac{4B'}{B} + \frac{C'}{C} + \frac{3}{r} \right) G^{12} + G^{22} \left( \frac{A^\theta}{A} + \frac{3B^\theta}{B} + \frac{C^\theta}{C} \right) - G^{33} \frac{CC^\theta}{r^2 B^2} = 0. \tag{2.20}$$

The notation of 0, 1 and 2 indicates the  $t, r$  and  $\theta$ . Terms belonging to the matter or effective part of the dynamical equations can be viewed separately by inserting the components of the Einstein tensor given in Eqs. (2.7)–(2.13). The dynamics of the gravitating axial system can be explored with the help of a perturbation scheme, which is useful in estimating the change in the system with the passage of time.

### 3 $f(R, T)$ model and perturbation approach

The selection of the model under observation is a crucial constituent of the analysis. Since we are dealing with the system analytically, the model selected would bring forth a fruitful mechanism for some particular form of  $f(R, T)$ . We found that the  $f(R, T)$  form suitable for a dynamical analysis is constrained to  $f(R, T) = f(R) + \lambda T$ , where  $\lambda$  is a positive constant and  $f(R)$  is an arbitrary function of the Ricci scalar. The origin of such a restriction is the fact that non-linear terms of the trace in  $f(R, T)$  complicate the formation of the modified field equations, which cannot be handled analytically. Such  $f(R, T)$  models bearing non-linear terms of the trace of energy momentum can be dealt with by using numerical techniques leading to more specific outcomes, whereas the findings of analytic approach yield more generic results. The viable  $f(R, T)$  model we have chosen is

$$f(R, T) = R + \alpha R^2 + \lambda T, \tag{3.21}$$

where any positive values can be assigned to  $\alpha$  and  $\lambda$ .

The onset of the modified field equations is non-linear in nature; the solution is still undetermined, which is why the perturbation approach is utilized to monitor the variations in the gravitating system with the time transition. All physical quantities are taken to be time independent initially, but the passage of time implicates a dependence on time as well. To introduce first order perturbations, we chose  $0 < \epsilon \ll 1$

$$A(t, r, \theta) = A_0(r, \theta) + \epsilon D(t)a(r, \theta), \tag{3.22}$$

$$B(t, r, \theta) = B_0(r, \theta) + \epsilon D(t)b(r, \theta), \tag{3.23}$$

$$C(t, r, \theta) = C_0(r, \theta) + \epsilon D(t)c(r, \theta), \tag{3.24}$$

$$\rho(t, r, \theta) = \rho_0(r, \theta) + \epsilon \bar{\rho}(t, r, \theta), \tag{3.25}$$

$$P_{xx}(t, r, \theta) = P_{xx0}(r, \theta) + \epsilon \bar{P}_{xx}(t, r, \theta), \tag{3.26}$$

$$P_{yy}(t, r, \theta) = P_{yy0}(r, \theta) + \epsilon \bar{P}_{yy}(t, r, \theta), \tag{3.27}$$

$$P_{zz}(t, r, \theta) = P_{zz0}(r, \theta) + \epsilon \bar{P}_{zz}(t, r, \theta), \tag{3.28}$$

$$P_{xy}(t, r, \theta) = P_{xy0}(r, \theta) + \epsilon \bar{P}_{xy}(t, r, \theta), \tag{3.29}$$

$$R(t, r, \theta) = R_0(r, \theta) + \epsilon D(t)e(r, \theta), \tag{3.30}$$

$$f(R, T) = [R_0(r, \theta) + \alpha R_0^2(r, \theta) + \lambda T_0(r, \theta)] + \epsilon D(t)e(r, \theta)[1 + 2\alpha R_0(r, \theta)], \tag{3.31}$$

$$f_R = 1 + 2\alpha R_0(r, \theta) + \epsilon 2\alpha D(t)e(r, \theta), \tag{3.32}$$

$$f_T = \lambda. \tag{3.33}$$

The first order perturbed Bianchi identities (2.18)–(2.20) imply

$$\left[ \dot{\bar{\rho}} + \left\{ \rho_0 \left( \frac{a}{A_0} + \frac{2\lambda_1 b}{B_0} + \frac{\lambda_1 c}{C_0} \right) + \frac{\lambda_1 b}{B_0} (P_{xx0} + P_{yy0}) + \frac{\lambda_1 c}{C_0} P_{zz0} + Z_{1p} \right\} \dot{D} \right] = 0, \tag{3.34}$$

