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On the functional renormalization group for the scalar field on curved background with non-minimal interaction

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Abstract The running of the non-minimal parameter ξ of the interaction of the real scalar field and scalar curvature is explored within the non-perturbative setting of the functional renormalization group (RG). We establish the RG flow in curved space-time in the scalar field sector, in particular derive an equation for the non-minimal parameter. The RG trajectory is numerically explored for different sets of initial data.

1 Introduction

The renormalization structure in a curved space-time is well known at both general and perturbative levels. In particular, it is well known that any theory which is renormalizable in flat space remains renormalizable in curved space [1] (see also [2] for a recent review). The necessary elements of the consistent quantum theory in a curved space-time are the purely gravitational vacuum action, which consists of the Einstein–Hilbert term with cosmological constant and also of the four fourth-derivative terms. On top of that, if the theory under discussion has scalar fields φ_i, new non-minimal terms of the form $\xi_{ij} R \varphi_i \varphi_j$ have to be included in the action. The renormalization group (RG) in curved space-time was introduced in [3–6] (see also [1]) as a useful tool to explore the scaling properties of the theory.

The renormalization and RG in curved space follow some important hierarchy, which means that: (i) The RG equations for the matter fields couplings and masses do not depend on ξ_{ij} and on the parameters of the vacuum action. More

general, these equations are not affected by the presence of external gravitational field.[1] (ii) The RG equations for ξ_{ij} depend on the matter fields couplings (but not on the masses of the fields, in the case of the minimal subtraction scheme of renormalization), but they do not depend on the parameters of the vacuum action.

(iii) The RG equations for the parameters of the vacuum action may depend on the couplings (beyond one-loop approximation) and on ξ_{ij}.

One has to note that the running of ξ_{ij} may have some important applications, especially to inflationary models such as Higgs inflation [10], because this running is closely related to the effective potential of the Higgs field in curved space [1] (see also [9]). The same concerns also other inflationary models, including the ones based on inflation, and Starobinsky inflation [11] and especially its modified version [12]. Therefore, it would be quite useful to know whether the non-minimal parameter can experience a strong running at some moment of the history of the universe. One of the possibilities to observe an intensive running of ξ is related to the non-perturbative effects in the framework of the functional RG (FRG) approach, developed by Polchinski and Wetterich [13–15] (see also [16] for a similar original derivation and [17–20] and [21] for reviews and an introduction to the subject). In the present Letter we present the FRG equations in curved space, in a background-independent covariant way similar to what has been done before for the perturbative RG in curved space. The FRG approach on a fixed de Sitter background has been previously considered in [22–24] and [25].[2]

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[1] This is not true if gravity is quantized [7,8], but we do not consider this part here.

[2] During the completion of this work the related preprint [28] has been published.

The present paper is essentially restricted to the case of a single scalar and hence to the equation for a single parameter ξ. We consider first the local potential approximation (LPA), dealing with the most simple theory with unbroken symmetry, and then we explore the more complicated case with the broken symmetry and wave-function renormalization. In fact, the extension of the RG flow for the broken phase is especially interesting, because the running of the non-minimal parameter in this case was not sufficiently well explored even in the perturbative approach. Some potentially interesting consequences of the RG flow for the non-minimal parameter are related to the scale dependence of the non-local parts of the induced gravitational action, which emerge due to the curvature dependence of the vacuum expectation value of the scalar field [26].

In the parallel work [27] we will also consider generalizations like the theory with a more general form $f(\phi)R$ of non-minimal interaction at quantum level and in different dimensions.

The paper is organized as follows. In Sect. 2 we describe the general scheme of FRG in a curved space-time and especially derivation of the RG equation for ξ. Section 3 is devoted to the numerical analysis of the equation for ξ. Section 4 describes the FRG in the scalar theory with broken symmetry. Finally, in Sect. 5 we draw our conclusions.

2 FRG for scalar field with non-minimal coupling

The renormalizable theory of a single scalar ϕ in a curved space starts from the classical action of the form

$$S = \int_x \left[-\frac{1}{2}\phi\triangle_g\phi + \frac{\xi}{2}R\phi^2 + V(\phi) \right] + S^{\text{grav}}[g], \qquad (1)$$

where we assumed Euclidean signature and use the notation $\int_x \equiv \int d^4x \sqrt{g(x)}$. Furthermore, $S^{\text{grav}}[g]$ corresponds to the vacuum action as described in the Introduction. We expect to discuss the FRG flow for the vacuum part in a separate article, so it will not be seriously dealt with in the present Letter or in [27]. $V(\phi)$ is a classical potential, which may be restricted to the form $(1/4!)\lambda\phi^4$ in the case we intend to remain within the scope of perturbatively renormalizable theories.

As usual in the RG approach, at quantum level all quantities start to depend on the scale which we identify as k. The practical use of the FRG approach implies the choice of the truncation scheme, which we choose in a most simple way, assuming that the effective average action is

$$\Gamma_k = \int_x \left[-\frac{Z_k}{2}\phi\triangle_g\phi + \frac{\xi_k}{2}R\phi^2 + u_k(\phi) \right] + \Gamma_k^{\text{grav}}[g]. \quad (2)$$

This truncation includes a scale-dependent effective potential u_k, a wave-function renormalization Z_k and the running

non-minimal parameter ξ_k, which does not depend on the momenta or on the field ϕ. The invariant cutoff action has the form

$$\Delta S_k = \frac{1}{2} \int_x \phi R_k(-\triangle_g)\phi, \qquad (3)$$

where

$$R_k(-\triangle_g) = Z_k r_k(-\triangle_g).$$

R_k is assumed to have the well-known properties of a cutoff function [17–20]. The anomalous dimension is defined as

$$\eta_k = -\frac{k\partial_k Z_k}{Z_k} = -\frac{\partial_t Z_k}{Z_k}, \quad \text{where } t = \log\frac{k}{\mu}. \qquad (4)$$

When the scale k runs from the UV-cutoff Λ to the IR, the dimensionless scale parameter t runs from $\log(\Lambda/\mu)$ to $-\infty$.