$$\begin{aligned} & \left[ \lambda_1 \bar{P}_{xx} + \lambda \bar{\rho} - 2(\lambda_1 P_{xx0} + \lambda \rho_0) \left( \frac{b}{B_0} + \frac{e\alpha}{I} \right) D \right]_{,1} \\ & + (\lambda_1 \bar{P}_{xx} + \lambda \bar{\rho}) \left( \frac{A'_0}{A_0} + \frac{3B'_0}{B_0} + \frac{C'_0}{C_0} + \frac{1}{r} \right) \\ & + \frac{1}{r^2} \left[ \lambda_1 \bar{P}_{xy} - 2 \left( \frac{2b}{B_0} + \frac{e\alpha}{I} \right) P_{xy0} D \right]_{,2} \\ & + \frac{\lambda_1 \bar{P}_{xy}}{r^2 B_0^2} \left( \frac{A_0^\theta}{A_0} + 4 \frac{B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) \\ & + (\lambda_1 \bar{P}_{yy} + \lambda \bar{\rho}) \left( \frac{1}{r} + \frac{B'_0}{B_0} \right) + (\lambda_1 \bar{P}_{zz} + \lambda \bar{\rho}) \frac{C'_0}{C_0} \\ & + D \left[ (\lambda_1 P_{xx0} + \lambda \rho_0) \left( \left( \frac{a}{A_0} \right)' + \left( \frac{c}{C_0} \right)' \right) \right. \\ & + 3 \left( \frac{b}{B_0} \right)' - \left( \frac{2b}{B_0} + \frac{e\alpha}{I} \right) \left( \frac{A'_0}{A_0} + \frac{3B'_0}{B_0} + \frac{C'_0}{C_0} + \frac{1}{r} \right) \left. \right) \\ & + (\lambda_1 P_{yy0} + \lambda \rho_0) \left( \left( \frac{b}{B_0} \right)' \left( \frac{2b}{B_0} + \frac{e\alpha}{I} \right) \frac{B'_0}{B_0} \right) \left( \frac{1}{r} + \frac{B'_0}{B_0} \right) \\ & + (\lambda_1 P_{zz0} + \lambda \rho_0) \left( \left( \frac{c}{C_0} \right)' - \left( \frac{2b}{B_0} + \frac{e\alpha}{I} \right) \frac{C'_0}{C_0} \right) \\ & + \lambda_1 P_{xy0} \left( \left( \frac{a}{A_0} \right)^\theta + 4 \left( \frac{b}{B_0} \right)^\theta + \left( \frac{c}{C_0} \right)^\theta \right. \\ & \left. - \left( \frac{2b}{B_0} + \frac{e\alpha}{I} \right) \frac{C_0^\theta}{C_0} \right) \left. \right] + Z_{2p} = 0, \tag{3.35} \end{aligned}$$

$$\begin{aligned} & \left[ \lambda_1 \bar{P}_{yy} + \lambda \bar{\rho} - 2(\lambda_1 P_{yy0} + \lambda \rho_0) \left( \frac{b}{B_0} + \frac{e\alpha}{I} \right) D \right]_{,2} \\ & + \left[ \frac{1}{r^2 B_0^4 I} \lambda_1 \bar{P}_{xy} \right]' + \bar{\rho} \frac{A_0^\theta}{A_0} + (\lambda_1 \bar{P}_{xx} + \lambda \bar{\rho}) \frac{B_0^\theta}{B_0} \\ & + \lambda_1 \bar{P}_{xy} \left( \frac{A'_0}{A_0} + \frac{4B'_0}{B_0} + \frac{C'_0}{C_0} + \frac{3}{r} \right) + (\lambda_1 \bar{P}_{yy} + \lambda \bar{\rho}) \\ & \times \left( \frac{A_0^\theta}{A_0} + 3 \frac{B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) + (\lambda_1 \bar{P}_{zz} + \lambda \bar{\rho}) \frac{C_0^\theta}{C_0} \\ & + D \left[ \rho_0 \left( \left( \frac{a}{A_0} \right)^\theta - 2 \left( \frac{b}{B_0} + \frac{e\alpha}{I} \right) \frac{A_0^\theta}{A_0} \right) + \lambda_1 P_{xy0} \right. \\ & \times \left( \left( \frac{a}{A_0} \right)' + \left( \frac{c}{C_0} \right)' + 4 \left( \frac{b}{B_0} \right)' - \left( \frac{4b}{B_0} + \frac{2e\alpha}{I} \right) \right. \\ & \times \left. \left( \frac{A'_0}{A_0} + \frac{4B'_0}{B_0} + \frac{C'_0}{C_0} + \frac{3}{r} \right) \right) + (\lambda_1 P_{xx0} + \lambda \rho_0) \\ & \times \left( 4 \left( \frac{b}{B_0} \right)^\theta - \left( \frac{2b}{B_0} + \frac{e\alpha}{I} \right) \frac{B_0^\theta}{B_0} \right) + (\lambda_1 P_{yy0} + \lambda \rho_0) \\ & \times \left( \left( \frac{a}{A_0} \right)^\theta + 3 \left( \frac{b}{B_0} \right)^\theta + \left( \frac{c}{C_0} \right)^\theta - \left( \frac{2b}{B_0} + \frac{e\alpha}{I} \right) \right. \end{aligned}$$

$$\begin{aligned} &\times \left( \frac{A_0^\theta}{A_0} + 3 \frac{B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) + (\lambda_1 P_{zz0} + \lambda \rho_0) \\ &\times \left[ \left( \frac{c}{C_0} \right)^\theta - \left( \frac{2b}{B_0} + \frac{e\alpha}{I} \right) \frac{C_0^\theta}{C_0} \right] + Z_{3p} = 0, \end{aligned} \quad (3.36)$$

$$\Gamma < \frac{\lambda N'_0 - \frac{3}{r} N_0 - 2\lambda(\rho_0 N_2)' - \frac{2}{r} (P_{xy0} N_2)^\theta + \lambda N_2 N_3 - \frac{2}{r} N_2 + \lambda P_{xy0} N_4 + Z_{2p}^N}{\lambda_1 (P_{xx0} N_1)' + \frac{\lambda_1}{r^2} (P_{xy0} N_1)^\theta - \frac{1}{r} N_1 (P_{xx0} + P_{yy0} + P_{zz0})}, \quad (4.40)$$

where  $Z_{1p}$ ,  $Z_{2p}$ , and  $Z_{3p}$  are given in the appendix. To have a simplification, we substitute  $I = 1 + 2\alpha R_0$  and  $J = e2\alpha R_0$ . The expression for the energy density  $\bar{\rho}$  is derived from Eq. (3.34) as

$$\begin{aligned} \bar{\rho} = & - \left\{ \rho_0 \left( \frac{a}{A_0} + \frac{2\lambda_1 b}{B_0} + \frac{\lambda_1 c}{C_0} \right) \right. \\ & \left. + \frac{\lambda_1 b}{B_0} (P_{xx0} + P_{yy0}) + \frac{\lambda_1 c}{C_0} P_{zz0} + Z_{1p} \right\} D. \end{aligned} \quad (3.37)$$

The energy density and pressure stresses are associated as [32,41]

$$\bar{P}_i = \Gamma \frac{p_{i0}}{\rho_0 + p_{i0}} \bar{\rho}, \quad (3.38)$$

where  $\Gamma$  describes the variation of different stresses with the energy density. The index has a variation as  $i = xx, yy, xy, zz$ , and Eq. (3.37) together with Eq. (3.38) leads to the corresponding perturbed stresses. Implementation of a linear perturbation on the Ricci scalar yields an ordinary differential equation having a solution of the following form:

$$D(t) = -e^{\sqrt{Z_4}t}. \quad (3.39)$$

The expression for  $Z_4$  is provided in the appendix; Eq. (3.39) is valid for overall positive values of  $Z_4$ .

### 4 N and pN approximation

This section presents the terms belonging to the N and pN limits with an instability criterion in terms of the adiabatic index. Making use of Eqs. (3.39) and (3.38) in Eq. (3.35) leads to the evolution equation. The N and pN approximations for the system considered are discussed in the following subsections.

#### 4.1 Newtonian approximation

To approximate the instability/stability range in the Newtonian regime, we let  $A_0 = 1$ ,  $B_0 = 1$ ,  $\rho_0 \gg p_{i0}$ ;  $i = xx, yy, xy, yy$ , and the Schwarzschild coordinates  $C_0 = r$ , the evolution equation along with these assumptions yields

where  $Z_{2p}^N$  corresponds to the N approximation terms of  $Z_{2p}$ , and

$$\begin{aligned} N_0 = & - \left\{ \rho_0 N_1 + \lambda_1 b (P_{xx0} + P_{yy0}) + \frac{\lambda_1 c}{r} P_{zz0} + Z_{1p}^N \right\}, \\ N_1 = & a + 2\lambda_1 b + \frac{\lambda_1 c}{r}, \quad N_2 = b + \frac{\alpha e}{I}, \\ N_3 = & a' + 4b' + 2 \left( \frac{c}{r} \right)', \quad N_4 = a^\theta + 4b^\theta + \frac{c^\theta}{r}. \end{aligned}$$

The inequality for  $\Gamma$  contains both material functions and effective part entries; the system remains stable as long as the inequality (4.40) holds. The terms appearing in the expression for  $\Gamma$  are presumed in such a way that all terms maintain positivity, and this requirement imposes some restrictions on the physical parameters. The constraints in the N approximation are

$$\begin{aligned} P_{xx0} + P_{yy0} + P_{zz0} &< \frac{\lambda_1 r}{N_1} (P_{xx0} N_1)' + \frac{1}{r^2}, \\ (P_{xy0} N_2)^\theta &< -2\lambda(\rho_0 N_2)'. \end{aligned}$$

Violations of these constraints imply instability in the sources and thus lead to gravitational collapse.

#### 4.2 Post Newtonian approximation

In the pN approximation, we assume  $A_0 = 1 - \frac{m_0}{r}$  and  $B_0 = 1 + \frac{m_0}{r}$ , and the corresponding inequality for the range of stability is

$$\Gamma < \frac{\lambda N'_{10} + N_9 N_{10} - 2\lambda(\rho_0 N_6)' - \frac{2}{r} (P_{xy0} N_6)^\theta + \lambda \rho_0 N_7 - \frac{3}{r} N_6 + \lambda P_{xy0} N_8 + Z_{2p}^{pN}}{\lambda_1 (P_{xx0} N_5)' + \frac{\lambda_1}{r^2} (P_{xy0} N_5)^\theta - \frac{1}{r} N_5 (P_{xx0} + P_{yy0} + P_{zz0}) + N_1 1}, \quad (4.41)$$

where

$$\begin{aligned} N_5 = & \left( \frac{ar}{r - m_0} + \frac{2\lambda_1 br}{r + m_0} + \frac{\lambda_1 c}{r} \right), \quad N_6 = \frac{2\lambda_1 br}{r + m_0} + \frac{e\alpha}{I}, \\ N_7 = & \left( \frac{ar}{r - m_0} \right)' + 4 \left( \frac{br}{r + m_0} \right)' + \left( \frac{2c}{r} \right)' \\ & - N_6 \left( \frac{2}{r} + \left( \frac{m_0}{r} \right)' \frac{3r}{r + m_0} \right), \end{aligned}$$

$$\begin{aligned}
 N_8 &= \left(\frac{ar}{r-m_0}\right)^\theta + \left(\frac{br}{r+m_0}\right)^\theta + \left(\frac{c}{r}\right)^\theta, \\
 N_9 &= \left(\frac{3}{r} + \left(\frac{m_0}{r}\right)' \frac{3r}{r+m_0}\right), \\
 N_{10} &= -\left\{\rho_0 N_5 + \frac{2\lambda_1 br}{r+m_0}(P_{xx0} + P_{yy0}) + \frac{\lambda_1 c}{r} P_{zz0} + Z_{1pN}\right\}, \\
 N_{11} &= \frac{P_{xy0} N_5}{(r+m_0)^2} \left(\left(\frac{ar}{r-m_0}\right)^\theta + \left(\frac{4br}{r+m_0}\right)^\theta\right).
 \end{aligned}$$

Likewise the metric coefficients and effective part terms in the Newtonian limit can be constrained to maintain stability of the self-gravitating system. The system is stable unless the above mentioned inequality holds, the system collapses when the ordering relation (4.41) breaks down. One can deduce the results of the GR approximations by choosing vanishing values of  $\lambda$  and  $\alpha$ .