The Wetterich equation for the scale dependent effective average action reads [14–16]

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\text{Tr}\left(\frac{\partial_t R_k}{\Gamma_k^{(2)}[\phi] + R_k}\right), \qquad (5)$$

where $\Gamma_k^{(2)}$ indicates a second variational derivative with respect to the scalar field and Tr includes the coincidence limit and covariant integration over the space-time variables. For the truncation (2) the l.h.s. becomes

$$\partial_t \Gamma_k = \int_x \left[-\frac{\eta_k Z_k}{2}\phi\triangle_g\phi + \frac{\partial_t\xi_k}{2}R\phi^2 + \partial_t u_k(\phi) \right]$$
$$+ \partial_t \Gamma_k^{\text{grav}}[g]. \qquad (6)$$

In order to derive the r.h.s. of (5), we need

$$\Gamma_k^{(2)} = -Z_k\triangle_g + \xi_k R + u_k''(\phi), \qquad (7)$$

where prime means simple derivative with respect to scalar field. The variation of the cutoff function can be cast into the form

$$\partial_t R_k = Z_k\left(\partial_t r_k - \eta_k r_k\right). \qquad (8)$$

Then the r.h.s. of the flow equation (5) takes the form

$$\frac{1}{2}\text{Tr}\left(\frac{\partial_t R_k}{\Gamma_k^{(2)}[\phi] + R_k}\right)$$
$$= \frac{1}{2}\text{Tr}\left[\frac{(\partial_t - \eta_k)r_k(-\triangle_g)}{(-\triangle_g + r_k(-\triangle_g)) + \xi_k Z_k^{-1}R + Z_k^{-1}u_k''(\phi)}\right]. \qquad (9)$$

Equation (5) with (6) and (9) represents the covariant flow equation corresponding to the truncation (2). It can be improved by including higher derivative terms into (6), but then the calculations of (9) should also be evaluated up to the corresponding higher order of approximation.

2.1 Elaborating the Wetterich equation

It proves useful to define

$$u_k(\phi) = \frac{m_k^2}{2}\phi^2 + w_k(\phi), \tag{10}$$

such that

$$u_k''(\phi) = m_k^2 + w_k''(\phi). \tag{11}$$

Thus we arrive at the following form of Eq. (9):

$$\frac{1}{2}\operatorname{Tr}\left(\frac{\partial_t R_k}{\Gamma_k^{(2)}[\phi] + R_k}\right) = \frac{1}{2}\operatorname{Tr}\left(\frac{B_k(-\triangle_g)}{P_k(-\triangle_g) + \Sigma_k}\right), \tag{12}$$

where we introduced the abbreviations

$$B_k(-\triangle_g) = (\partial_t - \eta_k)\, r_k(-\triangle_g), \tag{13}$$

$$P_k(-\triangle_g) = -\triangle_g + r_k(-\triangle_g) + \frac{m_k^2}{Z_k}, \tag{14}$$

$$\Sigma_k(\phi, R) = \frac{\xi_k}{Z_k}R + \frac{1}{Z_k}w_k''(\phi). \tag{15}$$

In order to analyze the flow equation in the truncation (2), we need to evaluate the expression (12) up to the first order in scalar curvature, while the terms with derivatives of curvature and higher powers of the curvature tensor can be disregarded. This means that we can effectively consider an approximation with constant R.

It is easy to note that the operators B_k and P_k commute. But for an inhomogeneous field and curvature the space-time-dependent Σ_k does not commute with B_k and P_k. But they commute in the constant curvature and constant ϕ approximation, the latter corresponds to the local potential approximation (LPA).

To simplify notations in what follows we skip the arguments of B_k, P_k, and Σ_k. Then the expansion of the r.h.s. of (12) into a power series in Σ_k gives

$$\begin{aligned}
\operatorname{Tr}\left(\frac{B_k}{P_k + \Sigma_k}\right) &= \operatorname{Tr}\left(\frac{B_k}{P_k(1 + P_k^{-1}\Sigma_k)}\right) \\
&= \operatorname{Tr}\left(B_k P_k^{-1}\frac{1}{1 + P_k^{-1}\Sigma_k}\right) \\
&= \operatorname{Tr}Q_{k,1} - \operatorname{Tr}\left(Q_{k,2}\Sigma_k\right) \\
&\quad + \operatorname{Tr}\left(Q_{k,2}\Sigma_k\frac{1}{P_k}\Sigma_k\right) + O(\Sigma_k^3), \quad (16)
\end{aligned}$$

where $Q_{k,m} = \frac{B_k}{P_k^m}$.

The first term on the r.h.s. is ϕ-independent and contributes only to the running in the vacuum sector Γ_k^{grav}. Until the cutoff action is specified, B_k and P_k are some unknown functions of $-\triangle_g$, which should be expanded to first order in the curvature tensor. To this end we shall apply the useful off-diagonal heat-kernel method, based on Laplace and Mellin transforms,

such that the operators in (16) can be derived from the heat kernel of the covariant Laplacian. The method is described in detail in [29–32] (see also [27]), so here we only sketch the main points of the derivation.