### 5 Summary and discussion

Observational signatures support the argument that gravitating sources might deviate from spherical symmetry incidentally. Thus non-spherical symmetries facilitate in examining realistic situations such as large scale structures, weak lensing, CMB, etc. Motivated by the significance of non-spherical symmetries, we intend to explore the impact of an axially symmetric gravitating source in the context of  $f(R, T)$  gravity. More particularly we are dealing with a restricted axial symmetry by ignoring meridional motions and rotation about the symmetry axis. The consequence of the restricted character of spacetime is that we have a vorticity-free case, because the absence of  $dt d\theta$  and  $dt d\phi$  terms indicates that vorticity of the gravitating source vanishes for an observer at rest. The metric under consideration is axially symmetric with three independent metric functions.

The implications of axial symmetry on gravitating system have been studied extensively in GR and modified theories of gravity. The alternative gravity theory we have chosen to establish the instability range is  $f(R, T)$  gravity, because the dynamical instability of axially symmetric sources in  $f(R, T)$  framework has not been ascertained yet. The model under study,  $f(R, T) = R + \alpha R^2 + \lambda T$ , is viable for positive values of  $\alpha$  and  $\lambda$ . The modified field equations are obtained by varying the action (3.27) for an anisotropic matter distribution. The components of the field equations (2.7)–(2.13) are used to arrive at conservation equations (2.18)–(2.20). These equations are of fundamental importance in the establishment of the instability range analytically.

The field equations are non-linear in nature; it is a difficult task to evaluate their general solution. To account for this issue, we consider a linear perturbation of the usual matter and dark source terms. The perturbed physical quantities

such as the energy density and anisotropic pressure stresses are extracted from the linearly perturbed components of the field equations, which are further inserted in the perturbed Bianchi identities to arrive at a collapse equation carrying both material and dark source ingredients. An ordinary differential equation is formed from the perturbed Ricci scalar, whose solution together with the evolution equation provides the adiabatic index.

The adiabatic index defines the range of instability for N and pN approximations, inducing some constraints on the physical quantities that are provided in the previous section. Corrections to GR and  $f(R)$  gravity can be determined by setting  $\alpha \rightarrow 0$ ,  $\lambda \rightarrow 0$  and  $\lambda \rightarrow 0$ , respectively.

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### Appendix

The following equations contain linearly perturbed terms of the conservation equations and the Ricci scalar, respectively:

$$\begin{aligned}
 Z_{1p} &= \frac{e}{2} - A_0^2 \left\{ \frac{1}{A_0^2 B_0^2 I^2} \left( (2\alpha e R_0)' \left( 1 - \frac{b}{B_0} \right) - 2\alpha e R_0 \frac{A_0'}{A_0} \right) \right\}_{,1} \\
 &\quad - \frac{A_0^2}{r^2} \left\{ \frac{2}{A_0^2 B_0^2 I^2} \left( (\alpha e R_0)^\theta \left( 1 - \frac{b}{B_0} \right) - (\alpha e R_0) \frac{A_0^\theta}{A_0} \right) \right\}_{,2} \\
 &\quad + \frac{\alpha^2 R_0^3}{I} + \frac{1}{B_0^2} \left[ \frac{(e^\theta (2\alpha e R_0))^\theta}{r^2} - 4\alpha \right. \\
 &\quad \times \left( (R_0 R_0') + \frac{(R_0 R_0^\theta)^\theta}{r^2} \right) \left( \frac{a}{A_0} + \frac{b}{B_0} + \frac{\alpha e R_0}{I} \right) \\
 &\quad + I' \left\{ \left( \frac{c}{C_0} \right)' - 2 \left( \frac{b}{B_0} \right)' - \frac{b}{B_0} \left( \frac{2A_0'}{A_0} + \frac{2B_0'}{B_0} - \frac{3}{r} \right) \right. \\
 &\quad \left. - \frac{c}{C_0} \left( \frac{A_0'}{A_0} - \frac{C_0'}{C_0} - \frac{1}{r} \right) + \frac{(2\alpha e R_0)}{I} \left( \frac{C_0'}{C_0} + \frac{2B_0'}{B_0} \right. \right. \\
 &\quad \left. \left. - \frac{3}{r} \right) \right\} + (e' (2\alpha e R_0))' + \frac{I^\theta}{r^2} \left\{ \left( \frac{c}{C_0} \right)^\theta - 2 \left( \frac{b}{B_0} \right)^\theta \right. \\
 &\quad \left. - \frac{b}{B_0} \left( \frac{2A_0^\theta}{A_0} + \frac{2B_0^\theta}{B_0} \right) - \frac{c}{C_0} \left( \frac{A_0^\theta}{A_0} - \frac{C_0^\theta}{C_0} \right) + \frac{(2\alpha e R_0)}{I} \right. \\
 &\quad \left. \times \left( \frac{C_0^\theta}{C_0} + \frac{2B_0^\theta}{B_0} \right) \right\} + (2\alpha e R_0)' \left( \frac{C_0'}{C_0} - \frac{2B_0'}{B_0} + \frac{1}{r} \right) \\
 &\quad + \frac{(2\alpha e R_0)^\theta}{r^2} \left( \frac{C_0^\theta}{C_0} - \frac{2B_0^\theta}{B_0} \right) + \left( \frac{2a}{A_0} + \frac{b}{B_0} \right) \\
 &\quad \times \left( I'' + \frac{I^{\theta\theta}}{r^2} \right) - \left( \frac{3A_0'}{A_0} + \frac{2B_0'}{B_0} + \frac{1}{r} + \frac{C_0'}{C_0} \right)
 \end{aligned}$$

$$\begin{aligned}
 & \times \times \left( \frac{(2\alpha e R_0)'}{I} \left( 1 - \frac{b}{B_0} \right) - \frac{A_0'}{A_0} \frac{(2\alpha e R_0)'}{I} \right) \\
 & + \left( \frac{A_0^\theta (2\alpha e R_0)'}{A_0 I} - \frac{(2\alpha e R_0)^\theta}{I} \left( 1 - \frac{b}{B_0} \right) \right) \\
 & \times \left( \frac{3A_0^\theta}{A_0} + \frac{C_0^\theta}{C_0} + \frac{2B_0^\theta}{B_0} \right) \Big], \tag{5.1} \\
 Z_{2p} = & \left[ \left[ \frac{1}{IB_0^2} \left\{ \frac{\ddot{D}}{DA_0^2} - \frac{1}{B_0^2} \left\{ \frac{eB_0^2}{2} + I' \left( \left( \frac{a}{A_0} \right)' \right. \right. \right. \right. \right. \right. \\
 & - \left( \frac{b}{B_0} \right)' + \left( \frac{c}{C_0} \right)' \Big\} + \left( J' - \frac{2b}{B_0} I' \right) \\
 & \times \left( \frac{A_0'}{A_0} + \frac{C_0'}{C_0} - \frac{B_0'}{B_0} - \frac{1}{r} \right) \\
 & + \frac{1}{r^2} \left( J^{\theta\theta} + \left( J^\theta - \frac{2b}{B_0} I^\theta \right) \left( \frac{A_0^\theta}{A_0} - \frac{3B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) \right. \\
 & \left. + \frac{2b}{B_0} I^{\theta\theta} + I^\theta \left( \left( \frac{a}{A_0} \right)^\theta + \left( \frac{c}{C_0} \right)^\theta - 3 \left( \frac{b}{B_0} \right)^\theta \right) \right] \Big]_{,1} \\
 & + \left[ \frac{1}{r^2 IB_0^4} \left\{ J^{\theta\theta} + \left( \frac{b}{B_0} \right)^\theta I' + J^\theta \left( \frac{B_0'}{B_0} + \frac{1}{r} \right) \right. \right. \\
 & \left. \left. - \left( \frac{b}{B_0} \right)' I^\theta \right\} \right]_{,2} \\
 & \times IB_0^4 - e \frac{B_0'}{B_0} + \frac{A_0'}{A_0} \left[ J'' + \frac{J^{\theta\theta}}{r^2} - \frac{2b}{B_0} \left( I'' + \frac{I^{\theta\theta}}{r^2} \right) \right. \\
 & + \left( J' - \frac{2b}{B_0} I' \right) \left( \frac{C_0'}{C_0} - \frac{2B_0'}{B_0} + \frac{1}{r} \right) \\
 & + I' \left( \left( \frac{c}{C_0} \right)' - \left( \frac{b}{B_0} \right)' \right) \\
 & + \frac{1}{r^2} \left\{ I^\theta \left( \left( \frac{c}{C_0} \right)^\theta - 2 \left( \frac{b}{B_0} \right)^\theta \right) \right. \\
 & \left. + \left( J^\theta - \frac{2b}{B_0} I^\theta \right) \left( \frac{C_0^\theta}{C_0} - \frac{2B_0^\theta}{B_0} \right) \right\} \\
 & + \left( \frac{(aA_0)'}{A_0^2} - \frac{2b A_0'}{B_0 A_0} \right) \\
 & \times \left( \frac{\alpha R_0^2 B_0^2}{2} + I'' + I' \left( \frac{C_0'}{C_0} - \frac{2B_0'}{B_0} + \frac{1}{r} \right) \right. \\
 & \left. + \frac{I^{\theta\theta}}{r^2} + \frac{I^\theta}{r^2} \left( \frac{C_0^\theta}{C_0} - \frac{2B_0^\theta}{B_0} \right) \right) \\
 & - \left\{ \frac{\alpha R_0^2 B_0^2}{2} + I' \left( \frac{A_0'}{A_0} + \frac{C_0'}{C_0} - \frac{B_0'}{B_0} - \frac{1}{r} \right) \right. \\
 & \left. + \frac{I^{\theta\theta}}{r^2} + \frac{I^\theta}{r^2} \left( \frac{A_0^\theta}{A_0} + \frac{C_0^\theta}{C_0} - \frac{3B_0^\theta}{B_0} \right) \right\} \\
 & \times \left( \left( \frac{a}{A_0} \right)' + 3 \left( \frac{b}{B_0} \right)' + \left( \frac{c}{C_0} \right)' \right) \\