The functions $Q_{k,m}$ of the covariant Laplacian in the Neumann series (16) admit representations in terms of the inverse Laplace transform,

$$Q_{k,m}(-\triangle_g) = \int_0^\infty \mathrm{d}t\, \mathcal{L}^{-1}[Q_{k,m}](t)\, \mathrm{e}^{t\triangle_g}, \tag{17}$$

where

$$\mathcal{L}[f](s) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-st} f(t). \tag{18}$$

In what follows we shall apply the useful formula [29–32]

$$\begin{aligned}
&\frac{1}{(4\pi)^{d/2}}\int_0^\infty \mathrm{d}t\, t^{-p}\mathcal{L}^{-1}[f](t) \\
&= \frac{1}{(4\pi)^{d/2}\Gamma(p)}\int_0^\infty \mathrm{d}s\, s^{p-1}f(s). \tag{19}
\end{aligned}$$

To evaluate the effective action in the given truncation (discussed below) we need the coincidence limit of the matrix elements $\langle x|Q_{k,m}|x'\rangle$. According to (17) we may use the heat-kernel expansion for small t [33],

$$\langle x|\mathrm{e}^{t\triangle_g}|x\rangle = \frac{1}{(4\pi t)^{d/2}}\big[a_0(x) + ta_1(x) + t^2 a_2(x) + \cdots\big] \tag{20}$$

to find a series expansion of $\langle x|Q_{k,m}|x\rangle$ in invariant powers of curvatures and their covariant derivatives. The first Schwinger–DeWitt coefficients $a_0, a_1, a_2, \ldots$ have the form

$$a_0 = 1, \quad a_1 = \frac{1}{6}R,$$

$$a_2 = \frac{1}{180}\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - R_{\mu\nu}R^{\mu\nu} + 6\triangle_g R + \frac{5}{2}R^2\right). \tag{21}$$

In a given truncation scheme we need only a_0 and a_1, but it is not difficult to keep also the next terms. Using the regulator function [34]

$$r_k(s) = (k^2 - s)\theta(k^2 - s), \tag{22}$$

one can arrive at the explicit expression

$$Q_{k,m}(s) = \frac{2k^2 - (k^2 - s)\eta_k}{M_k^m}\theta(k^2 - s),$$

$$M_k = k^2 + \frac{m_k^2}{Z_k}. \tag{23}$$

Let us note that the general review of the functions Q_n similar to the ones defined above can be found in Appendix A of the review paper on quantum gravity of Ref. [35].

Inserting the expansion (20) into (17), and using (19) we obtain

$$
\langle x | Q_{k,m}(-\triangle_g) | x \rangle
$$
$$
= \sum_{n=0}^{\infty} \int_0^{\infty} dt \, \mathcal{L}^{-1}[Q_{k,m}](t) \, \frac{1}{(4\pi t)^{d/2}} \, a_n(x) t^n
$$
$$
= \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \frac{a_n(x)}{\Gamma(d/2-n)} \int_0^{\infty} ds \, Q_{k,m}(s) \, s^{d/2-n-1},
$$
(24)

where the identification $p = d/2 - n$ has been used in Eq. (19) already.

In order to evaluate the integrals over s in Eq. (24), one can note that $Q_{k,m}$ in (19) are nonzero only on the interval $[0, k^2]$, which gives

$$
\int_0^{\infty} ds \, Q_{k,m}(s) s^{d/2-n-1}
$$
$$
= \left(1 - \frac{\eta_k}{d-2n+2}\right) \frac{1}{M_k^m} \frac{2k^{d-2n+2}}{d/2-n}. \quad (25)
$$

After integration we arrive at the result

$$
\langle x | Q_{k,m}(-\triangle_g) | x \rangle = \frac{2}{(4\pi)^{d/2}} \frac{1}{M_k^m}
$$
$$
\times \sum_n \left(1 - \frac{\eta_k}{d-2n+2}\right) \frac{a_n(x) \, k^{d-2n+2}}{\Gamma(d/2-n+1)}. \quad (26)
$$

As we have already mentioned, for our purposes it is sufficient to consider the $n = 0, 1$ terms in the last series. Expanding the r.h.s. of the flow equation (16) in powers of ϕ and the curvature up to the first order, in four dimensions we meet

$$
\frac{1}{2} \text{Tr} \, Q_{k,1}(-\triangle_g)
$$
$$
= \frac{1}{32\pi^2 M_k} \int_x \left[k^6 \left(1 - \frac{\eta_k}{6}\right) + \frac{k^4}{3}\left(1 - \frac{\eta_k}{4}\right) R + \dots \right].
$$
(27)

It is easy to see that these terms contribute only to the purely gravitational terms and hence are irrelevant for the running of ξ.

The second-order contribution is

$$
\frac{1}{2} \text{Tr} \left[Q_{k,2}(-\triangle_g) \Sigma_k \right] = \frac{1}{32\pi^2 M_k^2} \int_x \left[k^6 \left(1 - \frac{\eta_k}{6}\right) \Sigma_k \right.
$$
$$
\left. + \frac{k^4}{3}\left(1 - \frac{\eta_k}{4}\right) R \Sigma_k + \dots \right], \quad (28)
$$

where we again disregarded higher powers of the curvature.