 & - \left( \frac{A_0'}{A_0} + \frac{C_0'}{C_0} + \frac{3B_0'}{B_0} + \frac{1}{r} \right) \\
 & \times \left\{ I' \left( \left( \frac{a}{A_0} \right)' - \left( \frac{b}{B_0} \right)' + \left( \frac{c}{C_0} \right)' \right) \right. \\
 & + \left( \frac{A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{C_0'}{C_0} - \frac{1}{r} \right) \left( J' - \frac{2b}{B_0} I' \right) \\
 & + \frac{1}{r^2} \left( J^{\theta\theta} + \left( J^\theta - \frac{2b}{B_0} I^\theta \right) \right. \\
 & \times \left( \frac{A_0^\theta}{A_0} - \frac{3B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) + \frac{2b}{B_0} I^{\theta\theta} \\
 & \left. \left. + I^\theta \left( \left( \frac{a}{A_0} \right)^\theta + \left( \frac{c}{C_0} \right)^\theta - 3 \left( \frac{b}{B_0} \right)^\theta \right) \right) \right\} \\
 & - \left[ \left( \left( \frac{a}{A_0} \right)^\theta + \left( \frac{c}{C_0} \right)^\theta + 4 \left( \frac{b}{B_0} \right)^\theta \right) \right. \\
 & \times \left( I^{\theta\theta} + \frac{B_0^\theta}{B_0} I' + I^\theta \left( \frac{B_0'}{B_0} + \frac{1}{r} \right) \right) \\
 & \times \left( \frac{A_0^\theta}{A_0} + \frac{4B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) \left( \frac{B_0^\theta}{B_0} J' - J^{\theta\theta} - I' \left( \frac{b}{B_0} \right)^\theta \right. \\
 & \left. \left. - J^\theta \left( \frac{B_0'}{B_0} + \frac{1}{r} \right) + I^\theta \left( \frac{b}{B_0} \right)' \right) \right] \frac{1}{r^2} \\
 & - \left( \frac{B_0'}{B_0} + \frac{1}{r} \right) \left[ \frac{B_0^2}{A_0^2} \frac{\ddot{D}}{D} J - J'' + \frac{2b}{B_0} I'' \right. \\
 & - I' \left( \left( \frac{a}{A_0} \right)' - \left( \frac{b}{B_0} \right)' + \left( \frac{c}{C_0} \right)' \right) \\
 & - \left( J' - \frac{2b}{B_0} I' \right) \left( \frac{A_0'}{A_0} + \frac{C_0'}{C_0} - \frac{B_0'}{B_0} \right) \\
 & \left. + \frac{1}{r^2} \left( I^\theta \left( \left( \frac{a}{A_0} \right)^\theta + \left( \frac{c}{C_0} \right)^\theta - \left( \frac{b}{B_0} \right)^\theta \right) \right. \right. \\
 & \left. \left. - \left( J^\theta - \frac{2b}{B_0} I^\theta \right) \left( \frac{A_0^\theta}{A_0} - \frac{B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) \right) \right] \\
 & + \left( \frac{b}{B_0} \right)' \left[ \frac{LB_0^2}{2} + I' \left( \frac{A_0'}{A_0} + \frac{C_0'}{C_0} - \frac{B_0'}{B_0} \right) \right. \\
 & \left. - \frac{I^\theta}{r^2} \left( \frac{A_0^\theta}{A_0} - \frac{B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) - I'' \right] \\
 & + \frac{C_0'}{C_0} \left[ J'' - \frac{2b}{B_0} I'' + I' \left( \left( \frac{a}{A_0} \right)' - \left( \frac{2b}{B_0} \right)' \right) \right. \\
 & \left. + \left( J' - \frac{2b}{B_0} I' \right) \left( \frac{A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{1}{r} \right) \right. \\
 & \left. + \frac{1}{r^2} \left\{ J^{\theta\theta} + \left( J^\theta - \frac{2b}{B_0} I^\theta \right) \left( \frac{A_0^\theta}{A_0} - \frac{2B_0^\theta}{B_0} \right) \right\} \right]
 \end{aligned}$$