The derivation of the third term in the expansion (16) requires some commutations of Σ_k with P_k^{-1}, e.g.,

$$
\frac{1}{2} \text{Tr} \left(Q_{k,2} \Sigma_k \frac{1}{P_k} \Sigma_k \right)
$$
$$
= \frac{1}{2} \text{Tr} \left(Q_{k,2} \Sigma_k \left(\Sigma_k - \frac{1}{P_k}[P_k, \Sigma_k] \right) \frac{1}{P_k} \right)
$$
$$
= \frac{1}{2} \text{Tr} \left(Q_{k,3} \Sigma_k^2 \right) - \frac{1}{2} \text{Tr} \left(Q_{k,3} \Sigma_k \frac{1}{P_k}[P_k, \Sigma_k] \right). \quad (29)
$$

The last term containing the commutator of P_k and Σ_k gives rise to a running of the wave-function renormalization and will be dealt with in Sect. 4. It does not contribute to the running of u_k and ξ_k and thus can be neglected for the moment being. Thus we arrive at

$$
\frac{1}{2} \text{Tr} \left(Q_{k,3}(-\triangle_g) \Sigma_k^2 \right)
$$
$$
= \frac{1}{16\pi^2} \frac{1}{M_k^3} \sum_n \frac{k^{6-2n}}{\Gamma(3-n)} \left(1 - \frac{\eta_k}{6-2n}\right) \text{tr} \, (a_n \Sigma_k^2)
$$
$$
= \frac{1}{32\pi^2 M_k^3} \int_x \left[k^6 \left(1 - \frac{\eta_k}{6}\right) \Sigma_k^2 \right.
$$
$$
\left. + \frac{k^4}{3}\left(1 - \frac{\eta_k}{4}\right) R \, \Sigma_k^2 + \dots \right]. \quad (30)
$$

In the last expression we omitted most purely gravitational contributions (not all, since for example $R\Sigma_k$ contains still a term $\propto R^2$) and terms beyond the truncation scheme (2). Similarly, one obtains for constant field and curvature

$$
\frac{1}{2} \text{Tr} \left(Q_{k,2} \Sigma_k \left(P_k^{-1} \Sigma_k \right)^{m-1} \right) = \frac{1}{2} \text{Tr} \left(Q_{k,m+1} \Sigma_k^m \right)
$$
$$
= \frac{1}{32\pi^2 M_k^{1+m}} \int_x \left[k^6 \left(1 - \frac{\eta_k}{6}\right) \Sigma_k^m \right. \quad (31)
$$
$$
\left. + \frac{k^4}{3}\left(1 - \frac{\eta_k}{4}\right) R \, \Sigma_k^m + \dots \right]. \quad (32)
$$

In what follows we ignore the purely gravitational contribution $\Gamma_k^{\text{grav}}[g]$ in Eq. (6) and related RG flows. We expect to consider this subject in a separate article.

2.2 RG flow for couplings and non-minimal parameter

Inserting (27)–(31) into (16) and using the definition (15) yields

$$
\frac{1}{2} \text{Tr} \left(\frac{\partial_t R_k}{\Gamma_k^{(2)}[\phi] + R_k} \right)
$$
$$
= \frac{1}{32\pi^2 M_k} \int_x \left[k^6 \left(1 - \frac{\eta_k}{6}\right) + \frac{k^4}{3}\left(1 - \frac{\eta_k}{4}\right) R \right]
$$

$$\times \left[1 - \frac{\xi_k R + w_k''}{Z_k M_k} + \left(\frac{\xi_k R + w_k''}{Z_k M_k} \right)^2 \right.$$

$$\left. - \left(\frac{\xi_k R + w_k''}{Z_k M_k} \right)^3 + \cdots \right]$$

$$- \frac{1}{2Z_k^2} \, \mathrm{Tr} \left(Q_{k,3} w_k''(\phi) \frac{1}{P_k} \big[P_k, w_k''(\phi) \big] \right) + \cdots, \quad (33)$$

where terms containing R^2 and R^3 go beyond the truncation scheme and must be omitted.

One can assume that the classical potential for the scalar field at the cutoff is an even function. It follows from the flow equation (33) that the scale dependent effective potential remains even at all scales. Let us further assume that the $\mathbb{Z}_2$ symmetry is not spontaneously broken. Then the minimum of the effective potential is at $\phi = 0$ and we may expand w_k in (10) as

$$w_k(\phi) = \sum_{n=2}^{\infty} \frac{1}{(2n)!} \lambda_{(2n)k} \phi^{2n} = \frac{1}{24} \lambda_{4k} \phi^4 + \frac{1}{720} \lambda_{6k} \phi^6 + \cdots, \quad (34)$$

where $\lambda_{(2n)k}$ are scale-dependent coefficients. Their running is determined by the FRG equations, which will be now derived, along with the one for ξ_k.

For w_k in (34) the last term in (33) has the form $\phi^2 (\partial^2 \phi^2 + \ldots)$ and is beyond the truncation (2). As a result the wave function is not renormalized. Note that for a non-even cut-off potential with a ϕ^3 term, or for the flow in the broken phase, in which the even potential u_k is expanded about a nonzero mean field, there is a wave-function renormalization, as we shall see in Sect. 4. One may note that this situation does not depend on the presence of the curved background and can be already observed in scalar field theories in flat space-time [17–21].