$$\begin{aligned}
 &+ I^\theta \left[ \left( \frac{a}{A_0} \right)^\theta - 2 \left( \frac{b}{B_0} \right)^\theta - \frac{2b}{B_0} I^{\theta\theta} \right] \\
 &+ \left( \frac{(cC_0)'}{C_0^2} - \frac{2b}{B_0} \frac{C_0'}{C_0} \right) \left[ I' \left( \frac{A_0'}{A_0} - \frac{2B_0'}{B_0} + \frac{1}{r} \right) \right. \\
 &- \left. \frac{I^\theta}{r^2} \left( \frac{A_0^\theta}{A_0} - \frac{2B_0^\theta}{B_0} \right) + \frac{\alpha R_0^2 B_0^2}{2} + I'' + \frac{I^{\theta\theta}}{r^2} \right] \\
 &- \frac{\ddot{D}B_0^2}{DA_0^2 I} \left( J' - \frac{A_0'}{A_0} J - \frac{b}{B_0} I' \right), \tag{5.2}
 \end{aligned}$$

$$\begin{aligned}
 Z_{3p} = &Ir^2 B_0^4 \left[ \left[ \frac{1}{r^2 I B_0^4} \left\{ J^\theta + \left( \frac{b}{B_0} \right)^\theta I' \right. \right. \right. \\
 &+ \left. \left. J^\theta \left( \frac{B_0'}{B_0} + \frac{1}{r} \right) - \left( \frac{b}{B_0} \right)' I^\theta \right\} \right]_{,1} \\
 &+ \frac{\ddot{D}B_0^2}{DA_0^2 I} \left( \frac{A_0^\theta}{A_0} J + \frac{b}{B_0} I^\theta - J^\theta \right) \\
 &+ \left[ \frac{1}{Ir^2 B_0^4} \left\{ \frac{\ddot{D}B_0^2}{DA_0^2} J - J'' + \frac{2b}{B_0} I'' + \left( \frac{2b}{B_0} I' \right. \right. \right. \\
 &- \left. \left. J' \right) \left( \frac{A_0'}{A_0} + \frac{C_0'}{C_0} - \frac{B_0'}{B_0} \right) - I' \left( \left( \frac{a}{A_0} \right)' - \left( \frac{b}{B_0} \right)' \right. \right. \\
 &+ \left. \left. \left( \frac{c}{C_0} \right)' \right) + \frac{1}{r^2} \left( \left( \frac{2b}{B_0} I^\theta - J^\theta \right) \right. \right. \\
 &\times \left. \left. \left( \frac{A_0^\theta}{A_0} - \frac{B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) \right. \right. \\
 &- \left. \left. I^\theta \left( \left( \frac{a}{A_0} \right)^\theta + \left( \frac{c}{C_0} \right)^\theta - \left( \frac{b}{B_0} \right)^\theta \right) \right\} \right]_{,2} \\
 &- e \frac{B_0^\theta}{B_0} + \frac{A_0^\theta}{A_0} \left[ J'' + \frac{J^{\theta\theta}}{r^2} - \frac{2b}{B_0} \left( I'' + \frac{I^{\theta\theta}}{r^2} \right) \right. \\
 &+ \left. \left( J' - \frac{2b}{B_0} I' \right) \left( \frac{C_0'}{C_0} - \frac{2B_0'}{B_0} + \frac{1}{r} \right) \right. \\
 &+ \left. I' \left( \left( \frac{c}{C_0} \right)' - \left( \frac{b}{B_0} \right)' \right) \right. \\
 &+ \left. \frac{1}{r^2} \left\{ I^\theta \left( \left( \frac{c}{C_0} \right)^\theta - 2 \left( \frac{b}{B_0} \right)^\theta \right) + \left( J^\theta - \frac{2b}{B_0} I^\theta \right) \right. \right. \\
 &\times \left. \left. \left( \frac{C_0^\theta}{C_0} - \frac{2B_0^\theta}{B_0} \right) \right\} \right] + \left( \frac{(aA_0)^\theta}{A_0^2} - \frac{2b}{B_0} \frac{A_0^\theta}{A_0} \right) \\
 &\times \left( \frac{\alpha R_0^2 B_0^2}{2} + I'' + I' \left( \frac{C_0'}{C_0} - \frac{2B_0'}{B_0} + \frac{1}{r} \right) + \frac{I^{\theta\theta}}{r^2} \right. \\
 &+ \left. \frac{I^\theta}{r^2} \left( \frac{C_0^\theta}{C_0} - \frac{2B_0^\theta}{B_0} \right) \right) - \left( \frac{b}{B_0} \right)^\theta \left\{ \frac{\alpha R_0^2 B_0^2}{2} \right. \\
 &+ \left. I' \left( \frac{A_0'}{A_0} + \frac{C_0'}{C_0} - \frac{B_0'}{B_0} + \frac{1}{r} \right) + \frac{I^{\theta\theta}}{r^2} \right.
 \end{aligned}$$

$$\begin{aligned}
 &+ \left. \frac{I^\theta}{r^2} \left( \frac{A_0^\theta}{A_0} + \frac{C_0^\theta}{C_0} - \frac{3B_0^\theta}{B_0} \right) \right\} \\
 &- \frac{B_0^\theta}{B_0} \left\{ I' \left( \left( \frac{a}{A_0} \right)' - \left( \frac{b}{B_0} \right)' \right. \right. \\
 &+ \left. \left. \left( \frac{c}{C_0} \right)' \right) + \left( \frac{A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{C_0'}{C_0} - \frac{1}{r} \right) \right. \\
 &\times \left( J' - \frac{2b}{B_0} I' \right) + \frac{1}{r^2} \left( J^{\theta\theta} - \left( \frac{2b}{B_0} I^\theta - J^\theta \right) \right. \\
 &\times \left. \left( \frac{A_0^\theta}{A_0} - \frac{3B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) \right. \\
 &+ \left. \left. \frac{2b}{B_0} I^{\theta\theta} + I^\theta \left( \left( \frac{a}{A_0} \right)^\theta + \left( \frac{c}{C_0} \right)^\theta - 3 \left( \frac{b}{B_0} \right)^\theta \right) \right\} \right. \\
 &- \frac{1}{r^2} \left[ \left( \left( \frac{a}{A_0} \right)' + \left( \frac{c}{C_0} \right)' + \left( \frac{b}{B_0} \right)' \right) \right. \\
 &\times \left( I'^\theta + \frac{B_0^\theta}{B_0} I' + I^\theta \left( \frac{B_0'}{B_0} + \frac{1}{r} \right) \right) \\
 &- \left. \left( \frac{A_0'}{A_0} + \frac{4B_0'}{B_0} + \frac{C_0'}{C_0} \right) \left( \frac{B_0^\theta}{B_0} J' - J^{\theta\theta} - I' \left( \frac{b}{B_0} \right)^\theta \right. \right. \\
 &- \left. \left. J^\theta \left( \frac{B_0'}{B_0} + \frac{1}{r} \right) + I^\theta \left( \frac{b}{B_0} \right)' \right) \right] \\
 &+ \left( \frac{A_0^\theta}{A_0} + \frac{3B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) \left[ \frac{B_0^2}{A_0^2} \frac{\ddot{D}}{D} J + \frac{2b}{B_0} I'' \right. \\
 &- \left. I' \left( \left( \frac{a}{A_0} \right)' - \left( \frac{b}{B_0} \right)' + \left( \frac{c}{C_0} \right)' \right) \right. \\
 &- \left. J'' - \left( J' - \frac{2b}{B_0} I' \right) \left( \frac{A_0'}{A_0} + \frac{C_0'}{C_0} - \frac{B_0'}{B_0} \right) \right. \\
 &- \left. \frac{1}{r^2} \left( \left( J^\theta - \frac{2b}{B_0} I^\theta \right) \left( \frac{A_0^\theta}{A_0} - \frac{B_0^\theta}{B_0} \right) \right. \right. \\
 &+ \left. \left. \frac{C_0^\theta}{C_0} \right) - I^\theta \left( \left( \frac{a}{A_0} \right)^\theta + \left( \frac{c}{C_0} \right)^\theta - \left( \frac{b}{B_0} \right)^\theta \right) \right] \\
 &+ \left( \left( \frac{a}{A_0} \right)^\theta + \left( \frac{c}{C_0} \right)^\theta \right. \\
 &+ \left. 3 \left( \frac{b}{B_0} \right)^\theta \right) \left[ \frac{\alpha R_0^2 B_0^2}{2} + I' \left( \frac{A_0'}{A_0} + \frac{C_0'}{C_0} - \frac{B_0'}{B_0} \right) \right. \\
 &- \left. \frac{I^\theta}{r^2} \left( \frac{A_0^\theta}{A_0} - \frac{B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0} \right) - I'' \right] \\
 &- \frac{C_0^\theta}{C_0} \left[ J'' - \frac{2b}{B_0} I'' + I' \left( \left( \frac{a}{A_0} \right)' - \left( \frac{2b}{B_0} \right)' \right) \right. \\
 &+ \left. \left( J' - \frac{2b}{B_0} I' \right) \left( \frac{A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{1}{r} \right) \right.
 \end{aligned}$$



$$\begin{aligned}
 & + \frac{1}{r^2} \left\{ \left( J^\theta - \frac{2b}{B_0} I^\theta \right) \left( \frac{A_0^\theta}{A_0} - \frac{2B_0^\theta}{B_0} \right) \right. \\
 & + I^\theta \left( \left( \frac{a}{A_0} \right)^\theta - 2 \left( \frac{b}{B_0} \right)^\theta \right) \\
 & \left. + J^{\theta\theta} - \frac{2b}{B_0} I^{\theta\theta} \right\} + \left( \frac{(cC_0)^\theta}{C_0^2} - \frac{2b}{B_0} \frac{C_0^\theta}{C_0} \right) \\
 & \times \left[ \frac{\alpha R_0^2 B_0^2}{2} + I' \left( \frac{A_0'}{A_0} - \frac{2B_0'}{B_0} + \frac{1}{r} \right) \right. \\
 & \left. + I'' + \frac{I^{\theta\theta}}{r^2} - \frac{I^\theta}{r^2} r \left( \frac{A_0^\theta}{A_0} - \frac{2B_0^\theta}{B_0} \right) \right], \tag{5.3}
 \end{aligned}$$

$$\begin{aligned}
 Z_4 = & \frac{A_0^2}{2} \left( \frac{B_0 C_0}{b C_0 - c B_0} \right) \\
 & \times \left[ \frac{2}{B_0^2} \left\{ \frac{A_0' C_0'}{A_0 C_0} \left( \frac{a'}{A_0'} - \frac{a}{A_0} + \frac{c'}{C_0'} - \frac{c}{C_0} \right) + \frac{A_0''}{A_0} \right. \right. \\
 & \times \left( \frac{a''}{A_0''} - \frac{a}{A_0} \right) + \frac{B_0''}{B_0} \left( \frac{b''}{B_0''} - \frac{b}{B_0} \right) \\
 & + \frac{C_0''}{C_0} \left( \frac{c''}{C_0''} - \frac{c}{C_0} \right) - \frac{1}{r} \left( \frac{a}{A_0} - \frac{b}{B_0} - \frac{c}{C_0} \right)' \\
 & - \frac{2B_0'}{B_0} \left( \frac{b}{B_0} \right)' + \frac{2}{r^2} \left\{ \frac{2B_0^\theta}{B_0} \left( \frac{b}{B_0} \right)^\theta \right. \\
 & + \frac{A_0^{\theta\theta}}{A_0} \left( \frac{a^{\theta\theta}}{A_0^{\theta\theta}} - \frac{a}{A_0} \right) + \frac{B_0^{\theta\theta}}{B_0} \left( \frac{b^{\theta\theta}}{B_0^{\theta\theta}} - \frac{b}{B_0} \right) \\
 & + \frac{C_0^{\theta\theta}}{C_0} \left( \frac{c^{\theta\theta}}{C_0^{\theta\theta}} - \frac{c}{C_0} \right) + \frac{A_0^\theta C_0^\theta}{A_0 C_0} \\
 & \left. \left. \times \left( \frac{a^\theta}{A_0^\theta} - \frac{a}{A_0} + \frac{c^\theta}{C_0^\theta} - \frac{c}{C_0} \right) \right\} \right] - e - \frac{2bR_0}{B_0}. \tag{5.4}
 \end{aligned}$$

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