One can compare the ϕ-dependent terms in the expression (33) with Eqs. (6) and (34). In this way we arrive at the truncated flow equation for the scalar field theory non-minimally coupled to gravity,

$$- \frac{\eta_k Z_k}{2} \phi \triangle_g \phi + \frac{\partial_t \xi_k}{2} R \phi^2 + \frac{\phi^2}{2} \partial_t m_k^2$$

$$+ \frac{\phi^4}{24} \partial_t \lambda_{4k} + \frac{\phi^6}{720} \partial_t \lambda_{6k} = - \frac{1}{M_k^2} \frac{1}{32\pi^2 Z_k} \left[k^6 \left(1 - \frac{\eta_k}{6} \right) \right.$$

$$\times \left(\frac{1}{2} \lambda_{4k} \phi^2 + \frac{1}{24} \lambda_{6k} \phi^4 \right) + \frac{k^4}{3} \left(1 - \frac{\eta_k}{4} \right) R \lambda_{4k} \frac{\phi^2}{2} \right]$$

$$+ \frac{1}{M_k^3} \frac{1}{32\pi^2 Z_k^2} \left\{ k^6 \left(1 - \frac{\eta_k}{6} \right) \left[\xi_k R \lambda_{4k} \phi^2 + \frac{1}{4} (\lambda_{4k})^2 \phi^4 \right. \right.$$

$$\left. \left. + \frac{1}{24} \lambda_{4k} \lambda_{6k} \phi^6 \right] \right\} - \frac{1}{M_k^4} \frac{1}{32\pi^2 Z_k^3} \left\{ k^6 \left(1 - \frac{\eta_k}{6} \right) \frac{\lambda_{4k}^3}{8} \phi^6 \right\}$$

$$+ \cdots. \quad (35)$$

Now, we can compare coefficients on both sides of this equation:

– For the kinetic term $\phi \triangle_g \phi$ we observe that $Z_k = \text{constant}$ or $\eta_k = 0$, hence there is no wave-function renormalization for an even potential in the symmetric phase. One can fix then $Z_k = Z_\Lambda = 1$ at all scales k. Then, in particular, $M_k = k^2 + m_k^2$, and we denote $D_k = (k^2 + m_k^2)^{-1}$ for the sake of convenience.

– For the mass term the RG equation can easily be obtained from (35),

$$\partial_t m_k^2 = - \frac{1}{32\pi^2} k^6 D_k^2 \lambda_{4k}. \quad (36)$$

– For the first two interaction terms we have, with $Z_k \equiv 1$,

$$\partial_t \lambda_{4k} = - \frac{k^6}{32\pi^2} D_k^2 \left(\lambda_{6k} - 6 D_k \lambda_{4k}^2 \right), \quad (37)$$

$$\partial_t \lambda_{6k} = - \frac{k^6 D_k^2}{32\pi^2} \left(\lambda_{8k} - 30 D_k \lambda_{4k} \lambda_{6k} + 90 D_k^2 \lambda_{4k}^3 \right). \quad (38)$$

In order to keep our consideration simple, we shall truncate the Taylor expansion for the potential by disregarding all coefficients starting from $\lambda_{8,k}$, including setting $\lambda_{8,k} = 0$ in Eq. (38).

– Finally, the non-minimal term $\phi^2 R$ yields, for $Z_k \equiv 1$,

$$\partial_t \xi_k = \frac{k^6 D_k^2}{16\pi^2} \left(D_k \xi_k - \frac{1}{6k^2} \right) \lambda_{4k}. \quad (39)$$

In a perfect agreement with the general features of the perturbative RG in a curved space-time (as described in the Introduction), the equations for the couplings λ_{4k} and λ_{6k} do not depend on ξ_k, while the FRG equation for ξ_k depends on the couplings. At the same time, different from the minimal-subtraction based RG, here the β-functions for both couplings and non-minimal parameter do depend on the running mass of field m_k. In this respect the FRG equations resemble the physical, momentum-subtraction-based RG, developed for the interacting scalar field in a curved space-time in [36], but the mass dependence in the present FRG equations is much stronger.

3 FRG flow for couplings, mass and ξ

The RG equations (36)–(39) should be explored numerically. To this end we introduce the *dimensionless* quantities

$$m_t = \frac{m_k}{k}, \quad D_t = \frac{1}{1 + m_t^2},$$

$$\lambda_{4t} = \lambda_{4k}, \quad \lambda_{6t} = k^2 \lambda_{6k}, \quad \lambda_{8t} = k^4 \lambda_{8k}, \ldots \quad (40)$$
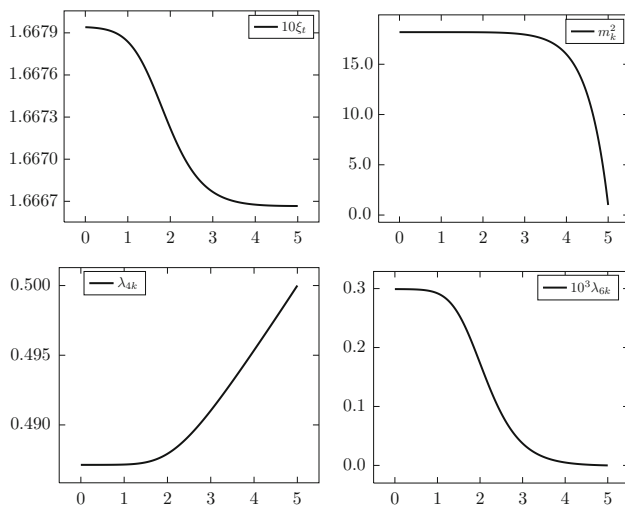
Fig. 1 Flow of the non-minimal parameter ξ_t and couplings m_k^2, λ_{4k}, λ_{6k} (in units of μ) with $t = \log(k/\mu)$ in the unbroken phase. The initial data at the cutoff $\Lambda = \mu e^5$ are $m^2 = \mu^2$, $\lambda_4 = 0.5$, $\lambda_6 = 0$, and $\xi = 1/6$

Fig. 2 Running of the non-minimal parameter ξ_t with $t = \log(k/\mu)$ for the same initial couplings as in Fig. 1, but with different initial values ξ in the vicinity of the conformal coupling $\xi_c = 1/6$. *Clockwise from top* $\xi_c - 2\delta$, $\xi_c - \delta$, $\xi_c + \delta$, and $\xi_c + 2\delta$ with $\delta = 1/24$

which are supposed to depend on the dimensionless parameter $t = \log(k/\mu)$, defined in (4). Then the equations become

$$\partial_t m_t^2 = -2m_t^2 - \frac{1}{32\pi^2} D_t^2 \lambda_{4t}, \tag{41}$$

$$\partial_t \lambda_{4t} = -\frac{1}{32\pi^2} D_t^2 \left(\lambda_{6t} - 6D_t \lambda_{4t}^2 \right), \tag{42}$$

$$\partial_t \lambda_{6t} = 2\lambda_{6t} - \frac{1}{32\pi^2} D_t^2 \left(\lambda_{8t} - 30 D_t \lambda_{4t} \lambda_{6t} + 90 D_t^2 \lambda_{4t}^3 \right). \tag{43}$$

Furthermore, the FRG equation for $\xi(t)$ in terms of the new variable has the form

$$\partial_t \xi_t = \frac{1}{16\pi^2} D_t^3 \left(\xi_t - \frac{1}{6D_t} \right) \lambda_{4t}. \tag{44}$$

It is easy to see that in the massless limit $D_t \to 1$ this equation reproduces the main features of the one-loop RG equation in the minimal-subtraction scheme, as is well known from [1,3].

The numerical analysis of these equations shows that the RG flow can be pretty different from the one for the perturbative one-loop RG running, mainly due to the mass dependence. As one can see from the plots presented at Fig. 1, the (dimensionful) mass grows quickly when one flows from the cutoff scale in the UV at $t = 5$ (corresponding to a cutoff value $\Lambda = e^5\mu$) to the IR at $t = 0$, which corresponds to $k = \mu$. As we have already noted above, in the non-perturbative FRG approach there is a sixth-power IR decoupling, which is very strong compared to the usual quadratic decoupling in the perturbative Appelquist and Carazzone theorem [37]. As a result, in the case under discussion one can observe that the running for all couplings and in particular ξ_t actually freezes at values $t \lesssim 1$ or equivalently at scales $k \lesssim \mu \cdot e$.

Figure 2 shows the running of the non-minimal parameter ξ_t for the same initial potential as in Fig. 1, but for varying initial ξ-values of at the UV-cutoff. Depending on the initial value the parameter may increase or decrease during the flow toward the infrared. But, in all considered cases, the values in the UV and IR are not very different. In the flows investigated (only some are displayed in the figure) the relative change was only about one percent.

4 Broken symmetry and FRG flow for anomalous dimension

As we already know, in the LPA truncation (2) there is no RG running for the anomalous dimension η_k for even potentials. At the same time, such running is present in scalar theory beyond one loop and it would be interesting to observe it within the FRG approach. One of the possibilities is related to theories with broken symmetries. This means we shall introduce the negative mass-squared in the classical action (1) and implement this information into the effective average action (2) by imposing the corresponding boundary condition at the cutoff scale $k = \Lambda$. Then the effective potential of the scalar field is not convex at intermediate scales and this must be taken into account, for in this case one has to consider oscillations near the non-symmetric minima of this potential.

In curved space the spontaneous symmetry breaking meets serious complications, because the position of such a minimum is not constant for a non-constant curvature. The situation was explored in detail in [26] and it was shown that the non-localities emerge in such a theory even at low orders in curvature. However, since our intention here is to consider relatively simple cases, let us consider the

zeroth-order approximation and, correspondingly, assume that the position of the minimum of the potential is homogeneous and curvature-independent, denoted by $\phi_{0,k}$, such that $u_k''(\phi_{0,k}) = m_k^2 \geq 0$ is the (physical) mass in the broken phase. Then one has to expand the effective potential as

$$u_k = \lambda_{0k} + \sum_{n \geq 2} \frac{\lambda_{nk}}{n!} (\phi - \phi_{0k})^n, \tag{45}$$

with small $\phi - \phi_{0k}$ and scale-dependent minimum ϕ_{0k}. Then

$$
\begin{aligned}
u_k'' &= \lambda_{2,k} + \sum_{n \geq 3} \frac{\lambda_{n,k}}{(n-2)!} (\phi - \phi_{0k})^{n-2} \\
&\equiv m_k^2 + w_k''.
\end{aligned} \tag{46}
$$

One can easily note that these definitions of m_k^2 and w_t are different from the previous ones, because the expansion is performed in the spontaneously broken phase. However, in the new notations Eqs. (13) and (15) have almost the same form as before in the old notations. But in the broken phase odd powers of $\phi - \phi_{0k}$ appear such that many more terms arise in the power series expansion of the *rhs* of the flow equation. When one calculates the *lhs* of the flow equation one must take into account that the minimum ϕ_{0k} of the scale dependent potential flows.

Thus, we continue by inserting the expansions (45) and (46) into the FRG equation with scale-dependent wave-function renormalization, in pretty much the same way as we did before. Finally, changing over to dimensionless quantities (40) and to dimensionless fields according to

$$\chi = k^{-1} Z_k^{1/2} \phi, \quad \chi_{0k} = k^{-1} Z_k^{1/2} \phi_{0k} \tag{47}$$

one can arrive at the FRG equations in the broken phase. To simplify the notation we use the following abbreviations:

$$A_t = \frac{1}{32\pi^2} \left(1 - \frac{\eta_t}{6}\right), \quad G_n = \frac{\lambda_n}{1 + m_t^2}. \tag{48}$$

The running of the (cosmological) constant λ_{0t} is given by

$$\partial_t \lambda_{0t} + 4\lambda_{0t} = A_t D_t, \tag{49}$$

and since it does not feed back into the running of the remaining couplings it will be discarded. Comparing terms linear in $\chi - \chi_{0t}$ yields the running of the mean field,

$$\partial_t \chi_{0t} + \left(1 + \frac{\eta_t}{2}\right) \chi_{0t} = A_t D_t \frac{G_3}{m_t^2}. \tag{50}$$

This flow equation ensures that χ_{0t} remains a minimum of the scale dependent potential at all scales. In writing the flow equations for the dimensionless couplings $m_t^2, \lambda_{3t}, \ldots, \lambda_{6t}$ in an expansion up to order 6 we use the flow equation (50) to simplify the resulting expressions. This way one arrives at

$$(\partial_t + 2 - \eta_t) m_t^2 = A_t D_t \left(\frac{\lambda_{3t}}{m_t^2} G_3 - G_4 + 2G_3^2\right), \tag{51}$$

$$
\begin{aligned}
&\left(\partial_t + 1 - \frac{3}{2}\eta_t\right) \lambda_{3t} \\
&= A_t D_t \left(\frac{\lambda_{3t}}{m_t^2} G_4 - G_5 + 6G_3 G_4 - 6G_3^3\right),
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
(\partial_t - 2\eta_t) \lambda_{4t} = A_t D_t \Big(&\frac{\lambda_{3t}}{m_t^2} G_5 - G_6 + 6G_4^2 \\
&+ 8G_3 G_5 - 36G_3^2 G_4 + 24G_3^4\Big),
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
\left(\partial_t - 1 - \frac{5}{2}\eta_t\right) \lambda_{5t} = A_t D_t \Big(&\frac{\lambda_{3t}}{m_t^2} G_6 + 20G_4 G_5 + 10G_3 G_6 \\
&- 90G_3 G_4^2 - 60G_3^2 G_5 + 240G_3^3 G_4 - 120G_3^5\Big),
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
(\partial_t - 2 - 3\eta_t) \lambda_{6t} = A_t D_t \Big(&20G_5^2 + 30G_4 G_6 - 90G_4^3 \\
&- 360G_3 G_4 G_5 - 90G_3^2 G_6 + 1080G_3^2 G_4^2 + 480G_3^3 G_5 \\
&- 1800G_3^4 G_4 + 720G_3^6\Big).
\end{aligned} \tag{55}
$$

In the two last flow equations the terms containing λ_{7t} and λ_{8t} are omitted in the sixth-order polynomial approximation.

On top of that, in curved space one meets the new equation for the non-minimal parameter,

$$
\begin{aligned}
(\partial_t - \eta_t) \xi_t = \frac{D_t}{16\pi^2} \Big[&\left(1 - \frac{\eta_t}{6}\right) \left(G_4 - 3G_3^2\right) D_t \xi_t \\
&- \frac{1}{6} \left(1 - \frac{\eta_t}{4}\right) \left(G_4 - 2G_3^2\right)\Big].
\end{aligned} \tag{56}
$$

In order to explore the system of equations (51)–(56) we need an additional equation which defines the scale dependence of η_t. In order to obtain this dependence – in our truncation it is induced by the last term in (29) or equivalently the last term in (33) – one has to remember that the running of all couplings and of Z_k (which defines η_t) does not depend on the presence of the curved space background. As a result we can use the well-known flat-space result for η_t derived in [21, 38] and recently explored in [39]. In terms of dimensionless quantities the result for the anomalous dimension reads

$$\eta_t = \frac{1}{32\pi^2} \frac{\left[u_t'''(\chi_{0t})\right]^2}{\left[1 + u_t''(\chi_{0t})\right]^4} = \frac{1}{32\pi^2} \frac{\lambda_{3t}^2}{(1 + m_t^2)^4}, \tag{57}$$

where χ_{0t} is the scale-dependent position of the minimum of the potential. Clearly, in the truncation scheme considered, the renormalization of the wave function only happens in the broken phase with nonzero coefficient λ_{3t}. It follows from (50) that symmetry can only be broken if λ_{3t} in the UV is nonzero, and this must be taken into account for by a numerical analysis of the system of equations (51)–(56).

The numerical results of such an analysis are presented at Figs. 3 and 4. The first plot of Fig. 3 shows the scale dependence of the minimizing value of the field ϕ_{0k} in the
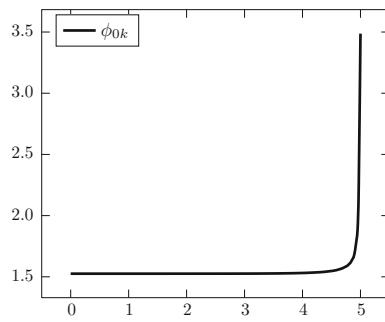
Fig. 3 Flow of the (dimensionful) minimum of the effective potential as a function of $t = \log(t/\mu)$. For $\phi_{0\Lambda} \approx 3.5\mu$ (and the same initial parameters as in Fig. 4) the system remains in the broken phase for all scales k
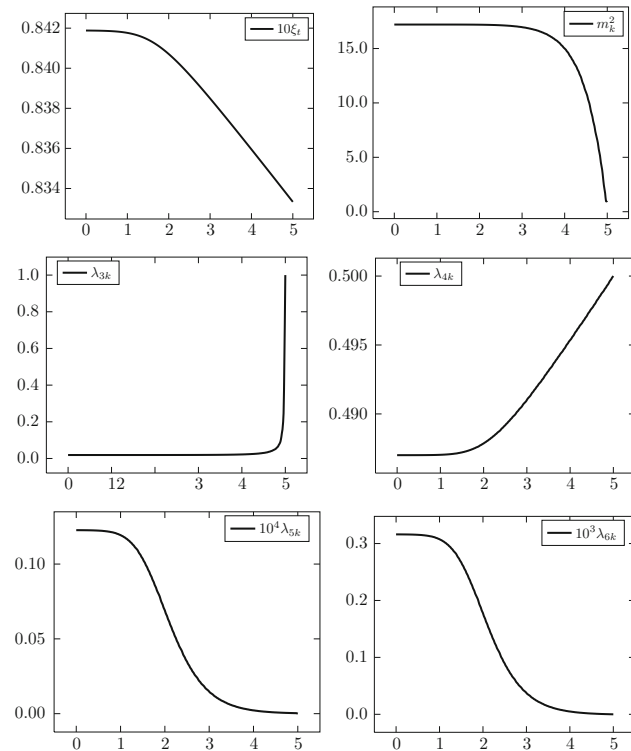
Fig. 4 Running of the non-minimal parameter ξ_t and couplings $m_k^2, \lambda_{3k}, \ldots, \lambda_{6k}$ (in units of μ) as functions of $t = \log(k/\mu)$. The initial data at the cutoff scale $k = \Lambda = \mu e^5$ are $m^2 = \mu^2$, $\lambda_4 = 0.5$, $\lambda_3 = \mu$, $\lambda_5 = \lambda_6 = 0$, and $\xi = 1/6$

broken phase. As expected, the minimum of the potential are driven closer to the origin by the quantum fluctuations. For the chosen initial parameters at the cutoff $\Lambda = \mu e^5$

$$\phi_0 \approx 3.5\mu, \quad m^2 = \mu^2, \quad \lambda_3 = \mu,$$
$$\lambda_4 = 0.5, \quad \lambda_5 = \lambda_6 = 0. \tag{58}$$

the system stays in the broken phase for all scales. This can also be seen from the running of the couplings shown in Fig. 4. The coupling λ_{3k} decreases rapidly when one moves from the UV to the IR. At the same time the higher couplings

λ_{5k} and λ_{6k} acquire nonzero values, although they remain small in the IR.

We solved the flow equations with vanishing η_t. Of course, one should choose the anomalous dimension self-consistently. But since at the scale $k = \mu$ we have $\lambda_3 \approx 0.0189$ and $m^2 \approx 15.2$, the first guess for the anomalous dimension

$$\eta_{k=\mu} \approx \frac{1}{32\pi^2} \frac{\lambda_{3\mu}^2}{(1 + m_\mu^2)^4} \approx 1.6 \times 10^{-11} \tag{59}$$

yields a tiny value in the infrared. Thus, we assume that $\eta_t = 0$ is a very good approximation.

In order to induce more dramatic qualitative changes in the flow of the non-minimal parameter, we considered other sets of initial parameters in the UV. For initial parameters which can be integrated to the infrared we did not observe a strong running of ξ. Thus we conclude that the qualitative form of the flow of ξ_t is not very sensitive to the initial parameters.

5 Conclusions

We have constructed and explored the FRG equations for a real scalar field in a curved space-time background, including the non-minimal parameter ξ of the non-minimal interaction between scalar field and curvature. The β-functions obtained within the very simple truncation scheme (2) reproduce several important features of the standard perturbative renormalization group, including what one can prove to be the non-perturbative universal properties of the RG flows. First of all, the FRG equations follow the well-known hierarchy of renormalization in curved space, as described by the points (i)–(iii) in the Introduction. Furthermore, the FRG trajectory for ξ corresponds to the equation which is linear in ξ, exactly as it should be at both perturbative and non-perturbative levels. In the massless case, the β-function for ξ has the conformal fixed point $\xi = 1/6$, which is typical for the one-loop case [1]. In our opinion, this may be the result of the restricted form of the truncation (2), because higher-loop corrections involve powers of $\log(\phi)$ in both potential and kinetic sectors, which are beyond the given approximation. It would be interesting to explore the inclusion of such terms in an FRG analysis; we expect to do this in a future work.

The most remarkable aspect of the RG flow for a massive theory is a strong sixth-power decoupling in the IR. As a result the running of all couplings and ξ actually stops very soon on the way from the cutoff scale in the UV down to the IR. We conclude that the desirable strong running of ξ cannot be achieved in the framework of scalar theory. At the same time, there are chances to achieve such an effect in the mixed theory with different mass scales, especially through the quantum effects of relatively light or massless particles.

An interesting extension of the RG flow in the theory with the non-minimal parameter is related to the broken phase, when one can also observe the wave-function renormalization and its effect on the RG trajectories for couplings and ξ. We have found that, regardless of the more complicated form of the RG flow, the qualitative form of the flow for ξ remains the same, in the sense that the numerical effect of the scale dependence is quite small in the scalar theory. One of the consequences is that the scale dependence in the non-local part of the induced action of gravity in the theory with spontaneous symmetry breaking will be also small. However, as discussed in [2], any form of scale dependence may have a significant impact on the induced cosmological constant term and especially on its non-local extensions. Therefore, the problem of the running of ξ from UV to IR deserves further detailed studies, especially in more general theories, which involve several mass scales.

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